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CESAR CAMACHO
ALCIDES LINS NETO

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THE TOPOLOGY OF INTEGRABLE DIFFERENTIAL FORMS NEAR A SINGULARITY

by César CAMACHO, Alcides LINS NETO

INTRODUCTION (1)

Here we consider integrable differential 1-forms ω defined in an open subset U of $\mathbf{R}^n$ or $\mathbf{C}^n$:

$$\omega = \sum_{i=1}^{n} a_i(x)\,dx_i, \qquad \omega \wedge d\omega \equiv 0.$$

The equation

$$\omega = a_1(x)\,dx_1 + \ldots + a_n(x)\,dx_n = 0$$

can be considered either as a total differential equation in the unknowns $x_1, \ldots, x_n$, or as a plane field π, of codimension one, outside the set of singularities:

$$\operatorname{Sing}(\omega) = \{p \in U \mid a_i(p) = 0 \text{ for all } i = 1, \ldots, n\}.$$

The solutions of this equation are the integral manifolds of π and by Frobenius' theorem define a codimension one foliation. This foliation plus $\operatorname{Sing}(\omega)$ will be called the singular foliation of ω. A natural problem to consider is the search for a significant class of integrable 1-forms ω for which the induced singular foliation is susceptible of a topological description. Next, one would wish to characterize among these forms those which are stable in the sense that all nearby integrable 1-forms induce foliations which are equivalent up to homeomorphism, the reason for this being to augment the set of integrable 1-forms whose induced singular foliation can be understood. Dimension two is special since the above equation becomes in this case an ordinary differential equation. The answer to this problem is then well-known: the first jet of ω at a singularity being hyperbolic characterizes the topology of the singular foliation associated to any 1-form close to ω.

(1) Most of the results in this paper were announced by M. René Thom at the meeting of the Académie des Sciences on March 3, 1980.

Starting in dimension three, however, the problem has a different nature, and, as we will see, the first jet of ω at a singularity does not give enough information about many integrable 1-forms.

A natural way of defining integrable 1-forms with singularity $p \in \mathbf{C}^n$ is to write $\omega = g\,df$ where f is a holomorphic function with a critical point p and g is holomorphic with $g(p) \neq 0$. We say that g^{-1} is an integrating factor of ω. The family of 1-forms defined in this way is quite general; in fact, it was shown by B. Malgrange [7], that the germ at $p \in \mathbf{C}^n$ of a holomorphic integrable 1-form ω, $\omega(p) = 0$, admits an integrating factor provided that the set of singularities of ω has codimension ≥ 3. It is easy to see in this case that ω is stable in the space of holomorphic 1-forms if and only if p is a nondegenerate critical point for f and so its topology is characterized by the 1-jet of ω at p. More general criteria for finding integrating factors have been studied by J. F. Mattei and R. Moussu [8], [10].

A different family of integrable 1-forms is the one induced by Lie group actions. For instance, the integrable 1-forms in $\mathbf{R}^3$ given by

$$\omega = \lambda_1 x_2 x_3 dx_1 + \lambda_2 x_1 x_3 dx_2 + \lambda_3 x_1 x_2 dx_3, \qquad \lambda_i \neq \lambda_j \text{ if } i \neq j$$

have as leaves the orbits of a linear action of the group $\mathbf{R}^2$. These 1-forms and their perturbations were thoroughly studied in [6]. They have a remarkable property: the first jet at $0 \in \mathbf{R}^3$, $j^1(\omega)_0$, vanishes, and this is a stable property under C^2-perturbations of ω which are null at $0 \in \mathbf{R}^3$.

That many integrable 1-forms arise from Lie group actions is a consequence of Theorems 1 and 2 of Chapter II, where we concentrate on dimension three.

Consider the power series development of a holomorphic 1-form ω with a singularity at $0 \in \mathbf{C}^3$:

$$\omega = \omega_k + \omega_{k+1} + \omega_{k+2} + \dots, \qquad k \geq 1,$$

where the coefficients of ω_j, $j \geq k$, are homogeneous polynomials of degree j. Then:

Suppose $0 \in \mathbf{C}^3$ *is an algebraically isolated zero of* $d\omega_k$ *and* $k \geq 3$. *Then there is a holomorphic change of coordinates* f *and holomorphic vector fields* X *and* Y *such that* $f^*\omega = \omega_k = i_X i_Y(dx_1 \wedge dx_2 \wedge dx_3)$ *and* $[X, Y] = Y$. *In fact* $X = \frac{1}{k+1} I$, $I(x) = x$, *and* $d\omega_k = i_Y(dx_1 \wedge dx_2 \wedge dx_3)$.

In other words, ω embeds in an action of the group of affine transformations of the complex line. When $k = 2$ one obtains that ω_k embeds in an action of the group $\mathbf{C}^2$. These homogeneous 1-forms ω_k, $k \geq 3$, are also stable in the following sense:

For any integrable 1*-form* η *sufficiently* C^{2k}*-close to* ω_k *near* $0 \in \mathbf{C}^3$, *there exists a point* $p(\eta)$ *near* $0 \in \mathbf{C}^3$, *such that the* $(k-1)$*-jet of* η *at* $p(\eta)$ *vanishes*, i.e. η *starts with order* k. *Moreover,* $p(\eta)$ *is continuous and* $p(\omega) = 0$.

This is Theorem 5 for dimension three; it gives an idea of how thin the space of differential 1-forms can become after the integrability condition is imposed. The corresponding versions of the above theorems in the C^∞ case in $\mathbf{R}^3$ are also valid. As a consequence of this we show in Theorem 6 that:

For any $k \geq 3$ *there exist homogeneous integrable* 1*-forms* ω_k *of degree* k *in* $\mathbf{R}^3$ *which are* C^{2k}*-structurally stable.*

By this we mean that all integrable 1-forms close to ω_k in the C^{2k}-topology are equivalent up to homeomorphism. However, ω_k is not stable in the C^{k-1}-topology. In fact, for any $\varepsilon > 0$ there are 1-forms $f.\omega_k$ which are $\varepsilon - C^{k-1}$-close to ω_k, where f is a C^∞ function vanishing in a small neighborhood of $0 \in \mathbf{R}^3$.

On the other hand, integrable 1-forms in the complex domain with a singularity at $0 \in \mathbf{C}^n$ are in general unstable for $n \geq 3$. One can see this for the 1-forms

$$\omega = \sum_{i=1}^{n} \lambda_i z_1 \dots \hat{z}_i \dots z_n dz_i, \qquad \lambda_i \notin \mathbf{R}\lambda_j \text{ if } i \neq j,$$

where $\hat{z}_i$ means that z_i is omitted in the product. Then:

The equivalence class of $\{\lambda_1, \dots, \lambda_n\} \subset \mathbf{C}$ *under the action of* $\mathrm{Gl}(2, \mathbf{R})$ *is the only topological invariant of the real codimension two foliation with singularities defined by* ω. This is Theorem 7.

The homogeneous forms considered up to this point constitute examples of *regular* forms, a notion which will be introduced now and which, as it turns out, endows a form with stability properties. We write H_k^p to denote the set of homogeneous p-forms of degree k on $\mathbf{C}^n$ and for $\omega \in H_k^1$ let

$$T_j^\omega : H_j^1 \to H_{k+j-1}^3 \quad \text{and} \quad S^\omega : \mathbf{C}^n \to H_{k-1}^1$$

be $T_j^\omega(\alpha) = \alpha \wedge d\omega + \omega \wedge d\alpha$ and $S^\omega(a) = L_a\omega$, the Lie derivative of ω along a. Then we say that $\omega \in H_k^1$ is regular if: *a)* ω is integrable, *b)* $\mathrm{Ker}(T_j^\omega) = \{0\}$ for $j \leq k-2$ and *c)* $\mathrm{Ker}(T_{k-1}^\omega) = \mathrm{Im}\, S^\omega$. Although this concept has a technical character, the mappings involved in its definition appear naturally in the integrability condition. For instance, if $\tilde{\omega} = \omega_0 + \omega_1 + \dots + \omega_k$ is a polynomial integrable 1-form, $T_{k-1}^{\omega_k}(\omega_{k-1}) = 0$ is the term of degree $2k-2$ of the equation $\tilde{\omega} \wedge d\tilde{\omega} = 0$. On the other hand, one can identify many integrable 1-forms which are regular. For instance, when $n = 3$ and $\omega \in H_k^1$ is such that $d\omega$ has an algebraically isolated zero at $0 \in \mathbf{C}^3$, then ω is regular. (Lemmas 2 and 3 of Chapter II.) Also the 1-form in $\mathbf{C}^n$

$$\omega = \sum_{i=1}^{n} \lambda_i z_1 \dots \hat{z}_i \dots z_n dz_i, \qquad \lambda_i \neq \lambda_j \text{ for } i \neq j,$$

is regular (Proposition 4), although $d\omega$ has no isolated zeros for $n \geq 3$.

Chapter III is devoted to the study of homogeneous regular forms and is preparatory to the stability theorem proved in Chapter IV, which goes as follows:

Let $\mathscr{R}_k(\mathbf{R}^n)$ be the set of regular homogeneous 1-forms of degree k. Define

$$\mathscr{R}_k^\ell(\mathbf{R}^n) = \{\omega \in \mathscr{R}_k(\mathbf{R}^n) \mid \dim \mathrm{Im}(S^\omega) = \ell\}.$$

Let $I^r(U)$, $r \geq 2k$, be the space of integrable 1-forms of class C^r endowed with the uniform C^r-topology and let $\omega \in I^r(U)$. A singularity $p \in U$ of ω is called *regular* of order $k \geq 1$ if the $k-1$ jet of ω at p vanishes, i.e. $j^{k-1}(\omega)_p \equiv 0$, and $j^k(\omega)_p$ is a regular homogeneous 1-form.

Let $M_k(\omega)$ be the set of regular singularities of ω in U and

$$M_k^\ell(\omega) = \{p \in M_k(\omega) \mid j^k(\omega)_p \in \mathscr{R}_k^\ell(\mathbf{R}^n)\}.$$

Then Theorem 5 asserts:

$M_k^\ell(\omega) \subset U$ *is an embedded submanifold of codimension* ℓ *and is stable in the following sense: for any relatively compact subset* $P \subset M_k^\ell(\omega)$, *there exists a neighborhood* N *of* ω, *such that if* $\eta \in N$ *then* $M_k^\ell(\eta)$ *has a relatively compact subset* $\widetilde{P}$ *diffeomorphic and close to* P.
A similar version holds for $\mathbf{C}^n$.

We start Chapter I with a recollection of the de Rham division theorem as it will be used frequently throughout this paper.

I. — PRELIMINARIES. THE DE RHAM DIVISION THEOREM

Throughout this paper we use the letter K to denote $\mathbf{R}$ or $\mathbf{C}$.

Let $\Lambda^p_A(n)$ be the set of germs at zero of differential p-forms of class C^∞ $(A = \infty)$, analytic or holomorphic $(A = H)$ in a neighborhood of zero in $\mathbf{R}^n$ or $\mathbf{C}^n$. Then $\Lambda^p_A(n)$ is a module with coefficients in $\Lambda^0_A(n)$.

Definition. — We say that $\omega \in \Lambda^1_A(n)$ has the *division property* (in $\Lambda_A(n)$) if for any $1 \leq p \leq n-1$ and $\alpha \in \Lambda^p_A(n)$ such that $\omega \wedge \alpha = 0$ there is $\beta \in \Lambda^{p-1}_A(n)$ such that $\alpha = \omega \wedge \beta$.

It is clear that ω has the division property if and only if the following sequence, where $\omega'(\alpha) = \omega \wedge \alpha$, is exact for $1 \leq p \leq n-1$

$$\Lambda^{p-1}_A(n) \xrightarrow{\omega'} \Lambda^p_A(n) \xrightarrow{\omega'} \Lambda^{p+1}_A(n).$$

Definition. — An r-tuple $(a_1, \ldots, a_r)$ of elements of $\Lambda^0_A(n)$ is called *regular* if (1) a_1 is not a zero divisor in $\Lambda^0_A(n)$ and (2) for any $1 \leq i \leq r-1$ the class of a_{i+1} in the quotient $\Lambda^0_A(n)/[a_1, \ldots, a_i]$ is not a zero divisor. Here $[a_1, \ldots, a_i]$ denotes the ideal generated in $\Lambda^0_A(n)$ by $a_1, \ldots, a_i$.

One says that a germ $\omega = \sum_{i=1}^{n} a_i dx_i \in \Lambda^1_A(n)$ defines a *regular sequence* if after reindexing the a_i, $(a_1, \ldots, a_n)$ is regular.

