PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

SYLVAIN E. CAPPELL
JULIUS L. SHANESON

The topological rationality of linear representations

Publications mathématiques de l'I.H.É.S., tome 56 (1982), p. 101-128
<http://www.numdam.org/item?id=PMIHES_1982__56__101_0>

THE TOPOLOGICAL RATIONALITY OF LINEAR REPRESENTATIONS

by Sylvain E. CAPPELL [1] *and* Julius L. SHANESON [2]

INTRODUCTION

A *real linear representation* of a group G is a map $\theta : G \times V \to V$, with V a real vector space, with $\theta(g, \ ) : V \to V$ a linear map for each $g \in G$, $\theta(e, \ ) =$ identity map of V for e the identity element of G, and $\theta(gh, x) = \theta(g, \theta(h, x))$, g and $h \in G$, $x \in V$; when G is a topological group we assume furthermore that θ is continuous. Two real representations θ_1 and θ_2 of the group G, $\theta_i : G \times V_i \to V_i$, with V_i real vector spaces, are said to be linearly (respectively, *nonlinearly or topologically*) equivalent if there is a linear isomorphism (resp., a *homeomorphism*) $f : V_1 \to V_2$ with $f(\theta_1(g, v)) = \theta_2(g, f(v))$, $v \in V_1$, $g \in G$. When there is such a linear (resp., a *nonlinear*) equivalence we write $\theta_1 = \theta_2$ (resp., $\theta_1 \sim \theta_2$). Clearly, if $\theta_1 = \theta_2$ and $\eta_1 = \eta_2$ (resp., $\theta_1 \sim \theta_2$ and $\eta_1 \sim \eta_2$) then $\theta_1 + \eta_1 = \theta_2 + \eta_2$ (resp., $\theta_1 + \eta_1 \sim \theta_1 + \eta_2$), where θ_1, θ_2, η_1 and η_2 are representations of G, and $\theta_i + \eta_i$ denote as usual the representation of G on the direct sum of the corresponding underlying vector spaces with the coordinatewise action of G.

This paper studies the *topological classification of finite dimensional representations of general finite groups, and of topological groups*. For finite groups, particularly complete results are obtained in a stable range, which is specified. A *topological rationality principle* for real linear representations of finite groups is stated in Theorem 1 and its Corollaries, for 2-groups, and in Theorems 2 and 3, for general finite groups. We then give a complete reduction of the topological classification of linear representations for compact Lie groups to that for finite groups. Similar results are also obtained for orthogonal (or unitary or just bounded) representations of noncompact groups with finitely many components.

In 1935, de Rham [DeR 1], at the International Topology Conference in Moscow, proposed the conjecture that for orthogonal matrices topological equivalence implies

(1) Research supported by NSF Grant No. MCS 76-06974. The hospitality of Université de Paris at Orsay and I.H.E.S. when this work was begun is gratefully acknowledged.

(2) Research supported by NSF Grant MCS 80-26053.

linear equivalence; this would have implied the analogous conjecture for representations of finite or compact groups. Earlier, Poincaré [P] had shown that for rotations of $\mathbf{R}^2$, topological equivalence is the same as linear equivalence; his results carry over to give the same result for linear representations of dimension less than or equal to 2 of finite or compact groups. De Rham [DeR 2] showed that topologically equivalent orthogonal matrices have the same eigenvalues (counted with multiplicity) which are *not* roots of unity; thus, he reduced the general problem of the topological classification of orthogonal matrices to the topological classification of representations of finite cyclic groups. Moreover, using Reidemeister torsion, de Rham showed that if the topological equivalence between two representations of a cyclic group satisfied the homogeneity and smoothness properties,

(i) it preserved the unit spheres of the representation space and
(ii) was a diffeomorphism on the unit sphere,

then the representations were linearly equivalent [DeR 2], [DeR 3], [R]. De Rham thus showed that for general representations of finite or compact groups, topological equivalences which were homogeneous and smooth, in the sense of (i) and (ii) implied linear equivalence; he made an analogous conclusion under such smoothness and homogeneity assumptions for the topological classification of orthogonal matrices. Kuiper and Robbin [KR] studied the general problem of the topological equivalence of matrices and showed that the key case was that of matrices with all the eigenvalues of modulus 1. Furthermore, they reduced this case to that of the topological classification of orthogonal matrices and thus, using de Rham's work, to the topological classification of representations of cyclic groups. They conjectured that in that case topological equivalence was the same as linear equivalence, which would imply an analogous conclusion for all matrices with eigenvalues of modulus 1 as well as for all representations of finite groups, and they adduced further evidence for this conjecture. It was known that topological equivalence and linear equivalence are the same for free representations of finite groups and some other classes of representations, by using the Atiyah-Singer fixed point theorem and Reidemeister torsion (algebraic K-theoretic) methods [AS], [AB], [M], [W 1], [KR], [CS 4]. A finite dimensional representation is called free (resp., pseudofree) if it restricts to a free action on the unit sphere (resp., the unit sphere minus a finite set of points) of the representation space. Using classifying space methods, R. Schultz [Sch] and D. Sullivan proved that topological equivalence and linear equivalence are the same for representations of p-groups, p an odd prime, as well as some other groups. In [CS 9] we verified the conjecture, that topological equivalence is the same as linear equivalence, for all representations of finite groups (and of orthogonal matrices, or of matrices with all eigenvalues of modulus 1) of dimension less than 6. However, the paper [CS 4] gave counterexamples to this conjecture of de Rham, and Kuiper and Robbin, and exhibited pseudofree 9-dimensional real representations of the cyclic group $\mathbf{Z}_{4k}$ which are topologically equivalent but not linearly equivalent, for each $k > 1$.

Homotopy theoretic classification problems for representations of groups, in analogy to de Rham's topological problem discussed above, were proposed by Adams and by Atiyah and Tall and studied in [AT] and by other authors. For connected compact Lie groups Lee and Wasserman [LW], and, for certain nonconnected compact Lie groups, Traczyk [Tr] and Kawakubo [Ka], showed that equivariant homotopy equivalence (and thus topological equivalence) of representations implied linear equivalence. Schultz [Sch], using his results on p-groups cited above, also showed that for certain compact Lie groups, including the connected groups, topological equivalence of representations implies linear equivalence [Sch]. (The treatment of compact Lie groups below specializes to an elementary demonstration of this for connected and certain other Lie groups; see Corollary (5.1).)

The present general results on topological classification of representations of finite groups are stated in § (1.1) below and on compact Lie groups in § (1.2). Here all representations will be finite dimensional. Any such representation of a finite group or, more generally, a compact Lie group, is linearly equivalent to an orthogonal representation; *i.e.*, we may assume that $\theta(g, \ )$ is an orthogonal transformation for any $g \in G$. Correspondingly, in the definition of linear equivalence we may assume that f is an orthogonal map, and thus a norm-preserving, linear isomorphism. The results on nonlinear equivalence in the present paper are also unchanged if we require that the representations be orthogonal and that the nonlinear equivalences f be norm-preserving and homogeneous, *i.e.*, that $||f(v)|| = ||v||$ and $f(\lambda v) = \lambda f(v)$ for $v \in V_1$ and $\lambda \in \mathbf{R}$, $\lambda > 0$. This is a consequence of the following elementary considerations. If θ_1 and θ_2 are orthogonal representations with $\theta_1 \sim \theta_2$, it is easy to replace the equivariant homeomorphism of their representation spaces by one which furthermore preserves the origin. It follows readily that there is a norm-preserving equivariant homeomorphism of the representation spaces of $(\theta_1 + \varepsilon)$ and $(\theta_2 + \varepsilon)$, ε the trivial one-dimensional representation [1]. Note that a norm-preserving equivariant homeomorphism of representation spaces is just the same as an equivariant *homeomorphism* of the unit spheres of the representation spaces. In view of this, it is instructive to contrast our results below on equivariant topological equivalence of unit spheres of representations with de Rham's result that equivariant *diffeomorphism* (or equivariant P.L. homeomorphism) of the unit spheres of the representation spaces, implies linear equivalence [2]. The investigation of the role of differentiability in such problems will be considered in a future study.

For G a finite group, or a compact Lie group, let R(G) denote the *real* representation ring of G (resp., let R Top(G) denote the quotient group of R(G)) consisting of

(1) However, there are pseudofree representations θ_1, θ_2 of Z_{4k}, each $k > 1$, with $\theta_1 \sim \theta_2$ for which there is *no norm-preserving* equivariant homeomorphism of the representation spaces of θ_1 and θ_2.

(2) It is obvious that if this is an equivariant diffeomorphism of the *whole* representation spaces then there is a linear equivalence; just differentiate at the origin. However, for some (though not all) of our examples below of equivariantly homeomorphic but not linearly equivalent representations, the homeomorphism can be taken to be a diffeomorphism except at the origin, which is fixed. In some other examples, the equivariant homeomorphism of the unit spheres can be made a diffeomorphism except at 2 points.

the free abelian group on the linear real representation, modulo the subgroup generated by elements of the form $\eta - \theta_1 - \theta_2$, where $\eta = \theta_1 + \theta_2$ (resp., where $\eta \sim \theta_1 + \theta_2$). Thus two real representations θ_1 and θ_2 of G represent the same element of R(G) (resp., of R Top(G)) if and only if $\theta_1 + \eta = \theta_2 + \eta$ (resp., $\theta_1 + \eta \sim \theta_2 + \eta$) for some real representation η. (The classical cancellation law asserts that $\theta_1 + \eta = \theta_2 + \eta$ is equivalent to $\theta_1 = \theta_2$. The analogous statement for nonlinear equivalence, that $\theta_1 + \eta \sim \theta_2 + \eta$ implies $\theta_1 \sim \theta_2$, is *false* (see [CS 4]); however, it will follow from results below that there is a cancellation law for nonlinear equivalence in many cases in a specified stable range.) It is well known that R(G) is a free abelian group with basis the irreducible real representations of G, and for finite G formulas for the number of these, and thus for rank(R(G)), in terms of intrinsic invariants of G are also well known.

Here we will, in particular, *study* R Top(G) *for general* finite groups, or compact Lie groups G. For a finite group G, an upper bound on rank (R Top(G)) is given in Corollary (3.1) below; for groups of order divisible by 4, this bound is often less than rank (R(G)). Moreover, for many (indeed, perhaps all) finite groups, Corollary (3.1) effectively computes rank (R Top(G)). These results incorporate stabilized versions of our *topological rationality principle* for representations of finite groups. For G a compact Lie group of positive dimension, R Top(G) is then computed by Corollaries (5.3) and (5.4).

An application to a generalization of a conjecture of P. A. Smith [Sm] on the representations obtained on the tangent spaces of the fixed points of smooth finite group actions on manifolds is made in § 2.

