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LYAPUNOV EXPONENTS, ENTROPY AND PERIODIC ORBITS FOR DIFFEOMORPHISMS

by A. KATOK (1) (2)

Dedicated to the memory of Rufus Bowen (1947-1978)

Introduction.

1. In this paper I study some dynamical properties of diffeomorphisms on compact manifolds by combining two different techniques, ε-trajectories and the Lyapunov characteristic exponents. These two approaches were developed separately and for different purposes. The technique of ε-trajectories introduced by Rufus Bowen ([1], [2], [3]) and D. V. Anosov ([4]; for proofs see [5], [6]) is based on the observation that assuming some hyperbolicity conditions, dynamical phenomena which are observed to almost occur for some diffeomorphism usually do occur for that diffeomorphism. Using this approach Bowen proved a number of profound results concerning the asymptotic growth and the limit distribution of periodic orbits for Axiom A diffeomorphisms and flows, uniqueness and the ergodic properties of equilibrium states and so on ([2], [3], [7]).

The second approach was developed by Ja. B. Pesin [8] for the study of ergodic properties (such as ergodicity, entropy, K-property, Bernoulli property) of smooth dynamical systems with an invariant measure equivalent to a Riemannian volume ([9], [10], [11]). Many of the ideas used in this cycle of papers had occurred in the earlier work of Brin and Pesin [12]. A large part of Pesin's arguments works without special assumptions about the invariant measure (D. Ruelle has also observed this fact in [21]). This section contains the description of the behavior of a diffeomorphism near a trajectory regular in the Lyapunov sense (for definitions of regularity, see [8], no. (0.3); [9], § 3; [13]) and the construction and the properties of invariant contracting and expanding manifolds,

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except for the absolute continuity ([8], §§ 1, 2). It is possible to consider a neighborhood of a regular trajectory with non-zero Lyapunov exponents as something similar to a neighborhood of a hyperbolic set, and use a technique very close to ε-trajectories.

2. To describe more carefully some problems which can be studied from this point of view, let us recall a few very basic notions about the Lyapunov exponents.

Let f be a diffeomorphism of a compact s-dimensional manifold M and $df: TM \to TM$ the derivative (linear part) of f. Let us fix a smooth Riemannian metric on M, *i.e.*, a scalar product (and consequently a norm) in every tangent space $T_x M$, $x \in M$, which depends on x in a differentiable way. The number

$$\chi^+(v, f) = \overline{\lim_{n\to\infty}} \frac{\ln \| df^n v \|}{n} \tag{0.1}$$

is called the upper Lyapunov exponent for the tangent vector $v \in TM$. The function χ^+ being defined on the tangent bundle TM takes on at most s values on each tangent space $T_x M$ and generates a filtration

$$L_1(x) \subset L_2(x) \subset \ldots \subset L_{r(x)}(x) = T_x M$$

of every such space. Namely, there are numbers $\chi_1(x) < \chi_2(x) < \ldots < \chi_{r(x)}(x)$ such that $\chi^+(v, f) = \chi_i(x)$ for $v \in L_i(x) \setminus L_{i-1}(x)$. The numbers $\chi_i(x)$ are called the upper Lyapunov exponents of f at the point x and the number $k_i(x) = \dim L_i(x) - \dim L_{i-1}(x)$ is called the multiplicity of the i-th exponent. None of these values depend on the choice of a Riemannian metric.

In general, the limit of $\frac{\ln \| df^n v \|}{n}$ may not exist but even the existence of such limits (which are called in that case the Lyapunov exponents) for all $v \in T_x M$ does not prevent a pathology in the asymptotic behavior of $(df^n)_x$ as n tends to infinity. Such a pathology is prevented by the conditions of regularity ([8], no. (0.3); [9], § 3) which in particular guarantee the existence of Lyapunov exponents. The multiplicative ergodic theorem proved by Oseledec [13] (for later proofs see [14], [15]) implies that for any Borel probability f-invariant measure μ the set of regular points has measure 1. Moreover, for almost every regular point x the exponents $\bar{\chi}_i(x)$ of f^{-1} at x and their multiplicities $\bar{k}_i(x)$ are equal to $-\chi_{r(x)-i+1}(x)$ and $k_{r(x)-i+1}(x)$ respectively, for $i = 1, \ldots, r(x)$.

The functions $r(x)$, $\chi_i(x)$, $k_i(x)$ are measurable and f-invariant with respect to any Borel invariant measure μ. Therefore, if μ is an ergodic measure, then the functions are constant almost everywhere. In this case we will denote these essential values of $r(x)$, $\chi_i(x)$, $k_i(x)$ by r^μ, χ_i^μ, k_i^μ respectively.

If all the functions $\chi_i(x)$ are different from zero μ-almost everywhere, then we will say that μ is a measure with non-zero Lyapunov exponents. In the case of ergodic μ this means that $\chi_i^\mu \neq 0$ for $i = 1, \ldots, r^\mu$.

The case of a measure with non-zero Lyapunov exponents is the center of our interest because the behavior of df along a regular trajectory with non-zero Lyapunov exponents is hyperbolic although the hyperbolicity is non-uniform (for an interesting discussion on this subject, see [9], § 1). To overcome non-uniformity, Pesin elaborated some technique which we reproduce partly in § 2 of this paper.

3. We apply the combined approach mentioned above to two special problems: (i) relationships between the Lyapunov exponents and the entropies of f (the topological entropy $h(f)$ and the measure-theoretical entropies $h_\mu(f)$ where μ is a Borel probability f-invariant measure), and (ii) connections between the properties of exponents and the periodic points of f.

The first problem was solved for C^2 diffeomorphisms when the measure μ is equivalent to a Riemannian volume. Namely, let $\chi^p(x) = \sum_{i:\chi_i(x)>0} k_i(x)\chi_i(x)$. Then:

$$h_\mu(f) = \int_M \chi^p(x)\,d\mu. \tag{0.2}$$

This result consists of two parts. The inequality

$$h_\mu(f) \leq \int_M \chi^p(x)\,d\mu \tag{0.3}$$

was proved in 1968 by G. A. Margulis for any C^1 diffeomorphism. This inequality was generalized to any Borel probability f-invariant measure; a similar estimation for $h(f)$ has also been found ([16], [17]).

The estimation of the entropy from below is a more delicate undertaking. It was proved for μ equivalent to a Riemannian volume by Pesin ([9], § 5, another proof is in [10]) who used hard machinery developed in [8] including the absolute continuity of systems of invariant manifolds. Pesin's proof essentially works for a $C^{1+\alpha}$ diffeomorphism ($\alpha > 0$) and for any measure μ such that the conditional measures on expanding manifolds are absolutely continuous with respect to the Riemannian volumes on those manifolds ([22]).

