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KARSTEN GROVE
STEPHEN HALPERIN

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CONTRIBUTIONS OF RATIONAL HOMOTOPY THEORY TO GLOBAL PROBLEMS IN GEOMETRY

by KARSTEN GROVE *and* STEPHEN HALPERIN

1. Introduction

In this paper we would like to draw attention to some very suggestive parallels between certain problems in global differential geometry, and some recent developments in rational homotopy theory. The problems in geometry centre around positively curved manifolds on the one hand, and the existence of isometry-invariant geodesics on the other. In the first case the relationship with rational homotopy theory is still conjectural; in the second it is precise, and we will use it to prove a very strong existence theorem.

Throughout, manifolds will always be assumed to be closed connected and simply connected.

A natural context for this discussion is provided by the following result of Felix and Halperin [4]; the class of n-manifolds M^n (or more generally the simply connected topological spaces of the rational homotopy type of a connected finite n-dimensional CW complex) is divided into two subclasses: either

(a) $\pi_p(M^n)$ is a finite group for all $p > 2n-1$, or

(b) the integers $\hat{\rho}_p = \sum_{q\leq p} \dim \pi_q(M) \otimes \mathbf{Q}$ grow exponentially in p (*i.e.* $\exists\, C > 1$, $\exists\, k \in \mathbf{N} : p > k \Rightarrow \hat{\rho}_p \geq C^p$).

Manifolds in class *(a)* are called *rationally elliptic*; the rest (in class *(b)*) are called *rationally hyperbolic*.

The " generic " manifold is rationally hyperbolic; rational ellipticity is a severely restrictive condition, albeit satisfied by all the simply connected homogeneous spaces. For instance if M^n is rationally elliptic then

(1.1) $\quad \dim \pi_*(M) \otimes \mathbf{Q} \leq n \quad$ ([7])

(1.2) $\quad \dim H_*(M; \mathbf{Q}) \leq 2^n \quad$ ([14])

and

(1.3) $\quad$ the Euler-Poincaré characteristic $\chi_M \geq 0 \quad$ ([13]).

Moreover

(1.4) $\chi_M > 0$ if and only if $H_p(M; \mathbf{Q}) = 0$ for all odd p ([13]).

By comparison if M^n admits a metric of non-negative sectional curvature then the classical conjecture of Chern-Hopf would imply $\chi_{M^n} \geq 0$ while a recent conjecture of Gromov [8] asserts that $\dim H_*(M^n; \mathbf{Q}) \leq 2^n$. (Gromov in fact proves that $\dim H_*(M^n; F) \leq C_n$, C_n only dependent on n.) The link between geometry and topology is provided by

Conjecture **(1.5)**. — *A closed* 1*-connected manifold* M, *which admits a metric of non-negative sectional curvature, is rationally elliptic.*

This conjecture has been attributed to Bott. Our interest in it was first stimulated by D. Toledo. An assertion equivalent to the conjecture is that the integers $\rho_p = \sum_{q \leq p} \dim H_q(\Omega M; \mathbf{Q})$ grow only sub-exponentially in p (*i.e.* $\forall C > 1$, $\forall k \in \mathbf{N}$, $\exists p > k : \rho_p < C^p$); in particular the principal result of [3] is a much weaker version of this conjecture.

In contrast with (1.5) we point out the existence of positively Ricci curved closed rationally hyperbolic manifolds. Indeed in [15] Hernández-Andrade constructs metrics of positive Ricci curvature on the Brieskorn varieties

$$V_m(4) = \{z \in \mathbf{C}^{m+1} \mid |z| = 1 \text{ and } z_1^4 + \ldots + z_{m+1}^4 = 0\},$$

for large m. The Brieskorn-Pham theorem (cf. [16]) implies that

$$\dim \pi_{m-1}(V_m(4)) \otimes \mathbf{Q} \geq 4 \quad \text{for } m \geq 3$$

whereas it follows from [7] that a rationally elliptic manifold V of dimension $2m - 1$ satisfies $\dim \pi_{m-1}(V) \otimes \mathbf{Q} \leq 2$ for $m \geq 3$ odd.

