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is not finitely generated**

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# HOMOLOGICAL EQUIVALENCE, MODULO ALGEBRAIC EQUIVALENCE, IS NOT FINITELY GENERATED

by HERBERT CLEMENS

## 0. Introduction

For a three-dimensional complex projective manifold  $V$ , let

$\mathcal{G}_h(V)$  = (group of algebraic one-cycles on  $V$  which are homologous to zero modulo those which are rationally equivalent to zero)

$\mathcal{G}_a(V)$  = (subgroup of  $\mathcal{G}_h(V)$  generated by cycles algebraically equivalent to zero).

In general there is a canonically defined Abel-Jacobi mapping

$$\Phi: \mathcal{G}_h(V) \rightarrow J(V) = \frac{(H^{3,0}(V) + H^{2,1}(V))^*}{H_3(V; \mathbf{Z})}$$

$$\sum n_i Z_i \mapsto \int_{\Gamma}$$

where  $\partial\Gamma = \sum n_i Z_i$ .

In [4], P. Griffiths proved that, if  $V$  is a generic quintic hypersurface in  $\mathbf{P}^4$ , then

(0.1)  $\Phi(\mathcal{G}_a(V)) = 0.$

Also he showed that there exist smooth quintics for which (0.1) holds but

$$\Phi(\mathcal{G}_h(V)) \neq 0.$$

The purpose of this paper is to prove the following:

*Theorem (0.2).* — *If  $V$  is a generic quintic threefold, then the vector space*

$$\Phi(\mathcal{G}_h(V)) \otimes_{\mathbf{Z}} \mathbf{Q}$$

*is not finite dimensional.*

Given Griffiths results about  $\Phi(\mathcal{G}_a(V))$  being generically zero, we obtain:

*Corollary (0.3).* — *If  $V$  is a generic quintic threefold, the abelian group*

$$\mathcal{G}_h(V)/\mathcal{G}_a(V)$$

*is not finitely generated.*

We should include a word of warning about the use of the term "generic". It should be taken to mean that there is a countable union of proper closed subvarieties of the set of all quintic threefolds such that the results hold for all  $V$  in the complement of that countable union.

There are three steps in the proof of (0.2). First we construct an infinite sequence of rational curves on a generic quintic  $V$  such that each curve has negative normal bundle. Then we deform to quintics with nodes in such a way that exactly one of the rational curves of the sequence comes to pass through a node. Finally we use the asymptotic properties of the Abel-Jacobi mapping developed in [1] and [2] to see that the image of the rational curve which comes to pass through the node "goes to infinity", while the images of all the other curves of the sequence "stay finite". By making this asymptotic construction, in turn, for different curves of the sequence, we will finally conclude the theorem.

### 1. Rational curves on quintic threefolds

Our initial purpose in this section will be to establish that, if

$$V \subseteq \mathbf{P}^4$$

is a generically chosen quintic threefold, then there exist infinitely many smooth rational curves  $C$  on  $V$  with normal bundle

$$\mathcal{N}_{C,V} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1).$$

We begin by constructing a smooth quartic surface

$$(1.1) \quad Q \subseteq \mathbf{P}^3$$

with the following properties:

- (1.2)      i)  $Q$  contains three lines  $L_0$ ,  $L_1$  and  $L_\infty$  such that  $L_0$  and  $L_1$  meet in a point  $x_0$  and  $L_\infty$  does not meet  $L_0$  or  $L_1$ .  
 ii) The plane through  $L_\infty$  and  $x_0$  meets  $Q$  residually in a non-singular cubic curve.

We construct  $Q$  by first defining

$$L_\infty \text{ given by } X_0 = X_3 = 0$$

$$L_0 \cup L_1 \text{ given by } X_1 = X_2^2 - X_3^2 = 0.$$

We then consider the fourth degree equation in

$$X = (X_0, X_1, X_2, X_3)$$

given by

$$(1.3) \quad 0 = X_0(X_1q(X) + (X_2^2 - X_3^2)\ell(X)) + X_3(X_1q'(X) + (X_2^2 - X_3^2)\ell'(X)),$$

where  $q$  and  $q'$  are homogeneous of degree 2, and  $\ell$  and  $\ell'$  are homogeneous of degree one. If these forms are chosen generically, Bertini's theorem allows the conclusion that the solution set to (1.3) can be singular only at points lying in the singular sets of each of the four hypersurfaces

$$X_0X_1 = 0, \quad X_0(X_2^2 - X_3^2) = 0, \quad X_3X_1 = 0, \quad X_3(X_2^2 - X_3^2) = 0.$$

Thus for generic  $q, q', \ell, \ell'$ , the solution set for (1.3) is a non-singular quartic  $Q$ . It is immediate to check that the remaining properties in (1.2) are satisfied.

*Lemma (1.4).* — *For  $Q$  as in (1.1)-(1.2),  $Q$  contains an infinite sequence of smooth rational curves of strictly increasing degrees.*

*Proof.* — Let

$$L_\infty + C_t, \quad t \in \mathbf{P}^1,$$

be the pencil of hyperplane sections of  $Q$  cut out by the planes containing  $L_\infty$ . Since  $Q$  is smooth, the linear system

$$L_\infty \cdot C_t, \quad t \in \mathbf{P}^1,$$

on  $L_\infty$  is without basepoints. So we have a well-defined morphism

$$(1.5) \quad Q \rightarrow \mathbf{P}^1$$

with fibre  $C_t$ . The fibre of (1.5) containing

$$x_0 = L_0 \cap L_1$$

is a smooth cubic by construction, so the generic fibre of (1.5) is a smooth cubic. We give these cubics a group structure by letting  $L_0$  define the "zero section" of (1.5). Then  $L_1$  defines another section of (1.5) which is the zero element of the fibre for exactly one value of  $t \in \mathbf{P}^1$ . So for generic  $t$ , the point  $L_1 \cap C_t$  is not a point of finite order on  $C_t$ . Then it is elementary to see that the sections of (1.5) given on smooth fibres  $C_t$  by  $n(L_1 \cdot C_t)$  for  $n \in \mathbf{Z}$  extend over the singular fibres to give everywhere defined sections

$$(1.6) \quad L_n \subseteq Q.$$

Furthermore all the  $L_n$  are distinct since there exist fibres  $C_t$  for which  $(L_1 \cdot C_t)$  is not a point of finite order. All  $L_n$  are smooth and, by the adjunction formula, the normal bundle of  $L_n$  in  $Q$  is given by

$$(1.7) \quad \mathcal{N}_{L_n, Q} \cong \mathcal{O}_{L_n}(-2).$$

Since the curves (1.6) form an infinite set of rigid curves in  $Q$ , their degrees cannot be bounded and so the lemma is proved.