Let us denote for a regular point $x \in M$ through E^u_x the subspace of T_xM corresponding to the positive Lyapunov exponents (see details in § 2), and through $\mathcal{J}^u(x)$ the Jacobian of df_x restricted to the subspace E^u_x (we assume that some Riemannian metric on M is fixed). Then:

$$\chi^p(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|\mathcal{J}^u(f^k x)|$$

and consequently:

$$\int \chi^p(x)\,d\mu = \int \ln|\mathcal{J}^u(x)|\,d\mu.$$

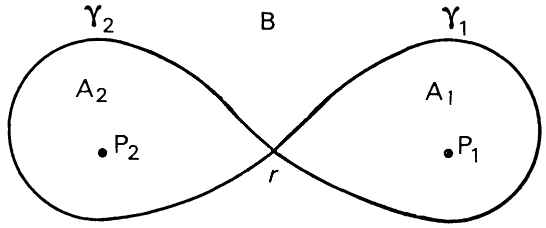
With this remark we can rewrite (0.3) as a kind of variational inequality:

$$h_\mu(f) - \int_M \ln|\mathcal{J}^u(x)|\,d\mu \leq 0.$$

Note that the function $\mathcal{J}^u(x)$ is defined on a set which has measure 1 with respect to any Borel probability f-invariant measure [1]. Measures with absolutely continuous conditional measures on unstable manifolds play the role of equilibrium states for the "potential" $|\mathcal{J}^u(x)|$. After the previous discussion it seems natural that the following relation

$$\sup_{\mu}\left(h_\mu(f)-\int_M \ln|\mathcal{J}^u(x)|\,d\mu\right)=0,$$

is true always. However, it can be shown that it is false. The counterexample was suggested by R. Bowen (an oral communication of D. Ruelle) and by the author. Let me describe this example.



Let f be a diffeomorphism of the two-dimensional sphere S^2 with three expanding fixed points p_1, p_2, q and one saddle point r. Suppose that the stable and unstable manifolds of the point r form two loops γ_1, γ_2 which divide S^2 into three regions A_1, A_2, B (see the picture, where q is a point at infinity). As $n \to +\infty$ every point from $A_1\setminus\{p_1\}$ tends to γ_1, from $A_2\setminus\{p_2\}$ to γ_2 and from $B\setminus\{q\}$ to $\gamma_1\cup\gamma_2$. Every probability invariant measure μ is concentrated on the four fixed points so that $\int\chi^p(x)d\mu>c>0$ while:

$$h_\mu(f)=0 \quad \text{and} \quad h(f)=0.$$

This example shows that the Lyapunov exponents of measures concentrated on periodic orbits may not have any influence on the entropy [2]. However, except for this case such an influence exists.

Corollary **(4.2)**. — *If a* $C^{1+\alpha}$ $(\alpha>0)$ *diffeomorphism f of a compact manifold has a Borel probability continuous (non-atomic) invariant ergodic measure with non-zero Lyapunov exponents then* $h(f)>0$.

4. Let us proceed now to the discussion about the exponents and periodic points. Let us denote by $\operatorname{Per} f$ the set of all periodic points of f and by $P_n(f)$ the number of periodic points with period n, *i.e.* the number of fixed points for f^n.

(1) If the subspace E^u_x is empty it is convenient to set $\mathcal{J}^u(x)=1$.

(2) M. Misiuzewicz observed (personal communication) that this diffeomorphism can be approximated in the C^k topology $k=1, 2, \dots$ by a diffeomorphism with the topological entropy bigger than $\frac{\log\alpha}{k}-\varepsilon$, where α is a bigger eigenvalue at the point r and ε is any positive number.

In the uniformly hyperbolic situation, *i.e.* for Axiom A diffeomorphisms, Bowen [1] proved that the asymptotical exponential growth of the number of periodic points was determined by the topological entropy:

$$\text{(0.4)} \qquad \overline{\lim_{n\to\infty}} \frac{\ln P_n(f)}{n} = h(f).$$

In § 1 we give a new definition of the measure-theoretical entropy $h_\mu(f)$ of a homeomorphism of a compact metric space. This definition is similar to the Bowen-Dinaburg definition of the topological entropy ([18], [19]). Using this definition we prove:

Theorem **(4.3)**. — *For a* $C^{1+\alpha}$ $(\alpha>0)$ *diffeomorphism f of a compact manifold and any Borel probability f-invariant measure* μ *with non-zero Lyapunov exponents*:

$$\overline{\lim_{n\to\infty}} \frac{\ln P_n(f)}{n} \geq h_\mu(f).$$

The upper bound of $h_\mu(f)$ is equal to $h(f)$. In the two-dimensional case any measure with positive entropy has non-zero Lyapunov exponents. Therefore, Theorem (4.3) implies the following relation between periodic points and topological entropy.

Corollary **(4.4)**. — *For any* $C^{1+\alpha}$ $(\alpha>0)$ *diffeomorphism f of a two-dimensional manifold*

$$\text{(0.5)} \qquad \overline{\lim_{n\to\infty}} \frac{\ln P_n(f)}{n} \geq h(f).$$

5. In the multi-dimensional case inequality (0.5) is not true for arbitrary diffeomorphism. Indeed it might be true generically, *i.e.* for any f from some dense G_δ set in the space $\mathrm{Diff}^r(M)$ of all C^r diffeomorphisms of M with C^r topology $(r\geq 1)$. Note that even in the two-dimensional case the answer is not known for $r=1$.

M. Herman asked whether, for diffeomorphisms, positive topological entropy was compatible with minimality or strict ergodicity. Corollary (4.4) gives negative answers to both questions in dimension 2. Recently Herman constructed a remarkable example of a minimal (but not strictly ergodic) diffeomorphism with positive topological entropy.

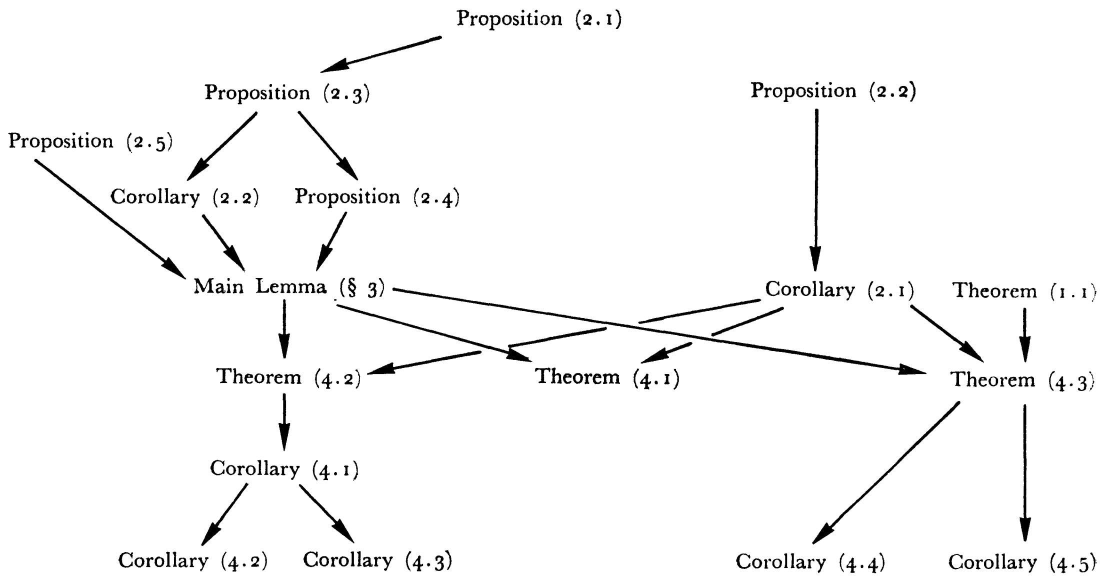
6. Let me mention one more result which like Corollary (4.4) does not include any mention of the Lyapunov exponents or even of measures.