The general treatment of topological classification of representations in this paper does not presume familiarity with any of the references cited in this introduction. In particular it is independent of (and does not supersede) the study of pseudofree representations of cyclic groups of [CS 4]. Except for Propositions 7 and 8, only basic results in the representation theory of groups are used. The proof of Proposition 7 uses the *h*-cobordism theorem; that of Proposition 8 on bundles over lens spaces uses methods from the study of non-simply connected manifolds.

CONTENTS

1. RESULTS ON TOPOLOGICAL EQUIVALENCE OF LINEAR REPRESENTATIONS

(1.1) Finite groups and topological rationality

First we consider finite 2-groups in Theorem 1 and its corollary and then representations of general finite groups in Theorems 2 and 3 and Corollaries (3.1) and (3.2). The representations of continuous groups, and in particular the compact Lie groups, are treated in § (1.2).

Suppose now that G is a finite group. By a *rational* representation of a group G is meant a representation over the field of rational numbers, $\mathbf{Q}$, *i.e.* $\theta : G \times V \to V$, V a vector space over $\mathbf{Q}$. The corresponding real representation is also called rational. Let $R_{\mathbf{Q}}(G)$ denote the representation-ring of G defined over the field $\mathbf{Q}$; it is a free abelian group on the irreducible rational representations of G and the number of these is well-known (see *e.g.* [CR], [Se]). We identify $R_{\mathbf{Q}}(G)$ with its image under the natural injection $R_{\mathbf{Q}}(G) \to R(G)$.

Let ρ_G denote the regular representation of G, the real valued functions on G regarded as a real representation. Trivially ρ_G is a rational representation and thus, for ψ a rational representation, $\psi + m\rho_G$ is a rational representation for any integer $m \geq 0$.

The first result describes a specific stable topological equivalence of a multiple of any real representation of a 2-group to a rational representation. By a p-group is meant any group of order a power of p, p a prime.

Theorem 1 (Topological Rationality Principle for 2-groups). — For G *a 2-group and* θ *a real linear representation of* G, *there is a unique rational representation* ψ *of* G *with* $2^{g+1}\theta + \rho_G$ *nonlinearly equivalent to the rational representation* $\psi + \rho_G$ *for* 2^g *the order of the largest cyclic subgroup of* G. *Moreover, for* θ_1 *and* θ_2 *real representations of* G *the following are equivalent:*

(i) $2^{g+1}\theta_1 + \rho_G \sim 2^{g+1}\theta_2 + \rho_G$

(ii) $n\theta_1 + \varphi \sim n\theta_2 + \varphi$, *for some integer* $n > 0$ *and some representation* φ *of* G

(iii) *for each cyclic subgroup* $H \subset G$,

$$\dim(V_1^H) = \dim(V_2^H)$$

where V_i^H *is the subspace of the underlying vector space of the representation space of* θ_i, *fixed by the action of* H

(iv) θ_1 *and* θ_2 *represent the same element of* $R\,\mathrm{Top}(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right)$.

Addendum. — *When* G *is furthermore abelian, the* $(g+1)$ *in Theorem 1 and Corollary* (1.1) *may be replaced by* g.

Corollary **(1.1).** — *Let* G *be a* 2*-group. Then* R Top(G) *is a direct sum of a free abelian group of rank* $q(G)$ *and a finite abelian* 2*-group,* $q(G) =$ *the number of conjugacy classes of cyclic subgroups of* G. *The composite map* $R_{\mathbf{Q}}(G) \to R\,\mathrm{Top}(G)$ *is an injection with cokernel a finite abelian* 2*-group of exponent less than* 2^{g+1} *and order less than* $2^{(g+1)r(G)}$, *with* $r(G)$ *the number of irreducible real representations of* G *and* 2^g *the order of the largest cyclic subgroup of* G.

Corollary **(1.2).** — *Let* G *be a* 2*-group. Then the following are equivalent:*

(i) *Topologically equivalent real representations of* G *are linearly equivalent.*

(ii) *For each* $g \in G$, g *is conjugate to* g^α *for some* $\alpha \equiv \pm 3 \pmod 8$.

The above results are in sharp contrast with the fact that for representations of p-groups with p odd, linear equivalence and nonlinear equivalence are the same [Sch].

For general 2-groups G, in addition to the upper bounds on the 2-torsion in R Top(G) given above, it is also often possible to obtain nontrivial lower bounds on 2-torsion (for all finite groups) by using homotopy-theoretic methods.

Example. — *For each cyclic group* $\mathbf{Z}_{8k}$, $k \geq 1$, *the* 2*-torsion in* R $\mathrm{Top}(\mathbf{Z}_{8k})$ *is not trivial.*

An explicit description is given below of the natural splitting, for 2-groups G, of the map

$$R_{\mathbf{Q}}(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right) \to R(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right)$$

from

$$R(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right) \to R\,\mathrm{Top}(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right) = R_{\mathbf{Q}}(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right).$$

Note that for any representations θ_1 and θ_2 of any finite group G, it is elementary that (i) or (ii) or (iv) of Theorem 1 imply (iii). Hence, for any finite group G for which the number of irreducible rational representations equals the number of irreducible real representations, $R(G) \to R\,\mathrm{Top}(G)$ is an isomorphism [Ka]. This applies, for example, to 2-groups without an element of order 8 [KR], or to symmetric groups.

Now we consider *arbitrary finite groups*. Let K be the subfield of R given by

$$K = \mathbf{Q}(\{(\mu + \mu^{-1}) \mid \mu \text{ is an odd root of } 1\}).$$

Let $R_K(G)$ denote the representation ring of G defined using the representations of G over the field K. This is a free abelian group on the irreducible representations of G over the field K and a formula for the number of these, in terms of intrinsic invariants of G, is well known. It is a classical fact (*e.g.* [CR], [Se]) that the natural inclusion for any G

$$R_K(G) \to R(G)$$

is an isomorphism when G is of odd order, or more generally when G has no element of order $4k$, $k > 1$.

For any algebraic number $x \in E$, E a finite field extension of K (*e.g.*, $E = K(x)$), set

$$\widehat{\mathrm{Tr}}(x) = \frac{1}{[E:K]} \mathrm{Tr}_{E/K}(x)$$

where $\mathrm{Tr}_{E/K}(x)$ denotes the trace of the K-linear map given by multiplication by x in E (see [L]). It is easy to check that $\widehat{\mathrm{Tr}}(x)$ is independent of the choice of E.

Trivially, the regular representation ρ_G, or any multiple of it, can be defined over the field K. For θ a real representation of the group G, let χ_θ denote the character of θ; *i.e.*, $\chi_\theta : G \to R$ is the function $\chi_\theta(g) = \mathrm{trace}(\theta(g, \ ))$. For any real representation θ we will specify at the end of this section, using the Brauer theory of representations, a positive integer m_θ; if G is abelian, or more generally supersolvable [1], set $m_\theta = 1$.

Theorem 2 (Topological Rationality Principle, first form). — Let θ be a real representation of G, a finite group. Let 2^g be the order of the largest cyclic 2-subgroup of G. Then $2^{g+2}\theta + m\rho_G$, for $m \geq m_\theta$, is nonlinearly equivalent to the unique representation φ defined over K with $\chi_\varphi = \widehat{\mathrm{Tr}}(\chi_{(2^{g+2}\theta + m\rho_G)})$. Moreover, for θ_1 and θ_2 two real representations of G,

(A) *if $\widehat{\mathrm{Tr}}(\chi_{\theta_1})(g) = \widehat{\mathrm{Tr}}(\chi_{\theta_2})(g)$ for all $g \in G$, then $2^{g+2}\theta_1 + m\rho_G \sim 2^{g+2}\theta_2 + m\rho_G$, for $m \geq \max(m_{\theta_1}, m_{\theta_2})$;*

(B) *and conversely, provided that for each cyclic subgroup of odd order $\mathbf{Z}_{2k+1} \subset G$,*

$$R(\mathbf{Z}_{2k+1}) \to R\ \mathrm{Top}(\mathbf{Z}_{2k+1})$$

is injective, if $n\theta_1 + \psi \sim n\theta_2 + \psi$ for some integer $n > 0$ and representation ψ of G, then $\widehat{\mathrm{Tr}}(\chi_{\theta_1})(g) = \widehat{\mathrm{Tr}}(\chi_{\theta_2})(g)$ for all $g \in G$.

Addenda. — (1) If G is a supersolvable finite group [1] there is the following stronger result. For each real representation θ of G there is a unique representation ψ of G defined over K with $\chi_\psi = \widehat{\mathrm{Tr}}(\chi_{2^{g+2}\theta}) = 2^{g+2}\widehat{\mathrm{Tr}}(\chi_\theta)$ and $2^{g+2}\theta + \rho_G$ topologically equivalent to the K-representation $\psi + \rho_G$.

(2) If G is also abelian (or more generally if G is supersolvable and has a 2-Sylow subgroup contained in the center of G), the $g+2$ in Theorem 2, Addendum 1 and Theorem 3 below can be replaced by $g+1$.

Now consider the composite map

$$R_K(G) \to R(G) \to R\ \mathrm{Top}(G).$$

[1] Or even if G is just a semidirect production of an abelian group by a supersolvable group.

Theorem 3 (Topological Rationality Principle, second form).

(A) *For any finite group* G, $R_K(G) \to R\,\mathrm{Top}(G)$ *has cokernel a finite abelian* 2*-group. Moreover, this cokernel has exponent less than* 2^{g+2} *and order less than* $2^{(g+2)\,r(G)}$ *where* 2^g *is the order of the largest cyclic* 2*-subgroup of* G *and* $r(G)$ *is the number of irreducible real representations of* G.

(B) $R_K(G) \to R\,\mathrm{Top}(G)$ *is also injective, provided that this is true for each cyclic subgroup of odd order* $\mathbf{Z}_{2k+1} \subset G$.

Explicitly, when for each cyclic subgroup $\mathbf{Z}_{2k+1}$ of odd order of G,

$$R(\mathbf{Z}_{2k+1}) \to R\,\mathrm{Top}(\mathbf{Z}_{2k+1})$$

is injective, the inverse of the injection

$$R_K(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right) \to R(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right)$$

given by $R(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right) \to R\,\mathrm{Top}(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right) \cong R_K(G) \otimes \mathbf{Z}\left(\frac{1}{2}\right)$ is described on the level of characters as $\chi \to \widehat{\mathrm{Tr}}(\chi)$. This applies, as described below, at least to all groups of order $2^a p^b$.