Next let  $K \subseteq \mathbf{P}^3$  be any plane, and consider the set of all quintic threefolds in  $\mathbf{P}^4$  having

$$(1.8) \quad Q + K$$

as the hyperplane section  $X_4 = 0$ . Again, using Bertini's theorem, the singular set of a generic quintic containing (1.8) must lie inside  $Q \cdot K$ . Suppose  $q(X) = 0$  defines  $Q$  in the hyperplane  $X_4 = 0$ , and  $\ell(X) = 0$  defines  $K$ , and  $Q$  and  $K$  meet transversely. It is easy to compute that, if  $H$  denotes the hyperplane given by  $X_4 = 0$ , and we assume that  $Q$ ,  $K$ , and the trace of  $f = 0$  on  $H$  meet transversely at a point, then the quintic hypersurface  $V$  given by

$$(1.9) \quad \ell(X)q(X) + X_4 f(X) = 0$$

has a node at the point. So for a generic quartic form  $f$ , the sixteen points where

$$(1.10) \quad \ell(X) = q(X) = X_4 = f(X) = 0$$

are ordinary nodes. An easy application of Bertini's theorem shows there are no other singularities for generic  $f$ .

Next consider the pencil of hyperplane sections

$$(1.11) \quad a\ell(X) - bX_4 = 0 \quad (a, b) \in \mathbf{P}^1$$

of  $V$ , *i.e.* the pencil with base locus  $K$ . If we denote the hyperplane (1.11) by  $H_{(a,b)}$ , then

$$(1.12) \quad H_{(a,b)} \cdot V = K + Q_{(a,b)},$$

where  $Q_{(a,b)}$  is given by the equation

$$(1.13) \quad 0 = b^5 q(X_0, \dots, X_3, 0) + b^4 a f(X_0, \dots, X_3, 0) + b^3 a^2 (\dots) + \dots$$

in  $\mathbf{P}^3$ . Our original quartic  $Q$  occurs as  $Q_{(0,1)}$ , and the rational family of quartic surfaces  $\{Q_{(a,b)}\}$ , considered as a curve in the projective space of all quartics in  $\mathbf{P}^3$ , can be made to assume any direction at  $Q$  by adjusting the choice of  $f(X)$ .

Now pick one of our rational curves  $L_n$  from (1.6). By adjusting the choice of  $K$  and the quartic form  $f$  used in building the quintic  $V$  containing  $L_n$ , we can insure that no node (1.10) of  $V$  lies on  $L_n$ . We next wish to concern ourselves with the computation of the normal bundle  $\mathcal{N}_{L_n, V}$  to  $L_n$  in  $V$ . Since  $L_n^2 = -2$  in  $Q$ , we have an exact sequence

$$(1.14) \quad 0 \rightarrow \mathcal{O}_{L_n}(-2) \rightarrow \mathcal{N}_{L_n, V} \rightarrow \mathcal{O}_{L_n} \rightarrow 0.$$

*Lemma (1.15).* — *For generic choice of  $f(X)$  in (1.9) the sequence (1.14) is not split. So, for generic  $V$  as above,*

$$\mathcal{N}_{L_n, V} \cong \mathcal{O}_{L_n}(-1) \oplus \mathcal{O}_{L_n}(-1).$$

*Proof.* — We resolve the singularities of  $V$  by forming

$$\hat{V} = \{((a, b), x) \in \mathbf{P}^1 \times V : x \in Q_{(a,b)}\}.$$

Then the normal bundle to  $Q$  in  $\hat{V}$  is  $\mathcal{O}_Q$ . By (1.14), in order to prove the lemma it suffices to show that, for appropriate choice of  $f(X)$  in (1.9), the bundle  $\mathcal{N}_{L_n, \hat{V}}$  has no non-zero section. But any section would give a section of  $\mathcal{N}_{Q, \hat{V}}|_{L_n}$  which extends to a non-zero section of  $\mathcal{N}_{Q, \hat{V}}$ . In other words,  $L_n$  would have to extend to a subvariety

$$(1.16) \quad L_n(\epsilon) \subseteq Q(\epsilon)$$

where  $Q(\epsilon)$  is the first order deformation of  $Q$  determined by the family  $\{Q_{(a,b)}\}$ .

To complete the proof of the lemma, we rephrase (1.16) in analytic terms. We have

$$\pi : \hat{V} \rightarrow \mathbf{P}^1$$

with fibre  $Q_{(a,b)}$ . Let  $\eta(a) \in H^2(Q_{(a,1)}; \mathbf{Z})$  be the (locally defined) flat section of  $R^2\pi_*(\mathbf{Z})$  whose value at  $a=0$  is the Chern class of  $L_n$ . Let  $\omega(a) \in H^{2,0}(Q_{(a,1)})$  be a locally defined holomorphic section of the Hodge subbundle

$$\mathcal{F}^2 \subseteq R^2\pi_*(\mathbf{C})$$

such that  $\omega(0) \neq 0$ . The inclusion (1.16) implies that  $\eta(a)$  is of type (1, 1) to first order at  $a=0$ , that is, the holomorphic function

$$h(a) = \int_{Q_{(a,1)}} \omega(a) \wedge n(a)$$

has  $\frac{\partial h}{\partial a}(0) = 0$ . Thus

$$\int_Q \frac{\partial \omega}{\partial a}(0) \wedge \eta(0) = - \int_Q \omega(0) \wedge \frac{\partial \eta}{\partial a}(0).$$

But  $\eta(a)$  is flat so that  $\frac{\partial \eta}{\partial a} = 0$  and we conclude

$$(1.16)' \quad \int_{L_n} \frac{\partial \omega}{\partial a}(0) = 0.$$

However no  $L_n$  is homologous to a multiple of the hyperplane section of  $Q$ , and we have already mentioned that, by choosing  $f$  appropriately in (1.9), the deformation  $Q(\epsilon)$  of  $Q$  can be made to be an arbitrary infinitesimal deformation of  $Q$  inside the space of all quartic hypersurfaces. By the local deformation theory of  $K_3$  surfaces, there exist deformations for which (1.16)' is violated, so it must be violated for a generic choice of  $f(X)$  in (1.9), and the proof of the lemma is complete.

Now suppose  $V_0$  is any (possibly singular) quintic hypersurface in  $\mathbf{P}^4$  with the property that there is a smooth rational curve

$$C \subseteq V_0$$

such that  $C$  contains no singular point of  $V_0$ . We wish to find various conditions to describe when the normal bundle  $\mathcal{N}_{C, V_0}$  has no sections. (Of course, [by (1.14),  $\mathcal{N}_{C, V_0}$  has no sections if and only if  $\mathcal{N}_{C, V_0} \cong \mathcal{O}_C(-1) + \mathcal{O}_C(-1)$ ].) Let  $d = \text{degree } C$ .