For rationally hyperbolic manifolds M an important invariant is the constant R_M given by

$$R_M^{-1} = \limsup [\dim H^p(\Omega M; \mathbf{Q})]^{1/p};$$

it is the radius of convergence of the Poincaré series $\Sigma \dim H^p(\Omega M; \mathbf{Q}) t^p$ of the loop space ΩM, and it is always less than 1 because of *(b)*. This invariant has been studied by Babenko [1] and Felix-Thomas [6]. We pose the following

Problem **(1.6)**. — *If g is a Riemannian metric on* M, *express* R_M *in terms of invariants of g.*

We turn now to our results on geodesics. Recall that a geodesic $c(t)$ on a Riemannian manifold is called *invariant* with respect to an isometry A if $c(t)$ is not constant and if $A \circ c(t) = c(t + a)$ for some constant a. If A has finite order then $c(t)$ is automatically closed. Two A-invariant geodesics are *geometrically distinct* if their images (as point sets) do not coincide. We establish

Theorem A. — *On a rationally hyperbolic Riemannian manifold* M *any isometry* A *has infinitely many geometrically distinct invariant geodesics.*

The conclusion of Theorem A does not hold in general for rationally elliptic manifolds: a rotation of the round two sphere has only one invariant geodesic and a generic "rotation" of S^3 has only two. Indeed it is not known if there must always be at least one invariant geodesic. We shall, however, show that this is the case for a large class of rationally elliptic manifolds (Theorem 2, § 3) and, as a corollary, deduce

Theorem B. — *On a* 1*-connected closed Riemannian manifold of odd dimension every isometry has an invariant geodesic.*

The transition from geometry to rational homotopy which is necessary here is accomplished by Tanaka's optimal extension [20] of the main result in [12] on the one hand, and by that of [11] on the other. These, together with non triviality theorems for Whitehead products [5], provide the ingredients for the proof of Theorem A (§ 2). For our results on isometry-invariant geodesics on rationally elliptic manifolds (§ 3) we rely on [10], [13] and [7].

2. Invariant Geodesics on rationally hyperbolic Manifolds

Throughout A denotes an isometry on a fixed (closed 1-connected) Riemannian n-manifold M^n. The space M_A^I of A-invariants paths on M is defined as the paths $\sigma : I \to M$ satisfying $\sigma(1) = A(\sigma(0))$ with the uniform topology. The homotopy type of this space depends only on the homotopy class of A, and the existence theorems for A-invariant geodesics depend only on the homotopy type of M_A^I (cf. [9], [10], [20]).

Since the isometries of M form a compact Lie group [17], A is homotopic to an element of finite order, and so it is sufficient to consider the case that A has order k.

In this case we know from [11] that A yields an automorphism (which we also denote by A) of order k on the Sullivan minimal model over $\mathbf{Q}$, $(\Lambda X_M, d)$, of M, and this automorphism can be taken to preserve X_M. Here ΛX_M is the free graded commutative algebra over the graded space X_M, and the standard isomorphisms

$$H(\Lambda X_M) \cong H^*(M; \mathbf{Q}); \qquad X_M^p \cong \operatorname{Hom}_{\mathbf{Z}}(\pi_p(M), \mathbf{Q})$$

are A-equivariant [11]. Note also that, in view of the second isomorphism, M is *rationally elliptic* if $\dim X_M < \infty$ and *rationally hyperbolic* if $\dim X_M = \infty$.

By a recent theorem of Tanaka [20] generalizing an earlier result of [12] an isometry A has infinitely many invariant geodesics if M_A^I has an unbounded sequence of rational Betti numbers. The main result of [11] states on the other hand that this is the case unless $\dim(X_M^A)^{\text{even}} \leq \dim(X_M^A)^{\text{odd}} \leq 1$, where $X_M^A \subseteq X_M$ is the subspace of vectors fixed by A. In particular if $\dim X_M^A = \infty$ then A has infinitely many invariant geodesics.