Corollary (**3.1**). — *For* G *a finite group, let* $s(G)$ *be the number of equivalence classes of elements of* G *under the equivalence relation* $\approx$, *where* $g \approx h$ *if* g *or* g^{-1} *is conjugate in* G *to* h^{1+2jb} *for some* j, b *the largest odd divisor of the exponent of* g. *Then,*

(A) $\mathrm{rank}\,R\,\mathrm{Top}(G) \leq s(G)$;

(B) $\mathrm{rank}\,R\,\mathrm{Top}(G) \geq s(G)$ *provided that for each cyclic subgroup of odd order* $\mathbf{Z}_{2k+1} \subset G$,

$$R(\mathbf{Z}_{2k+1}) \to R\,\mathrm{Top}(\mathbf{Z}_{2k+1})$$

is injective.

Recall that for p an odd prime and $a > 0$,

$$\mathrm{rank}(R(\mathbf{Z}_{2^a p^b})) = 2^{a-1} p^b + 1.$$

Example. — *For* p *an odd prime,* $\mathrm{rank}(R\,\mathrm{Top}(\mathbf{Z}_{2^a p^b})) = \dfrac{(a+1)(p^b+1)}{2}$; *the torsion subgroup of* $R\,\mathrm{Top}(\mathbf{Z}_{2^a p^b})$ *is a* 2*-group of order less than* $2^{a2^a p^b}$.

Recall that for any character χ of a complex representation of a finite group G, χ takes values in the field $Q\{\mu \mid \mu \text{ is a root of } 1\}$; hence, the real part of χ, $\mathrm{Re}(\chi)$ takes values in the field $Q\{\mu + \bar{\mu} \mid \mu \text{ is a root of } 1\}$.

Corollary (1.2) above determined for which 2-groups topological equivalence and linear equivalence of representations are the same. Consider this problem for general finite groups.

Corollary **(3.2)**. — *Let* G *be a finite group.*

(A) *Then,* (i) *implies* (ii) *where:*

(i) *Real representations of* G *which are topologically equivalent are linearly equivalent.*

(ii) *For every character* χ *of a complex representation of* G, $\mathrm{Re}(\chi)$ *takes values in the field* $K = \mathbf{Q}\{\mu + \bar{\mu} \mid \mu$ *is an odd root of* $1\}$.

(B) *Conversely, provided that for each odd order cyclic subgroup* $\mathbf{Z}_{2k+1} \subset G$,

$$R(\mathbf{Z}_{2k+1}) \to R\,\mathrm{Top}(\mathbf{Z}_{2k+1})$$

is injective, (ii) *implies* (i).

Parts (A) of Theorems 2, 3, (3.1) and (3.2) assert the existence of many topological equivalences of representations. Parts (B) assert that in a stable range, subject to a condition on the odd order cyclic subgroups $\mathbf{Z}_{2k+1}$ of G, these are all the topological equivalences up to a certain 2-group. The hypothesis in Parts (B) on the injectivity of $R(\mathbf{Z}_{2k+1}) \to R\,\mathrm{Top}(\mathbf{Z}_{2k+1})$ has been proved at least for $2k+1$ an odd prime-power [Sch] and perhaps other cases. Thus, all parts of Theorems 2, 3, (3.1) and (3.2) apply completely to all groups of order $2^a p^b$, p a prime.

Added Note. — Two pairs of researchers, W. Pardon and W. C. Hsiang, and H. I. Madsen and M. Rothenberg have announced that, in fact, for all groups of odd order, topological equivalence of representations implies linear equivalence. Assuming this, the hypothesis, in parts B of Theorems 2 and 3 and their Corollaries, on $R\,\mathrm{Top}(\mathbf{Z}_{2k+1})$, would obviously be always satisfied, and thus could just be dropped.

It would be interesting to complete the topological classification of representations of finite groups outside our stable range. If this were done for cyclic groups, it would complete the topological classification of orthogonal matrices [DeR 2], [KR], [CS 2, 3, 4, 9]. For special classes of representations of cyclic groups this was done in [CS 4]. Carrying this out for general representations of finite groups would give, in particular, a complete calculation of the torsion of R Top(G). We conjecture that the size of the torsion of $R\,\mathrm{Top}(\mathbf{Z}_{2^a p^b})$ depends heavily on the parity of the class number of the cyclotomic field of p-th roots of unity; in particular we think it may behave differently for $\mathbf{Z}_{116}$ than it does for $\mathbf{Z}_n$, $n < 112$ (cf. [CS 10]). A penetrating attack, using representation theory, topology and number theory is called for to complete the solution of the *topological canonical form problem*, analogous to classical Jordan canonical form for matrices.

Note that (i) of Theorem 1 and Part (A) of Theorem 2, as well as the addenda to these results, can be viewed as, in particular, describing a stable range for the topological classification of the linear representations of a finite group G. The number m_θ used there in giving a lower bound to this stable range could be specified as follows. Write $\theta_{\mathbf{C}} = a - b$, where $\theta_{\mathbf{C}}$ is the complexification of the representation θ and a and b are sums of representations induced from 1-dimensional complex representations of subgroups of G; the existence of a and b is a consequence of the Brauer Theory of representations [CR], [Se]. Set [1] $m_\theta = 1 + 2^{g+2}(\dim b)$; clearly when G is abelian, or more generally, supersolvable, we may take $m_\theta = 1$.

The proofs of the results of this section begin with a special case, Proposition 7, which is stated in § 3 and proved there using geometrical topology. In § 4 this result, together with representation-theoretic methods, are used in proving Theorem 2. The remaining results on finite groups are then derived.

(1.2) Continuous Groups

Now we consider finite-dimensional representations of topological groups with finitely many components. Let θ be a representation of such a group G, with character denoted χ_θ. Let G_0 be the connected component of the identity element of G. The fixed points θ^{G_0} of the action of G_0 on the representation space of θ form a subrepresentation of θ. Moreover, θ^{G_0} can be regarded as a representation of the *finite* quotient group G/G_0. The topological classification of orthogonal representations of G will be reduced to those of the finite group G/G_0, which was treated in § (1.1).

Theorem 4. — Let θ be an orthogonal (or unitary, or just a bounded) representation of a topological group G. *Suppose* G *has finitely many connected components and let* G_0 *denote the connected component of the identity. Then a representation η of* G *is topologically equivalent to θ if and only if the following three conditions hold*:

(i) *η is a bounded representation*;

(ii) *there is an equation of characters*

$$\chi_\theta - \chi_{\theta^{G_0}} = \chi_\eta - \chi_{\eta^{G_0}};$$

(iii) *the representations θ^{G_0} and η^{G_0} of the finite group G/G_0 are topologically equivalent.*

Recall that a representation η is said to be bounded if $\|\eta(g, x)\|$ is bounded for $\|x\| = 1$, $g \in G$. (The choice of the norm $\| \ \|$ on the representation space of η does not matter.)

Corollary (4.1). — Let G *be a topological group with finitely many components. Let* G_0 *denote the connected component of the identity. Suppose that for representations of the finite group* G/G_0 *topological equivalence implies linear equivalence. Then the same is true for orthogonal (or unitary, or just bounded) representations of* G.

Example. — For $A = (A_1, \ldots, A_n)$ a sequence of commuting m by m real matrices, let $\varphi_A = \varphi$, $\varphi : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$, *e.g.* $\varphi(v, t) \in \mathbf{R}^m$ for $v \in \mathbf{R}^m$, $t = (t_1, \ldots, t_n) \in \mathbf{R}^n$ be the (unique) solution of the differential equations $\partial\varphi/\partial t_i = A_i \varphi$, $i = 1$ to n, with

[1] It is easy, using more elaborate procedures, to often specify lower values for m_θ than that given here.

initial condition $\varphi(v, 0) = v$. Here φ_A and φ_B, where $B = (B_1, \ldots, B_n)$, are said to be topologically equivalent if there is a homeomorphism $f: \mathbf{R}^m \to \mathbf{R}^m$ (which may be assumed to preserve the origin) with $\varphi_A(f(v), t) = f(\varphi_B(v, t))$. If the matrices A_i are skew-symmetric (or, more generally, are diagonalizable and have purely imaginary eigenvalues) by applying the above corollary to $G = \mathbf{R}^n$, we see that φ_A and φ_B are topologically equivalent if and only if there is a nonsingular matrix C with

$$CA_iC^{-1} = B_i, \quad \text{for } 1 \leq i \leq n. \tag{1}$$

However, standard elementary examples with hyperbolic flows (*e.g.* in [A]) show that when A has nonimaginary eigenvalues, equation (1) often fails. For a complete earlier discussion of the case $n = 1$ from another perspective, see the result of Kuiper [K 1]; some other related results are in [Ma], [CKP], [I], [K 2], [K 3].

We are now able to state quite complete results for the compact Lie groups.

Theorem 5. — Let G *be a compact Lie group with* G_0 *the connected component of the identity. Let* θ_1 *and* θ_2 *be linear representations of* G. *Then* θ_1 *is topologically equivalent to* θ_2 *if and only if*

(i) $\chi_{\theta_1} - \chi_{\theta_1^{G_0}} = \chi_{\theta_2} - \chi_{\theta_2^{G_0}}$ *and*

(ii) *the representations* $\theta_1^{G_0}$ *and* $\theta_2^{G_0}$ *of the finite group* G/G_0 *are topologically equivalent.*

Thus, this result reduces the topological classification of linear representations of compact Lie groups to that of finite groups, which has already been treated above. Condition (i) means that if we decompose $\theta_i = \theta_i^{G_0} + \theta_i'$, $i = 1, 2$, then θ_1' and θ_2' are linearly similar.

The proof of Theorem 5 and its corollaries and the result of de Rham on which it is based are elementary. A related homotopy theoretic result from which it also follows was given in [Tr] along with Corollary (5.1).

Corollary **(5.1)**. — *Let* G *be a compact Lie group with* G_0 *the connected component of the identity. Then the following are equivalent statements:*

(i) *Representations of* G *which are topologically equivalent are linearly equivalent.*

(ii) *Representations of the finite group* G/G_0 *which are topologically equivalent are linearly equivalent.*

The conditions for (ii) to be satisfied were discussed in Corollaries (1.2) and (3.2) above. For example, for representations of compact Lie groups with fewer than 8 components, linear equivalence and topological equivalence are the same. For various classes of Lie groups, that topological equivalence implies linear equivalence was given in [LW], [Sch], [Tr], [Ka].

Call an element g of a compact connected Lie group G *hyper-regular* if the closure of the group generated by g is a maximal torus of G. The hyper-regular elements of G are easily seen to be dense; their complement has measure 0.

Corollary (5.2). — *Let* G *be a compact connected Lie group and* g *a hyper-regular element of* G. *If* θ *and* η *are representations of* G *with* $\theta(g, \ )$ *and* $\eta(g, \ )$ *topologically equivalent matrices, then* θ *and* η *are linearly equivalent representations of* G.