Let  $\mathcal{E}_X$  denote the tangent sheaf to  $X$ , and consider the commutative diagram with exact rows and columns:

$$(1.17) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_{V_0} & = & \mathcal{O}_{V_0} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\mathcal{E}}_{V_0} & \longrightarrow & \mathcal{O}_{V_0}(1)^5 & \xrightarrow{(*)} & \mathcal{O}_{V_0}(5) \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{E}_{V_0} & \longrightarrow & \mathcal{E}_{P^4} & \longrightarrow & \mathcal{O}_{V_0}(5) \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where, if  $V_0$  is defined by  $F_0(X) = 0$ , the mapping  $(*)$  is given by

$$(\sigma_j)_{j=0, \dots, 4} \mapsto \sum_j \sigma_j \frac{\partial F_0}{\partial X_j}.$$

We want to use the diagram (1.17) to get an explicit description of  $\mathcal{N}_{C, V_0}$ . To this end, suppose that  $C$  is parametrized by

$$(1.18) \quad \begin{array}{l} \alpha: \mathbf{P}^1 \rightarrow C \\ (t_0, t_1) \mapsto (\alpha_j(t_0, t_1))_{j=0, \dots, 4}. \end{array}$$

Then we have a second commutative diagram with exact rows and columns

$$(1.19) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_C & = & \mathcal{O}_C & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\mathcal{E}}_C & \xrightarrow{(**)} & \tilde{\mathcal{E}}_{V_0} \otimes \mathcal{O}_C & \longrightarrow & \mathcal{N}_{C, V_0} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{E}_C & \longrightarrow & \mathcal{E}_{V_0} \otimes \mathcal{O}_C & \longrightarrow & \mathcal{N}_{C, V_0} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where the mapping  $(**)$  is given by

$$\sigma \frac{\partial}{\partial t_0} + \tau \frac{\partial}{\partial t_1} \mapsto \left( \sigma \frac{\partial \alpha_j}{\partial t_0} + \tau \frac{\partial \alpha_j}{\partial t_1} \right)_{j=0, \dots, 4}.$$

Pulling diagrams (1.17) and (1.19) back to  $\mathbf{P}^1$  via  $\alpha$  in (1.18), we see that a section of  $\mathcal{N}_{\mathbb{C}, \mathbb{V}_0}$  is given by a set

$$(1.20) \quad (\sigma_j(t_0, t_1))_{j=0, \dots, 4}$$

of five homogeneous forms of degree  $d$  subject to the condition that

$$(1.21) \quad \sum_j \sigma_j(t) \frac{\partial F_0}{\partial X_j}(\alpha(t)) \equiv 0.$$

The expression (1.20) gives the zero section of  $\mathcal{N}_{\mathbb{C}, \mathbb{V}_0}$  if and only if there exist linear forms  $\sigma(t), \tau(t)$  such that

$$(1.22) \quad \sigma_j(t) = \sigma(t) \frac{\partial \alpha_j}{\partial t_0}(t) + \tau(t) \frac{\partial \alpha_j}{\partial t_1}(t), \quad j = 0, \dots, 4.$$

We are now reduced to a computation in linear algebra. If we write out the condition (1.21) in powers  $t_0^a t_1^b$ , with  $a + b = 5d$ , we obtain a system of  $(5d + 1)$  equations in the  $5(d + 1)$  unknowns  $\sigma_{a'b'}^j$  where

$$\sigma_j(t) = \sum_{a'+b'=d} \sigma_{a'b'}^j t_0^{a'} t_1^{b'}.$$

This system is given by a  $(5d + 1) \times 5(d + 1)$  matrix

$$(1.23) \quad M(\alpha, F_0)$$

whose entries are homogeneous of degree one in the coefficients of  $F_0(X)$  and homogeneous of degree 4 in the coefficients of  $\alpha(t)$ . The equations (1.22) give a 4-dimensional set of solutions to the system of equations so that:

*Lemma (1.24).* —  $\mathcal{N}_{\mathbb{C}, \mathbb{V}_0}$  has no non-trivial sections if and only if,

$$\mathcal{N}_{\mathbb{C}, \mathbb{V}_0} \cong \mathcal{O}_{\mathbb{C}}(-1) + \mathcal{O}_{\mathbb{C}}(-1)$$

if and only if the matrix  $M(\alpha, F_0)$  in (1.23) is of maximal rank, if and only if the forms

$$\frac{\partial F_0}{\partial X_j}(\alpha(t))$$

of degree  $4d$  generate the homogeneous forms of degree  $5d$  on  $\mathbf{P}^1$ .

We are now reduced to a constant count in order to achieve the result claimed in the first paragraph of § 1. We view

$$(1.25) \quad F(\alpha_0(t), \dots, \alpha_4(t)) \equiv 0$$

as a system of  $(5d + 1)$  equations on

$$\mathbf{P}^{\binom{5+4}{4}-1} \times \mathbf{P}^{5(d+1)-1},$$

where the coordinates of the first projective space are the coefficients of the variable quintic form  $F(X_0, \dots, X_4) = 0$ , and the coordinates of the second projective space



are the coefficients of the variable homogeneous forms  $\alpha_j(t)$  of degree  $d$ . Elementary dimension theory says that every component of the solution set  $\mathcal{S}$  of (1.25) has dimension greater than or equal to  $\binom{5+d}{4} + 2$ . So this must be true at the solution

$$\alpha: \mathbf{P}^1 \rightarrow L_n \subseteq V$$

occurring in Lemma (1.15). However, since the normal bundle to  $L_n$  in  $V$  has no sections the fibre dimension of the projection

$$(1.26) \quad \mathcal{S} \rightarrow \mathbf{P}^{\binom{5+d}{4}-1}$$

at this solution is exactly 3. In fact the fibre is parametrized by  $\mathbf{PGL}(2)$ . So by looking at dimensions the map (1.26) must be surjective. Thus:

*Theorem (1.27).* — *If  $d(n)$  is the degree of the rational curve  $L_n$  occurring in (the proof of) Lemma (1.4), the generic quintic threefold contains a smooth rational curve  $C$  of degree  $d(n)$  with normal bundle*

$$\mathcal{O}_C(-1) + \mathcal{O}_C(-1).$$

In fact, using Lemma (1.24), we can explicitly “solve” the equation (1.25) by power series methods near a solution  $(C, V_0)$  with  $\mathcal{N}_{C, V_0} = \mathcal{O}_C(-1) + \mathcal{O}_C(-1)$ . For, if we consider the linear variation of quintics

$$(1.28) \quad F_\lambda(X) = F_0(X) + \lambda F(X),$$

we can repeatedly differentiate the equation  $F_\lambda(\alpha(\lambda, t)) \equiv 0$  with respect to  $\lambda$ . Evaluating the resulting equations at  $\lambda = 0$ , we obtain equations

$$(1.29) \quad \sum_{j=0}^4 \frac{\partial^n \alpha_j}{\partial \lambda^n}(t) \frac{\partial F_0}{\partial X_j}(\alpha(t)) = \text{polynomial expression in} \\ \frac{\partial^k F_0}{\partial X_{i_1} \dots \partial X_{i_k}}(\alpha(t)), \quad \frac{\partial^l F}{\partial X_{i_1} \dots \partial X_{i_l}}(\alpha(t)) \quad \text{and} \quad \frac{\partial^m \alpha_j}{\partial \lambda^m}(t) \quad \text{for } m < n.$$

Since the matrix  $M(\alpha, F_0)$  in (1.23) is of maximal rank, we can solve (1.29) successively for the terms of

$$\alpha(\lambda, t) = \alpha(t) + \lambda \alpha'(t) + \lambda^2 \alpha''(t) + \dots$$