In [4] is introduced the notion of rational category of a minimal model, which is finite for models whose cohomology is finite dimensional. The category $\mathrm{cat}_0(M)$ of $(\Lambda X_M, d)$ is the Lusternik-Schnirelmann category of the rationalization M_0 of M ([4]; Theorem (4.7)) and satisfies $\mathrm{cat}_0(M^n) \leq \mathrm{cat}(M^n) \leq n$.

Theorem A in the introduction now clearly follows from

Theorem 1. — Let A *be an automorphism of finite order* k *of a minimal model* $(\Lambda X, d)$ *of finite rational category. Suppose* A *preserves* X *and* $\dim X = \infty$. *Then* $\dim X^A = \infty$.

Proof. — By induction on k. Write $k = pk'$ with $p > 1$ a prime and put $B = A^p$. Then by induction X^B is infinite dimensional. In [11] is described a differential d in ΛX^B and a surjective morphism $\pi : \Lambda X \to \Lambda X^B$ of minimal models. It follows in particular from [4; Theorem (5.1)] that then $\mathrm{cat}_0(\Lambda X^B, d) < \infty$.

Moreover since A induces an automorphism of ΛX^B of order p, and π is equivariant, it suffices to consider the case where k is a prime p.

In this case a recent result of [5; Theorem (1.1)] asserts that the sequence of integers $\dim X^{2\ell+1}$ is unbounded. We can thus pick ℓ so that

$$\dim X^{2\ell+1} > p \cdot m, \qquad m = \mathrm{cat}(\Lambda X, d).$$

Now extend the coefficient field to $\mathbf{C}$ and let $\{x_i\}$ be a basis of X of homogeneous eigenvectors for A, and note that the eigenvalues are contained among the p-th roots of unity, $1, \lambda_2, \ldots, \lambda_p$.

Our condition on $X^{2\ell+1}$ implies that the eigenspace corresponding to some λ_i has dimension bigger than m. If $\lambda_i = 1$, then $\dim(X^A)^{\mathrm{odd}} > \mathrm{cat}(\Lambda X^A, d)$ and [4; Theorem VII] implies that $\dim X^A = \infty$.

Now suppose $\lambda_i \neq 1$. The graded space $L = \Sigma L_j$ defined by $L_j = \mathrm{Hom}(X^{j+1}; \mathbf{C})$ has a canonical structure of a graded Lie algebra, as determined by the quadratic part of the differential in ΛX (cf. [18], [19], [5]). In particular A acts on L by a Lie algebra automorphism of order p. Our hypothesis $\lambda_i \neq 1$ gives $m+1$ linearly independent eigenvectors $\alpha_\nu \in L_{2\ell}$ with $A(\alpha_\nu) = \lambda_i \alpha_\nu$. A second result of [5; Theorem (2.2)] now implies the existence of a $\beta \in L$ such that for all s there is a sequence $\nu_1, \ldots, \nu_s$ with $1 \leq \nu_i \leq m$ and

$$\beta_s = [\alpha_{\nu_s}, [\alpha_{\nu_{s-1}}, [\ldots[\alpha_{\nu_1}, \beta]] \neq 0.$$

Clearly β may be taken to be an eigenvector of A with eigenvalue λ, say. Because p is a prime, λ_i is a primitive root of unity and so $\lambda_i^r = \lambda^{-1}$ for some r. It follows that $\beta_{r+jp} \in L^A$, $j = 0, 1, 2, \ldots$ and so also $\dim X^A = \infty$, q.e.d.

The main result of [11] can be used also to exhibit rationally elliptic manifolds on which any isometry has infinitely many invariant geodesics. We do not, however, have anything like a classification of these manifolds.

3. Invariant Geodesics on rationally elliptic Manifolds

With our results for rationally hyperbolic manifolds in mind we confine ourselves in this section to rationally elliptic manifolds.

Recall [10] that an isometry A on a 1-connected closed Riemannian manifold M has at least one invariant geodesic provided M_A^I is not contractible, or equivalently $\mathrm{id} - A_* : \pi_*(M) \to \pi_*(M)$ is not an isomorphism.