Theorem (de Rham [4]). — *If* $\omega \in \Lambda^1_A(n)$ *defines a regular sequence then* ω *has the division property.*

Definition. — Let $\omega = \sum_{i=1}^{n} a_i dx_i \in \Lambda^1_A(n)$, $a_i(0) = 0$ for $1 \leq i \leq n$. We say that zero is an *algebraically isolated zero* of ω if the vector space $\Lambda^0_A(n)/[a_1, \ldots, a_n]$ has finite dimension.

A proof of the following theorem and its corollary can be found in [10] or [11].

Theorem. — *Let* $\omega \in \Lambda^1_A(n)$, $\omega(0) = 0$, *with an algebraically isolated zero at* 0. *Then* ω *has the division property.*

Corollary (Parametric division). — *Let* $\omega_y = \sum_{i=1}^{n} a_i(x, y) dx_i$, *where* $(x, y) \in K^n \times K^m$ *and* a_i *is analytic,* $1 \leq i \leq n$. *Suppose that* $0 \in K^n \times K^m$ *is an algebraically isolated zero of* ω_0. *If* α_y *is a p-form* $1 \leq p \leq n-1$ *in* K^n *depending analytically on the parameter* $y \in K^m$

and $\alpha_y \wedge \omega_y = 0$ *then there exists an analytic* $(p-1)$*-form* β_y, *depending analytically on the parameter* y, *such that* $\alpha_y = \omega_y \wedge \beta_y$, *for any* y *in a neighborhood of* 0.

Another fact which will be used is the following:

Proposition. — *Let* $\omega \in \Lambda^1_H(n)$, $a_i(0) = 0$ for $1 \leq i \leq n$. *Let* $\tilde{\omega}$ *be the complexification of* ω. *Then* $0 \in K^n$ *is an algebraically isolated zero of* ω *if and only if* $0 \in \mathbf{C}^n$ *is a topologically isolated zero of* $\tilde{\omega}$.

The proof can be found also in [10] pg. 181.

We proceed to show a dual version of the de Rham division theorem.

Definition. — Let $X = \sum_{i=1}^{n} X_i \dfrac{\partial}{\partial x_i}$, $X(0) = 0$, be the germ at zero of a C^∞, analytic or holomorphic vector field in K^n. We say that X has an *algebraically isolated zero* at $0 \in K^n$ if the vector space $\Lambda^0_\infty(n)/[X_1, \ldots, X_n]$ (or $\Lambda^0_H(n)/[X_1, \ldots, X_n]$) has finite dimension.

Definition. — The vector field X has the *division property* if for any $1 \leq p \leq n-1$ and $\alpha \in \Lambda^p_A(n)$ such that $i_X(\alpha) = 0$ there is $\beta \in \Lambda^{p+1}_A(n)$ such that $\alpha = i_X(\beta)$.

By i_X we denote the interior product $i_X(\alpha)(v_1, \ldots, v_{p-1}) = \alpha(X, v_1, \ldots, v_{p-1})$, where $\alpha \in \Lambda^p_A(n)$ and $v_1, \ldots, v_{p-1}$ are vector fields.

Theorem. — *Let* X *be a* C^∞ *or holomorphic vector field in* $\mathbf{R}^n$ *or* $\mathbf{C}^n$ *with an algebraically isolated zero at* 0. *Then* X *has the division property.*

Proof. — Let $* : \Lambda^p_A(n) \to \Lambda^{n-p}_A(n)$ be the Hodge star operator. If

$$\eta = \sum_{1 \leq i_1 < \ldots < i_p \leq n} a_{i_1 \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p} \in \Lambda^p_A(n)$$

then

$$*\eta = \sum_{1 \leq j_1 < \ldots < j_{n-p} \leq n} (\operatorname{sgn} \sigma) a_{i_1 \ldots i_p} dx_{j_1} \wedge \ldots \wedge dx_{j_{n-p}}$$

where $(i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$ is a permutation σ of $(1, \ldots, n)$ and $\operatorname{sgn} \sigma = 1$ if σ is even, $\operatorname{sgn} \sigma = -1$ if σ is odd. Then the following diagram commutes

$$\begin{array}{ccccc}
\Lambda^{p-1}_A(n) & \xrightarrow{\omega'} & \Lambda^p_A(n) & \xrightarrow{\omega'} & \Lambda^{p+1}_A(n) \\
\downarrow * & & \downarrow * & & \downarrow * \\
\Lambda^{n-p+1}_A(n) & \xrightarrow{i_X} & \Lambda^{n-p}_A(n) & \xrightarrow{i_X} & \Lambda^{n-p-1}_A(n)
\end{array}$$

where $1 \leq p \leq n-1$, $*\omega = i_X(dx_1 \wedge \ldots \wedge dx_n)$ and $\omega'(\alpha) = \omega \wedge \alpha$. It follows that ω has an isolated zero at 0. By the de Rham theorem the first horizontal sequence is exact. This implies that the second sequence is exact. ■

II. — INTEGRABLE 1-FORMS IN DIMENSION THREE

Here we consider integrable 1-forms ω in K^3. For such forms we can write

$$d\omega = Y_1 dx_2 \wedge dx_3 + Y_2 dx_3 \wedge dx_1 + Y_3 dx_1 \wedge dx_2$$

or

$$d\omega = i_Y(dx_1 \wedge dx_2 \wedge dx_3) \quad \text{where} \quad Y = \sum_{i=1}^{3} Y_i \frac{\partial}{\partial x_i}.$$

The vector field Y is called the *rotation* of ω and will be denoted by rot ω. We say that $0 \in K^3$ *is an algebraically isolated zero of* $d\omega$ if it is an algebraically isolated zero for rot ω. When this happens we say that ω is *simple at* $0 \in K^3$. It is easy to verify that this condition is independent of the coordinate system.

From the integrability condition $\omega \wedge d\omega \equiv 0$ one obtains

$$i_Y(\omega) d\omega - \omega \wedge i_Y(d\omega) = 0.$$

Thus

$$i_Y(\omega) d\omega = 0$$

and if ω is simple at $0 \in K^3$ we have $i_Y(\omega) = 0$, i.e. rot ω is tangent to the leaves of ω.

1. Integrable 1-forms and Lie group actions.

An integrable 1-form is called homogeneous when all its coefficients are homogeneous polynomials of the same degree.

Theorem 1. — Any integrable 1*-form* ω_k *homogeneous of degree* k, $k \geq 3$, *and simple at* $0 \in K^3$ *can be written*

$$\omega_k = i_X i_Y(\Omega), \qquad \Omega = dx_1 \wedge dx_2 \wedge dx_3$$

where $Y = \text{rot}(\omega_k)$ *and* $X(x_1, x_2, x_3) = \frac{1}{k+1}(x_1, x_2, x_3)$.

Proof. — One has $i_Y \omega_k = 0$. Then by the de Rham theorem $\omega_k = i_Y \alpha = i_X i_Y \Omega$ where X is linear, $X(x) = A.x$. Moreover,

$$i_Y \Omega = d\omega_k = d(i_X i_Y \Omega) = L_X(i_Y \Omega) - i_X d(i_Y \Omega) = L_X(i_Y \Omega)$$

or

$$i_Y \Omega = i_{[X,Y]} \Omega + i_Y L_X \Omega.$$

Now,

$$L_X \Omega = \frac{d}{dt}(e^{tA})^* \Omega \Big|_{t=0} = \frac{d}{dt}(\det e^{tA}) \Big|_{t=0} . \Omega = \text{tr}(A) . \Omega.$$

Therefore $i_Y \Omega = i_{[X,Y]} \Omega + \mathrm{tr}(A) . i_Y \Omega$ and so

$$[X, Y] = \lambda Y, \quad \lambda = 1 - \mathrm{tr}(A) \quad \text{and} \quad X(p) = A.p.$$

Since $[I, Y] = (k-2)Y$ where $I(x) = x$, one has

$$\left[X - \frac{\lambda}{k-2} I, Y\right] = 0.$$

The following lemma shows that $X = \frac{\lambda}{k-2} I$. Since $\lambda = 1 - \mathrm{tr}(A) = 1 - \frac{3\lambda}{k-2}$ one obtains $X = \frac{1}{k+1} I$. ■

Remark. — When $k = 2$ the same proof shows that $\omega_2 = i_X i_Y(\Omega)$ where X and Y are commutative vector fields. In fact, if $k = 2$, X and Y are linear and $[X, Y] = \lambda Y$. So $XY - YX = \lambda Y$ and $\mathrm{tr}(X - YXY^{-1}) = 3\lambda$. Thus $\lambda = 0$.

Lemma 1. — *Let* Y *be a homogeneous vector field of degree* $k - 1 \geq 2$ *in* K^n *such that* $0 \in K^n$ *is an algebraically isolated zero for* Y. *Let* B *be a linear vector field such that* $[B, Y] = 0$. *Then* $B \equiv 0$.

Proof. — Assuming B and Y complex, let v be an eigenvector of B with eigenvalue μ. Since

$$DY(v) . B(v) - BY(v) = 0 \text{ for any } v$$

we have
$$\mu DY(v) . v = BY(v).$$

By the homogeneity of Y, $DY(v) . v = (k-1) Y(v)$. So

$$BY(v) = \mu(k-1) Y(v).$$

Since $0 \in \mathbf{C}^n$ is an isolated zero of Y, $Y(v)$ is an eigenvector of B with eigenvalue $\mu(k-1)$.

Similarly $Y^j(v)$ is an eigenvector of B with eigenvalue $\mu(k-1)^j$. Since $k - 1 \geq 2$ this implies that $\mu = 0$. So all eigenvalues of B vanish. Therefore $B^{\ell+1} = 0$ for some $0 \leq \ell \leq n-1$. We proceed to show that $\ell = 0$.

If on the contrary $\ell > 0$, there exists $z \in \mathbf{C}^n$ such that $B^\ell z \neq 0$. Since $Y(e^{tB} z) = e^{tB} Y(z)$, we have

$$Y\left(z + tBz + \ldots + \frac{t^\ell}{\ell!} B^\ell z\right) = \left(I + tB + \ldots + \frac{t^\ell}{\ell!} B^\ell\right) Y(z)$$

or dividing by $t^{(k-1)\ell}$

$$Y\left(\frac{1}{t^\ell}\left(z + \ldots + \frac{t^{\ell-1}}{(\ell-1)!} B^{\ell-1} z\right) + \frac{1}{\ell!} B^\ell z\right) = \frac{1}{t^{k\ell-\ell}}\left(I + \ldots + \frac{t^\ell}{\ell!} B^\ell\right) Y(z).$$

Taking limits as $t \to \infty$, we obtain $Y(B^\ell z) = 0$ which is absurd. Then $\ell = 0$. ■

Remark. — The existence of homogeneous integrable 1-forms ω, simple at $0 \in K^3$, can be shown as follows. For any $k \geq 3$ find a volume preserving vector field Y, i.e. $L_Y\Omega = 0$, such that Y is homogeneous of degree k and has $0 \in K^3$ as an algebraically isolated zero. Then the form $\omega = i_I i_Y \Omega$ satisfies

$$d\omega = d i_I i_Y \Omega = L_I i_Y \Omega = i_{[I,Y]}\Omega + i_Y L_I \Omega.$$

Using $L_I\Omega = 3\Omega$, obtain

$$d\omega = i_{[I,Y]}\Omega + i_{3Y}\Omega = i_{(k+1)Y}\Omega.$$

Therefore $\mathrm{rot}(\omega) = (k+1)Y$.

2. Finite Determinacy.

Here we consider integrable 1-forms in K^3 which can be written as $\omega = \omega_k + R$, where $\lim_{x\to 0} |x|^{-k}R(x) = 0$, and $\omega_k = j^k(\omega)_0$. Clearly ω_k is integrable. We say that $0 \in K^3$ is a *simple singularity of order k* of ω when $0 \in K^3$ is an algebraically isolated zero for $d\omega_k$.

Theorem 2. — Let ω be an integrable 1-form of class C^r defined in an open set $U \subset K^3$ ($r = \infty$ or analytic if $K = \mathbf{R}$ and r = holomorphic if $K = \mathbf{C}$). Suppose that $0 \in K^3$ is a simple singularity or order $k \geq 3$ of ω, where $j^k(\omega)_0 = \omega_k$. Then there exists a C^r local diffeomorphism f such that $f(0) = 0$ and $f^*(\omega) = \omega_k$ (1).