The resulting solution is unique up to the action of  $\mathbf{GL}(2)$ , that is, if  $\beta(\lambda, t)$  is another solution then there exists a holomorphic function  $\sigma_\lambda$  with values in  $\mathbf{GL}(2)$  such that

$$\alpha(\lambda, t) = \beta(\lambda, \sigma_\lambda(t)).$$

## 2. Rational curves through nodes

We return to the quartic surface  $Q \subseteq \mathbf{P}^3$  defined in (1.3) and to the threefolds

$$(2.1) \quad \ell(X)q(X) + X_4 f(X) = 0$$

as in (1.9). Here  $q(X) = q(X_0, \dots, X_3)$  defines  $Q$  and  $\ell(X)$  and  $f(X)$  are arbitrary forms of degrees one and four respectively. The sixteen nodes of the three-fold (2.1) are always on  $Q$  but can be moved on  $Q$  by changing their defining equations  $\ell(X) = f(X) = 0$ . So given  $n_0 \in \mathbf{Z}$ , we can arrange that our threefold (2.1) have exactly one node on  $L_{n_0}$  and that no other  $L_n$  pass through any node, where  $\{L_n\}_{n \in \mathbf{Z}}$  is the family of rational curves described in (1.6). By choosing  $\ell(X)$  and  $f(X)$  generically with this property, we can in fact insure, as we have seen in § 1, that for our particular choice of threefold  $V_0$  in (2.1), for all  $n \neq n_0$ ,

$$(2.2) \quad \mathcal{N}_{L_n, V_0} \cong \mathcal{O}_{L_n}(-1) + \mathcal{O}_{L_n}(-1).$$

As before we let  $F_0(X)$  denote the polynomial (2.1) and  $V_0$  denote the associated quintic hypersurface. Let  $C_0$  denote the rational curve  $L_{n_0}$  in  $V_0$ .

*Lemma (2.3).* — *For generic choice of  $V_0$  with a node on  $C_0$ , the curve  $C_0$  is rigid in  $V_0$ .*

*Proof.* — Let  $\tilde{V}$  be the threefold obtained by blowing up the nodes of  $V_0$ , and let  $E$  be the exceptional divisor of  $\tilde{V}$  over the distinguished node  $p_0$  of  $V_0$ . Let  $\tilde{C}$  be the proper transform of  $C_0$ . The negativity of the normal bundle to  $E$  in  $\tilde{V}$  gives that the only deformations of

$$\tilde{C} + (\text{curve in } E)$$

in  $\tilde{V}$  which do not have a component lying in  $E$  are deformations of  $\tilde{C}$  or  $\tilde{C} + L_1$  or  $\tilde{C} + L_2$ , where  $L_1$  and  $L_2$  are the lines on  $E$  passing through  $(\tilde{C} \cap E)$ . If  $\hat{V}$  is as in the proof of Lemma (1.15) we have

$$\varphi_1: \tilde{V} \rightarrow \hat{V},$$

a mapping which, say, blows down the ruling containing  $L_1$ . A deformation of  $\tilde{C}$  or  $\tilde{C} + L_1$  would give rise to a deformation of  $C_1 = \varphi_1(\tilde{C})$  in  $\hat{V}$ . But we have seen in the proof of Lemma (1.15) that for generic choice of  $f(X)$  in (2.1),

$$\mathcal{N}_{C_1, \hat{V}} = \mathcal{O}_{C_1}(-1) \oplus \mathcal{O}_{C_1}(-1)$$

making deformations of  $C_1$  impossible. Next we let

$$V_2 = (\text{smooth threefold obtained from } \tilde{V} \text{ by blowing down the ruling of } E \text{ which contains } L_2).$$

We claim that, if  $C_2$  denotes the image of  $\tilde{C}$  in  $V_2$ , again

$$\mathcal{N}_{C_2, V_2} \cong \mathcal{O}_{C_2}(-1) \oplus \mathcal{O}_{C_2}(-1)$$

in the generic case. To see this, first notice that the sheaf  $\mathcal{N}_{\tilde{C}, \tilde{V}}$  injects into each of the two sheaves,  $\mathcal{N}_{C_1, \hat{V}}$  and  $\mathcal{N}_{C_2, V_2}$ , under the natural maps. The “downward modification” given by restricting from  $\mathcal{N}_{C_1, \hat{V}}$  to the subsheaf  $\mathcal{N}_{\tilde{C}, \tilde{V}}$  will have the effect of lowering the degree of one of the two summands in the splitting of  $\mathcal{N}_{C_1, \hat{V}}$  by one. Which summand is reduced in degree depends on the direction in which the center

of the blowing up  $\tilde{V} \rightarrow \hat{V}$  meets  $C_1$  at  $p_0$ . The summand of  $\mathcal{N}_{C_1, \hat{V}}$  which passes through  $p_0$  in that distinguished direction is the one which is *not* altered. Conversely, the "upward modification" given by passing from  $\mathcal{N}_{\tilde{C}, \tilde{V}}$  to the sheaf  $\mathcal{N}_{C_2, V_2}$  containing it will have the effect of raising the degree of the "non-distinguished summand" by one.

So suppose we can find a *special* case in which

$$\text{i) } \mathcal{N}_{C_1, \hat{V}} \cong \mathcal{O} + \mathcal{O}(-2)$$

but where

ii) the fibre of the  $\mathcal{O}$  summand of  $\mathcal{N}_{C_1, \hat{V}}$  at the point over  $p_0$  gives a direction *different* from that of  $\varphi_1(L_2)$ .

Then blowing up along  $\varphi_1(L_2)$  gives  $\mathcal{N}_{\tilde{C}, \tilde{V}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2)$ . Since, in this decomposition, only the direction of the factor  $\mathcal{O}(-1)$  is uniquely determined at each point, there is only one upward modification of  $\mathcal{N}_{\tilde{C}, \tilde{V}}$  which will give  $\mathcal{O} \oplus \mathcal{O}(-2)$ , all others giving  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ . Since the direction of  $L_2$  at the point of  $\tilde{C}$  over  $p_0$  is not that of the factor  $\mathcal{O}(-1)$  of  $\mathcal{N}_{\tilde{C}, \tilde{V}}$ , the upward modification in this case will yield

$$\mathcal{N}_{C_2, V_2} \cong \mathcal{O}(-1) + \mathcal{O}(-1).$$

But if this last happens for a special case, it must happen in the generic case by semi-continuity.

So we need only find a  $V_0$  for which i) and ii) hold. That is, we need to find a  $V_0$  such that, for the pencil of quartic surfaces  $Q_{(a,b)} \subseteq V_0$ , we have that:

- i)  $C_0 \subseteq Q$  deforms to first order in the pencil,
- ii) the first order deformation of  $C_0$  does not pass through the basepoint  $p_0$  of the pencil.

It is possible to achieve this for the following reasons:

i) Since it is only one condition on deformations of the quartic surface  $Q \subseteq \mathbf{P}^3$  that  $C_0$  deforms to first order, a constant count shows that if

$$q(X_0, \dots, X_3) + \lambda f(X_0, \dots, X_3, 0)$$

is a generic such deformation, then (2.1) can still be taken to have 16 distinct ordinary nodes as its only singularities.

ii) It therefore suffices to find *some* deformation such that  $p_0$  does not lie on the first order deformation of  $C_0$ . Then the same will be true for the generic deformation i).

iii) One can construct such a deformation by taking an element  $\xi$  of the Lie algebra of  $\mathbf{PGL}(4)$  which moves  $p_0$  in a direction tangent to  $Q$  but normal to  $C_0$  and choosing a linear deformation which is tangent the one parameter family  $\exp(\lambda\xi)Q$ ,  $\lambda \in \mathbf{C}$ .