Corollary **(4.3)**. — *If f is a* $C^{1+\alpha}$ $(\alpha>0)$ *diffeomorphism of a compact two-dimensional manifold and* $h(f)>0$ *then f has a hyperbolic periodic point with a transversal homoclinic point and consequently there exists an f-invariant hyperbolic set* Γ *such that the restriction of f into* Γ *is topologically conjugate to a topological Markov chain (subshift of finite type) and* $h(f|_\Gamma)>0$.

This fact may be considered as a topological counterpart for the following statement which is an immediate corollary of Theorems (7.2), (7.9), (8.1) from Pesin's work [9]:

If f is a C^2 diffeomorphism of a compact two-dimensional manifold with smooth invariant measure μ and $h_\mu(f)>0$ then there exists a set Γ of positive measure such that $f|_\Gamma$ is metrically isomorphic to an ergodic Markov chain, *i.e.* a Markov chain which is ergodic as a measure-preserving transformation.

7. The relationships between the statements in this paper may be represented by the following diagram:



1. Definition of measure-theoretical entropy through d_n^f metrics.

Let X be a compact metric space with the distance function $d(\cdot,\cdot)$, $f: X \to X$ a homeomorphism of X, and d_n^f an increasing system of metrics on X defined by:

$$d_n^f(x,y) = \max_{0 \le i \le n-1} d(f^i x, f^i y).$$

Dinaburg [19] and Bowen [18] showed independently that the topological entropy $h(f)$ can be described through asymptotic behavior of the ε-entropy of the space X provided by the metrics d_n^f, namely:

$$h(f) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\ln N_f(n,\varepsilon)}{n}$$

where $N_f(n,\varepsilon)$ is a minimal number of ε-balls in the d_n^f metric covering the space X. We are going to define the entropy $h_\mu(f)$ with respect to Borel probability f-invariant ergodic measure μ by a similar manner. The metric entropy turns out to be the asymptotic value of the same kind with some subsets of positive measure instead of the whole space X. Namely, for $\varepsilon>0$, $\delta>0$, let us denote by $N_f(n, \varepsilon, \delta)$ the minimal number of ε-balls in the d_n^f-metric which cover the set of measure more than or equal to $1-\delta$.

Theorem (**1.1**). — *For every* $\delta>0$:

$$h_\mu(f)=\lim_{\varepsilon\to 0}\varliminf_{n\to\infty}\frac{\ln N_f(n,\varepsilon,\delta)}{n}=\lim_{\varepsilon\to 0}\varlimsup_{n\to\infty}\frac{\ln N_f(n,\varepsilon,\delta)}{n}.$$

Proof. — The easy part of the proof is to show that the quantity in the right part of the formula does not exceed $h_\mu(f)$. To prove this it is enough to show that:

$$\varlimsup_{n\to\infty}\frac{\ln N_f(n,\varepsilon,\delta)}{n}\leq h_\mu(f) \tag{1.1}$$

for every $\varepsilon>0$, $\delta>0$.

Let us choose a finite measurable partition ξ of X such that the maximal diameter of elements of ξ is less than $\varepsilon/2$. Then, each element of the partition

$$\xi_{-n}=\xi\vee f^{-1}\xi\vee\ldots\vee f^{-n+1}\xi$$

lies inside some ε-ball in the metric d_n^f. Let:

$$A_{n,\varepsilon,\gamma}=\{x\in X : x\in c_n(x),\ c_n(x)\in\xi_{-n},\ \mu(c_n(x))>\exp-n(h_\mu(f,\xi)+\gamma)\}.$$

Since f is ergodic with respect to the measure μ then by Macmillan's theorem, for every $\gamma>0$, $\mu(A_{n,\varepsilon,\gamma})\to 1$ as $n\to\infty$. Consequently, for sufficiently large n, we have $\mu(A_{n,\varepsilon,\gamma})>1-\delta$. The set $A_{n,\varepsilon,\gamma}$ contains at most $\exp n(h_\mu(f,\xi)+\gamma)$ elements of the partition ξ_{-n} and can be covered by the same number of ε-balls in the metric d_n^f. Thus, for every $\gamma>0$:

$$\varlimsup_{n\to\infty}\frac{\ln N_f(n,\varepsilon,\delta)}{n}\leq h_\mu(f,\xi)+\gamma.$$

Since γ can be taken arbitrarily small and $h_\mu(f,\xi)\leq h_\mu(f)$ we obtain (1.1).

For the second half of the theorem we have to recall several definitions and facts about Hamming metrics.

Let:

$$\Omega_{N,n}=\{\omega=(\omega_0,\ldots,\omega_{n-1}) : \omega_i\in\{1,\ldots,N\},\ i=0,1,\ldots,n-1\}$$

where N and n are positive integers. The Hamming metric $\rho^H_{N,n}$ on $\Omega_{N,n}$ is defined by:

$$\rho^H_{N,n}(\omega,\bar{\omega})=\frac{1}{n}\sum_{i=0}^{n-1}(1-\delta_{\omega_i\bar{\omega}_i})$$

where $\delta_{k\ell}$ is a Kronecker symbol:

$$\delta_{k\ell}=\begin{cases}0 & \text{if } k\neq\ell\\ 1 & \text{if } k=\ell.\end{cases}$$

For $\omega \in \Omega_{N,n}$, $r > 0$, we denote by $B^H(\omega, r)$ the closed r-ball in the metric $\rho^H_{N,n}$ with the center in ω. The standard combinatorial arguments show that the number of points in $B^H(\omega, r)$ depends only on r, N, n and is equal to:

$$(1.2) \qquad B(r, N, n) = \sum_{m=0}^{[nr]} (N-1)^m \binom{n}{m}.$$

It is easy to show using Stirling's formula that for $0 < r < \frac{N-1}{N}$:

$$(1.3) \qquad \lim_{n\to\infty} \frac{\ln B(r, N, n)}{n} = r\ln(N-1) - r\ln r - (1-r)\ln(1-r).$$

If $\xi = (c_1, \ldots, c_N)$ is a finite ordered measurable partition of X we can, for every positive integer n, define the map $\varphi^n_{f,\xi} : X \to \Omega_{N,n}$ by $\varphi^n_{f,\xi}(x) = (k_0(x), \ldots, k_{n-1}(x))$ where $f^i x \in c_{k_i(x)}$.