Remark. — The theorem is also true for $k = 2$ in the following case. Let $\omega_2 = j^2(\omega)_0$ and $Y = \mathrm{rot}\,\omega_2$. In this case Y is a linear vector field in K^3 and ω_2 can be written as $\omega_2 = i_X i_Y(\Omega)$, where X is linear. If we assume that there is a linear combination $aX + bY$ satisfying non resonance conditions in the C^∞ case or Siegel's conditions in the analytic case (cf. [13] and [12]), then Theorem 3 is true for ω.

Proof of Theorem 2. — Let $\tilde{Y} = \mathrm{rot}(\omega)$. Since $i_{\tilde{Y}}(\omega) = 0$ there is, by the de Rham division theorem, a 2-form η such that $\omega = i_{\tilde{Y}}(\eta)$. But $-\eta = i_{\tilde{X}}(\Omega)$, where $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. Therefore $\omega = i_{\tilde{X}} i_{\tilde{Y}}(\Omega)$ and similarly $\omega_k = i_X i_Y(\Omega)$ where by Theorem 1 $X(x) = \frac{1}{k+1}x$ and $Y = \mathrm{rot}\,\omega_k$. Let f be a local diffeomorphism $f(0) = 0$, such that $j^1(f)_0 =$ identity and $f^*(\tilde{X}) = X$. Now $f^*(\omega) = i_X i_{\overline{Y}}(\Omega)$ where $\overline{Y} = \det(Df).f^*(\tilde{Y}) = \mathrm{rot}(f^*\omega)$. This implies as in Theorem 1 that $[X, \overline{Y}] = \frac{k-2}{k+1}\overline{Y}$ or $[I, \overline{Y}] = (k-2)\overline{Y}$, where $I(x) = x$. Therefore $\overline{Y}$ is homogeneous of degree $k-1$. Moreover $j^{k-1}(\overline{Y})_0 = j^{k-1}(Y)_0 = Y$, because $j^1(f)_0 =$ identity. Therefore $f^*\omega = \omega_k$. ∎

(1) A particular version of this theorem was obtained independently by Cerveau and Moussu.

3. Polynomial Integrable Forms.

Here we study integrable 1-forms whose coefficients are polynomials of degree $k \geq 3$. Such a form is written

$$\omega = \omega_0 + \omega_1 + \ldots + \omega_k$$

where ω_j is homogeneous of degree j.

The main result of this section is that under the hypothesis that $0 \in K^3$ is an algebraically isolated zero of $d\omega_k$ then ω is, modulo a translation, a homogeneous form of degree k.

Lemma 2. — Let ω_{k-1} be a homogeneous form of degree $k-1$ and ω_k as above. Then

$$\omega_{k-1} \wedge d\omega_k + \omega_k \wedge d\omega_{k-1} = 0 \tag{1}$$

if and only if $\omega_{k-1} = L_a(\omega_k)$ for some $a \in K^3$. Here $L_a(\omega_k)$ is the Lie derivative of ω_k in the direction of the constant vector field a.

Proof. — By Theorem 1 we have $\omega_k = i_X i_Y(\Omega)$ where $d\omega_k = i_Y(\Omega)$. Then $i_X(d\omega_k) = \omega_k$. From (1) we obtain

$$i_X(\omega_{k-1})d\omega_k - \omega_{k-1} \wedge \omega_k - \omega_k \wedge i_X(d\omega_{k-1}) = 0. \tag{2}$$

Using the interior product i_Y in (2) we get

$$-i_Y(\omega_{k-1})\omega_k + i_Y i_X(d\omega_{k-1})\omega_k = 0.$$

Then
$$i_Y(\omega_{k-1} - i_X d\omega_{k-1}) = 0.$$

This means that $\omega_{k-1} - i_X d\omega_{k-1} = i_Y \alpha$ for some $\alpha \in \Lambda^2(K^3)$. Now, $\alpha = -i_v(\Omega)$, therefore

$$\omega_{k-1} - i_X d\omega_{k-1} = i_v d\omega_k$$

where v is constant.

We obtain:

$$i_X \omega_{k-1} = i_X i_v(d\omega_k) = -i_v(\omega_k)$$

and
$$d(i_X \omega_{k-1}) = -d i_v(\omega_k).$$

Then
$$\omega_{k-1} - i_X(d\omega_{k-1}) - d(i_X \omega_{k-1}) = i_v(d\omega_k) + d(i_v \omega_k),$$

i.e.
$$\omega_{k-1} - L_X \omega_{k-1} = L_v \omega_k.$$

But $X = \frac{1}{k+1} I$, so $L_X \omega_{k-1} = \frac{k}{k+1}\omega_{k-1}$ and

$$\omega_{k-1} = L_a \omega_k, \quad a = (k+1)v.$$

Conversely, since $\omega_k \wedge d\omega_k = 0$ we have

$$L_a \omega_k \wedge d\omega_k + \omega_k \wedge d(L_a \omega_k) = 0.$$

Therefore if $\omega_{k-1} = L_a \omega_k$ we obtain (1). ■

Lemma 3. — *Let ω_k be as above and let ω_j be a homogeneous form of degree j, $0 \leq j \leq k-2$ such that*

(3) $$\omega_j \wedge d\omega_k + \omega_k \wedge d\omega_j = 0.$$

Then $\omega_j = 0$.

Proof. — As in Lemma 2 we have $i_Y(\omega_j - i_X d\omega_j) = 0$ and then

$$\omega_j - i_X(d\omega_j) = i_Y \alpha, \qquad \alpha \in \Lambda^2(\mathbf{R}^3).$$

However, since Y is homogeneous of degree $k-1$ and $\omega_j, i_X(d\omega_j)$ are of degree $j < k-1$, we have

$$\omega_j - i_X(d\omega_j) = 0.$$

Consequently, $i_X(\omega_j) = 0$ and $di_X \omega_j = 0$. Therefore $\omega_j = L_X \omega_j$. Since $X = \frac{1}{k+1} I$, $L_X \omega_j = \frac{j+1}{k+1} \omega_j$. So $\omega_j = 0$. ∎

Lemma 4. — *Let ω_k be a homogeneous differential form of degree k in K^n and $f_b(x) = x + b$. Then*

(4) $$f_b^*(\omega_k) = \omega_k + L_b(\omega_k) + \tilde{\omega}_{k-2} + \ldots + \tilde{\omega}_0$$

where $\tilde{\omega}_j$ is homogeneous of degree j.

Proof. — Let $\omega = \sum_{i=1}^{n} P_i(x) dx_i$ where each $P_i(x)$ is homogeneous of degree k. Then

$$P_i(x+b) = P_i(x) + DP_i(x).b + \frac{1}{2} D^2 P_i(x).b^2 + \ldots + \frac{1}{k!} D^k P_i(x).b^k,$$

$$f_b^*(\omega_k) = \sum_{i=1}^{n} P_i(x+b) dx_i = \sum_{i=1}^{n} \sum_{j=0}^{k} \frac{1}{j!} D^j P_i(x).b^j dx_i$$

$$= \sum_{j=0}^{k} \sum_{i=1}^{n} \frac{1}{j!} D^j P_i(x).b^j dx_i = \omega_k + \sum_{i=1}^{n} DP_i(x).b dx_i + R_{k-2}$$

$$= \omega_k + L_b \omega_k + R_{k-2},$$

where R_{k-2} is a polynomial form of degree $k-2$. ∎

Theorem 3. — *Let ω be a polynomial integrable form of degree k in K^3. Write ω as a sum of homogeneous forms ω_j:*

$$\omega = \omega_0 + \omega_1 + \ldots + \omega_{k-1} + \omega_k$$

and assume that ω_k is simple at $0 \in K^3$.

Then there is $a \in K^3$ such that $\omega = f_a^*(\omega_k)$, $f_a(x) = x + a$.

Proof. — Since $\omega \wedge d\omega = 0$, we have

$$\omega_k \wedge d\omega_k = 0 \qquad \text{and} \qquad \omega_k \wedge d\omega_{k-1} + \omega_{k-1} \wedge d\omega_k = 0.$$

By (1), $\omega_{k-1} = L_a(\omega_k)$. From (4) we obtain

$$f_b^*(\omega) = \sum_{j=0}^{k-2} f_b^* \omega_j + \omega_{k-1} + R'_{k-2} + \omega_k + L_b \omega_k + R''_{k-2}.$$

Taking $b = -a$, we get

$$f_b^*(\omega) = \omega_k + \overline{\omega}_{k-2} + \ldots + \overline{\omega}_0$$

where $\overline{\omega}_j$ has degree j. By the integrability of ω

$$\overline{\omega}_{k-2} \wedge d\omega_k + \omega_k \wedge d\overline{\omega}_{k-2} = 0.$$

Then by (3), $\overline{\omega}_{k-2} = 0$.

Similarly $\overline{\omega}_{k-3} = \ldots = \overline{\omega}_0 = 0$. Then $f^*_{-a}(\omega) = \omega_k$. ■

Remark. — Observe that the main properties about ω_k that we have used in the proof of Theorem 3 are:

(5) ω_k is integrable.

(6) If α is a homogeneous 1-form of degree $j \leq k-1$ such that $\alpha \wedge d\omega_k + \omega_k \wedge d\alpha = 0$ then $\alpha = 0$ if $j \leq k-2$ and $\alpha = L_a(\omega_k)$ for some $a \in K^3$, if $j = k-1$.

In § 1, Chapter III, we shall see examples of homogeneous 1-forms in K^n, $n > 3$, which satisfy conditions (5), (6). This motivates the following definition.

Definition. — Let ω_k be a homogeneous 1-form of degree k in K^n. We say that ω_k is *regular* if it satisfies conditions (5) and (6) above.

With the same proof of Theorem 3, we have:

Theorem 3'. — Let $\omega = \omega_0 + \ldots + \omega_k$ *be a polynomial integrable 1-form in* K^n, *where* ω_j *is homogeneous of degree j and* ω_k *is regular. Then there is* $a \in K^n$ *such that* $\omega = f_a^*(\omega_k)$, $f_a(x) = x + a$.

III. — REGULAR INTEGRABLE FORMS

The notion of regularity plays a fundamental role in the study of stability properties of integrable forms. In this chapter we derive its main properties.

1. Regular Homogeneous 1-forms.

Let E be a vector space over the field K ($K = \mathbf{R}$ or $\mathbf{C}$) and let η be a p-form on E. We say that η is homogeneous of degree k if there exists a linear coordinate system on E in which η is expressed as a homogeneous p-form of degree k, i.e. all coefficients of the expression of η in this coordinate system are homogeneous polynomials of degree k. Of course, if η is homogeneous of degree k in some linear coordinate system, then the same is true for all linear coordinate systems on E. We denote by $H_k^p(E)$, or simply H_k^p, the set of all homogeneous p-forms of degree k on E.

The condition of regularity, given before can be expressed as follows. Let $\omega \in H_k^1$. Consider the linear operators $T_j^\omega : H_j^1 \to H_{k+j-1}^3$ and $S^\omega : K^n \to H_{k-1}^1$ defined by $T_j^\omega(\alpha) = \alpha \wedge d\omega + \omega \wedge d\alpha$ and $S^\omega(a) = L_a(\omega)$. Then ω is regular if and only if ω is integrable and satisfies the following conditions

$$\text{(7)} \qquad \operatorname{Ker}(T_j^\omega) = \{0\} \quad \text{if } 0 \leq j \leq k-2.$$

$$\text{(8)} \qquad \operatorname{Ker}(T_{k-1}^\omega) = \operatorname{Im}(S^\omega).$$

Observe that the integrability condition, $\omega \wedge d\omega = 0$, implies that

$$L_a(\omega) \wedge d\omega + \omega \wedge d(L_a(\omega)) = 0, \qquad a \in K^n,$$

i.e. $\operatorname{Im}(S^\omega) \subset \operatorname{Ker}(T_{k-1}^\omega)$ for every integrable $\omega \in H_k^1$.

We use the following notations:

$$\mathscr{R}_k(E) = \mathscr{R}_k = \{\omega \in H_k^1 \mid \omega \text{ is regular}\}$$
$$\mathscr{R}_k^\ell(E) = \mathscr{R}_k^\ell = \{\omega \in \mathscr{R}_k \mid \dim(\operatorname{Im}(S^\omega)) = \ell\}.$$

Now we can state the results.

Proposition 1. — *$\mathscr{R}_k^\ell(E)$ is open in the set of integrable homogeneous 1-forms of degree k for any $k \geq 1$.*

Definition. — Let $\omega \in H_k^1(E)$. We say that ω *can be written with $m \leq n$ variables* if there is a linear coordinate system $x = (x_1, \ldots, x_n)$ in E, such that $\omega = \sum_{i=1}^{m} p_i(x_1, \ldots, x_m)dx_i$

in this coordinate system, that is, ω does not depend on $x_{m+1}, \ldots, x_n$. The *rank* of ω ($\mathrm{rank}(\omega)$) is the minimum number of variables in which ω can be written.