This completes the proof of Lemma (2.3).

Since  $C_0$  is rigid in  $V_0$ , the "constant count" argument in (1.25) ff. gives that the curve  $C_0 \subseteq V_0$  must have deformations in each linear pencil

$$(2.4) \quad \{V_\lambda\}_{\lambda \in \mathbf{C}}$$

given by the family of equations

$$F_0(X) + \lambda F(X) = 0, \quad \lambda \in \mathbf{C},$$

for any choice of quintic form  $F(X)$ . We choose  $F(X)$  generically—*i.e.* so that its zero set is smooth, meets  $V_0$  transversely, and contains none of the nodes of  $V_0$ . As in § 1, we look for solutions

$$(2.5) \quad F_0(\alpha(\lambda, t)) + \lambda F(\alpha(\lambda, t)) \equiv 0.$$

This time we will center our attention at

$$(1, 0, \dots, 0) \in \mathbf{P}^4$$

which we assume to be the node of  $V_0$ , and will use coordinates  $t = (t_0, t_1)$  such that when  $\lambda = t_1 = 0$

$$\alpha(\lambda, t) = (1, 0, \dots, 0).$$

But now setting  $t = (1, 0)$  and looking at the left-hand-side of (2.5) as a polynomial in  $\lambda$  we obtain a contradiction since  $F_0(\alpha(\lambda, (1, 0)))$  has no constant or first-order term in  $\lambda$  whereas  $F(\alpha(\lambda, (1, 0)))$  has constant term

$$F(1, 0, \dots, 0) \neq 0.$$

We remedy this by “base extension”, that is, we write

$$(2.6) \quad \begin{aligned} \lambda &= \mu^2, \\ F_0(\beta(\mu, t)) + \mu^2 F(\beta(\mu, t)) &\equiv 0 \end{aligned}$$

with  $\beta(0, (1, 0)) = (1, 0, \dots, 0)$ . As in § 1, we successively differentiate (2.6) with respect to  $\mu$  and set  $\mu = 0$ . The first time we do this, we obtain the equation

$$(2.7) \quad \sum_{j=0}^4 \frac{\partial F_0}{\partial X_j}(\beta(0, t)) \frac{\partial \beta_j}{\partial \mu}(0, t) \equiv 0.$$

The equation (2.7) is homogeneous of degree  $5d$  in  $(t_0, t_1)$ . As in § 1, we write (2.7) out in powers of  $t_0, t_1$ . The coefficient of  $t_0^{5d}$  is zero, and so we get  $5d$  equations in  $5(d+1)$  unknowns. As in (1.21), solutions to (2.7),

$$\sigma_j(t) = \frac{\partial \beta_j}{\partial \mu}(0, t),$$

give sections of the normal sheaf  $\mathcal{N}_{C_0, V_0}$ . Since only a four-dimensional set of solutions correspond to the action of  $\mathbf{GL}(2)$ , there must be a solution to (2.7) which gives a non-zero section of the normal sheaf  $\mathcal{N}_{C_0, V_0}$ .

Suppose now that (2.7) had a set of solutions of dimension  $\geq 6$ . We write

$$(2.8) \quad \begin{aligned} \frac{\partial \beta_j}{\partial \mu}(0, t) &= b_j t_0^d + \dots \\ \frac{\partial F_0}{\partial X_j}(\beta(0, t)) &= f_j t_0^{4d-1} t_1 + \dots \end{aligned}$$

Now  $f_0 = 0$ , and by our assumption that  $V_0$  has only an ordinary node at  $(1, 0, \dots, 0)$ , the  $b_j$  must satisfy a non-trivial relation

$$\sum_{j=1}^4 f_j b_j = 0.$$

So, under the natural mapping

$$\Gamma(\mathcal{N}_{C_0, V_0}) \rightarrow \mathbf{C}^3$$

which is given by evaluating sections at  $(t_0, t_1) = (1, 0)$ , the images of our sections

$$\sigma_j(t) = \frac{\partial \beta_j}{\partial \mu}(0, t)$$

go onto at most a two-dimensional subspace of the image vector space. This means that for  $\hat{V}$  as in the proof of Lemma (2.3), it is at most one condition that a section of  $\mathcal{N}_{C_0, V_0}$  lift to a section of  $\mathcal{N}_{C_1, \hat{V}}$  under the natural map

$$\begin{array}{ccc} \hat{V} & \longrightarrow & V_0 \\ \cup | & & \cup | \\ C_1 & \longrightarrow & C_0. \end{array}$$

This is because the composition

$$(2.9) \quad \mathcal{N}_{C_1, \hat{V}} \rightarrow \mathcal{N}_{C_0, V_0} \rightarrow \mathbf{C}^3,$$

restricted to the point of  $C_1$  lying over  $(1, 0, \dots, 0) \in C_0$ , has one-dimensional image, and any analytic section of  $\mathcal{N}_{C_0, V_0}$ , defined locally near  $(1, 0, \dots, 0)$  and passing through the image of (2.9) at  $C_0$ , lifts to a locally defined section of  $\mathcal{N}_{C_1, \hat{V}}$ . Now we have arranged our choice of  $V_0$  so that

$$H^0(C_1; \mathcal{N}_{C_1, \hat{V}}) = 0,$$

so it cannot be that

$$\dim H^0(C_0; \mathcal{N}_{C_0, V_0}) \geq 2.$$

So (2.7) can have at most a *five-dimensional* set of solutions, four dimensions of which come from the action of the group  $\mathbf{GL}(2)$ .

Notice that we could as well have made this last argument with  $\mathcal{N}_{C_1, V_1}$  instead of  $\mathcal{N}_{C_1, \hat{V}}$  since, as we saw in the proof of Lemma (2.3), both normal bundles can be taken to be equal to  $\mathcal{O}(-1) + \mathcal{O}(-1)$ . Since no non-trivial section of  $\mathcal{N}_{C_0, V_0}$  lifts to either  $\mathcal{N}_{C_1, \hat{V}}$  or  $\mathcal{N}_{C_1, V_1}$ , we can conclude that sections of  $\mathcal{N}_{C_0, V_0}$  for which

$$\sum_{j,k} \frac{\partial^2 F_0}{\partial X_j \partial X_k}(1, 0, \dots, 0) b_j b_k = 0$$

must have  $b_1 = \dots = b_4 = 0$ . This is because  $(f_1, \dots, f_4)$  in (2.8) is the direction of  $C_0$  in the tangent cone to  $V_0$  at  $(1, 0, \dots, 0)$ , so that the common solutions to

$$\sum_{j=1}^4 f_j b_j = 0$$

and the above quadratic equation give precisely the union of the images of  $\mathcal{N}_{C_1, \hat{v}}$  and  $\mathcal{N}_{C_1, v_1}$  in  $\mathcal{N}_{C_0, v_0}$ .