The pre-image of the metric $\rho^H_{N,n}$ defines a semi-metric on X which we denote by $d^{f,\xi}_n$.

Now we proceed to the proof of the inequality:

$$(1.4) \qquad h_\mu(f) \leq \underline{\lim}_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\ln N_f(n, \varepsilon, \delta)}{n}.$$

Obviously, the theorem follows from inequalities (1.1) and (1.4).

We can assume that the measure μ is everywhere dense in X, *i.e.*, the measure of any non-empty open subset of X is positive. The general case is reduced to this particular case by replacing X by its closed subset supp μ.

For a partition ξ of X, let us denote by $\partial\xi$ the union of the boundaries ∂c of all elements $c \in \xi$ and let:

$$U_\gamma(\xi) = \bigcup_{c\in\xi} (U_\gamma(c)),$$

where γ is a positive number and:

$$U_\gamma(c) = \{x \in c : \exists y \in X \setminus c, d(x, y) < \gamma\}.$$

Since $\bigcap_{\gamma>0} U_\gamma(\xi) = \partial\xi$ then $\lim_{\gamma\to 0} \mu(U_\gamma(\xi)) = \mu(\partial\xi)$.

Let us fix some finite ordered measurable partition ξ of X such that $\mu(\partial\xi) = 0$. Let $\varepsilon > 0$ be small and find $\gamma \in (0, \varepsilon)$ such that $\mu(U_\gamma(\xi)) < \frac{\varepsilon^2}{4}$. If $x, y \in X$ and $d^f_n(x, y) < \gamma$ then for every $i : 0 \leq i \leq n-1$ either the points $f^i x$ and $f^i y$ belong to the same element of ξ or both of them belong to the set $U_\gamma(\xi)$. Let us denote for brevity the characteristic functions of the set $U_\gamma(\xi)$ by χ_γ and let:

$$B_{n,\varepsilon} = \left\{x \in X : \sum_{i=0}^{n-1} \chi_\gamma(f^i x) < \frac{n\varepsilon}{2}\right\}.$$

Since $\int_X \chi_\gamma d\mu < \frac{\varepsilon^2}{4}$ and f preserves the measure μ we have:

$$\frac{n\varepsilon^2}{4} \geq \int_X \sum_{i=0}^{n-1} \chi_\gamma(f^i x)\, d\mu \geq \int_{X\setminus B_{n,\varepsilon}} \sum_{i=0}^{n-1} \chi_\gamma(f^i x)\, d\mu$$

$$\geq \frac{n\varepsilon}{2}\mu(X\setminus B_{n,\varepsilon})$$

so that $\mu(X\setminus B_{n,\varepsilon}) < \frac{\varepsilon}{2}$.

If $x \in B_{n,\varepsilon}$ and $d_n^f(x,y) < \gamma$ then $d_n^{f,\xi}(x,y) < \frac{\varepsilon}{2}$; *i.e.*, an intersection of any γ-ball in the metric d_n^f with the set $B_{n,\varepsilon}$ is contained in some $\varepsilon/2$-ball in the semi-metric $d_n^{f,\xi}$.

Let us consider a system $\mathfrak{U}$ of γ-balls in the d_n^f-metric containing $N_f(n, \gamma, \delta)$ balls, and covering the set F_n such that $\mu(F_n) \geq 1-\delta$. Then $\mu(F_n \cap B_{n,\varepsilon}) > 1-\delta-\frac{\varepsilon}{2}$. Suppose that $\varepsilon < \frac{1-\delta}{2}$ so that $\mu(F_n \cap B_{n,\varepsilon}) > \frac{1-\delta}{2}$. Since the intersection of every ball from $\mathfrak{U}$ with $B_{n,\varepsilon}$ is contained in some $\varepsilon/2$-ball in $d_n^{f,\xi}$ then there exists a system of $N_f(n, \gamma, \delta)$ balls in $d_n^{f,\xi}$ of radius $\varepsilon/2$ which covers a set of measure bigger than $\frac{1-\delta}{2}$. Using Macmillan's theorem we deduce that for a sufficiently large n some part of that set of measure bigger than $\frac{1-\delta}{4}$ consists of elements of ξ_{-n} and the measure of each element is less than $\exp -n(h_\mu(f, \xi)-\varepsilon)$. Consequently, the number of such elements is more than:

$$\frac{(1-\delta)\exp(n(h_\mu(f,\xi)-\varepsilon))}{4}.$$

Thus, we have:

$$N_f(n, \gamma, \delta) > \frac{\exp n(h_\mu(f,\xi)-\varepsilon)}{B\left(\frac{\varepsilon}{2}, |\xi|, n\right)} \cdot \left(\frac{1-\delta}{4}\right). \tag{1.5}$$

Combining (1.5) and (1.3) we obtain:

$$\varliminf_{n\to\infty} \frac{\ln N_f(n, \gamma, \delta)}{n} \geq h_\mu(f,\xi) - \varepsilon(1+\ln(N-1)) + \varepsilon\ln\varepsilon + (1-\varepsilon)\ln(1-\varepsilon).$$

Since $\gamma < \varepsilon$ and ε can be chosen arbitrarily small we have:

$$\lim_{\varepsilon\to 0} \varliminf_{n\to\infty} \frac{\ln N_f(n, \varepsilon, \delta)}{n} \geq h_\mu(f,\xi)$$

for every partition ξ such that $\mu(\partial\xi) = 0$. We can find a partition with this property and with sufficiently small diameter (recall that every open set has positive measure) and consequently with the entropy $h_\mu(f,\xi)$ arbitrarily close to $h_\mu(f)$. Inequality (1.4) is proved. ■

Remark. — M. Misiurewicz suggested another proof of Theorem (1.1) similar in ideas to his proof of the variational principle. This proof avoids the use of combinatorial arguments (formulae (1.2) and (1.3)).

2. Behavior of a diffeomorphism near regular trajectories.

In the first half of this section we collect preparatory material about the behavior of any $C^{1+\alpha}$ (for some $\alpha>0$) diffeomorphism in a neighborhood of a regular trajectory with non-zero Lyapunov exponents. The main conclusion is that after some non-autonomous change of coordinates the diffeomorphism becomes uniformly hyperbolic in a neighborhood of the trajectory and the size of this neighborhood oscillates very slowly (Proposition (2.3)). Actually, we slightly modify notations, definitions and results from Section 1 of Pesin's work [8], especially Theorems (1.5.1) and (1.6.1). In the second half of the section we derive some consequences from the hyperbolicity.