Corollary (5.3). — *Suppose* G *is a compact Lie group. Set* K *equal to the finite group which is the quotient of* G *by the connected component of the identity. Then*

$$\mathrm{R\,Top}(\mathrm{G}) = \mathrm{R}(\mathrm{G})/(\mathrm{Ker}(\mathrm{R}(\mathrm{K}) \to \mathrm{R\,Top}(\mathrm{K}))).$$

Recall that as K is a finite group, R Top(K) and thus the kernel

$$\mathrm{Ker}(\mathrm{R}(\mathrm{K}) \to \mathrm{R\,Top}(\mathrm{K}))$$

was calculated in Theorem 3 and its corollary above.

Corollary (5.4). — *Given* $\mathrm{G} = \mathrm{G}_0 \times \mathrm{K}$, G_0 *a connected compact Lie group and* K *a finite group. Set* $\mathrm{R}(\mathrm{G}) = \mathbf{Z} \oplus \widetilde{\mathrm{R}}(\mathrm{G})$, $\widetilde{\mathrm{R}}(\mathrm{G})$ *the reduced representation group. Then* $\mathrm{R\,Top}(\mathrm{G}) \cong (\widetilde{\mathrm{R}}(\mathrm{G}) \otimes \mathrm{R}(\mathrm{K})) \oplus \mathrm{R\,Top}(\mathrm{K})$ *(where* R Top(K) *is as in Theorem 3 and Corollary* (3.1)*).*

Using furthermore our result that topological similarity and linear similarity of orthogonal matrices are the same in dimension less than 6 [CS 9], we have:

Corollary (5.5). — *Let* θ_1 *and* θ_2 *be representations of the compact Lie group* G *with* G_0 *the connected component of the identity of* G. *Suppose* $\theta_1 \sim \theta_2$ *and* $\dim(\theta_i^{\mathrm{G}_0}) < 6$. *Then* θ_1 *and* θ_2 *are linearly equivalent.*

It is trivial that if θ_1 and θ_2 are linear real representations of a finite or compact Lie group G and their restrictions to each cyclic subgroup H of G are linearly equivalent then θ_1 and θ_2 are themselves linearly equivalent. D. Kazhdan has asked us if our methods could verify if this remains valid when linear equivalence is replaced by topological equivalence of representations. From Corollary (5.2) this is clearly valid for compact connected groups. A counterexample for finite groups will be given, and related problems investigated, in [CS 7].

2. AN APPLICATION TO SMOOTH GROUP ACTIONS MANIFOLDS

Let $f: \Sigma \to \Sigma$ be a diffeomorphism of period n on a (homology) sphere Σ. A well known conjecture of P. A. Smith [Sm], [AB] asserted that if f has isolated fixed points, the representations of $\mathbf{Z}_n$ obtained on the tangent spaces of the fixed points of f would be linearly equivalent. In [CS 6] we gave counterexamples to this; however, we conjectured in general that at least for those diffeomorphisms, or group actions, which we called " of Smith type ", the representations would be *topologically* equivalent. An action of G is said to be " of Smith type " if for each cyclic subgroup H of G, the set of fixed points of H on Σ is discrete or connected (1). (If Σ is a (mod p) homology sphere every diffeomorphism of period a power of the prime p is, by Smith theory, of Smith type) (2).

Petrie [Pe 2], [Pe 3] had earlier announced examples of smooth actions of some highly noncyclic groups on spheres with isolated fixed points for which the representations at fixed points were not linearly equivalent. Those examples were not " of Smith type ".

In [CS 6] we proved our revised form of this conjecture of P. A. Smith for periodic diffeomorphisms of homology spheres which are free outside a subset of dimension less than 2 in Σ. Here we prove a general stabilized form of this revised conjecture, modulo 2-torsion in R Top(G).

Theorem 6. — Let x and y be fixed points of a smooth action of a finite group G *on a* (mod 2) *homology sphere* Σ. *Assume that for each cyclic subgroup* H *of* G, *the fixed point set of* H *is discrete or connected* (3). *Then letting* θ_x *and* θ_y *denote the representations of* G *on the tangent space of* Σ *at* x *and* y,

$$\theta_x = \theta_y \quad \textit{in } \mathrm{R\,Top(G)} \otimes \mathbf{Z}\left(\frac{1}{2}\right).$$

The hypothesis in Theorem 6 on the fixed points of cyclic subgroups, *i.e.* that the action be of " Smith type ", cannot be eliminated. This can be seen by considering Petrie's examples of some actions of noncyclic groups on spheres [Pe 2], [Pe 3] mentioned above.

---

(1) That is, for each $g \in \mathrm{G}$, the set of fixed points of g is discrete or connected.

(2) Similarly, if f has isolated fixed points and is of period pq, p and q prime, on a (mod pq) homology sphere Σ, then f is of Smith type.

(3) It suffices to replace " or connected " by " or x and y are in the same connected component of the fixed points of H ".

Note that by Smith theory the hypothesis on cyclic subgroups H is often satisfied, *e.g.* for H a 2-group, or the action having only isolated fixed points with G of order pq with p and q distinct primes, etc.

Remark. — In Theorem 6, if furthermore G is a group without elements of order $4k$, $k \geq 2$ (or, more generally, a group in which each $g \in \mathrm{G}$ of exponent $4k$ with $k \geq 2$ is such that g or g^{-1} is conjugate to g^m for each $m \equiv 1 \pmod{2b}$, $0 < m < 4k$, b the largest odd factor of k), then $\theta_x = \theta_y$ in $\mathrm{R(G)}$. This is an extension of the results of [AS], [AB], [M], [Sa] on some actions of groups of odd order. On the other hand we have examples [CS 6] of periodic diffeomorphisms of period 2^r on homotopy spheres, $r \geq 3$, with two isolated fixed points and free outside a circle containing these two points, with $\theta_x \neq \theta_y$ in $\mathrm{R(G)}$.

The conclusion of Theorem 6 could be refined to state that $2^{g+2}(\theta_x - \theta_y) = 0$ in $\mathrm{R\,Top(G)}$, where 2^g is the order of the largest cyclic 2-subgroup of G ([1]). We still conjecture the topological equivalence $\theta_x \sim \theta_y$, or at least the stable topological equivalence, $\theta_x = \theta_y$ in $\mathrm{R\,Top(G)}$. (However, we have examples, even for G cyclic, for which the unit spheres of the representation spaces are not equivariantly homeomorphic.)

Theorem 6 was announced without proof in [CS 5]. Its proof, using Theorem 2, is given at the end of § 4 below.

([1]) When G is supersolvable with central 2-Sylow subgroup, or is a 2-group (resp., an abelian 2-group), the 2^{g+2} can be improved to 2^{g+1} (resp., 2^g).

3. GEOMETRICAL TOPOLOGY FOR A SPECIAL CASE

Let σ_k denote the 2-dimensional real representation of the cyclic group $\mathbf{Z}_{2m}$ given by

$$\sigma_k(g) = \begin{pmatrix} \cos\frac{\pi k}{m} & \sin\frac{\pi k}{m} \\ -\sin\frac{\pi k}{m} & \cos\frac{\pi k}{m} \end{pmatrix}$$

for g the generator of $\mathbf{Z}_{2m}$. Let δ_{-1} denote the unique nontrivial one-dimensional real representation of $\mathbf{Z}_{2m}$.

This section is devoted to proving the following preliminary proposition, which starts the argument used in section 4. This proposition could also be obtained from our general detailed results on pseudofree representations of cyclic groups [CS 4] but the proof given here of this needed special case is simpler and much shorter.

Proposition 7. — Suppose m is even. Then $4\sigma_k + \delta_{-1} \sim 4\sigma_{m+k} + \delta_{-1}$.

This will be derived from the next proposition. Let $L_k(2m)$ denote the 7-dimensional lens space which is the quotient of the unit sphere of the representation space of $4\sigma_k$ by the action of $\mathbf{Z}_{2m}$. Let E_k denote the total space of the unique nontrivial interval bundle over $L_k(2m)$. There are canonical isomorphisms $\pi_1(L_k(2m)) \cong \pi_1(E_k) \cong \mathbf{Z}_{2m}$. Note that the boundary of E_k is a 2-fold cover of $L_k(2m)$, and thus $\partial E_k \cong L_k(m)$.

An h-cobordism of the compact manifolds with boundary M_0^n and M_1^n is a triple $(W^{n+1}; M_0, M_1)$ with W a compact manifold and $M_0 \cup M_1 \subset \partial W$ with $\partial M_i \to (\partial W - \text{interor } (M_0 \cup M_1))$ and $M_i \to W$ homotopy equivalences, for $i = 0, 1$.

Proposition 8. — There is an h-cobordism $(W^9; E_k, E_{m+k})$ *inducing the canonical isomorphisms* $\pi_1(E_k) \cong \mathbf{Z}_{2m} \cong \pi_1(E_{m+k})$.

Here we only check and need that W is a topological manifold. With more work, this proposition could be verified in the differentiable category [cf. CS 6] thus eliminating the apparent use of the theory of topological manifolds.

The proof of Proposition 8 beings with the construction of a cobordism from E_k to E_{m+k} using the following lemma.

Lemma 8A. — Suppose m is even. There is a homotopy equivalence $f : L_{m+k}(2m) \to L_k(2m)$ *compatible with the given identifications* $\pi_1(L_{m+k}(2m)) \cong \mathbf{Z}_{2m} \cong \pi_1(L_k(2m))$. *Moreover, f is normally cobordant to the identity map of* $L_k(2m)$.

Proof of Lemma 8A. — The criteria for the existence of a homotopy equivalence f of lens spaces, as recalled for example in [M], is easily verified:

$$(m+k)^4 \equiv k^4 (\mathrm{mod}\ 2m).$$

Note that the map induced by f on twofold covers, $\hat{f}\colon L_{k+m}(m) \to L_k(m)$ is homotopic to the linear diffeomorphism of these lens spaces.

Next to see that f is topologically normally cobordant to the identity, recall that a PL (resp., topological) normal cobordism class of a lens space is determined by the even and odd components, respectively of the normal invariant [W 1; § 14]. In the present instance, as the restrictions of the representations σ_k and σ_{k+m} to $\mathbf{Z}_m \subset \mathbf{Z}_{2m}$ are equal, the map induced by f on twofold covers, $\hat{f}\colon L_{k+m}(m) \to L_k(m)$ is evidently homotopic to a linear diffeomorphism; in particular, $\hat{f}$ has zero normal invariant. But, to determine the odd part of the normal invariant, we may replace f by its 2-fold cover $\hat{f}$. Hence, the odd part of the normal invariant of f is zero. For studying the even part of the normal invariant, we may replace $L_k(2m)$ by its largest odd-fold covering space. Thus we may as well assume $2m = 2^a$, $a \geq 2$. When $a = 2$, f is homotopic to a linear diffeomorphism; suppose $a \geq 3$. The PL (resp., topological) normal cobordism class of such a 7-dimensional lens space is determined [W 1; § 14] by well defined splitting invariants, which are

(i) Kervaire-Arf invariants, a_1 and a_2 in $\mathbf{Z}_2$, of submanifolds of dimension 2 and 6, and
(ii) a signature invariant $s_1 \in \mathbf{Z}_{4m}$ (resp., $\mathbf{Z}_{2m}$) of a generalized submanifold in dimension 4.