So there is a four-dimensional set of simultaneous solutions to (2.7) and the inhomogeneous quadratic equation

$$(2.10) \quad \sum_{j,k} \frac{\partial^2 F_0}{\partial X_j \partial X_k} (1, 0, \dots, 0) b_j b_k = -2F(1, 0, \dots, 0)$$

(see (2.8) and (2.6)). Now if we differentiate (2.6) twice with respect to  $\mu$  and set  $\mu = 0$ , we obtain

$$(2.11) \quad \sum_{j=0}^4 \frac{\partial F_0}{\partial X_j} (\beta(0, t)) \frac{\partial^2 \beta_j}{\partial \mu^2} (0, t) \equiv - \sum_{j,k} \frac{\partial^2 F_0}{\partial X_j \partial X_k} (\beta(0, t)) \frac{\partial \beta_j}{\partial \mu} (0, t) \frac{\partial \beta_k}{\partial \mu} (0, t) - 2F(\beta(0, t)).$$

By (2.10) the coefficient of  $t_0^{5d}$  in (2.11) is zero so (2.11) becomes a system of  $5d$  equations in  $(5d + 5)$  unknowns. This system is of maximal rank since we have seen above that

$$\sum_{j=0}^4 \frac{\partial F_0}{\partial X_j} (\beta(0, t)) \sigma_j(t) \equiv 0$$

has exactly a five-dimensional set of solutions.

So there is a four-dimensional linear set of simultaneous solutions to (2.11) and any equation of the form

$$(2.12) \quad \sum_{j,k} \frac{\partial^2 F_0}{\partial X_j \partial X_k} (1, 0, \dots, 0) \frac{\partial^2 \beta_j}{\partial \mu^2} (0, (1, 0)) \frac{\partial \beta_k}{\partial \mu} (0, (1, 0)) = \text{constant}$$

for given values of  $\frac{\partial \beta_k}{\partial \mu} (0, t)$ . As before, there will be an appropriate choice of constant in (2.12) which makes consistent the system of equations obtained by differentiating (2.6) three times and setting  $\mu = 0$ .

We repeat the process to solve successively for

$$\frac{\partial^n \beta_j}{\partial \mu^n} (0, t), \quad j = 0, \dots, 4.$$

As before our solutions are determined modulo  $\mathbf{GL}(2)$  except, this time, at the first stage, the fact that (2.10) is quadratic will mean that we have two deformations of  $C_0$ . This is because, if

$$\beta(\mu, t) = \alpha(t) + \mu\beta'(t) + \mu^2\beta''(t)$$

is a solution of (2.6) modulo  $\mu^3$ , then so is

$$\beta(\mu, (t_0, t_1)(I + \nu\sigma))$$

for any  $(2 \times 2)$ -matrix  $\sigma$  and small  $\nu$ . So differentiating

$$(2.13) \quad F_0(\beta(\mu, (t_0, t_1)(I + \nu\sigma))) + \mu^2 F(\beta(\mu, (t_0, t_1)(I + \nu\sigma)))$$

with respect to  $\mu$  and  $\nu$  and setting  $\mu = \nu = 0$  we obtain that

$$(2.14) \quad \beta'(t_0, t_1) + (t_0, t_1)\sigma \begin{pmatrix} \frac{\partial \alpha}{\partial t_0}(t_0, t_1) \\ \frac{\partial \alpha}{\partial t_1}(t_0, t_1) \end{pmatrix}$$

solves (2.7). Differentiating (2.13) twice with respect to  $\mu$  and once with respect to  $\nu$  and setting  $\mu = \nu = 0$ , we obtain that the coefficients of  $t_0^d$  in (2.14) also solve (2.10). Thus the simultaneous solutions to (2.7) and (2.10) must actually be made up of two parallel linear sets of dimension four. So  $\alpha(t) = \beta(0, t)$  will have *two* prolongations  $\beta(\mu, t)$  modulo the action of  $\mathbf{GL}(2)$ .

In other words, the rational curve  $C_0 \subseteq V_0$  fits into a locally defined family

$$C_{\lambda, \mu} \subseteq V_\lambda$$

in (2.4).

### 3. Impossibility of finite generation

We return again to the equation

$$(3.1) \quad \ell(X)q(X) + X_4 f(X) = 0$$

in (2.1). Choosing  $f(X)$  outside a countable union of hypersurfaces in the set of all quintics we arrange that the curve  $S \subseteq Q$  defined by

$$(3.2) \quad X_4 = q(X) = f(X) = 0$$

is non-singular and meets each curve  $L_n$ ,  $n \in \mathbf{Z}$ , transversely (see (1.6)), and that

$$S \cap L_m \cap L_n = \emptyset$$

for all  $m, n \in \mathbf{Z}$ ,  $m \neq n$ . Also we choose  $f(X)$  as in Lemma (1.15) so that *no*  $L_n$  stays algebraic to first order in the direction  $f(X_0, X_1, X_2, X_3, 0)$ .

Next choose a linear pencil of hyperplanes

$$(3.3) \quad \ell_s(X) = s_0 \ell_0(X) + s_1 \ell_1(X),$$

$s = (s_0, s_1) \in \mathbf{P}^1$  such that no hyperplane  $\ell_s(X) = 0$  contains more than one point of the countable set

$$(3.4) \quad S \cap \left( \bigcup_{\mathbf{Z}} L_n \right)$$

and such that any  $\ell_s(X)$  containing a point of (3.4) meets  $Q$  transversely. Again it is a question of choosing  $\ell_s(X)$  outside a countable union of hypersurfaces in the set of all pencils. Now form the two-parameter family of quintics

$$V(s, \lambda), \quad \lambda \in \mathbf{C}, \quad s \in \mathbf{P}^1,$$

given by

$$(3.5) \quad \ell_s(X)q(X) + X_4f(X) + \lambda F(X) = 0$$

for  $F(X)$  generic. (Compare (3.5) with (3.1) and (1.28).) Notice that  $Q \subseteq V(s, 0)$  for all  $s \in \mathbf{P}^1$ . Also

$$\mathcal{N}_{L_n, V(s, 0)} \cong \mathcal{O}_{L_n}(-1) + \mathcal{O}_{L_n}(-1)$$

whenever  $L_n$  does not contain one of the singular points of  $V(s, 0)$ .

We pick a basepoint  $\hat{s} \in \mathbf{P}^1$  such that the hyperplane  $\ell_{\hat{s}}(X) = 0$  contains no point of the set (3.4). Suppose now that we had a relation

$$(3.6) \quad \sum_{|n| \leq N} c_n L_n \approx 0$$

which held on some open set  $U$  containing  $(\hat{s}, 0)$  in  $\mathbf{P}^1 \times \mathbf{C}$ . Here  $\approx$  denotes rational equivalence, and the  $c_n$  are integers. For each  $n$  with

$$|n| \leq N,$$

let  $s_n \in \mathbf{P}^1$  be such that  $V(s_n, 0)$  has a node on  $L_n$  (and hence on no other  $L_m$ ). We can choose paths  $\gamma_n$  in  $\mathbf{P}^1$  with

$$\partial \gamma_n = s_n - s_0$$

such that, if  $s \in (\gamma_n - \{s_n\})$ , then, for all  $m$  with  $|m| < N$ ,

$$(3.7) \quad L_m \text{ contains no singular point of } V(s, 0).$$

So also  $\mathcal{N}_{L_m, V(s, 0)} \cong \mathcal{O}_{L_m}(-1) + \mathcal{O}_{L_m}(-1)$  if  $s \in \gamma_n$ ,  $n \neq m$ , or if  $s \in (\gamma_n - \{s_n\})$ ,  $n = m$ .