Let f be a $C^{1+\alpha}$ ($\alpha>0$) [1] diffeomorphism of a compact Riemannian s-dimensional manifold M. Let us denote for $\chi>0$, $\ell>1$ by $\Lambda_{\chi,\ell}$ the set of all points $x\in M$ with the following properties: there exists a decomposition $T_xM=E^s_x\oplus E^u_x$ such that for every $n\in\mathbf{Z}^+$, $m\in\mathbf{Z}$ we have for all vectors $v\in df^m E^s_x$:

$$||df^n_{f^m x}v||\leq \ell \exp -n\chi \exp(\chi\alpha 10^{-3}(n+|m|))||v||$$

$$||df^{-n}_{f^m x}v||\geq \ell^{-1} \exp n\chi \exp(-\chi\alpha 10^{-3}(n+|m|))||v||$$

for $v\in df^m E^u_x$:

$$||df^n_{f^m x}v||\geq \ell^{-1} \exp n\chi \exp(-\chi\alpha 10^{-3}(n+|m|))||v||$$

$$||df^{-n}_{f^m x}v||\leq \ell \exp -n\chi \exp(\chi\alpha 10^{-3}(n+|m|))||v||$$

and for the angle $\gamma(x)$ between the subspaces E^s_x and E^u_x:

$$\gamma(f^m x)\geq \ell^{-1} \exp -\alpha\chi 10^{-3}|m|$$

(cf. [8], (1.3.5)-(1.3.7)).

Let for an integer k with $0\leq k\leq s$:

$$\Lambda^k_{\chi,\ell}=\{x\in\Lambda_{\chi,\ell}:\ \dim E^s_x=k\}.$$

Obviously if $\chi_1\leq\chi_2$, $\ell_1\geq\ell_2$ then $\Lambda^k_{\chi_1,\ell_1}\supset\Lambda^k_{\chi_2,\ell_2}$.

Proposition **(2.1)** (cf. [8], Theorem (1.3.1)):

(i) *The sets* $\Lambda^k_{\chi,\ell}$ *are closed;*

(ii) *The subspaces* E^s_x, E^u_x *depend on x continuously on the set* $\Lambda^k_{\chi,\ell}$.

(iii) *For every integer q and* $\ell\geq 1$ *there exists* $L=L(q,\ell)$ *such that* $f^q(\Lambda^k_{\chi,\ell})\subset\Lambda^k_{\chi,L}$.

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[1] If f belongs to the class C^r, $r\geq 2$, we will take $\alpha=1$ in all formulas including α.

The statement (i) is obvious from the definitions. The proofs of statements of Theorem (1.3.1) in [8], similar to (ii) and (iii), work automatically in our case.

Let us denote:

$$\bigcup_{\ell>1} \Lambda^k_{\chi,\ell} = \Lambda^k_\chi, \quad \bigcup_{k,\chi} \Lambda^k_\chi = \Lambda.$$

Proposition **(2.2)**. — *Let x be a regular point for f with the Lyapunov exponents $\chi_1(x), \ldots, \chi_{r(x)}(x)$ different from zero, $\chi(x) = \min_{1\leq i\leq r(x)} |\chi_i(x)|$ and $k(x) = \sum_{i:\chi_i(x)<0} k_i(x)$ be the number of negative exponents with their multiplicities. Then $x \in \Lambda^{k(x)}_{\chi(x),\ell}$ for some $\ell \geq 1$.*

The proof is the same as in [8], Theorem (1.2.1) and below. This proposition and the multiplicative ergodic theorem imply the following statement.

Corollary **(2.1)**. — *For any Borel probability f-invariant measure μ with non-zero Lyapunov exponents $\mu(\Lambda) = 1$. If moreover μ is an ergodic measure, then $\mu(\mu^k_\chi) = 1$, where:*

$$\chi = \min_i |\chi^\mu_i|, \quad k = \sum_{i:\chi^\mu_i<0} k^\mu_i. \tag{2.1}$$

The next step is the definition of the so-called Lyapunov Riemannian metric near regular points which allows us to consider the linear parts of f along the trajectory of such a point as uniformly hyperbolic operators. This construction is contained in Theorems (1.5.1) and (1.6.1) of [8]. We summarize their content in a slightly modified way:

Proposition **(2.3)**. — *There exists a number $r_0 > 0$ which depends only on f such that for every point $x \in \Lambda^k_\chi$ we can find a neighborhood $B(x)$ and a diffeomorphism $\Phi_x : B^k_{r_0} \times B^{s-k}_{r_0} \to B(x)$ (B^i_r — Euclidean r-ball around the origin in $\mathbf{R}^i$) with the following properties.*

(i) *The image of the standard Euclidean metric in $B^k_{r_0} \times B^{s-k}_{r_0}$ is a Riemannian metric $\langle\cdot,\cdot\rangle'_x$ in $B(x)$ which generates the norm $\|\ \|'_x$ in each tangent space T_yM, $y \in B(x)$ connected with the norm $\|\ \|$, generated by the given Riemannian metric, by the following inequalities:*

$$K_1 \leq \frac{\|\ \|'_x}{\|\ \|} \leq K_2 A(x)$$

where K_1, K_2 are absolute constants and $A(x)$ is a Borel function of x such that for any integer m:

$$A(f^m x) \leq A(x) . \left(\min\left(\left(\frac{3}{2}\right)^{\alpha|m|}, \exp 2\chi\alpha . 10^{-3}|m|\right)\right). \tag{2.2}$$

and:

$$\sup_{x\in\Lambda^k_{\chi,\ell}} A(x) = A^k_{\chi,\ell} < \infty. \tag{2.3}$$

(ii) *The map:*

$$f_x = \Phi^{-1}_{f(x)} \circ f \circ \Phi_x : B^k_{r_0} \times B^{s-k}_{r_0} \to \mathbf{R}^k \times \mathbf{R}^{s-k} = \mathbf{R}^s$$

has the form:

$$f_x(u, v) = (A_x u + h_{1x}(u, v), B_x v + h_{2x}(u, v))$$

where $h_{1x}(0, 0) = h_{2x}(0, 0) = 0$, $dh_{1x}(0, 0) = dh_{2x}(0, 0) = 0$ *and:*

$$\text{(2.4)} \qquad \begin{aligned} &\|A_x\| < \exp\frac{-99}{100}\chi \\ &\|B_x^{-1}\|^{-1} < \exp\frac{-99}{100}\chi \end{aligned}$$

(all norms here and below in this section are Euclidean).