But, the splitting invariants of the PL normal cobordism class of $\hat{f}$ are those of f, but with s_1 taken mod $2m$. Hence, $a_1 = 0 = a_2$ and $s_1 = 0\ (\mathrm{mod}\ 2m)$. Therefore the topological splitting invariants of f in $\mathbf{Z}_2$, $\mathbf{Z}_2$, $\mathbf{Z}_{2m}$ are 0, and f is topologically normally cobordant to the identity of $L_k(2m)$.

Proof of Proposition 8. — Lemma 8A above produces a topological normal cobordism V from $L_k(2m)$ to $L_{m+k}(2m)$ given by a map of triples

$$(V; L_k(2m), L_{m+k}(2m)) \to (L_k(2m) \times I; L_k(2m) \times 0, L_k(2m) \times 1)$$

with the restrictions to the boundary components being homotopy equivalences.

We may assume, if necessary by taking a connected sum of V with a Milnor manifold, that the signature of V is zero. This normal cobordism represents an element $\sigma(V)$ of the surgery obstruction group to obtaining a homotopy equivalence, $\sigma(V) \in L_8^h(\mathbf{Z}_{2m})$. Taking induced nontrivial line bundles over $L_k(2m)$ and V, we obtain a normal cobordism of the identity of E_k to the homotopy equivalence $(E_{k+m}, \partial E_{k+m}) \to (E_k, \partial E_k)$. To replace this normal cobordism by an h-cobordism, it suffices to check that its surgery obstruction in the surgery obstruction group for homotopy equivalences $\psi!\,\sigma(V) \in L_9^h(\mathbf{Z}_{2m}^-, \mathbf{Z}_m)$ is zero. Here $\psi!$ is the map $\psi!\colon L_8^h(\mathbf{Z}_{2m}) \to L_9^h(\mathbf{Z}_{2m}^-, \mathbf{Z}_m)$ corresponding to taking induced line bundles (cf. [CS 1] for a discussion of $\psi!$ in the context of simple homotopy equivalences); the notation $\mathbf{Z}_{2m}^-$ denotes $\mathbf{Z}_{2m}$ with the nontrivial orientation character.

Now, Lemma 8B below states that $(\sigma(V)) = \alpha(y)$ for some $y \in L_8^s(\mathbf{Z}_{2m})$, where α is the natural map from the surgery obstruction group for simple homotopy equivalences, $\alpha : L_8^s(\mathbf{Z}_{2m}) \to L_8^h(\mathbf{Z}_{2m})$. Moreover, in the diagram

$$(1) \qquad L_8^s(\mathbf{Z}_{2m}) \xrightarrow{\psi!} L_9^s(\mathbf{Z}_{2m}^-, \mathbf{Z}_m) \xrightarrow{\partial} L_8^s(\mathbf{Z}_m),$$

where $\partial \circ \psi! = t$ is the transfer homomorphism to the 2-fold cover, comparison of the exact sequences (1.3) and (3.6) of [CS 1] show that $\operatorname{Ker}(\psi!) = \operatorname{Ker}(t)$. (Both kernels are shown there to be the image of the same map from a generalized Browder-Livesay group.) Therefore, in the diagram

$$\begin{array}{ccccc} L_8^s(\mathbf{Z}_{2m}) & \xrightarrow{\psi!} & L_9^s(\mathbf{Z}_{2m}^-, \mathbf{Z}_m) & \xrightarrow{\partial} & L_8^s(\mathbf{Z}_m) \\ \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle\alpha} \\ L_8^h(\mathbf{Z}_{2m}) & \xrightarrow{\psi!} & L_9^h(\mathbf{Z}_{2m}^-, \mathbf{Z}_m) & \xrightarrow{\partial} & L_8^h(\mathbf{Z}_m) \end{array} \quad , \quad t = \partial \circ \psi!$$

to show that $\psi!(\sigma(V)) = 0$, it suffices to check that $\psi!(y) = 0$ and thus that $t(y) = 0$. But $L_8^s(\mathbf{Z}_m)$ is detected by two invariants;

(1) the signature invariant $L_8^s(\mathbf{Z}_m) \to L_8^s(e) = \mathbf{Z}$, and

(2) a multi-signature invariant, to the reduced representation ring of $\mathbf{Z}_m$, of the Atiyah-Singer type [Pe 1], [W 1], [W 2].

These invariants are actually defined even on $L_8^h(\mathbf{Z}_m)$. Thus it suffices to check that these invariants of $t(\sigma(V)) \in L_8^h(\mathbf{Z}_m)$ are zero.

Clearly, these invariants of the transfer $t(\sigma(V))$ are the multisignature and signature invariants of the 2-fold cover $\hat{V}$ of V; here $\hat{V}$ is a normal cobordism of ∂E_k to ∂E_{k+m}. Note that ∂E_k (resp., ∂E_{k+m}) is the 2-fold cover $L_k(m)$ (resp., $L_{k+m}(m)$) of $L_k(2m)$ (resp., $L_{k+m}(2m)$); the given homotopy equivalence of ∂E_{k+m} to ∂E_k can be identified with $\hat{f}$, with $\hat{f}$ as in the proof of Lemma 8A above homotopic to a linear diffeomorphism. But by a standard application of the Atiyah-Singer G-signature theorem [AS], [Pe 1], [W 1, § 13, 14] the multi-signature invariant of a cobordism $\hat{V}$ is given by a difference of the Atiyah-Singer ρ-invariants of the lens spaces on the boundary of $\hat{V}$. As $\partial E_k = L_k(m) = L_{k+m}(m) = \partial E_{k+m}$, the difference is zero and hence the multi-signature invariant of $\hat{V}$ is zero.

Last, letting I denote signature or index, we check that $I(\hat{V})$ is zero. Recall that $I(V) = 0$. Now it is a well known consequence of the Thom-Hirzebruch signature formula that the signature is multiplicative for covering spaces of closed manifolds.

The two components of the boundary of $\hat{V}$ (resp., V) $L_k(m)$ and $L_{k+m}(m)$ (resp., $L_k(2m)$ and $L_{k+m}(2m)$) are diffeomorphic (resp., diffeomorphic by a map which changes the generator of $\pi_1(L_k(2m))$. Let $\hat{V}'$ (resp., V') be the closed manifold obtained by identification of boundary components using this diffeomorphism. Clearly, as lens spaces are rational homology spheres, $I(V) = I(V')$, $I(\hat{V}) = I(\hat{V}')$. As $\hat{V}'$ is the 2-fold cover of the closed manifold V', $I(\hat{V}') = 2I(V')$. Thus, $I(\hat{V}) = I(\hat{V}') = 2I(V') = 2I(V) = 0$.

Lemma 8B. — $\sigma(V) \in$ *Image*$(L_8^s(\mathbf{Z}_{2m}) \xrightarrow{\alpha} L_8^h(\mathbf{Z}_{2m}))$.

Proof of Lemma 8B. — From the Rothenburg exact sequence [Sh]

$$L_8^s(\mathbf{Z}_{2m}) \xrightarrow{\alpha} L_8^h(\mathbf{Z}_{2m}) \xrightarrow{\beta} H^8(\mathbf{Z}_2 ; \mathrm{Wh}(\mathbf{Z}_{2m}))$$

it suffices to check that $\beta(\sigma(V)) = 0$. Now, as the determinant map det is an isomorphism and γ is injective,

$$\mathrm{Wh}(\mathbf{Z}_{2m}) \xrightarrow{\det} U(\mathbf{Z}[\mathbf{Z}_{2m}])/\{\pm t^i\} \xrightarrow{\gamma} U(\mathbf{Z}[\mathbf{Z}_{2m}]/(\Sigma))/\{\pm t^i\}$$

the $\mathbf{Z}_2$-action on $\mathrm{Wh}(\mathbf{Z}_{2m})$ is trivial (see [M]), U = units, $\Sigma = (1 + t + \ldots + t^{2m-1})$. Thus, $H^8(\mathbf{Z}_2 ; \mathrm{Wh}(\mathbf{Z}_{2m})) \cong \mathrm{Wh}(\mathbf{Z}_{2m})/2(\mathrm{Wh}(\mathbf{Z}_{2m}))$. The definition of β gives that $\beta(\sigma(V))$ is represented by the Whitehead torsion $\tau(f)$ of the homotopy equivalence of the 7-dimensional boundary components of V, $L_{m+k}(2m) \xrightarrow{f} L_k(2m)$. But, letting r denote Reidemeister torsions [M], as $L_k(2m)$ (resp., $L_{k+m}(2m)$) was constructed as the quotient of the unit sphere of the representation space $4\sigma_k$ (resp., $4\sigma_{k+m}$),

$$r(L_k(2m)) = (1 - t^k)^4, \quad r(L_k(2m)) = (1 - t^{m+k})^4,$$

thus, $$\gamma(\det(\tau(f)) = r(L_k(2m))/r(L_k(m)) = ((1-t)^{m+k}/(1-t)^k)^4.$$

We leave to the reader the exercise of checking that $((1-t)^{m+k}/(1-t)^k)^2 \in \mathrm{Image}(\gamma)$. Hence, $\gamma(\det(\tau(f)) = \gamma(\det(2c))$ for some c and therefore $\tau(f) \in 2\mathrm{Wh}(\mathbf{Z}_{2m})$ and $\beta(\sigma(V)) = 0$.