The relation (3.6) would have to continue itself along  $\gamma_n$  to a relation

$$(3.8) \quad c_n L_n(\lambda^{1/2}) + \sum_{\substack{|m| \leq N \\ m \neq n}} c_m L_m(\lambda) \approx 0$$

valid in  $V_{(s_n, \lambda)}$  for small  $\lambda$  by § 1 and § 2. So now, as in § 2, we make the base extension  $\mu^2 = \lambda$  to get a family of threefolds

$$(3.9) \quad W_\mu = V_{(s_n, \mu^2)}$$

parametrized by  $\mu$  in an ambient fourfold with sixteen ordinary nodes coming from the sixteen nodes of  $V_{(s_n, 0)}$ . We blow up these nodes in the fourfold to obtain a smooth ambient space in which the family (3.9) specializes to the normal crossing variety

$$W_0 = \tilde{V} + \sum_{j=1}^{16} E_j$$



in which the  $E_j$  are disjoint threefolds, each isomorphic to a smooth quadric in  $\mathbf{P}^4$ ,  $\tilde{V}$  is the proper transform of  $V_{(s_n, 0)}$  and

$$\tilde{V}.E_j = \text{smooth quadric surface} \cong \mathbf{P}^1 \times \mathbf{P}^1.$$

So the left-hand-side of (3.8) gives a “normal function”

$$Z(\mu) = c_n L_n(\mu) + \sum_{m \neq n} c_m L_m(\mu^2).$$

We pick, as in [1]; (3.22)-(3.23), a continuously varying three-chain  $\Gamma\left(\frac{\log \mu}{2\pi i}\right)$

$$\partial\Gamma\left(\frac{\log \mu}{2\pi i}\right) = Z(\mu)$$

and wish to compute the monodromy

$$(3.10) \quad \Gamma\left(\frac{\log \mu}{2\pi i} + 1\right) - \Gamma\left(\frac{\log \mu}{2\pi i}\right).$$

The first step is to compute the limiting position of  $Z(\mu)$  in  $W_0$  as  $\mu$  approaches zero. As in [2]; Lemma (2.2), the resulting formula is

$$\lim_{\mu \rightarrow 0} Z(\mu) = c_n(\tilde{L}_n + M) + \sum_{m \neq n} c_m L_m,$$

where  $\tilde{L}_n$  is the proper transform of  $L_n$  and  $M$  is a line in the quadric, say  $E_1$ , corresponding to the node through which  $L_n$  passes. This is because  $L_n$  has the node as a simple point. (See the proof of Lemma (2.2) in [2].) To compute the monodromy of this situation, we proceed just as in (5.23) of [3]. The vanishing cycle (3.10) is the image under a Gysin mapping of a two-cycle  $\gamma$  in  $E_1 \cap \tilde{V}$  which is homologous, in  $E_1$ , to  $c_n M$ . The following sequence of four figures may help visualize the situation:

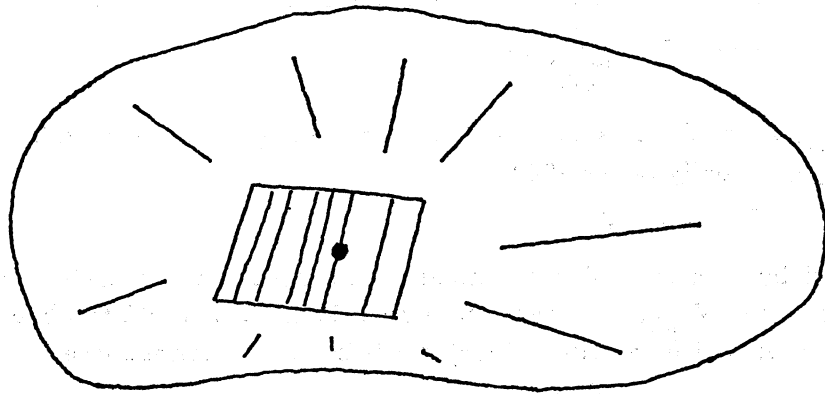


FIG. 1. — Family  $L_n(\mu)$  in variety  $\mu^2 = x_1^2 + \dots + x_4^2$

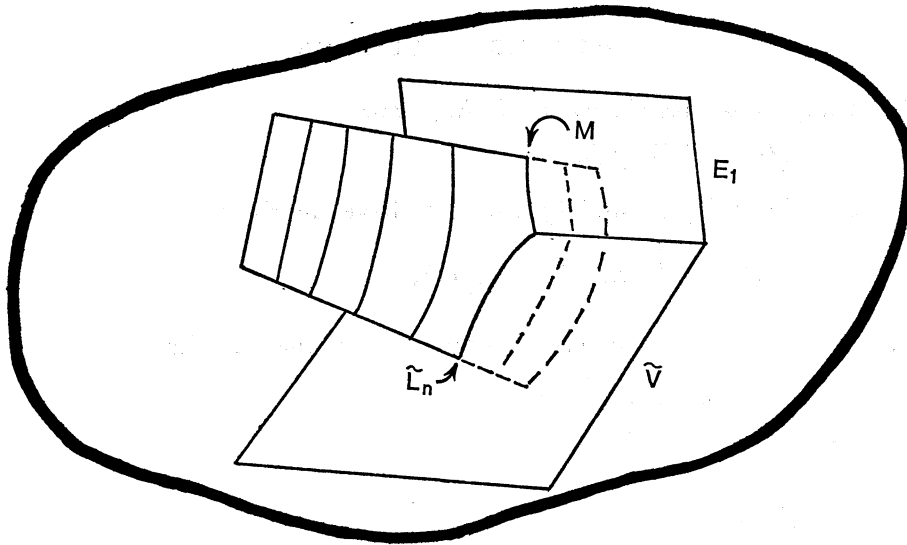


FIG. 2. — Lift of family  $L_n(\mu)$  after blowing up node in Figure 1

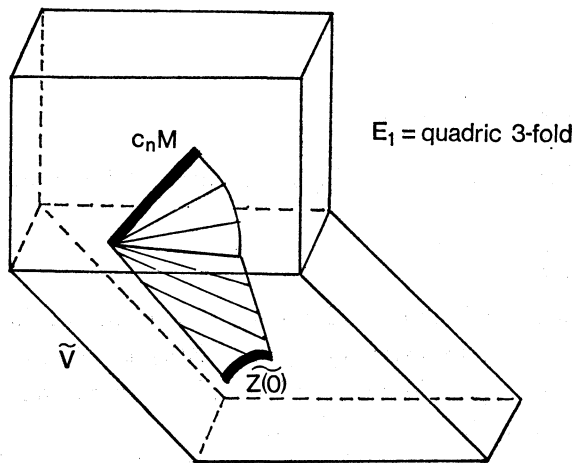


FIG. 3. —  $\lim_{\mu \rightarrow 0} \Gamma \left( \frac{\log \mu}{2\pi i} \right) \subseteq W_0 = \tilde{V} + E_1 + \dots$