Proposition 2. — Let $\omega \in \mathscr{R}_k^\ell$. *Then* $\mathrm{rank}(\omega) = \ell$.

Proposition 3. — Let $\omega \in \mathrm{H}_k^1(\mathrm{K}^n)$ *be integrable and* $d\omega \not\equiv 0$. *Let* $\ell \geq 2$. *Then* $\omega \in \mathscr{R}_k^\ell(\mathrm{K}^n)$ *if and only if there exists an* ℓ*-dimensional subspace* $\mathrm{E} \subset \mathrm{K}^n$ *such that the restriction* $\omega/\mathrm{E} \in \mathscr{R}_k^\ell(\mathrm{E})$.

Corollary. — Let $\omega \in \mathscr{R}_k^\ell(\mathrm{K}^m)$ *with* $d\omega \not\equiv 0$. *If* $f: \mathrm{K}^n \to \mathrm{K}^m$ *is linear and surjective then* $f^*(\omega) \in \mathscr{R}_k^\ell(\mathrm{K}^n)$.

Before proving the results we give some examples.

Example 1. — Let $\omega = df$ where $f: \mathrm{K}^n \to \mathrm{K}$ is homogeneous of degree $k+1$. Then $\omega \in \mathscr{R}_k$ if and only if $k = 1$ and f is a non-degenerated quadratic form in K^n. This is true because if $k \geq 2$ then any form $\omega_k = df$ admits perturbations of lower order (see Theorem 3′).

Example 2. — Let ω be an integrable homogeneous 1-form of degree $k \geq 3$ in K^3 such that $0 \in \mathrm{K}^3$ is an algebraically isolated zero of $d\omega$. Let $f: \mathrm{K}^n \to \mathrm{K}^3$ be defined by $f(x_1, \ldots, x_n) = (x_1, x_2, x_3)$ and $\omega^* = f^*(\omega)$. Then, by Proposition 3, $\omega^* \in \mathscr{R}_k^3(\mathrm{K}^n)$.

Example 3. — Homogeneous 1*-forms defined by linear* K^{n-1} *actions on* K^n.

Let ω be the homogeneous 1-form of degree $n-1$ defined by

$$\omega = \sum_{i=1}^{n} a_i x_1 \ldots \hat{x}_i \ldots x_n \, dx_i. \tag{9}$$

Where $a_i \in \mathbf{C}$ and the symbol $\hat{x}_i$ means omission of x_i in the product. Every form of type (9) is integrable and in fact they are induced by $\mathbf{C}^{n-1}$ linear actions on $\mathbf{C}^n$, in the following sense.

Let $\mathrm{X}_1, \ldots, \mathrm{X}_{n-1}$ be linear commutative vector fields in $\mathbf{C}^n$. Assume

$$\mathrm{X}_j(x_1, \ldots, x_n) = (\alpha_j^1 x_1, \ldots, \alpha_j^n x_n), \quad \alpha_j^i \in \mathbf{C}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n-1.$$

Define $\omega = i_{\mathrm{X}_1 \wedge \ldots \wedge \mathrm{X}_{n-1}}(\Omega)$, where $\Omega = dx_1 \wedge \ldots \wedge dx_n$. Then it is easy to see that ω has an expression like in (9), where $a_i = \pm \det(\mathrm{A}_i)$ and A_i is the $(n-1) \times (n-1)$ minor obtained from $(\alpha_j^\ell)_{1 \leq j \leq n-1}^{1 \leq \ell \leq n}$ by deleting the i-th column. This case corresponds to the canonical form for an open and dense set of linear $\mathbf{C}^{n-1}$ actions on $\mathbf{C}^n$. In the case of linear $\mathbf{R}^{n-1}$ actions on $\mathbf{R}^n$ we can in the same way induce an integrable 1-form ω and the canonical form is

$$\omega = f(x, u, v)\left[\sum_{i=1}^{k} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} (u_j^2 + v_j^2)^{-1} [(\alpha_j u_j + \beta_j v_j) du_j + (-\beta_j u_j + \alpha_j v_j) dv_j]\right] \tag{10}$$

for an open and dense set of actions, where $x = (x_1, \ldots, x_k)$, $u = (u_1, \ldots, u_\ell)$, $v = (v_1, \ldots, v_\ell)$, $k + 2\ell = n$ and $f(x, u, v) = x_1 \ldots x_k(u_1^2 + v_1^2) \ldots (u_\ell^2 + v_\ell^2)$.

If we complexify (10) then it can be reduced to the form (9) which is easier in handling algebraic computations. So we assume in the real case that ω is complexified and is like in (9).

Observe that (9) or (10) can be considered as 1-forms in K^m, where $m \geq n$. We have the following.

Proposition 4. — Let $\omega = \sum_{i=1}^{n} a_i x_1 \ldots \hat{x}_i \ldots x_n dx_i$ *be a 1-form in* $\mathbf{C}^m$, *where* $m \geq n \geq 2$. *If* $a_i \neq a_j \neq 0$ *for* $i \neq j$, $0 \leq i, j \leq n$, *then* $\omega \in \mathscr{R}_{n-1}^n(\mathbf{C}^m)$.

(1.1) *Proof of Proposition 1.* — Let $\omega \in \mathscr{R}_k^\ell(E)$ and consider as before the operators $T_j^\omega(\alpha) = \alpha \wedge d\omega + \omega \wedge d\alpha$, $\alpha \in H_j^1$ and $S^\omega(a) = L_a(\omega)$, $a \in K^n$. By definition we have that

$$\text{(7)} \qquad \operatorname{Ker}(T_j^\omega) = \{0\} \quad \text{if } 0 \leq j \leq k-2 \quad \text{and}$$

$$\text{(8)} \qquad \operatorname{Ker}(T_{k-1}^\omega) = \operatorname{Im}(S^\omega), \qquad \dim(\operatorname{Im}(S^\omega)) = \ell.$$

We want to prove that (7) and (8) are true for all $\eta \in H_k^1$, integrable, sufficiently near ω.

First of all observe that the maps $\eta \mapsto T_j^\eta$ and $\eta \mapsto S^\eta$ are continuous. Since the set of one to one linear operators of H_j^1 into H_{k+j-1}^3 is open in the set of all linear operators we get that if η is sufficiently near ω then $\operatorname{Ker}(T_j^\eta) = \{0\}$ for $0 \leq j \leq k-2$, so that (7) is true for η.

Let us consider (8). Since $\eta \mapsto T_{k-1}^\eta$ and $\eta \mapsto S^\eta$ are continuous, for η sufficiently near ω, we get

$$\text{(8 a)} \qquad \dim \operatorname{Ker}(T_{k-1}^\eta) \leq \dim \operatorname{Ker}(T_{k-1}^\omega) = \ell \quad \text{and}$$

$$\text{(8 b)} \qquad \dim \operatorname{Im}(S^\eta) \geq \dim \operatorname{Im}(S^\omega) = \ell.$$

Now, if η is integrable, then $\operatorname{Im}(S^\eta) \subset \operatorname{Ker}(T_{k-1}^\eta)$, so that

$$\ell = \dim \operatorname{Ker}(T_{k-1}^\omega) \geq \dim \operatorname{Ker}(T_{k-1}^\eta) \geq \dim \operatorname{Im}(S^\eta) \geq \dim \operatorname{Im}(S^\omega) = \ell.$$

Hence $\operatorname{Im}(S^\eta) = \operatorname{Ker}(T_{k-1}^\eta)$ and $\dim \operatorname{Im}(S^\eta) = \ell$. ■

(1.2) *Proof of Proposition 2.* — By example 1 we can suppose that $d\omega \not\equiv 0$.

Let $\omega \in \mathscr{R}_k^\ell(K^n)$ and $a \in \operatorname{Ker}(S^\omega) - \{0\}$. We proceed to prove that in this case ω can be written with $n-1$ variables. By a linear change of variables we can suppose that $a = \partial/\partial x_n$. If $\omega = \sum_{i=1}^{n} p_i(x) dx_i$, $L_a(\omega) = 0$ implies that $\frac{\partial p_i}{\partial x_n} = 0$, therefore $p_i = p_i(x_1, \ldots, x_{n-1})$, $1 \leq i \leq n$. We have to prove that $p_n \equiv 0$. We write $z = (x_1, \ldots, x_{n-1})$, $y = x_n$, $p_n = p$ and $\alpha = \sum_{i=1}^{n-1} p_i(z) dx_i$, so that $\omega = \alpha + p(z) dy$.

Since α does not depend on y we get $pd\alpha = dp \wedge \alpha$ and $\alpha \wedge d\alpha = 0$. From this get $pd\omega = dp \wedge \omega$. We need a lemma.

Lemma 5. — Let p be a homogeneous polynomial of degree j, $0 \leq j \leq k$. If $\omega \in \mathscr{R}_k(K^n)$, $d\omega \not\equiv 0$ and $pd\omega = dp \wedge \omega$ then $p \equiv 0$.

Proof. — If $j = 0$ we get $pd\omega = 0$ and since $d\omega \not\equiv 0$ then $p = 0$. If $0 < j \leq k$ then the equation $pd\omega = dp \wedge \omega$ implies that $dp \wedge d\omega = 0$ or $dp \wedge d\omega + \omega \wedge d(dp) = 0$ and since ω is regular we get $dp = 0$ if $0 < j \leq k - 1$ and $dp = L_v(\omega)$, $v \in K^n$, if $j = k$. In the case $0 < j \leq k - 1$ we get $p = 0$ because p is homogeneous. Let us consider the case $j = k$. In this case $L_v(\omega) = dp$ implies that $L_v(d\omega) = 0$ and $pd\omega = dp \wedge \omega$ implies

$$\begin{aligned} L_v(p)d\omega = L_v(pd\omega) &= L_v(dp \wedge \omega) \\ &= d(L_v(p)) \wedge \omega + dp \wedge L_v(\omega) = d(L_v(p)) \wedge \omega. \end{aligned}$$

Since $L_v(p)$ has degree $k - 1$ it follows that $L_v(p) = 0$.

Now let $X(x) = \sum_{i=1}^{n} x_i \partial/\partial x_i$. Then $dp \wedge \omega = pd\omega$ implies that

$$kp\omega - qdp = i_X(dp \wedge \omega) = i_X(pd\omega) = pi_X(d\omega)$$

where $q = i_X(\omega)$ is a homogeneous polynomial of degree $k + 1$. We have

$$i_X(d\omega) = L_X(\omega) - d(i_X(\omega)) = (k + 1)\omega - dq$$

because ω is homogeneous of degree k. Therefore

$$p\omega = pdq - qdp.$$

Applying L_v to both members of the equation we get

$$pdp = pdr - rdp$$

where $r = L_v(q)$ is a homogeneous polynomial of degree k. Applying i_X to both members of the equation we have $kp^2 = pi_X(dr) - ri_X(dp) = p(kr) - r(kp) = 0$. Hence $p = 0$. This finishes the proof of the lemma. ∎

By Lemma 5 we get $\omega = \alpha = \sum_{i=1}^{n-1} p_i(x_1, \ldots, x_{n-1})dx_i$.

Now $\ell = \dim \operatorname{Im}(S^\omega) = \operatorname{codim} \operatorname{Ker}(S^\omega)$, so that applying Lemma 5 inductively it is possible to find a linear coordinate system $(x_1, \ldots, x_n)$ such that $\omega = \sum_{i=1}^{\ell} \omega^i(x_1, \ldots, x_\ell)dx_i$. On the other hand, suppose we could write ω with $m < \ell$ variables. In this case it is easy to see that $\operatorname{codim} \operatorname{Ker}(S^\omega) \leq m < \ell$, which is a contradiction. Hence $\operatorname{rank}(\omega) = \ell$. ∎

(1.3) *Proof of Proposition 3.* — Suppose first that $\omega \in \mathscr{R}_k^\ell(K^n)$. In this case, by Proposition 2, ω can be written with ℓ variables, $\omega = \sum_{i=1}^{\ell} p_i(x_1, \ldots, x_\ell)dx_i$. Take

$E = \{(x_1, \ldots, x_\ell, 0, \ldots, 0) \mid x_1, \ldots, x_\ell \in K\}$. So ω/E has the same expression as ω and it is not difficult to verify that $\omega/E \in \mathscr{R}_k^\ell(E)$.