Proof of Proposition 7. — Proposition 7 follows from Proposition 8 and an application, after taking products with S^1, of the s-cobordism theorem. This part of the argument is similar to one of [CS 4]. Let W^9 be the h-cobordism from E_k to E_{m+k} respecting generators of π_1, produced by Proposition 8, and let $\partial_0 W$ be the induced h-cobordism on the boundary between the double covers $L_k(m)$ and $L_{m+k}(m)$. These two lens spaces are the same and hence by a standard argument (given in [M] for the smooth case and easily extended to the topological case, cf. [CS 4, Prop. 2.9]), the h-cobordism of linear lens spaces $\partial_0 W$ is an s-cobordism. Thus, by the s-cobordism theorem, there is a homeomorphism

$$\psi_0 : \partial_0 W \to L_k(m) \times [0, 1]$$

with $\psi_0(m) = (x, 0)$, $x \in L_k(m)$. Taking products with S^1, we have for Whitehead torsions,

$$\tau((W \times S^1; (E_k \cup \partial_0 W) \times S^1) = 0$$

as taking products with S^1 kills torsion. Hence there is a homeomorphism

$$\psi_1 : W \times S^1 \to E_k \times [0, 1] \times S^1$$

with $$\psi_1(x, t) = (\psi_0(x), t), \quad t \in S^1, \quad x \in \partial_0 W,$$

and $$\psi_1(x, t) = (x, 0, t), \quad x \in E_k, \quad t \in S^1.$$

Let $$\widetilde{\psi}_1 : W \times \mathbf{R} \to E_k \times [0, 1] \times \mathbf{R},$$

on the infinite cyclic covering spaces, be the unique map covering ψ_1 with $\widetilde{\psi}_1(x, t) = (x, 0, t)$ for $x \in E_k$, $t \in \mathbf{R}$; by uniqueness of lifts $\psi_1(x, t) = (\psi_0(x), t)$, $t \in \mathbf{R}$, $x \in \partial_0 W$, as well. Let the homeomorphisms

$$\psi : E_{k+m} \times \mathbf{R} \to E_k \times \mathbf{R} = E_k \times 1 \times \mathbf{R}$$

be the restriction of $\widetilde{\psi}_1$. Clearly ψ preserves generators of π_1. Note that if $x \in L_{k+m}(m) = \partial E_{k+m}$,

$$(1) \qquad \psi(x, t) = (\varphi(x), t)$$

with $\varphi : L_{k+m}(m) \to L_k(m)$ given by the restriction of ψ_0.

The unit sphere of the underlying representation space of $4\sigma_k + \delta_{-1}$ (resp., $4\sigma_{m+k} + \delta_{-1}$) can be decomposed equivariantly as

$$S^8 = (S^7 \times [-1, 1]) \cup D^8 \times \{\pm 1\}$$

where the summands meet in $S^7 \times \{\pm 1\}$, and D^8 is the unit ball of the representation space of $4\sigma_k$ (resp., $4\sigma_{m+k}$), with $\partial D^8 = S^7$, and $[-1, 1]$ is the unit ball of the representation space of δ_{-1}. Similarly, we can decompose the complements of the origin in the representation space of $4\sigma_k + \delta_{-1}$ (resp., $4\sigma_{k+m} + \delta_{-1}$) as

$$(2) \qquad \mathbf{R}^9 - 0 = S^8 \times \mathbf{R} = (S^7 \times [-1, 1]) \times \mathbf{R} \cup (D^8 \times \{\pm 1\} \times \mathbf{R}).$$

Here the $\mathbf{R}$-coordinate is a radial coordinate on which the action of $\mathbf{Z}_{2m}$ is trivial. Of course, $S^7 \times [-1, 1]$ is equivariantly the universal cover of E_k (resp., E_{m+k}) with respect to the usual action of the covering translation group $\mathbf{Z}_{2m}$; similarly $S^7 \times [-1, 1] \times \mathbf{R}$ is the universal cover of $E_k \times \mathbf{R}$ (resp., $E_{m+k} \times \mathbf{R}$). Hence ψ lifts to an equivariant homeomorphism of covering spaces of $E_{m+k} \times \mathbf{R}$ and $E_k \times \mathbf{R}$,

$$h : S^7 \times [-1, 1] \times \mathbf{R} \to S^7 \times [-1, 1] \times \mathbf{R}.$$

By (1) above and uniqueness of lifting in covering spaces, there is a homeomorphism $g : S^7 \to S^7$ so that, for $u = \pm 1$,

$$h(z, u, t) = (g(z), u, t), \qquad z \in S^7, \ t \in \mathbf{R}.$$

Let $\bar{g} : D^8 \to D^8$ be the homeomorphism obtained by radial extension of g. Then define $f_0 : \mathbf{R}^9 - \{0\} \to \mathbf{R}^9 - \{0\}$ using the decomposition of (2) above, by

$$f_0 \mid S^7 \times [-1, 1] \times \mathbf{R} = h$$

and $$f_0 \mid D^8 \times \{\pm 1\} \times \mathbf{R} \quad \text{by} \quad f_0(z, u, t) = (\bar{g}(z), u, t).$$

Extend f_0 to $f : \mathbf{R}^9 \to \mathbf{R}^9$ by setting $f(0) = 0$. Then using (2) above, f is seen to be an equivariant homeomorphism of the representation spaces of $4\sigma_k + \delta_{-1}$ and $4\sigma_{k+m} + \delta_{-1}$.

4. PROOFS OF RESULTS ON FINITE GROUPS AND LIE GROUPS

Let K denote, as in § 1, the field

$$K = \mathbf{Q}[\{\mu + \mu^{-1} \mid \mu \text{ is an odd root of } 1\})$$

and recall that for $x \in F$, a finite extension of K, we set

$$\widehat{\mathrm{Tr}}(x) = \frac{1}{[F; K]} \mathrm{Tr}_{F/K}(x).$$

We begin the proof of Theorem 2 with the following lemma. The results of § 3 are used in this section only in the proof of this lemma.

Lemma 9. — For each real representation θ *of the cyclic group* $\mathbf{Z}_n$, $n = 2^a b$, *with* b *odd, there is a representation* ψ *of* $\mathbf{Z}_n$ *defined over* K *with character* $\chi_\psi = 2^{a+1}\widehat{\mathrm{Tr}}(\chi_\theta)$ *and* $2^{a+1}\theta + \rho \sim \psi + \rho$, ρ *the regular representation of* $\mathbf{Z}_n$.

Addendum to Lemma 9. — If n *is a power of* 2, $n = 2^a$, *the* $a + 1$ *in the conclusion of the lemma can be replaced by* a.

We argue by induction on n. If $n \not\equiv 0 \pmod 4$ or $n = 4$ it is well known [CR], [Se] that every real representation of G is linearly equivalent to a representation defined over K and so there is nothing to prove. Assume now that $n = 2^a b$, $a \geq 2$ and $a \neq 2$ or $b > 1$. The lemma is true, from the inductive hypothesis, for representations θ in the image of $\tau_m^* : R(\mathbf{Z}_m) \to R(\mathbf{Z}_n)$, where $\tau_m : \mathbf{Z}_n \to \mathbf{Z}_m$, $m < n$. Thus we need only consider the irreducible representations

$$\sigma_j : \mathbf{Z}_n \to SO_2$$

given by

$$\sigma_j(h) = \begin{pmatrix} \cos \dfrac{2\pi j}{n} & \sin \dfrac{2\pi j}{n} \\ -\sin \dfrac{2\pi j}{n} & \cos \dfrac{2\pi j}{n} \end{pmatrix}$$

for $0 < j < 2^{a-1}b$, j prime to $n = 2^a b$, and h the generator of $\mathbf{Z}_n$. But from Proposition 7 of § 3, $8\theta_j + \delta_{-1} \sim 4\sigma_j + 4\sigma_{2^{a-1}b-j} + \delta_{-1}$ where δ_{-1} is the nontrivial one-dimensional representation. Then $8\sigma_j + \rho \sim \gamma + \rho$, where $\gamma = 4\sigma_j + 4\sigma_{2^{a-1}b-j}$, and $\widehat{\mathrm{Tr}}(\chi_{8\sigma_j}) = \widehat{\mathrm{Tr}}(\chi_\gamma)$. However, γ is induced from the representation $4\sigma_j$ of $\mathbf{Z}_{2^{a-1}b}$.

Now, as observed for example by Schultz, for θ and η representations of H, a subgroup of the finite group G, and $\mathrm{Ind}(\theta)$ and $\mathrm{Ind}(\theta)$ denoting the induced representations of G, then $\theta \sim \eta$ implies $\mathrm{Ind}(\theta) \sim \mathrm{Ind}(\eta)$. This is obvious if we regard $\mathrm{Ind}(\theta)$ as the H-equivariant functions from G to the underlying vector space of the representation θ.

Now, by the inductive hypothesis on n, the representations σ_j of $\mathbf{Z}_{2^{a-1}b}$ satisfy the conclusion of the lemma. Hence so do the representations σ_j of $\mathbf{Z}_{2^a b}$.

The proof of the addendum to the lemma is the same induction but starting with the fact that every real representation of $\mathbf{Z}_{2^a}$, $a < 3$, is easily seen to be defined over the field K.

Proof of Theorem 2. — First we show that $\gamma = (2^{g+2}\theta + m_\theta \rho_G)$ is topologically equivalent to the realification of a representation φ defined over K with $\chi_\varphi = \widehat{\mathrm{Tr}}(\chi_\gamma)$. Recall that from the Brauer theory (as in [Se]), we may write, letting $\varphi_{\mathbf{C}}$ denote the complexification of,

$$\theta_{\mathbf{C}} = a - b$$

where a and b are complex representations of G which are sums of representations induced from 1-dimensional complex representations of subgroups of G. Thus we may write

$$\theta_{\mathbf{C}} + b = a$$

and thus the equation of real representations

$$2\theta + \beta = \alpha,$$

where $\beta = b + \bar{b}$, $\alpha = a + \bar{a}$, $\bar{u} =$ the complex conjugate of u. Hence,

$$(1) \qquad 2^{g+2}\theta + 2^{g+1}\beta + \rho_G = 2^{g+1}\alpha + \rho_G.$$

Now as 1-dimensional complex representations factor through cyclic groups, we may apply the above Lemma 9 to get

$$(2) \qquad 2^{g+1}\alpha + \rho_G \sim \varphi_1 + \rho_G$$

$$(3) \qquad 2^{g+1}\beta + \rho_G \sim \varphi_2 + \rho_G$$

where φ_i is the realification of a representation defined over K, with $\chi_{\varphi_1} = 2^{g+1}\widehat{\mathrm{Tr}}(\chi_\alpha)$, $\chi_{\varphi_2} = 2^{g+1}\widehat{\mathrm{Tr}}(\chi_\beta)$. Substituting (2) and (3) in (1) gives

$$(4) \qquad (2^{g+2}\theta + \varphi_2 + \rho_G) \sim (\varphi_1 + \rho_G)$$

with $\widehat{\mathrm{Tr}}$ applied to the characters of both sides giving the same function. We may write $\varphi_2 = (\dim \varphi_2)\rho_G - \varphi_3$, φ_3 some representation defined over K. Adding φ_3 to both sides of (4) gives the result.