Let us set $\lambda(\chi) = \max\left(\frac{1}{2}, \exp\frac{-99}{100}\chi\right)$. *Then, for* $z = (u, v)$, $h_x(z) = (h_{1x}(z), h_{2x}(z))$:

$$\text{(2.5)} \qquad \|(dh_x)_{z_1} - (dh_x)_{z_2}\| \leq M\lambda(\chi)\|z_1 - z_2\|^\alpha,$$

with an absolute constant M.

(iii) *The metric* $\langle \cdot, \cdot \rangle_x'$ *depends on* x *continuously on any set* $\Lambda^k_{\chi,\ell}$.

(iv) *For any* $z \in M$ *the decomposition*

$$T_z M = d\Phi_x \mathbf{R}^k \times d\Phi_x \mathbf{R}^{s-k}$$

depends continuously on x *for such* $x \in \Lambda^k_{\chi,\ell}$ *that* $z \in B(x)$.

Although the last two statements are not contained in the cited theorems they follow easily from the definition of the metrics $\langle\ \rangle_x'$ ([8], formula (1.5.8)) and Proposition (2.1) (ii).

Now we are going to diminish the size of the neighborhoods $B(x)$ to provide the hyperbolicity of f in the reduced neighborhoods. The new neighborhood $C(x)$ for $x \in \Lambda^k_{\chi,\ell}$ has the form $C(x) = \Phi_x(B^k_{\varepsilon(x)} \times B^{s-k}_{\varepsilon(x)})$ where:

$$\varepsilon(x) = \frac{(1-\lambda(\chi))^{2/\alpha}}{100^{1/\alpha}}(2M)^{-1/\alpha}(A(x))^{-1/\alpha}.$$

We shall call the neighborhood $C(x)$ the standard x-box. From inequality (2.2) we have for any integer m:

$$\text{(2.6)} \qquad \varepsilon(f^m x) \leq \varepsilon(x)\left(\min\left(\left(\frac{3}{2}\right)^{|m|}, \exp 2\chi . 10^{-3}|m|\right)\right).$$

Furthermore, for $z = (u, v) \in \Phi_x^{-1}(C(x))$ we obtain from (2.5) the following estimation of the non-linear part of f_x:

$$\text{(2.7)} \qquad \|(dh_x)_z\| \leq MA(x)\|z\|^\alpha \leq \frac{(1-\lambda(\chi))^2}{100}.$$

For $x \in \Lambda^k_{\chi,\ell}$ we have from (2.3):

$$\text{(2.8)} \qquad \varepsilon(x) \geq \frac{(1-\lambda(\chi))^{2/\alpha}}{100^{1/\alpha}}(2M)^{-1/\alpha}(A^k_{\chi,\ell})^{-1/\alpha} = \varepsilon(k, \chi, \ell) > 0.$$

We shall need the following uniformly shrinked version of the standard boxes. Namely, let us fix a number $h: 0<h\leq 1$ and set for $x\in\Lambda^k_{\chi,\ell}$:

$$C(x,h)=\Phi_x(B^k_{h\varepsilon(x)}\times B^{s-k}_{h\varepsilon(x)}).$$

Let us denote by $U^{\gamma,\delta,h}_x$ $(0<\gamma<1, \delta\geq 0, 0<h\leq 1)$ the following class of $(s-k)$-dimensional submanifolds of the neighborhood $C(x,h)$:

$$U^{\gamma,\delta,h}_x=\{\Phi_x(\text{graph }\varphi): \varphi\in C^1(B^{s-k}_{h\varepsilon(x)}, B^k_{h\varepsilon(x)}), \|\varphi(0)\|\leq\delta, \|d\varphi\|\leq\gamma\}.$$

Obviously if $\gamma_1\geq\gamma_2$, $\delta_1\geq\delta_2$ then $U^{\gamma_1,\delta_1,h}_x\supset U^{\gamma_2,\delta_2,h}_x$. We define in a similar way the class $S^{\gamma,\delta,h}_x$ of k-dimensional submanifolds of $C(x,h)$:

$$S^{\gamma,\delta,h}_x=\{\Phi_x(\text{graph }\varphi): \varphi\in C^1(B^k_{h\varepsilon(x)}, B^{s-k}_{h\varepsilon(x)}), \|\varphi(0)\|\leq\delta, \|d\varphi\|\leq\gamma\}.$$

The following proposition shows that for a properly chosen number γ and for any sufficiently small δ the classes $U^{\gamma,\delta,h}_x$ are f-invariant and every manifold from such a class is expanding with respect to the Lyapunov metric $\langle\cdot,\cdot\rangle'_x$. For $\chi>0$, let:

$$\gamma(\chi)=\frac{1-\lambda(\chi)}{20}.$$

Proposition (**2.4**). — *Suppose that* $x\in\Lambda^k_{\chi,\ell}$, $\delta\leq\dfrac{h\varepsilon(x)}{2}$ *and* $N\in U^{\gamma(\chi),\delta,h}_x$. *Then:*

(i) $fN\cap C(f(x))\in U^{\lambda(\chi)\cdot\gamma(\chi),\,\delta\cdot\left(\frac{1+\lambda(\chi)}{2}\right),\,h}_{f(x)}$;

(ii) *for any two points* $y_1, y_2\in N$:

$$d'_{f(x)}(f(y_1),f(y_2))>\left(\frac{1}{2}+\frac{1}{2\lambda(\chi)}\right)d'_x(y_1,y_2)$$

where $d'_x(\cdot,\cdot)$ *is the distance function generated by the metric* $\langle\cdot,\cdot\rangle'_x$.

Remark. — We can assume that the constant M is large enough so that

$$f(C(x))\subset B(f(x)).$$

Proof. — It is more convenient to work in the Euclidean space $\mathbf{R}^s$ rather than in the neighborhoods $C(x)$ and $C(f(x))$. So, we take a map $\varphi\in C^1(B^{s-k}_{h\varepsilon(x)}, B^k_{h\varepsilon(x)})$ such that $\varphi(0)\leq\delta$, $\|d\varphi\|\leq\gamma(\chi)$ and show that the set

$$f_x(\text{graph }\varphi)\cap(B^k_{h\varepsilon(f(x))}\times B^{s-k}_{h\varepsilon(f(x))})$$

can be represented in the form graph $\tilde\varphi$, where:

$$\tilde\varphi\in C^1(B^{s-k}_{h\varepsilon(f(x))}, B^k_{h\varepsilon(f(x))}),$$
$$\|d\tilde\varphi\|\leq\lambda(\chi)\cdot\gamma(\chi) \quad\text{and}\quad \tilde\varphi(0)\leq\delta\left(\frac{1+\lambda(\chi)}{2}\right).$$

This fact implies the statement (i) of the proposition. To prove the second statement we shall show that for any $v_1, v_2\in B^{s-k}_{h\varepsilon(x)}$:

$$\|f_x(\varphi(v_1),v_1)-f_x(\varphi(v_2),v_2)\|>\left(\frac{1}{2}+\frac{1}{2\lambda}\right)\|\varphi(v_1)-\varphi(v_2),v_1-v_2)\|.$$