Suppose now that there exists an ℓ-dimensional plane $E \subset K^n$, such that $\omega/E \in \mathscr{R}_k^\ell(E)$. We can suppose $E = \{(x, 0) \mid x \in K^\ell\}$. Let $E_j = \{(x, 0) \mid x \in K^{\ell+j}\}$ and $\eta_j = \omega/E_j$. The idea is to prove that $\eta_j \in \mathscr{R}_k^\ell(E_j)$, by induction on $j = 0, \ldots, n-\ell$.

For $j = 0$ the assertion is clear. Suppose the assertion true for $j \geq 0$ and let us prove that it is true for $j+1$. First of all we prove that η_{j+1} can be written with $\ell + j$ variables. If $x \in E_{j+1}$ we write $x = (z, y)$ where $z \in E_j$ and $y \in K$. In this coordinate system η_{j+1} can be written as

$$\eta_{j+1} = \alpha_k + y\alpha_{k-1} + \ldots + y^k\alpha_0 + p(z,y)dy$$

where p is a homogeneous polynomial of degree k and α_i is a homogeneous 1-form of degree i which does not depend on y and dy.

Then $\eta_{j+1}/E_j = \alpha_k = \eta_j \in \mathscr{R}_k^\ell(E_j)$, by induction. Set $a = \partial/\partial y = (0, 1)$ and let $g_t : E_j \to E_{j+1}$ be defined by $g_t(z) = z + ta$. Then $g_t^*(\eta_{j+1}) = \alpha_k + \ldots + t^k\alpha_0 = \beta_t$.

Since α_k is regular, by Theorem 3′ there exists $v \in E_j$ such that $h^*(\beta_1) = \alpha_k$, where $h(z) = z + v$. We define $f: E_{j+1} \to E_{j+1}$ by $f(z,y) = (z + yv, y)$. Then it is not difficult to see that $f^*(\eta_{j+1}) = \alpha_k + q(z,y)dy$, where q is homogeneous of degree k. We write $q(z,y) = q_k(z) + yq_{k-1}(z) + \ldots + y^k q_0$ where q_i is homogeneous of degree i, $0 \leq i \leq k$. Now the integrability condition applied to $f^*(\eta_{j+1})$ implies that

$$q_i d\alpha_k = dq_i \wedge \alpha_k, \quad 0 \leq i \leq k.$$

Suppose first that $d\alpha_k \not\equiv 0$. In this case, by Lemma 5, we get $q_i = 0$, $0 \leq i \leq k$, and then $f^*(\eta_{j+1}) = \alpha_k$.

If $d\alpha_k \equiv 0$ then, by Example 1, $k = 1$ and $\alpha_k = dg$ where g is a non-degenerate quadratic form. In this case $\omega = dg + \Delta = \sum_{r,s} a_{rs}x_r dx_s$, where $\Delta/E_j = 0$ and the matrix (a_{rs}) has $\operatorname{rank} \geq \ell + j \geq \ell \geq 2$. If $\Delta \not\equiv 0$ we have in fact $\operatorname{rank}(a_{rs}) \geq 3$. The idea is to show that in this case $d\Delta = 0$ so that $d\omega = 0$ which contradicts the hypothesis. In fact, suppose that we had $d\Delta \not\equiv 0$. In this case $d\omega_0 \not\equiv 0$ and it can be shown that the matrix (a_{rs}) has rank at most 2 (see [5] or [9]). This proves that $d\alpha_k \not\equiv 0$ in any case.

By the above argument we can suppose that $\eta_{j+1} = \alpha_k$, does not depend on y or dy. Let $\beta \in H_m^1(E_{j+1})$ be such that

$$(*) \qquad \beta \wedge d\alpha_k + \alpha_k \wedge d\beta = 0.$$

We write

$$\beta = \beta_m + y\beta_{m-1} + \ldots + y^m\beta_0 + q(z,y)dy$$

where q has degree m and $\beta_i \in H_i^1(E_j)$. Then it is not difficult to see that $(*)$ implies that $\beta_i \wedge d\alpha_k + \alpha_k \wedge d\beta_i = 0$ for $0 \leq i \leq m$. Since $\alpha_k \in \mathscr{R}_k(E_j)$ we get $\beta_i = 0$ for

$0 \leq i \leq k-2$ and $\beta_{k-1} = L_v(\alpha_k)$ (if $m = k-1$). Consequently, for $m < k-1$ we have $\beta = q(z,y)dy$ and for $m = k-1$ we have $\beta = L_v(\alpha_k) + q(z,y)dy$. We write

$$q(z,y) = q_m(z) + yq_{m-1}(z) + \ldots + y^m q_0$$

where q_i is homogeneous of degree i. Now, equation $(*)$ implies that

$$q_i d\alpha_k + \alpha_k \wedge dq_i = 0, \quad 0 \leq i \leq m.$$

Since $m \leq k-1$ we get by Lemma 5 that $q_i = 0$ so that $\beta = L_v(\alpha_k)$ if $m = k-1$ and $\beta = 0$ if $m < k-1$. This ends the proof. ■

(1.4) *Proof of Proposition 4.* — Let $\omega = \sum_{i=1}^{n} a_i x_1 \ldots \hat{x}_i \ldots x_n dx_i$ where $a_i \neq a_j \neq 0$, $1 \leq i, j \leq n$, be considered as a 1-form in $\mathbf{C}^m$, $m \geq n$. By Proposition 3 it is sufficient to consider the case $m = n$.

Let α be a homogeneous 1-form of degree j such that

$$(*) \qquad \alpha \wedge d\omega + \omega \wedge d\alpha = 0.$$

We want to prove that $\alpha = 0$ if $0 \leq j \leq n-3$ and $\alpha = L_v(\omega)$ if $j = n-2$. We can write $\alpha = \sum_{i=1}^{n} \sum_{|\sigma|=j} b_i^\sigma x^\sigma dx_i$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$, $x^\sigma = x_1^{\sigma_1} \ldots x_n^{\sigma_n}$ and $|\sigma| = \sigma_1 + \ldots + \sigma_n$. Let us write α in another way:

$$\alpha = \sum_{i=1}^{n} \sum_{|\sigma|=j+1} \sigma_i C_i^\sigma x^{\sigma - e_i} dx_i$$

where $C_i^\sigma = 0$ if $\sigma_i = 0$, $\sigma_i C_i^\sigma = b_i^{\sigma - e_i}$ if $\sigma_i > 0$ and $\sigma - e_i = (\sigma_1, \ldots, \sigma_i - 1, \ldots, \sigma_n)$.

Differentiating α we get

$$d\alpha = \sum_{k<\ell} \sum_{|\sigma|=j+1} \sigma_k \sigma_\ell C_{k\ell}^\sigma x^{\sigma - e_k - e_\ell} dx_k \wedge dx_\ell$$

where $C_{k\ell}^\sigma = C_\ell^\sigma - C_k^\sigma$. In the same way we can write

$$\omega = x_1 \ldots x_n \sum_{i=1}^{n} a_i \frac{dx_i}{x_i}$$

and
$$d\omega = x_1 \ldots x_n \sum_{k<\ell} a_{k\ell} \frac{dx_k \wedge dx_\ell}{x_k x_\ell}$$

where $a_{k\ell} = a_\ell - a_k$. Now equation $(*)$ implies that

$$0 = \alpha \wedge d\omega + \omega \wedge d\alpha = x_1 \ldots x_n \sum_{\substack{i<k<\ell \\ |\sigma|=j+1}} e_{ik\ell}^\sigma x^{\sigma - e_i - e_k - e_\ell} dx_i \wedge dx_k \wedge dx_\ell$$

where $e_{ik\ell}^\sigma = f_{ik\ell}^\sigma + f_{k\ell i}^\sigma + f_{\ell ik}^\sigma$, $f_{ik\ell}^\sigma = \sigma_i C_i^\sigma a_{k\ell} + \sigma_k \sigma_\ell a_i C_{k\ell}^\sigma$. Then $e_{ik\ell}^\sigma = 0$, $1 \leq i, k, \ell \leq n$.

Now suppose that $0 \leq j \leq n-3$. In this case for any σ with $|\sigma| = j+1$ there exist $k \neq \ell$ in $\{1, \ldots, n\}$ such that $\sigma_k = \sigma_\ell = 0$, because $\sigma_1 + \ldots + \sigma_n \leq n-2$. Therefore we get for such σ

$$0 = e_{ik\ell}^\sigma = \sigma_i C_i^\sigma a_{k\ell} = \sigma_i C_i^\sigma (a_\ell - a_k).$$

Since $a_\ell - a_k \neq 0$, we have $C_i^\sigma = 0$, $i = 1, \ldots, n$, which implies that $\alpha = 0$.

If $j = n-2$ and σ is such that $\sigma_k = \sigma_\ell = 0$, $k \neq \ell$, we get in the same way $C_i^\sigma = 0$. Therefore if $C_i^\sigma \neq 0$, σ must be of the form $\sigma = (1, \ldots, 1) - e_\ell$ for some ℓ and in this case we have

$$0 = e_{i\ell k}^\sigma = C_i^\sigma a_{k\ell} + C_k^\sigma a_{\ell i} + a_\ell C_{ik}^\sigma.$$

Therefore $a_i C_k^\sigma = a_k C_i^\sigma$, $1 \leq i, k \leq n$, which means that the vector $C^\sigma = (C_1^\sigma, \ldots, C_n^\sigma)$ is a scalar multiple of the vector $a = (a_1, \ldots, a_n)$, say $C^\sigma = \lambda_\ell a$, where $\sigma = (1, \ldots, 1) - e_\ell$. Therefore we can write

$$\alpha = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial x_j} (x_1 \ldots \hat{x}_i \ldots x_n) \right) dx_i = L_v(\omega)$$

where $v = \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial x_j}$, as can be verified directly. ∎

2. Reduction of variables for analytic integrable 1-forms.

Let ω be an integrable 1-form defined in an open set $U \subset K^n$. Given an open set $V \subset U$, we say that ω *can be written with* $\ell \leq n$ *variables* in V if there is a diffeomorphism $f : V \to f(V) \subset K^n$ such that $f(p) = p$ and

$$f^*(\omega) = \sum_{i=1}^{\ell} \omega_i(x_1, \ldots, x_\ell) dx_i.$$

The *rank of* ω *at* p is the minimum number of variables in which ω can be written in a neighborhood of p. We use the notation $\text{rank}_p(\omega)$ for the rank of ω at p.

Geometrically the fact that $\text{rank}_p(\omega) = m < n$, means that the foliation defined by ω is locally equivalent at p to the product of a codimension one singular foliation in K^m by K^{n-m}.

Examples

1) If ω is a regular homogeneous 1-form of degree k, then $\text{rank}_0(\omega) = \dim \text{Im}(S^\omega)$, where $S^\omega(a) = L_a(\omega)$, $a \in K^n$ (cf. Prop. 2).

2) If ω is an integrable 1-form such that $\omega_p = 0$ and $d\omega_p \neq 0$, then $\text{rank}_p(\omega) = 2$ (cf. [5], [9]).

3) Let $\omega \in I^r(U)$ $(r \geq 4)$ and suppose that there exist $p \in U$ and a 3-dimensional plane $F \subset K^n$ such that $p \in F$ and p is a hyperbolic singularity of the vector field $\text{rot}(\omega/F)$. Then $\text{rank}_p(\omega) = 3$. The proof can be found in [6].

It is an open question to know whether a 1-form $\omega \in I^r(U)$ with $J^{k-1}(\omega)_p \equiv 0$ and $J^k(\omega)_p \in R_k^\ell$ can be reduced to ℓ variables near p. Along this direction we have the following result.

Theorem 4. — *Let* ω *be an analytic integrable* 1*-form defined in an open set* $U \subset K^n$. *Suppose that there exist* $p \in U$ *and a* 3*-dimensional plane* $F \subset K^n$ *such that* $p \in F$ *and* p *is an algebraically isolated singularity of the vector field* $\operatorname{rot}(\omega/F)$. *Then* $\operatorname{rank}_p(\omega) = 3$.

Proof. — The idea is to prove that if ω can be written with ℓ variables, $4 \leq \ell \leq n$, then it can be written with $\ell - 1$ variables. More specifically, if $f^*(\omega) = \sum_{i=1}^{\ell} \omega_i(x_1, \ldots, x_\ell)\, dx_i$ for a diffeomorphism $f: V \to f(V) \subset K^n$, $f(0) = 0$, then we shall construct a diffeomorphism $g: V' \to g(V')$, $g(0) = 0$, of the form $g(x_1, \ldots, x_n) = (g_1(x_1, \ldots, x_\ell), x_{\ell+1}, \ldots, x_n)$ where $g_1 : V_1 \subset K^\ell \to K^\ell$ and such that

$$(f \circ g)^*(\omega) = g^*(f^*(\omega)) = \sum_{i=1}^{\ell-1} \widetilde{\omega}_i(x_1, \ldots, x_{\ell-1})dx_i.$$

So we can suppose that $\ell = n$ and all the steps of the induction procedure will be similar to this case. We can suppose also that $p = 0$ and $F = \{(x, 0) \in K^n \mid x \in K^3\}$.