Now we prove Part A of Theorem 2. From what we have just shown, for $\gamma_i = 2^{g+2}\theta_i + m\rho_G$, $m = \max(m_{\theta_1}, m_{\theta_2})$, we have $\gamma_i \sim \varphi_i$, φ_i defined over K with $\widehat{\mathrm{Tr}}(\chi_{\gamma_i}) = \chi_{\varphi_i}$. But then φ_1 and φ_2 are representations over K with the same character. Hence $\varphi_1 = \varphi_2$ and thus $\gamma_1 \sim \varphi_1 = \varphi_2 \sim \gamma_2$.

To prove Part B of Theorem 2 it suffices to show, under the hypothesis on odd cyclic subgroups of G, that if $\theta_1 \sim \theta_2$ then $\widehat{\mathrm{Tr}}(\chi_{\theta_1}) = \widehat{\mathrm{Tr}}(\chi_{\theta_2})$. It suffices to check this on cyclic subgroups of G; so set $G = \mathbf{Z}_n$. We may also inductively assume the result for all proper subgroups of $\mathbf{Z}_n$. Thus we may assume that

$$\widehat{\mathrm{Tr}}(\chi_{\theta_1})(f) = \widehat{\mathrm{Tr}}(\chi_{\theta_2})(f) \tag{5}$$

for $f \in \mathbf{Z}_n$ and *not* a generator. We must only show the same formula for f a generator. The case n odd is covered by the hypothesis. Assume n is even. Now we may decompose any real representation of $\mathbf{Z}_n$, $\theta_i = \alpha_i + \beta_i$, where α_i is a sum of representations of $\mathbf{Z}_n$ which factor through various $\mathbf{Z}_m$, $m < n$ and m dividing n, and β_i is a sum of 2-dimensional real representations σ_j which send the generator h of $\mathbf{Z}_n$ to $\begin{pmatrix} \cos\frac{2\pi j}{n} & \sin\frac{2\pi j}{n} \\ -\sin\frac{2\pi j}{n} & \cos\frac{2\pi j}{n} \end{pmatrix}$, where j is prime to n. By considering the points of the representation space of θ_i which are fixed by the action of some nontrivial subgroup of $\mathbf{Z}_n$, we see that, as $\theta_1 \sim \theta_2$, we have $\alpha_1 \sim \alpha_2$.

Hence, from our inductive hypothesis for $m < n$,

$$\widehat{\mathrm{Tr}}(\chi_{\alpha_1}) = \widehat{\mathrm{Tr}}(\chi_{\alpha_2}) \tag{6}$$

it remains only to check that for n even and f a generator of $\mathbf{Z}_n$, $\widehat{\mathrm{Tr}}(\chi_{\beta_1})(f) = \widehat{\mathrm{Tr}}(\chi_{\beta_2})(f)$.

When $n \equiv 2 \pmod 4$, writing $n = 2k$, k odd, we have

$$\chi_{\sigma_j}(f) = -\chi_{\sigma_j}((k+1)f)$$

and hence,

$$\chi_{\beta_i}(f) = -\chi_{\beta_i}((k+1)f) \qquad i = 1, 2. \tag{7}$$

As $(k+1)f$ is not a generator of $\mathbf{Z}_n$, from (5), $\widehat{\mathrm{Tr}}(\chi_{\theta_1}((k+1)f)) = \widehat{\mathrm{Tr}}(\chi_{\theta_2}((k+1)f))$ and hence, from (6), $\widehat{\mathrm{Tr}}(\chi_{\beta_1}((k+1)f)) = \widehat{\mathrm{Tr}}(\chi_{\beta_2}(k+1)f)$. Substituting this in (7) gives that $\widehat{\mathrm{Tr}}(\chi_{\beta_1})(f) = \widehat{\mathrm{Tr}}(\chi_{\beta_2})(f)$.

When $n \equiv 0 \pmod 4$ we show that $\widehat{\mathrm{Tr}}(\chi_{\beta_1})(f) = 0 = \widehat{\mathrm{Tr}}(\chi_{\beta_2}(f))$, f a generator of $\mathbf{Z}_n$. Set $f = kh$, h the canonical generator of $\mathbf{Z}_n$ and k an integer prime to n. Then β_i was a sum of the σ_j, $i = 1, 2$, and

$$\begin{aligned}\widehat{\mathrm{Tr}}(\chi_{\sigma_j})(kh) &= \widehat{\mathrm{Tr}}\left(2\cos\frac{2\pi jk}{n}\right)\\ &= \widehat{\mathrm{Tr}}(\mu + \bar{\mu}), \quad \mu \text{ a primitive } n\text{-th root of unity,}\\ &= \frac{1}{c}\cdot \sum_{\substack{1 \leqslant j < n\\ j \text{ prime to } n}} \mu^j, \quad c = [\mathrm{K}(\mu + \bar{\mu}) : \mathrm{K}],\\ &= \frac{1}{c}.0 = 0,\end{aligned}$$

as n is divisible by 4.

The addendum 1 to Theorem 2 is proved similarly to Theorem 2, but using further the fact that any irreducible complex representation of G, a supersolvable group, is induced from a 1-dimensional representation of a subgroup. The proof of addendum 2 is similar, but using the fact (left as an exercise) that if G has in addition a central 2-Sylow subgroup, every irreducible real representation is induced by a representation, of some subgroup, which factors through a cyclic group.

Proofs of Theorems 3 and 1. — Theorem 3 (and (3.2)) is immediate from Theorem 2. Corollary (3.1) then follows by using standard representation-theoretic methods of computing the number of irreducible representations over a field (see [CR], [Se]), in this case applied to K. Corollary (1.2) can be obtained from Corollary (3.1) using the elementary fact that, in a 2-group, (ii) of (3.1) implies that for each $g \in \mathrm{G}$, g^α is conjugate to g or g^{-1} for all odd β. To obtain Theorem 1 from these results, note that by standard representation theory, for G a 2-group, $\mathrm{R}_\mathrm{Q}(\mathrm{G}) \to \mathrm{R}_\mathrm{K}(\mathrm{G})$ is an isomorphism and a rational representation is determined by the dimensions of the fixed points of the cyclic subgroups. Moreover, the coefficient 2^{g+2} in Theorem 2 is improved to 2^{g+1} in Theorem 1 by using the addendum to Lemma 9 of this section.

Next we prove the results on compact Lie groups.

Proof of Theorem 5. — It is easy to see that conditions (1) and (2) imply that $\theta_1 \sim \theta_2$. Just decompose the G representations θ_i as $\theta_i = \theta_i^{\mathrm{G}_0} + \theta_i'$, $i = 1, 2$. Then condition (1) implies that $\theta_1^{\mathrm{G}_0} \sim \theta_2^{\mathrm{G}_0}$ and (2) implies that θ_1' and θ_2', as they have the same characters, are linearly equivalent [Ad] and, *a fortiori*, topologically equivalent. Hence, θ_1 and θ_2 are topologically equivalent.

Next we show the converse, that $\theta_1 \sim \theta_2$ implies (1) and (2). It is easy to see that a topological equivalence of θ_1 to θ_2 carries the subspace $\theta_1^{\mathrm{G}_0}$ to $\theta_2^{\mathrm{G}_0}$. Condition (2) is thus immediate. Now decompose, as above, $\theta_i = \theta_i^{\mathrm{G}_0} + \theta_i'$, $i = 1, 2$. Clearly $(\theta_i')^{\mathrm{G}_0} = 0$. We will complete the argument by showing that $\theta_1' = \theta_2'$.

Consider the difference of characters $\chi_{\theta_1} - \chi_{\theta_2}$. For each $g \in G$, $\theta_1(g)$ and $\theta_2(g)$ are orthogonal matrices. By an elementary argument of [DeR 2] (cf. [KR]) as $\theta_1(g)$ and $\theta_2(g)$ are furthermore topologically equivalent, those eigenvalues of $\theta_1(g)$ and of $\theta_2(g)$ which are *not* roots of unity are the same, counted with multiplicity. In particular, for each $g \in G$, $\chi_{\theta_1}(g) - \chi_{\theta_2}(g)$ has value a sum of roots of unity. Moreover, $\chi_{\theta_1} - \chi_{\theta_2} = (\chi_{\theta_1^0} - \chi_{\theta_2^0}) + (\chi_{\theta_1'} - \chi_{\theta_2'})$, $\theta_i^0 = \theta_i^{G_0}$, and clearly, as $\theta_i^{G_0}$ may be regarded as a representation of the finite group G/G_0, the eigenvalues of $(\theta_i^{G_0})(g)$ are all roots of unity, for $g \in G$. Hence $\chi_{\theta_1'}(g) - \chi_{\theta_2'}(g)$ is a sum of roots of unity, for $g \in G$. But the function $\chi_{\theta_1'} - \chi_{\theta_2'}$ thus maps continuously the group G, regarded as a space, to the set of algebraic numbers. Hence, $\sigma = \chi_{\theta_1'} - \chi_{\theta_2'}$ is constant on connected components of G; it is, of course, zero on the component of the identity element of G, as $\dim \theta_1' = \dim \theta_2'$. Thus, σ can be regarded as defined by a class function $G/G_0 \to \mathbf{R}$ which vanishes on the identity element and has equal values on inverse elements. Writing such a class function of the finite group G/G_0 in terms of characters of representations, we see that $(\theta_1' - \theta_2')$ has the character of an element of $R(G/G_0) \otimes \mathbf{R}$. But as an element of $R(G) \otimes \mathbf{R}$ is determined by its character [Ad],

$$(\theta_1' - \theta_2') \in \mathrm{Image}(R(G/G_0) \otimes \mathbf{R} \to R(G) \otimes \mathbf{R}).$$

But this image is detected under the map $R(G) \to R(G/G_0)$ sending $\theta \to \theta^{G_0}$. As $(\theta_1')^{G_0} - (\theta_2')^{G_0} = 0 - 0 = 0$, we see that $\theta_1' = \theta_2'$.

Proof of Corollaries (5.2) *to* (5.5). — All except (5.2) and (5.4) are immediate. from Theorem 5 and standard facts about compact Lie groups. To prove (5.4), use additionally the fact that for representations of dimension less than 6 of a finite group, topological equivalence implies linear equivalence [CS 9].

To prove (5.2), using a maximal torus, reduce to the case where G is a torus. In that case, as g is hyper-regular it follows that $\theta(h, \ )$ and $\eta(h, \ )$ are topologically equivalent for each $h \in G$ respectively. Then use an argument similar to that of the proof of Theorem 5.

Proof of Theorem 4. — First of all it is easy to see that a representation which is topologically equivalent to a bounded representation is bounded; bounded representations are characterized by the 0-vector having a neighborhood whose image under the action of G is a subset of a compact set. Now, standard methods in representation theory show that bounded representations are linearly equivalent to orthogonal representations. As such orthogonal representations behave very much as in the compact case, the proof proceeds along much the same lines as that of Theorem 5.