It follows from this that if A has no invariant geodesics then for all finite dimensional A-stable subspaces $Y \subseteq X_M$

$$\det(\mathrm{id}_Y - A|_Y) = 1$$

and hence, in particular (cf. [10], Corollary (1.8)), each such Y has even dimension. (Recall that ΛX_M denotes the Sullivan minimal model over $\mathbf{Q}$ and that $A : X_M \to X_M$ has finite order.)

Theorem B of the introduction follows from Theorem A and

Theorem 2. — Suppose M *is a rationally elliptic Riemannian n-manifold which has an isometry with no invariant geodesics. Then*

(i) *n is even.*
(ii) *If $\chi_M \neq 0$ then $n \equiv 0(4)$ and the Poincaré polynomial $f(t) = \Sigma \dim H^p(M; \mathbf{Q}) t^p$ is the Poincaré polynomial of a space $S \times S$, where S is rationally elliptic. In particular $\chi_M = f(1)$ is a square.*

Proof. — Put $\rho_p = \dim X_M^p$ and note by our remarks above with $Y = X_M^p$ that it is an even integer. Hence by [13]

$$n = \sum_i (2i - 1)(\rho_{2i-1} - \rho_{2i})$$

is also even, and congruent (mod. 4) to $\sum_i \rho_{2i-1} - \rho_{2i}$. Again by [13] this sum is zero if $\chi_M \neq 0$. Finally, if $\chi_M \neq 0$ it follows from [13] that

$$f(t) = \prod_i (1 - t^{2i})^{\rho_{2i-1} - \rho_{2i}}.$$

Write $\sigma_i = \frac{1}{2} \rho_i$. It follows from [7] that

$$g(t) = \prod_i (1 - t^{2i})^{\sigma_{2i-1} - \sigma_{2i}}$$

is the Poincaré polynomial of a rationally elliptic space S. Clearly $f(t) = g(t)^2$ and the theorem follows.

Remark 1. — From Poincaré duality it follows that the Lefschetz number L_A of an isometry A on a closed 1-connected odd-dimensional manifold is even. A result of [9] states on the other hand that if A has no invariant geodesics then A has exactly one fix point. This gives a proof of Theorem B with out using rational homotopy theory.

Remark 2. — By using the full structure of the A-equivariant minimal model $(\Lambda X_M, d)$ we can exhibit many more examples of manifolds on which any isometry has invariant geodesics. On the other hand each of the minimal models $(\Lambda X_{S^3\times S^3}, d)$ and $(\Lambda X_{S^3\times S^3\times S^3\times S^3}, d)$ have automorphisms A of finite order (6 and 12 respectively) satisfying $\det(\mathrm{id}_Y - A|_Y) = 1$ for each A-invariant subspace $Y \subset X_M$. There are also non product models ΛX with finite order automorphisms A satisfying this condition. Take *e.g.* $X = X^2 \oplus X^5 \oplus X^7$ with a basis $\{x_1, x_2\}$ for X^2, $\{y_1, y_2\}$ for X^5 and $\{z_1, z_2\}$ for X^7. Define the differential d as follows

$$dx_1 = 0, \quad dy_1 = x_1^3 + x_1x_2^2 - x_1^2x_2, \quad dz_1 = x_1^3x_2^2(x_1 - x_2)^2$$

$$dx_2 = 0, \quad dy_2 = x_2^3 + x_2x_1^2 - x_1x_2^2, \quad dz_2 = x_1^2x_2^3(x_1 - x_2)^2$$

and define the action of A by

$$Ax_1 = x_2, \quad Ay_1 = y_2, \quad Az_1 = z_2$$

$$Ax_2 = x_2 - x_1, \quad Ay_2 = y_2 - y_1, \quad Az_2 = z_2 - z_1.$$

Then $H(\Lambda X, d)$ has finite dimension and by a theorem of Barge-Sullivan [2] there is a closed manifold M $(\dim M = 22)$ with $(\Lambda X, d)$ as model. We do not, however, know if A is induced by an action on M.

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Matematisk Institut, Københavns Universitet,
Mathematics Department, University of Toronto.

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