In order to prove that ω can be written with $n - 1$ variables we shall construct an analytic vector field X in a neighborhood W of 0, such that X is transversal to the plane $\widetilde{F} = \{(x, 0) \mid x \in K^{n-1}\}$ and $i_X(d\omega) = 0$.

Suppose for a moment that we have constructed such a vector field. Let $V \subset K^n$ be a neighborhood of $0 \in K^n$, and $f: V \to f(V) \subset W$ be a diffeomorphism such that $f^*(X) = \partial/\partial x_n = e_n$. If $\eta = f^*(\omega)$, then we have $i_{e_n}(d\eta) = 0$ and the integrability condition $\eta \wedge d\eta = 0$ implies that $i_{e_n}(\eta) = 0$ and so

$$L_{e_n}(\eta) = i_{e_n}(d\eta) + d(i_{e_n}(\eta)) = 0.$$

Therefore the coefficients of η do not depend on x_n, so that $\eta = \sum_{i=1}^{n} \eta_i(x_1, \ldots, x_{n-1})dx_i$. Using that $i_{e_n}(\eta) = 0$ we get $\eta_n \equiv 0$. Hence ω can be written with $n - 1$ variables.

Now we construct the vector field X. Suppose $X = \sum_{i=1}^{3} \Delta_i\, \partial/\partial x_i + \partial/\partial x_n$. The condition $i_X(d\omega) = 0$ is equivalent to

$$(*) \qquad \Delta_1 \omega_{1j} + \Delta_2 \omega_{2j} + \Delta_2 \omega_{3j} + \omega_{nj} = 0, \qquad 1 \leq j \leq n$$

where $d\omega = \sum_{1 \leq i < j \leq n} \omega_{ij}\, dx_i \wedge dx_j$, $\omega_{ij} = -\omega_{ji}$.

Now observe that the three conditions

$$(**) \qquad \begin{cases} -\Delta_2 \omega_{12} + \Delta_3 \omega_{31} = \omega_{1n} \\ \Delta_1 \omega_{12} - \Delta_3 \omega_{23} = \omega_{2n} \\ -\Delta_1 \omega_{31} + \Delta_2 \omega_{23} = \omega_{3n} \end{cases}$$

are equivalent to the conditions (*).

In fact, to obtain (**) it is sufficient to make $j = 1, 2, 3$ in (*). On the other hand, if the conditions (**) are true, it is sufficient to apply the relation $d\omega \wedge d\omega = 0$ to obtain (*) for $j \geq 4$. For more details see [6].

Now we write the conditions (**) in another way. Let Y be the vector field $\omega_{23}\,\partial/\partial x_1 + \omega_{31}\,\partial/\partial x_2 + \omega_{12}\,\partial/\partial x_3$ and α be the 2-form

$$-\Delta_1\,dx_2 \wedge dx_3 - \Delta_2\,dx_3 \wedge dx_1 - \Delta_3\,dx_1 \wedge dx_2.$$

Then (**) is equivalent to $\zeta = i_Y(\alpha)$ where $\zeta = \sum_{i=1}^{3} \omega_{in}\,dx_i$. Therefore to obtain Δ_1, Δ_2 and Δ_3 it is sufficient to prove that there exists a 2-form α such that $\zeta = i_Y(\alpha)$. Since $i_Y(\zeta) = \omega_{23}\,\omega_{1n} + \omega_{31}\,\omega_{2n} + \omega_{12}\,\omega_{3n} = 0$ (because $d\omega \wedge d\omega = 0$) the proof of the theorem is reduced to the parametric version of the de Rham division theorem, which can be applied in this case because $0 \in K^n$ is an algebraically isolated zero of $Y(x, 0) = \mathrm{rot}(\omega/F)$. This finishes the proof. ■

IV. — STABILITY OF INTEGRABLE FORMS

1. Stability of regular points.

Let ω be an integrable 1-form of class C^r defined in an open subset $U \subset K^n$ (C^r = holomorphic if $K = \mathbf{C}$), where $r \geq k$. Then we can consider the k-jet of ω at $p \in U$, $j^k(\omega)_p$, as a polynomial 1-form so that

$$j^k(\omega)_p = \omega_0 + \omega_1 + \ldots + \omega_k$$

where ω_j is a homogeneous 1-form of degree j. If $j^{k-1}(\omega)_p = 0$ then it is not difficult to see that $\omega_k = j^k(\omega)_p$ is a homogeneous 1-form of degree k.

Definition. — Let $\omega \in I^r(U)$, $r \geq 1$. A singularity p of ω is called *regular of order* $k \geq 1$ if $j^{k-1}(\omega)_p = 0$ and $j^k(\omega)_p$ is a regular homogeneous 1-form.

Write $M_k(\omega)$ to denote the set of regular singularities of order k of ω and

$$M_k^\ell(\omega) = \{p \in M_k(\omega) \mid j^k(\omega)_p \in \mathscr{R}_k^\ell(K^n)\}.$$

Denote by $C^s(M, N)$ the set of all C^s maps from the manifold M to N, endowed with the C^s-uniform topology.

Theorem 5. — Let $\omega \in I^r(U)$, $r \geq 2k$. Then $M_k^\ell(\omega)$ is an embedded codimension ℓ submanifold of class C^{r-k+1} (holomorphic if $K = \mathbf{C}$). Moreover, if we fix a relatively compact open subset $P \subset M_k^\ell(\omega)$, then there exist neighborhoods $N \subset I^r(U)$ of ω and V, $P \subset V \subset U$, such that for any $\eta \in N$ there exists an embedding $h_\eta : P \to U$ of class C^{r-k+1} such that $h_\eta(P) = M_k^\ell(\eta) \cap V$. The map $\eta \mapsto h_\eta \in C^{r-k+1}(P, U)$ can be chosen so that it is continuous.

(1.1) *Proof of Theorem 5.* — Let J^ℓ be the space of 1-forms in K^n whose coefficients are polynomials of degree $\leq \ell$. A form $x \in J^\ell$ can be written as a sum $x = x_0 + x_1 + \ldots + x_\ell$ where x_i is a homogeneous form of degree i. Given x and y in J^{2k} define

$$F_m(x,y) = \sum_{\substack{i+j=m \\ i,j \geq 0}} (x_i \wedge dy_j + y_j \wedge dx_i), \qquad m \leq 2k.$$

Notice that if ω is an integrable form and $x = j^{2k}(\omega)_p$ then $F_m(x, x) = 0$ for, $1 \leq m \leq 2k$.

Define also

$$F(x,y) = (F_1(x,y), \ldots, F_{2k}(x,y))$$

and

$$G(x,y) = (F_k(x,y), \ldots, F_{2k-1}(x,y)).$$

Let $\pi : J^{2k} \to J^{k-1}$ be the projection defined as

$$\pi(x_0 + \ldots + x_{2k}) = x_0 + \ldots + x_{k-1}$$

and consider the algebraic variety

$$V_{2k} = \{x \in J^{2k} \mid F(x, x) = 0\}$$

and its projection $V_{2k}^{k-1} = \pi(V_{2k}) = V$. The tangent space to V_{2k} at $x \in V_{2k}$ is by definition

$$T_x(V_{2k}) = \{\dot{x} \in J^{2k} \mid F(\dot{x}, x) = 0\}.$$

Lemma 6. — Let $\omega \in I^r(U)$, $p \in M_k(\omega)$ and $j^{2k}(\omega)_p = x^0 = \omega_k + \omega_{k+1} + \ldots + \omega_{2k}$, *where* ω_j *is homogeneous of degree* j, $k \leq j \leq 2k$. *Then*

$$\pi(T_{x^0}(V_{2k})) = \operatorname{Im}(S_p) = \{\pi(\dot{x}) \mid G(\dot{x}, x^0) = 0\}.$$

By S_p we denote the operator $S_p(a) = L_a(\omega_k)$, $a \in K^n$.

Proof. — Let $\dot{x} = \dot{x}_0 + \ldots + \dot{x}_{2k}$. Then

$$F_m(\dot{x}, x^0) = \sum_{\substack{i+j=m \\ i,j \geq 0}} (\dot{x}_i \wedge d\omega_j + \omega_j \wedge d\dot{x}_i) \qquad m = 1, \ldots, 2k.$$

For $m < k$ we have $F_m(\dot{x}, x^0) \equiv 0$.

For $m = k$, $F_m(\dot{x}, x^0) = \dot{x}_0 \wedge d\omega_k$.

Then $F_k(\dot{x}, x^0) = 0$ implies by assumption that $\dot{x}_0 = 0$. Since

$$F_{k+1}(\dot{x}, x^0) = \dot{x}_1 \wedge d\omega_k + \omega_k \wedge d\dot{x}_1$$

again $F_{k+1}(\dot{x}, x^0) = 0$ implies $\dot{x}_1 = 0$. So by induction we obtain

$$\dot{x}_0 = \dot{x}_1 = \dot{x}_2 = \ldots = \dot{x}_{k-2} = 0.$$

Finally

$$F_{2k-1}(\dot{x}, x^0) = \dot{x}_{k-1} \wedge d\omega_k + \omega_k \wedge d\dot{x}_{k-1}$$

then if $F_{2k-1}(\dot{x}, x^0) = 0$ we have $\dot{x}_{k-1} = L_a \omega_k$, $a \in K^n$. This implies that

$$\{L_a \omega_k \mid a \in K^n\} = \pi(T_{x^0}(V_{2k})) = \{\pi(\dot{x}) \mid G(\dot{x}, x^0) = 0\}. \quad \blacksquare$$

Lemma 7. — Let $p \in M_k^\ell(\omega)$ *and* $j^{2k}(\omega)_p = x^0 = \omega_k^0 + \ldots + \omega_{2k}^0 \in V_{2k}$. *Let* $F = \pi(T_{x^0}(V_{2k}))$. *Then* $\dim F = \ell$ *and if* $E \subset J^{k-1}$ *is a codimension* ℓ *subspace such that* $J^{k-1} = E \oplus F$ *then there is* $\mu > 0$ *such that if* $|x - x^0| < \mu$, $x \in V_{2k}$ *and* $\pi(x) \in E$ *then* $\pi(x) = 0$.

Proof. — Let $x \in V_{2k}$ be written as $x = x^0 + \Delta x^0$. Then

$$F(x^0 + \Delta x^0, x^0 + \Delta x^0) = 2F(\Delta x^0, x^0) + F(\Delta x^0, \Delta x^0) = 0$$

and

$$2G(\Delta x^0, x^0) + G(\Delta x^0, \Delta x^0) = 0.$$

We define

$$H(z, \bar{z}) = 2G(z + \bar{z}, x^0) + G(z + \bar{z}, z + \bar{z})$$

where $z \in E$ and $\bar{z} = x_k + \ldots + x_{2k}$. Clearly $H(z, \bar{z}) = 0$ when $\Delta x^0 = z + \bar{z}$. Then we have from the definition of G that $H(0, \bar{z}) = 0$ and $\partial_1 H(0, 0) . \dot{z} = 2G(\dot{z}, x^0)$.

Since $\pi(x^0) = 0$ then $G(\dot{z}, x^0) = 0$ implies $F(\dot{z}, x^0) = 0$ and so $\dot{z} \in F$. Consequently $\partial_1 H(0, 0)$ is one to one and so H is locally one to one as a function of $z \in E$. This means that if $|z|, |z'| < \mu$ and $|\bar{z}| < \mu$ with $z = x_0 + \ldots + x_{k-1}$, $z' = x'_0 + \ldots + x'_{k-1}$, $\bar{z} = x_k + \ldots + x_{2k}$ and $H(z, \bar{z}) = H(z', \bar{z})$ then $z = z'$. In particular if $|z| < \mu$, $|\bar{z}| < \mu$ and $H(z, \bar{z}) = 0 = H(0, \bar{z})$ then $z = 0$. Therefore if $\Delta x^0 = z + \bar{z}$, i.e. if $x = z + \bar{z} + x^0 \in V_{2k}$, $|z|, |\bar{z}| < \mu$ and $\pi(x) \in E$, then $z = \pi(x) = 0$.