Let $w=(w_1, w_2)$ be a tangent vector to the manifold graph φ at the point $(\varphi(v), v)$. Then:

(2.9) $$\|w_1\| \leq \gamma(\chi) \cdot \|w_2\|.$$

Let us consider the vector:

$$(df_x)_{(\varphi(v),v)}(w_1, w_2) = (A_x w_1 + (dh_{1x})_{(\varphi(v),v)}(w_1, w_2), B_x w_2 + (dh_{2x})_{(\varphi(v),v)}(w_1, w_2)) = (\widetilde{w}_1, \widetilde{w}_2).$$

From (2.7), (2.9) and (2.4) we have:

$$\|\widetilde{w}_1\| \leq \lambda(\chi)\|w_1\| + \frac{(1-\lambda(\chi))^2}{100}(\|w_1\| + \|w_2\|)$$
$$\leq \left(\lambda(\chi)\cdot\gamma(\chi) + \frac{(1-\lambda(\chi))^2}{100}(1+\gamma(\chi))\right)\|w_2\|;$$

$$\|\widetilde{w}_2\| \geq \lambda^{-1}(\chi)\|w_2\| - \frac{(1-\lambda(\chi))^2}{100}(\|w_1\| + \|w_2\|)$$
$$\geq \left(\lambda^{-1}(\chi) - \frac{(1-\lambda(\chi))^2}{100}(1+\gamma(\chi))\right)\|w_2\|.$$

We want to show that:

(2.10) $$\|\widetilde{w}_1\| \leq \lambda(\chi)\cdot\gamma(\chi)\cdot\|\widetilde{w}_2\|.$$

To do that it is enough to prove the following inequality:

$$\lambda(\chi)\cdot\gamma(\chi) + \frac{(1-\lambda(\chi))^2}{100}(1+\gamma(\chi)) \leq \lambda(\chi)\cdot\gamma(\chi)\left(\lambda^{-1}(\chi) - \frac{(1-\lambda(\chi))^2}{100}(1+\gamma(\chi))\right).$$

Let us omit the dependence of λ and γ on χ in the subsequent computation. We have $\left(\text{recall that } \gamma = \frac{1-\lambda}{20}\right)$:

$$\frac{\lambda(1-\lambda)}{20} + \frac{(1-\lambda)^2}{100}\left(1+\frac{1-\lambda}{20}\right) < \frac{\lambda(1-\lambda)}{20} + \frac{(1-\lambda)^2}{50} = \frac{1-\lambda}{20}\left(\lambda + \frac{2}{5}(1-\lambda)\right)$$
$$< \frac{1-\lambda}{20}\left(\lambda + \frac{3}{5}(1-\lambda)\right) = \frac{1-\lambda}{20}\left(1 - \frac{2}{5}(1-\lambda)\right)$$
$$= \frac{1-\lambda}{20} - \frac{(1-\lambda)^2}{50}$$
$$< \frac{1-\lambda}{20} - \frac{(1-\lambda)^2}{100}\left(1+\frac{1-\lambda}{20}\right)$$
$$< \lambda\left(\frac{1-\lambda}{20}\right)\left(\lambda^{-1} - \frac{(1-\lambda)^2}{100}\left(1+\frac{1-\lambda}{20}\right)\right).$$

Inequality (2.10) shows that the tangent space to the manifold $f_x(\text{graph } \varphi)$ at every point belongs to the cone:

$$K_{\lambda(\chi)\gamma(\chi)} = \{(w_1, w_2) : \|w_1\| \leq \lambda(\chi) \cdot \gamma(\chi) \cdot \|w_2\|\}.$$

Thus $f_x(\text{graph } \varphi) = \text{graph } \widetilde{\varphi}$ where $\widetilde{\varphi}$ is a map defined on some subset of the ball $B^{s-k}_{r_0}$ (see remark) and $\|d\widetilde{\varphi}\| \leq \lambda(\chi) \cdot \gamma(\chi)$.

We want to show that the domain of the map $\widetilde{\varphi}$ contains the ball $B^{s-k}_{h\varepsilon(f(x))}$. It is easy to see that this domain coincides with the image of the map:

$$\pi_\varphi = \pi_2 \circ f_x \circ (\varphi \times \text{id}) : B^{s-k}_{h\varepsilon(x)} \to \mathbf{R}^{s-k}$$

where:

$$\pi_2 : \mathbf{R}^s = \mathbf{R}^k \times \mathbf{R}^{s-k} \to \mathbf{R}^{s-k}$$

is the natural projection. The explicit expression for π_φ is:

(2.11) $$\pi_\varphi v = B_x v + h_{2x}(\varphi(v), v).$$

The map π_φ is expanding since we have, from (2.4) and (2.7) ($v_1, v_2 \in B^{s-k}_{h\varepsilon(x)}$):

$$\begin{aligned}\|\pi_\varphi v_1 - \pi_\varphi v_2\| &\geq \|B_x(v_1 - v_2)\| - \|h_{2x}(\varphi(v_1), v_1) - h_{2x}(\varphi(v_2), v_2)\| \\ &\geq \lambda^{-1}(\chi)\|v_1 - v_2\| - \frac{(1-\lambda(\chi))^2}{100}(\|v_1 - v_2\| + \|\varphi(v_1) - \varphi(v_2)\|) \\ &\geq \left(\lambda^{-1} - \frac{(1-\lambda(\chi))^2}{50}\right)\|v_1 - v_2\| > \left(\frac{1}{2} + \frac{1}{2\lambda(\chi)}\right)\|v_1 - v_2\| \\ &> \|v_1 - v_2\|.\end{aligned} \tag{2.12}$$

Suppose that $v \in \partial B^{s-k}_{h\varepsilon(x)}$, *i.e.*, $\|v\| = h\varepsilon(x)$. Then the substitution of $v_1 = v$, $v_2 = 0$ gives us:

$$\begin{aligned}\|\pi_\varphi v\| &\geq \left(\lambda^{-1}(\chi) - \frac{(1-\lambda(\chi))^2}{50}\right)\|v\| > (\lambda(\chi))^{-1/495}\|v\| \\ &= (\exp 2\chi 10^{-3})\varepsilon(x)h\end{aligned}$$

so that by (2.6):

$$\|\pi_\varphi v\| > h\varepsilon(f(x)).$$

On the other hand, it follows from (2.11), (2.7) and (2.6) that:

$$\|\pi_\varphi 0\| < h\varepsilon(f(x))$$

so that the image of the boundary of the disc $B^{s-k}_{h\varepsilon(x)}$ lies outside the disc $B^{s-k}_{h\varepsilon(f(x))}$ while the image of its center lies inside. Since π_φ is an expanding map we can conclude that:

$$\pi_\varphi(B^{s-k}_{h\varepsilon(x)}) \supset B^{s-k}_{h\varepsilon(f(x))}$$

i.e., the domain of the map $\widetilde{\varphi}$ contains the ball $B^{s-k}_{h\varepsilon(f(x))}$.