Proof of Theorem 6. — We first show that when G has no element of order $4k$ with $k \geq 2$, $\theta_x = \theta_y$ in $R(G)$. It clearly suffices to prove this for G a cyclic group $\mathbf{Z}_n$ with $n \not\equiv 0 \pmod 4$ or $n = 4$. Arguing by induction, we may, by considering first the factors of n, assume that $\chi_{\theta_x} = \chi_{\theta_y}$ on any proper subgroup of $\mathbf{Z}_n$. Now, by results of [O] the fixed points of a periodic map f on a (mod 2) homology sphere has Euler characteristic the Lefschetz number of f. Hence, when the fixed point set of an element of $\mathbf{Z}_n$ is discrete, there are 0 or 2 of them. If 0, there is nothing to prove. If 2 and n is odd, as a mod 2 homology sphere is, in particular, a rational homology sphere, we are essentially in the case discussed in [Sa], using the methods of [AS], [AB], [M]. When $n = 2k$, k odd, or $n = 4$, let V be the fixed point set of $\mathbf{Z}_2 \subset \mathbf{Z}_{2^a b}$. By Smith theory, V is a (mod 2) homology sphere. Letting α_x (resp., α_y) denote the representation of G on the tangent space of V at x (resp., y) we may write $\theta_u = \alpha_u + \beta_u$, $u = x$ or y, where α_u is a representation of $\mathbf{Z}_{2^a b}$ factoring through $\mathbf{Z}_{2^{a-1} b}$ and β_u is a sum of representations σ_j sending the generator of $\mathbf{Z}_n$ to

$$\begin{pmatrix} \cos\frac{2\pi j}{n} & \sin\frac{2\pi j}{n} \\ -\sin\frac{2\pi j}{n} & \cos\frac{2\pi j}{n} \end{pmatrix}, \quad \text{for } j \text{ odd.}$$

Now by induction $\alpha_x = \alpha_y$. To see that $\beta_x = \beta_y$, note that for $n \equiv 2 \pmod 4$ or for $n = 4$, any sum of representations of the form σ_j is determined by its restriction to $\mathbf{Z}_{n/2}$, whence the result.

Now to complete the proof of the theorem, we show that in general $\widehat{\mathrm{Tr}}(\chi_{\theta_x}) = \widehat{\mathrm{Tr}}(\chi_{\theta_y})$ and apply Theorem 2. Again, we may assume $G = \mathbf{Z}_n$, $n = 2^a b$, b odd, and argue by induction on n. The cases $a = 0$ or 1 have already been considered. Also, by induction, we may assume $\widehat{\mathrm{Tr}}(\chi_{\theta_x})(f) = \widehat{\mathrm{Tr}}(\chi_{\theta_y})(f)$ for f *not* a generator of $\mathbf{Z}_n$. Define V as above as the fixed point set of the subgroup $\mathbf{Z}_2 \subset \mathbf{Z}_{2^a b}$; by Smith theory, this is a mod 2-homology sphere and again write $\theta_u = \alpha_u + \beta_u$, $u = x$ or y, α_u the representation of G on the tangent space of V at u. Again by induction, $\widehat{\mathrm{Tr}}(\chi_{\alpha_x}) = \widehat{\mathrm{Tr}}(\chi_{\alpha_y})$. So we need only check that

$$(1) \qquad \widehat{\mathrm{Tr}}(\chi_{\beta_x})(f) = \widehat{\mathrm{Tr}}(\chi_{\beta_y})(f)$$

for f any generator of $\mathbf{Z}_n$, $n = 2^a b$, $a \geq 2$ and β_x or β_y a sum of representations of the form σ_j defined as above. But, the same kind of calculation as that made at the end of the proof of Theorem 2 shows that $\widehat{\mathrm{Tr}}(\chi_{\sigma_j})(f) = 0$, f a generator of $\mathbf{Z}_n$; thus both sides of (1) vanish identically for f a generator.

BIBLIOGRAPHY

[A] V. I. Arnold, *Ordinary Differential Equations*, Cambridge, MIT Press, 1973.

[Ad] J. F. Adams, *Lectures on Lie Groups*, New York, Benjamin, Inc., 1969.

[AB] M. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes: II. Applications, *Ann. of Math.*, **88** (1968), 451-491.

[AS] M. ATIYAH and I. SINGER, The index of elliptic operators, III, *Ann. of Math.*, **87** (1968), 546-604.

[Bre 1] G. BREDON, Representations at fixed points of smooth actions of compact groups, *Ann. of Math.*, **89** (1969), 512-532.

[Bre 2] — *Introduction to compact transformation groups*, New York, Academic Press, 1972.

[CKP] C. CAMACHO, N. KUIPER, J. PALIS, The topology of holomorphic flows with singularity, *Publ. Math. I.H.E.S.*, **48** (1979), 5-38.

[CR] C. CURTIS and I. REINER, *Representation theory of finite groups and associative algebras*, New York, Interscience-Wiley, 1962.

[CS 1] S. E. CAPPELL and J. L. SHANESON, Pseudo-free group actions I, *Proc. of the Aarhus Topology Conference of 1978, Springer Lecture Notes in Math.*, **763** (1979), 395-447.

[CS 2] — Linear Algebra and Topology, *Bull. Amer. Math. Soc., New Series*, **1** (1979), 685-687.

[CS 3] — Nonlinear similarity of matrices, *Bull. Amer. Math. Soc., New Series*, **1** (1979), 899-902.

[CS 4] — Nonlinear similarity, *Ann. of Math.*, **113** (1981), 315-355.

[CS 5] — Fixed points of periodic smooth maps, *Proc. National Acad. of Sci. USA*, **77** (1980), 5052-5054.

[CS 6] — Fixed points of periodic differentiable maps (to appear).

[CS 7] — The geometry of linear representations of groups and subgroups (to appear in *Amer. Jour. of Math.*).

[CS 8] — Matrices, topology, and elementary number theory (to appear).

[CS 9] — Nonlinear similarity and linear similarity are the same in dimensions less than 6 (to appear).

[CS 10] — Class numbers and periodic smooth maps (to appear).

[DeR 1] G. de RHAM, Sur les nouveaux invariants topologiques de M. Reidemeister, *International Conference of Topology* (Moscow, 1935), *Recueil Mathématique*, Moscow, 1936, t. I (43), 737-743.

[DeR 2] — Reidemeister's torsion invariant and rotation of S^n, *International Conf. on Differential Analysis*, Bombay Colloquium of 1964, Oxford Univ. Press, 1964, 27-36.

[DeR 3] G. de RHAM, S. MAUMARY, and M. A. KERVAIRE, Torsion et type simple d'homotopie, *Lecture Notes in Math.*, **48**, Springer-Verlag, 1967.

[I] Ju. S. IL'IAŠENKO, Global and local aspects of the theory of complex differential equations, *Proc. Int. Cong. of Math.*, Helsinki, 1978, 821-826.

[Ka] K. KAWAKUBO, Weyl group actions and equivariant homotopy equivalence, *Proc. of Amer. Math. Soc.*, **80** (1980), 172-176.

[Kla] B. KLARES, *Classification topologique des n-tuples de champs de vecteurs holomorphes commutatifs sur* $P_{n+1}(\mathbf{C})$, Thèse, Université de Strasbourg, 1980, partie II.

[K 1] N. H. KUIPER, The topology of a solution of a differential equation on $\mathbf{R}^n$, *Proc. Int. Congress on Manifolds*, Tokyo, 1973, 195-203.

[K 2] — La topologie des singularités hyperboliques des actions de $\mathbf{R}^2$, *Astérisque*, **59-60** (1978), 131-150.

[K 3] — The topology of linear $\mathbf{C}^m$-flows on $\mathbf{C}^n$, to appear in the *Proceedings of a Conference on Dynamical Systems*, Rio de Janeiro, July 1981.

[KR] N. KUIPER and J. W. ROBBIN, Topological classification of linear endomorphisms, *Inventiones Math.*, **19** (1973), 83-106.

[L] S. LANG, *Algebra*, Addison-Wesley, 1965.

[LW] C. LEE and A. WASSERMAN, On the groups JO(G), *Memoirs Amer. Math. Soc.*, Vol. 2, Issue 1, No. 159, Amer. Math. Soc., Providence, R. I., 1975.

[M] J. MILNOR, Whitehead torsion, *Bull. Amer. Math. Soc.*, **72** (1966), 358-426.

[O] R. OLIVER, Fixed point sets of group actions on finite acyclic complexes, *Comment. Math. Helv.*, **50** (1975), 155-177.

[Pe 1] T. PETRIE, The Atiyah-Singer invariant, the Wall groups $L_n(\ ,1)$, and the function $(te^x+1)/(te^x-1)$, *Ann. of Math.*, **92** (1970), 174-187.

[Pe 2] — G-Surgery I-A survey, *Springer Lecture Notes*, **664** (1978), 197-233.

[Pe 3] — Three Theorems on Transformation Groups, *Springer Lecture Notes*, **763** (1979), 549-572.

[P] H. POINCARÉ, Sur les courbes définies par les équations différentielles, *Œuvres de H. Poincaré*, Vol. I, Paris, Gauthier-Villars (1926).

[R] M. G. ROTHENBURG, Torsion invariants and finite transformation groups, *Symposia in Pure Math.*, XXXII, part 1, 267-312, Amer. Math. Soc., 1978.

[Sa] C. Sanchez, Actions of groups of odd order on compact, orientable manifolds, *Proc. Amer. Math. Soc.*, **54** (1976), 445-448.

[Sch] R. Schultz, On the topological classification of linear representations, *Topology*, **16** (1977), 263-270.

[Se] J.-P. Serre, *Représentations linéaires des groupes finis*, Paris, Hermann (1967).

[Sh] J. L. Shaneson, Wall's surgery obstruction groups for $\mathbf{Z} \times G$, *Ann. of Math.*, **90** (1969), 296-334.

[Sm] P. A. Smith, New results and old problems in finite transformation groups, *Bull. Amer. Math. Soc.*, **66** (1960), 401-415.

[Tr] P. Traczyk, On the G-homotopy equivalence of spheres of representations, *Math. Zeitschrift*, **161** (1978), 257-261.

[W 1] C. T. C. Wall, *Surgery on compact manifolds*, New York, Academic Press, 1970.

[W 2] — Classification of Hermitian forms, VI, group rings, *Ann. of Math.*, **103** (1976), 1-80.

Courant Institute of Mathematical Sciences
251 Mercer Street
New York, N.Y. 10012

Department of Mathematics
Rutgers University
New Brunswick, N.J. 08903

Manuscrit reçu le 2 novembre 1979,
Révisé le 2 mars 1981.