Lemma 8. — Let $\omega \in I^r(U)$ *and* $p \in M^\ell_k(\omega)$. *Let* $j^{k-1}(\omega) : U \to J^{k-1}$ *be the* $(k-1)$*-jet section of* ω. *Then there is a neighborhood* V *of* p *such that* $M^\ell_k(\omega) \cap V = (j^{k-1}(\omega))^{-1}(0) \cap V$. *Moreover* $M^\ell_k(\omega) \cap V$ *is an embedded* C^{r-k+1} *submanifold of codimension* ℓ *of* V (*holomorphic if* $K = \mathbf{C}$).

Proof. — Define $h : U \to J^{2k}$ by $h(q) = j^{2k}(\omega)_q$. Then

$$h(p) = \omega_k + \ldots + \omega_{2k} = x^0 \in V_{2k}.$$

Consider the map $g : U \to J^{k-1}$ given by $g = \pi \circ h$, where $\pi : J^{2k} \to J^{k-1}$ is the natural projection. Then $g(p) = 0$ and $Dg(p) . v = \pi(Dh(p) . v) = L_v(\omega_k)$, $v \in K^n$. Since $J^{k-1} = E \oplus F$, $F = \pi(T_{x^0}(V_{2k})) = \mathrm{Im}(S_p)$, it is clear that g intersects E transversely at $0 \in J^{k-1}$. Therefore if $W \subset J^{k-1}$ is a small neighborhood of $0 \in J^{k-1}$ and $V = g^{-1}(W)$ then $g^{-1}(E) \cap V = g^{-1}(E \cap W)$ is a C^{r-k+1} codimension ℓ submanifold of U. By Lemma 7 we know that if W is small enough, then $j^{k-1}(\omega)_q \in \pi(V_{2k}) \cap E \cap W$ if and only if $j^{k-1}(\omega)_q = 0$. This implies that $g^{-1}(E) \cap V = (j^{k-1}(\omega))^{-1}(0) \cap V = g^{-1}(0) \cap V$, so that $(j^{k-1}(\omega))^{-1}(0) \cap V$ is a codimension ℓ submanifold of V. Since $\mathscr{R}^\ell_k(K^n)$ is open in the set of homogeneous integrable 1-forms, then

$$(j^{k-1}(\omega))^{-1}(0) \cap V = M^\ell_k(\omega) \cap V$$

if V is small enough. This ends the proof. ■

Now Theorem 5 follows from the transversality theory.

For $\eta \in I^r(U)$, define $g_\eta : U \to J^{k-1}$ by $g_\eta(p) = j^{k-1}(\eta)_p$.

Lemma 9. — Let $\omega \in I^r(U)$ *and* $p_0 \in M^\ell_k(\omega)$. *Then there exist neighborhoods* V *of* p_0 *and* N *of* ω *in* $I^r(U)$, *such that for any* $\eta \in N$, $g_\eta^{-1}(0) \cap V = M^\ell_k(\eta) \cap V$ *is a codimension* ℓ *submanifold of* U. *Moreover if* Q *is an* ℓ*-dimensional submanifold transversal to* $M^\ell_k(\omega)$ *at* p_0, *then* $M^\ell_k(\eta) \cap V \cap Q$ *contains exactly one point* $h(\eta) \in M^\ell_k(\eta) \cap V \cap Q$. *The point* $h(\eta)$ *is characterized by the property* $h(\eta) \in M^\ell_k(\eta / Q \cap V)$.

Proof. — The first part of the lemma follows easily form the transversality theory and Lemma 7. Let $E \subset J^{k-1}$ be such that $E \oplus F = J^{k-1}$, $F = \mathrm{Im}(Dg_\omega(p_0))$. Then

g_ω intersects E transversely at p_0, so there exist neighborhoods V of p_0 and N of ω such that if $\eta \in N$ then g_η/V intersects E transversely. Using Lemma 7 it is not difficult to see that $g_\eta^{-1}(E) \cap V = M_k^\ell(\eta) \cap V$.

Now observe that if $\omega_k = j^k(\omega)_{p_0}$, then $\omega_k/T_{p_0}(Q) = j^k(\omega/Q)_{p_0}$. But $\omega_k \in \mathscr{R}_k^\ell(K^n)$ so that, by Proposition 3, $\omega_k/T_{p_0}(Q) \in \mathscr{R}_k^\ell(T_{p_0}(Q))$. Therefore $p_0 \in M_k^\ell(\omega/Q)$. Let us denote $\omega/Q = \tilde{\omega}$. Then Lemma 9 is reduced to the following:

Lemma 10. — Let $\tilde{\omega} \in I^r(Q)$ *where* $\dim(Q) = \ell$. *Suppose that* $p_0 \in M_k^\ell(\tilde{\omega})$. *Then there exist neighborhoods* V *of* p_0 *and* $\tilde{N}$ *of* $\tilde{\omega}$ *such that if* $\eta \in \tilde{N}$ *then* η *has a unique singularity* $p(\eta) \in M_k^\ell(Q) \cap \tilde{V}$. *Moreover the map* $\eta \in I^r(Q) \to p(\eta) \in \tilde{V}$ *is continuous.*

Proof. — We can assume $p_0 = 0 \in K^n$. For $\eta \in I^r(Q)$ let $g_\eta : Q \to J^{k-1}$ be defined by $g_\eta(p) = j^{k-1}(\eta)_p$. Then $F = \operatorname{Im}(Dg_\omega(0))$ has dimension ℓ. If $E \subset J^{k-1}$ is such that $J^{k-1} = E \oplus F$, then $g_{\tilde{\omega}}$ intersects E transversely in a unique point. Therefore there exist neighborhoods $\tilde{V}$ of 0 and $\tilde{N}$ of $\tilde{\omega}$ such that if $\eta \in \tilde{N}$ then $g_\eta/\tilde{V}$ intersects E transversely in a unique point $p(\eta) = g_\eta^{-1}(E) \cap \tilde{V}$. ■

End of the proof of Theorem 5. — Let P be a relatively compact subset of $M_k^\ell(\omega)$ and consider a tubular neighborhood $\pi : W \to P$. We can suppose that the fibers $Q_p = \pi^{-1}(p)$, $p \in P$, are C^∞.

Given $p \in \overline{P}$, let V_p and N_p be as in Lemma 9. Since $\overline{P}$ is compact, we take $p_1, \ldots, p_m$ such that $\bigcup_{i=1}^m V_{p_i} \supset P$. Let $V = W \cap (\bigcup_{i=1}^m V_{p_i})$ and $N = \bigcap_{i=1}^m N_{p_i}$. Take the restriction $\tilde{\pi} = \pi/V$ and the fibers $\tilde{Q}_p = \tilde{\pi}^{-1}(p)$, $p \in P$. Now, if $\eta \in N$ then, by construction, for any $p \in P$, there exists a unique point $q = h(\eta, p) \in \tilde{Q}_p$, such that $j^{k-1}(\eta/\tilde{Q}_p)_p = 0$. Define $h_\eta(p) = h(\eta, p)$. By Lemma 9, $j^{k-1}(\eta)_{h(\eta,p)} = 0$, therefore $h_\eta(P) = M_k^\ell(\eta) \cap V$. Now, Lemma 10 implies that $\eta \mapsto h_\eta \in C^{r-k+1}(P, U)$ is continuous and Theorem 5 is proved. ■

2. Structural Stability.

Here we consider a class of integrable forms in $\mathbf{R}^3$ which are locally structurally stable.

Theorem 6. — Let ω *be a* C^r *integrable* 1*-form defined in an open set* $U \subset \mathbf{R}^3$, *where* $r \geq 2k$. *Let* $p \in U$ *be a simple singularity of order* $k \geq 3$ *of* ω. *Suppose that* $\omega_k = j^k(\omega)_p$ *is such that* ω_k/S^2 *defines a structurally stable singular foliation on* S^2 *where* S^2 *is the unit sphere in* $\mathbf{R}^3$. *Then the germ of* ω *at* p *is* C^r*-structurally stable.*

We observe that the case $k = 2$ was already studied in [6].

Proof. — First of all we note that by Theorem 5 there exist neighborhoods W of p and N of ω in the C^r topology such that for any $\eta \in N$, η has a unique singularity of order k, $p(\eta) \in W$. If N is small enough then for any $\eta \in N$, $j^k(\eta)_{p(\eta)} = \eta_k$ is such that $0 \in \mathbf{R}^3$ is an algebraically isolated singularity of $\mathrm{rot}(\eta_k)$ and so $p(\eta)$ is a simple singularity of η. We can suppose that $p(\eta) = 0$. So it is enough to prove that if $\omega = \omega_k + R$ and $\eta = \eta_k + \widetilde{R}$, where $\lim_{x \to 0} |x|^{-k} R(x) = \lim_{x \to 0} |x|^{-k} \widetilde{R}(x) = 0$, then ω and η are locally equivalent at $0 \in \mathbf{R}^3$, provided that ω_k is close to η_k.

By the hypothesis, there is $\rho > 0$ small such that if S^2_ρ denotes the sphere of radius ρ centered at $0 \in \mathbf{R}^3$, then ω/S^2_ρ and η/S^2_ρ are topologically equivalent. This follows from the fact that ω_k/S^2_ρ is structurally stable. Let $h : S^2_\rho \to S^2_\rho$ be an equivalence between ω/S^2_ρ and η/S^2_ρ. Now the idea is to extend h to $B = \{x \in \mathbf{R}^3 \mid |x| \leq \rho\}$ using vector fields tangent to the leaves of ω and η.

By Theorem 1 we know that $\omega_k(I) = \eta_k(I) = 0$ where $I(x) = \sum_{i=1}^{3} x_i \partial/\partial x_i$. Using this it is possible to construct vector fields X and $\widetilde{X}$ in B such that $0 \in \mathbf{R}^3$ is a sink for both of them and $\omega(X) = \eta(\widetilde{X}) = 0$. Let X_t and $\widetilde{X}_t$ be the flows of X and $\widetilde{X}$ respectively. Given $x \in B$, let $t < 0$ be such that $X_t(x) \in S^2_\rho$. We define $h(x) = \widetilde{X}_{-t}(h(X_t(x)))$. It is not difficult to see that h is an equivalence between ω and η. This finishes the proof. ■

Remark. — It is not difficult to see that for any $k \geq 3$ there exist 1-forms ω_k as in Theorem 6. In fact, for $k = 3$ the set of structurally stable homogeneous 1-forms is dense in the space of homogeneous simple forms of degree 3 [14].

V. — REGULAR HOLOMORPHIC FORMS

1. Homogeneous 1-forms.

In contrast with the real case, integrable forms in the complex domain are in general not structurally stable in $\mathbf{C}^n$, $n \geq 3$. For one of the families of regular forms given in this paper this remark follows from the more general theorem:

Theorem 7. — Consider the integrable form ω *in* $\mathbf{C}^n$, $n \geq 3$

$$\omega = \sum_{i=1}^{n} \lambda_i z_1 \ldots \hat{z}_i \ldots z_n dz_i$$

such that

$$\lambda_i \notin \mathbf{R}\lambda_j \quad \text{for } i \neq j.$$

Then the equivalence class of

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbf{C}$$

under the action of $\mathrm{Gl}(2, \mathbf{R})$ *is the sole topological invariant of the real codimension two foliation with singularities defined by* ω *on* $\mathbf{C}^n$ (1).

Proof. — The leaves of ω are the same as the orbits of the $\mathbf{C}^{n-1}$-action φ generated by the commuting vector fields

$$Z_j(z) = \lambda_1^{-1} z_1 \frac{\partial}{\partial z_1} - \lambda_j^{-1} z_j \frac{\partial}{\partial z_j} \qquad j = 2, \ldots, n.$$

In fact, one has $(\lambda_1 \lambda_2 \ldots \lambda_n)^{-1} \omega = i_{Z_2 \wedge \ldots \wedge Z_n}(dz_1 \wedge \ldots \wedge dz_n)$. Let $\tilde{\varphi}$ be the $\mathbf{C}^{n-1}$-action induced in the same way by the form $\tilde{\omega} = \sum_{i=1}^{n} \tilde{\lambda}_i z_1 \ldots \hat{z}_i \ldots z_n dz_i$ and suppose there is an isomorphism $u \in \mathrm{Gl}(2, \mathbf{R})$ such that $u(\lambda_i) = \tilde{\lambda}_i$ for $i = 1, \ldots, n$. This induces an isomorphism $\tilde{u}: (\mathbf{R}^2)^{n-1} \to (\mathbf{R}^2)^{n-1}$ by $\tilde{u} = (u, \ldots, u)$ ($n - 1$ times). Putting $\mathbf{C}^n = \bigoplus_{i=1}^{n} E_i$ where $E_i = \{z \in \mathbf{C}^n \mid z_j = 0 \text{ for } j \neq i\}$ we obtain that $\tilde{u}$ preserves the isotropy groups, i.e.: $\tilde{u}(G_i) = \tilde{G}_i$, $i = 1, \ldots, n$, where

$$G_i = \{g \in \mathbf{C}^{n-1} \mid \varphi(g, p) = p \in E_i - \{0\}\},$$
$$\tilde{G}_i = \{g \in \mathbf{C}^{n-1} \mid \tilde{\varphi}(g, p) = p \in E_i - \{0\}\}.$$

---

(1) This theorem was obtained independently by B. Klares in the context of $\mathbf{C}^{n-1}$- linear actions on $\mathbf{C}^n$.