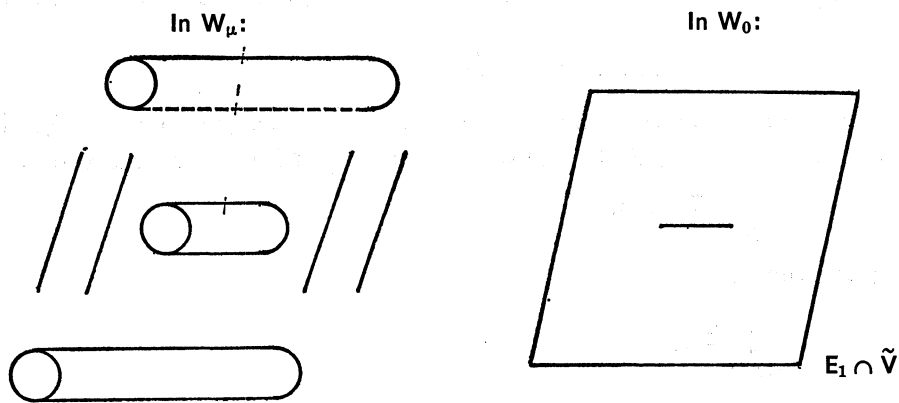


FIG. 4. — The vanishing cycle over  $\left[ \lim_{\mu \rightarrow 0} \Gamma \left( \frac{\log \mu}{2\pi i} \right) \right] \cap \tilde{V} \cap E_1$

The family (3.9) has sixteen vanishing cycles

$$(3.11) \quad \sigma_j(\mu) \in H_3(W_\mu; \mathbf{Z}),$$

one for each node of  $V_{(s_n, 0)}$ , and (3.10) must have the form

$$(3.12) \quad c_n \sigma_1(\mu) + \sum_{j=2}^{16} 2a_j \sigma_j(\mu), \quad a_j \in \mathbf{Z}.$$

The vanishing cycles have one relation coming from the fact that

$$\text{rank } H_4(V_{(s_n, 0)}; \mathbf{Z}) = 2$$

(see [3]; (1.27)). This relation is

$$(3.13) \quad \sum_{j=1}^{16} \sigma_j(\mu) = 0.$$

The relation (3.13) says that, in the Neron model for the family of intermediate Jacobians,  $J(W_\mu)$ ,  $\mu$  near 0, the fibre  $J(W_0)$  has as its group of components

$$(3.14) \quad \frac{\sum_{j=1}^{16} \mathbf{Z} \sigma_j}{\mathbf{Z}(\sum_{j=1}^{16} \sigma_j) + \sum_{j=1}^{16} 2\mathbf{Z} \sigma_j}$$

(see [1], § 3). By Theorem (3.23) of [1], the normal function

$$(3.15) \quad \int_{\Gamma(\frac{\log \mu}{2\pi i})} \in J(W_\mu)$$

gives a well-defined section whose extension over  $\mu = 0$  lies in the component of  $J(W_0)$  given by (3.12). So the assumption (3.6) that  $Z(\mu)$  is rationally equivalent to zero implies that the section (3.15) is the zero section which in turn implies that  $c_n \sigma_1$  represents zero in the group (3.14). So  $c_n$  must be *even*.

We repeat the argument for each  $n$  with  $|n| \leq N$  to get that

$$\sum_{|n| \leq N} c_n L_n \approx 0 \quad \text{implies all } c_n \text{ even.}$$

Now suppose  $d_n = \text{degree } L_n$  and that the subgroup of  $J(V_{(s, \lambda)})$  generated by the images

$$5\Phi(L_n) - d_n \Phi(\text{plane section})$$

had  $\leq M$  generators, or even that, when we tensor this group with  $\mathbf{Q}$ , there are  $\leq M$  generators. We would then have that, if  $N \geq M/2$ , there must be a relation

$$\sum_{|n| \leq N} c_n L_n \approx 0.$$

So each  $c_n$  is even and

$$(3.16) \quad \sum_{|n| \leq N} \frac{c_n}{2} L_n$$

gives a point of order 2 in  $J(V_{(s,\lambda)})$ . Let

$$S \subseteq \mathbf{Z} - \{-N, \dots, N\}$$

be another set of more than  $M$  integers. Again there must be a relation

$$\sum_{n \in S} c_n L_n \approx 0$$

and again  $\sum_{n \in S} \frac{c_n}{2} L_n$

is of order two in  $J(V_{(s,\lambda)})$ . Repeating the argument for successive disjoint sets in  $\mathbf{Z}$ , we will eventually find two disjoint sets giving the *same* point of order two in  $J(V_{(s,\lambda)})$ .

So either there is an  $S \subseteq \mathbf{Z} - \{-N, \dots, N\}$  giving rise to the same point of order 2 in  $J(V_{(s,\lambda)})$  that (3.16) does, in which case we get

$$(3.17) \quad \sum_{|n| \leq N} \frac{c_n}{2} L_n + \sum_{n \in S} \frac{c_n}{2} L_n$$

giving zero in  $J(V_{(s,\lambda)})$ , or we can eliminate (3.16) from the set of points of order 2 of  $J(V_{(s,\lambda)})$  and apply the same argument all over again, with  $\mathbf{Z} - \{-N, \dots, N\}$  replacing  $\mathbf{Z}$  and the set

$$(3.18) \quad J(V_{(s,\lambda)})_{(2)} - \{\text{point (3.16)}\}$$

replacing the set  $J(V_{(s,\lambda)})_{(2)}$  of points of order two. But (3.17) means that, if  $|n| \leq N$  then  $c_n$  is divisible by four. This goes exactly as (3.6)-(3.15). Since some  $c_n$ ,  $|n| \leq N$ , is not infinitely divisible, there must exist  $N' \geq N$  such that, if

$$S' \subseteq (\mathbf{Z} - \{-N', \dots, N'\})$$

then  $\sum_{n \in S'} \frac{c_n}{2} L_n$  can never give the same point of order two as (3.16).

Now replace  $\mathbf{Z}$  by  $(\mathbf{Z} - \{-N', \dots, N'\})$  and  $J(V_{(s,\lambda)})_{(2)}$  by (3.18) and repeat the reasoning. Eventually we will have eliminated all points of order two in  $J(V_{(s,\lambda)})_{(2)}$  from consideration and obtain a contradiction to the assumption that the image of the Abel-Jacobi map is finitely generated in  $J(V_{(s,\lambda)})$  or even the assumption that this group, when tensored with  $\mathbf{Q}$  is finitely generated. So the theorem announced in the introduction is proved.

*Note:* The referee has pointed out that the rather cumbersome argument of the last three paragraphs can be replaced by the following lemma from linear algebra: *Let  $\Gamma$  be an abelian group with*

$$\Gamma_{\text{torsion}} \subseteq (\mathbf{Q}/\mathbf{Z})^r.$$

*Then*

$$\dim_{\mathbb{F}_2}(\Gamma/2\Gamma) \leq \dim_{\mathbf{Q}}(\Gamma \otimes \mathbf{Q}) + r.$$

*Proof:*  $\Gamma \hookrightarrow \mathbf{Q}^d \times (\mathbf{Q}/\mathbf{Z})^r$ . If  $\Gamma_1$  is the pull-back of  $\Gamma$  in  $\mathbf{Q}^{d+r}$ , then  $\Gamma_1/2\Gamma_1$  surjects onto  $\Gamma/2\Gamma$  and one is reduced to the torsion-free case.

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