This allows us to define a conjugacy between φ and $\widetilde{\varphi}$ on each E_i:

$$h_i \circ \varphi(g, p) = \widetilde{\varphi}(\widetilde{u}(g), h_i(p)), \quad p \in E_i.$$

From this and the linearity of φ and $\widetilde{\varphi}$ it follows easily that $h = (h_1, \ldots, h_n)$ is a conjugacy between φ and $\widetilde{\varphi}$, i.e.

$$h\varphi(g, p) = \widetilde{\varphi}(\widetilde{u}(g), h(p)) \quad p \in \mathbf{C}^n.$$

Conversely, suppose there is a local homeomorphism, $h : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ around $0 \in \mathbf{C}^n$, which is a topological equivalence between $\omega = \sum_{i=1}^n \lambda_i z_1 \ldots \hat{z}_i \ldots z_n dz_i$ and $\omega' = \sum_{i=1}^n \lambda_i' z_1 \ldots \hat{z}_i \ldots z_n dz_i$. Let $F_i = \{z \in \mathbf{C}^n \mid z_i = 0\}$ and $F_{ij} = \{z \in \mathbf{C} \mid z_i = z_j = 0\}$. Then $\operatorname{Sing}(\omega) = \operatorname{Sing}(\omega') = \bigcup_{i<j} F_{ij}$ and we can assume without loss of generality that $h(F_{ij}) = F_{ij}$. Let $\widetilde{F}_i = F_i - \bigcup_{j \neq i} F_{ij}$. Then $\widetilde{F}_i$ is a leaf of both ω and ω' homeomorphic to $\mathbf{R}^{n-1} \times T^{n-1}$. So its holonomy is a linear action of $\mathbf{Z}^{n-1}$ in the transverse section $\Sigma_i = \{(1, \ldots, 1, z_i, \ldots, 1) \mid z_i \in \mathbf{C}\}$. By hypothesis the holonomy of $\widetilde{F}_i$ is not trivial. On the other hand, the holonomy of a leaf of ω or ω' contained in $\mathbf{C}^n - \bigcup_{i=1}^n F_i$ is trivial. So $h(\widetilde{F}_i) = \widetilde{F}_k$ for some k, and since $h(\widetilde{F}_{ij}) = \widetilde{F}_{ij}$ for $i \neq j$, then $h(\widetilde{F}_i) = \widetilde{F}_i$ and $h(F_i) = F_i$. The holonomy of $\widetilde{F}_i$ is generated along the curves $\gamma_{ij} : S^1 \to \widetilde{F}_i$, $\gamma_{ij}(\theta) = (1, \ldots, 1, \underbrace{0}_{i}, 1, \ldots, \underbrace{e^{2\pi i\theta}}_{j}, 1, \ldots, 1)$ and since $h(F_{ij}) = F_{ij}$ then

$$h_*([\gamma_{ij}]) = [\gamma_{ij}] \quad \text{for all } i \neq j.$$

For simplicity we assume that $h(p_i) = p_i$ where $p_i = (1, \ldots, 1, 0, 1, \ldots, 1)$, 0 in the i-th place. Then h induces by projection along the leaves of ω', a germ of a homeomorphism $h_i : (\Sigma_i, 0) \to (\Sigma_i, 0)$ conjugating the holonomies of ω and ω'. If $f_j, f_j' : \Sigma_i \to \Sigma_i$ are the generators of the holonomies of ω and ω' relative to γ_{ij} we must have $f_j(z_i) = \exp(-2\pi i\lambda_j/\lambda_i) . z_i$, $f_j'(z_i) = \exp(-2\pi i\lambda_j'/\lambda_i') . z_i$ and $h_i \circ f_j = f_j' \circ h_i$ for all $j \neq i$. By the first part of the theorem we can take $\lambda_1 = \lambda_1' = 1$ and $\lambda_2 = \lambda_2' = i$. We show now that $\lambda_3 = \lambda_3', \ldots, \lambda_n = \lambda_n'$. We write $\lambda_j = x_j + iy_j$ and $\lambda_j' = x_j' + iy_j'$. The holonomy of $\widetilde{F}_1$ is generated by:

$$\text{for } \omega\text{:} \quad f_2(z_1) = e^{2\pi} . z_1, \quad f_j(z_1) = e^{2\pi y_j} . e^{-2\pi i x_j} . z_1, \quad j \geq 3;$$

$$\text{for } \omega'\text{:} \quad f_2'(z_1) = e^{2\pi} . z_1, \quad f_j'(z_1) = e^{2\pi y_j'} . e^{-2\pi i x_j'} . z_1, \quad j \geq 3.$$

We need the following lemma.

Lemma 11. — *Let* $h : \mathbf{C} \to \mathbf{C}$, $h(1) = 1$, *be a homeomorphism such that for any* $(m_1, m_2) \in \mathbf{Z}^2$ *and* $z \in \mathbf{C}$

$$h(\mu_1^{m_1} \mu_2^{m_2} z) = \mu_1'^{m_1} \mu_2'^{m_2} h(z)$$

where $\mu_j \neq 0 \neq \mu'_j$ *for* $j = 1, 2$. *Then*

$$\frac{\log |\mu_2|}{\log |\mu_1|} = \frac{\log |\mu'_2|}{\log |\mu'_1|}$$

provided that $|\mu_1| \neq 1$.

Proof. — First observe that $G = \{\mu_1^{m_1} \mu_2^{m_2} \mid m_1, m_2 \in \mathbf{Z}\}$ is a subgroup of the multiplicative group $\mathbf{C} - \{0\}$. Therefore either G is discrete or 1 is an accumulation point of G. In the first case it is not difficult to see that there exists $(m, n) \in \mathbf{Z}^2 - \{0\}$ such that $\mu_1^m \mu_2^n = 1 = \mu_1'^m \mu_2'^n$, so that

$$-\frac{m}{n} = \frac{\log |\mu_2|}{\log |\mu_1|} = \frac{\log |\mu'_2|}{\log |\mu'_1|}.$$

In the second case there exists a sequence $(m_k, n_k) \in \mathbf{Z}^2 - \{0\}$ such that

$$\lim_{k \to \infty} (|m_k| + |n_k|) = \infty \quad \text{and} \quad \lim_{k \to \infty} \mu_1^{m_k} \mu_2^{n_k} = 1.$$

Then $$h(1) = \lim_{k \to \infty} h(\mu_1^{m_k} \mu_2^{n_k}) = \lim_{k \to \infty} \mu_1'^{m_k} \mu_2'^{n_k} = 1.$$

This implies

$$\lim_{k \to \infty} (m_k \log |\mu_1| + n_k \log |\mu_2|) = \lim_{k \to \infty} (m_k \log |\mu'_1| + n_k \log |\mu'_2|) = 0$$

and $$\lim_{k \to \infty} -\frac{m_k}{n_k} = \frac{\log |\mu_2|}{\log |\mu_1|} = \frac{\log |\mu'_2|}{\log |\mu'_1|}. \quad \blacksquare$$

Since $h_1 \circ f_k = f'_k \circ h_1$, taking $k = 2$, $\mu_1 = e^{2\pi}$, $k = j \geq 3$, $\mu_2 = \exp(-2\pi i \lambda_j)$, we obtain by the lemma that:

$$\frac{\log(e^{2\pi y_j})}{\log(e^{2\pi})} = \frac{\log(e^{2\pi y'_j})}{\log(e^{2\pi})}.$$

So $y_j = y'_j$ for all $j \geq 3$.

We use the same argument for $\widetilde{F}_2$. The holonomy of $\widetilde{F}_2$ is given by

$$\text{for } \omega: \quad g_1(z_2) = e^{-2\pi}. z_2, \quad g_j(z_2) = e^{-2\pi x_j}. e^{-2\pi i y_j}. z_2, \quad j \geq 3$$

$$\text{for } \omega': \quad g'_1(z_2) = e^{-2\pi}. z_2, \quad g'_j(z_2) = e^{-2\pi x'_j}. e^{-2\pi i y'_j}. z_2, \quad j \geq 3.$$

Since $h_2 \circ g_k = g'_k \circ h_2$, taking $k = 2$, $\mu_1 = e^{-2\pi}$, and $k = j \geq 3$, $\mu_2 = \exp(-2\pi\lambda_j)$, we obtain

$$\frac{\log(e^{-2\pi x_j})}{\log(e^{-2\pi})} = \frac{\log(e^{-2\pi x'_j})}{\log(e^{-2\pi})}.$$

So $x_j = x'_j$ for $j \geq 3$ and this means $\lambda_j = \lambda'_j$ for all $j \geq 3$. $\blacksquare$

2. Topological determination in three dimensions.

Proposition 5. — *Let* ω *be a holomorphic integrable form with a singularity at* $0 \in \mathbf{C}^3$ *such that in a neighborhood of* $0 \in \mathbf{C}^3$ ω *is written as*

$$\omega = \lambda_1 z_2 z_3 dz_1 + \lambda_2 z_1 z_3 dz_2 + \lambda_3 z_1 z_2 dz_3 + \mathrm{R}(z)$$

where $\lim_{z \to 0} |z|^{-2} \mathrm{R}(z) = 0$ *and* $\lambda_i \notin \mathbf{R}\lambda_j$ *for* $i \neq j$. *Then* ω *is topologically equivalent to* $\omega_2 = j^2(\omega)_0$ *near* $0 \in \mathbf{C}^3$.

Proof. — The proof consists in finding an equivalence between rot ω and rot ω_2 sending leaves of ω to leaves of ω_2. The vector field rot ω is in the Siegel domain, i.e. the convex hull of the eigenvalues of $j^1(\text{rot}\ \omega)_0$ contains $0 \in \mathbf{C}$. Modulo a holomorphic change of coordinates (see [3]) we can assume that the coordinate 2-planes $\{z \in \mathbf{C}^3 \mid z_i = 0\}$ are all invariant by rot ω. The integral complex curves of rot ω passing through points z with $z_i \neq 0$ for all i, are closed subsets of $\mathbf{C}^3$ at a positive distance from $0 \in \mathbf{C}^3$. The intersection of these integrals with each $\mathrm{C}_i = \{z \in \mathbf{C}^3 \mid |z_i| = 1\}$ gives a real 1-foliation with a closed integral $\gamma_i = \{z \in \mathbf{C}^3 \mid |z_i| = 1 \text{ and } z_j = 0 \text{ for } j \neq i\}$ which is hyperbolic of saddle type for all i. From this it is clear that the integral of rot ω passing through a point $z \in \mathrm{B} = \{z \mid |z_i| \leq 1\}$ leaves as intersection with $\partial\mathrm{B}$ a closed curve provided $0 < |z_i| < 1$ for $1 \leq i \leq 3$.

Let $\mathrm{S} = \mathrm{C}_1 \cap \mathrm{C}_2$ and $\mathrm{S}_0 = \mathrm{S} \cap \{z \mid z_3 = 0\}$. It follows from [3] that any homeomorphism $h : \mathrm{S} \to \mathrm{S}$ with $h/\mathrm{S}_0 = \text{identity}$ can be extended to a topological equivalence between rot ω and rot ω_2. In our case the foliation induced by ω/S is completely characterized by the holonomy of S_0, i.e. by a $\mathbf{Z}^2$-action φ_3 on $\mathbf{C}$ whose linear part is the $\mathbf{Z}^2$-action ρ_3 generated by the diffeomorphisms

$$f_1(z_3) = \exp\left(2\pi i \frac{\lambda_1}{\lambda_3}\right) z_3, \qquad f_2(z_3) = \exp\left(2\pi i \frac{\lambda_2}{\lambda_3}\right) z_3.$$

So it is enough that h be a conjugacy between φ_3 and ρ_3. Since such an h clearly exists the proof is finished. ■

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Instituto de Matemática Pura e Aplicada
Estrada Da. Castorina, 110
22.460, Rio de Janeiro, RJ, Brasil.

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