Let v_0 be the solution of the equation:

$$f_x(\varphi(v_0), v_0) = (\widetilde{\varphi}(0), 0).$$

The equation can be rewritten in the form $\pi_\varphi v_0 = 0$ or:

$$v_0 = -B_x^{-1}(h_{2x}(\varphi(v_0), v_0)).$$

Let us omit the dependence of λ and γ on χ as above.

From (2.4) and (2.7) we obtain the following estimation of $||v_0||$:

$$||v_0|| \leq \frac{\lambda(1-\lambda)^2}{100}(\delta + (1+\gamma)||v_0||)$$

or:

$$\text{(2.13)} \qquad ||v_0|| \leq \frac{\delta\lambda(1-\lambda)}{100}.$$

Furthermore:

$$\tilde{\varphi}(0) = A_x(\varphi(v_0)) + h_{1x}(\varphi(v_0), v_0)$$

so that we can estimate $||\tilde{\varphi}(0)||$, namely:

$$\begin{aligned} ||\tilde{\varphi}(0)|| &\leq \lambda(||\varphi(0)|| + \gamma||v_0||) + \frac{(1-\lambda)^2}{100}(||\varphi(0)|| + (1+\gamma)||v_0||) \\ &\leq \delta\left(\lambda + \frac{\lambda^2(1-\lambda)\gamma}{100} + \frac{(1-\lambda)^2}{100} + \frac{\lambda(1-\lambda)(1+\gamma)}{100}\right) \\ &= \delta\left(\lambda + \frac{1-\lambda}{2}\left(\frac{\lambda^2\gamma}{50} + \frac{1-\lambda}{50} + \frac{\lambda(1+\gamma)}{50}\right)\right) \\ &\leq \delta\left(\lambda + \frac{1-\lambda}{2}\right) = \delta\left(\frac{1+\lambda}{2}\right). \end{aligned}$$

For these estimations we used inequalities (2.4), (2.7), (2.10) and (2.13).

To finish the proof of (i) we have to show only that for $v \in B^{s-k}_{h\varepsilon(f(x))}$:

$$||\tilde{\varphi}(v)|| \leq h\varepsilon(f(x)).$$

For:

$$\begin{aligned} ||\tilde{\varphi}(v)|| &\leq ||\tilde{\varphi}(0)|| + ||d\tilde{\varphi}|| \cdot ||v|| < \left(\frac{1+\lambda}{2}\right)\delta + \lambda\gamma h\varepsilon(f(x)) \\ &< \left(\frac{1+\lambda}{2}\right)\frac{h\varepsilon(x)}{2} + \frac{h\varepsilon(f(x))}{80} < h\varepsilon(f(x)). \end{aligned}$$

We used the inequalities $\lambda\gamma = \dfrac{\lambda(1-\lambda)}{20} \leq \dfrac{1}{80}$ and $\dfrac{\varepsilon(f(x))}{\varepsilon(x)} > \dfrac{2}{3}$.

Let us proceed to the proof of (ii). We have:

$$\begin{aligned}
&||f_x(\varphi(v_1), v_1)-f_x(\varphi(v_2), v_2)||\\
&\quad\geq ||B_x(v_1-v_2)||-||A_x(\varphi(v_1)-\varphi(v_2))||-||h_x(\varphi(v_1), v_1)-h_x(\varphi(v_2), v_2)||\\
&\quad\geq \lambda^{-1}||v_1-v_2||-\lambda\gamma||v_1-v_2||-\frac{(1-\lambda)^2}{100}(1+\gamma)\,||v_1-v_2||\\
&\quad\geq \left(\lambda^{-1}-\frac{\lambda(1-\lambda)}{20}-\frac{(1-\lambda)^2}{50}\right)||v_1-v_2||>\left(\lambda^{-1}-\frac{1-\lambda}{20}\right)||v_1-v_2||\\
&\quad> \left(\frac{1}{2\lambda}+\frac{1}{2}+\frac{1-\lambda}{10\lambda}\right)||v_1-v_2||>\left(\frac{1}{2\lambda}+\frac{1}{2}\right)(1+\gamma)\,||v_1-v_2||\\
&\quad\geq \left(\frac{1}{2\lambda}+\frac{1}{2}\right)||(\varphi(v_1), v_1)-(\varphi(v_2), v_2)||. \quad \blacksquare
\end{aligned}$$

We do not formulate explicitly the result similar to Proposition (2.4) for the classes $S_x^{\gamma(x),\delta,h}$. In the next section we will use both of these results.

In general we do not guarantee that the number $\varepsilon(x)$ and the map Φ_x depend on x continuously on the sets $\Lambda^k_{\chi,\ell}$. Indeed, Proposition (2.3) (iii) and (iv) provide continuous dependence of the classes $U_x^{\gamma,\delta,h}$ and $S_x^{\gamma,\delta,h}$ on these sets if we consider instead of manifolds from the classes their pieces of fixed size.

Let us denote the neighborhood:

$$\Phi_x(B^k_{h\varepsilon(k,\chi,\ell)/2}\times B^{s-k}_{h\varepsilon(k,\chi,\ell)/2})$$

of a point $x\in\Lambda^k_{\chi,\ell}$ by $C(x, h, k, \chi, \ell)$ (cf. (2.8)). Sometimes for convenience of notation (if the numbers k, χ, ℓ are fixed) we shall write ε instead of $\varepsilon(k, \chi, \ell)$ and $\hat{C}(x, h)$ instead of $C(x, h, k, \chi, \ell)$.

Furthermore, we shall call any manifold of the form $N\cap\hat{C}(x, h)$ where:

$$N\in U_x^{\gamma(x),\frac{h\varepsilon}{4},h}$$

an admissible (u, h)*-manifold near* x and any manifold of the form:

$$N\cap\hat{C}(x, h) \quad \text{where} \quad N\in S_x^{\gamma(x),\frac{h\varepsilon}{4},h}$$

an admissible (s, h)*-manifold near* x.

Let $d(\cdot, \cdot)$ be the distance function generated by the given Riemannian metric on M. We have from Propositions (2.1) and (2.3):

Corollary **(2.2)**. — *For any* k, $\chi>0$, $\ell>0$, $\beta<\frac{1}{4}$, $0<h\leq 1$ *there exists a number* $\varkappa=\varkappa(k, \chi, \ell, \beta, h)$ *such that if* $x, y\in\Lambda^k_{\chi,\ell}$, $d(x,y)<\varkappa$, $N\in U_y^{4\beta\gamma(x),h\beta\varepsilon(k,\chi,\ell),h}$ *(resp.* $N\in S_y^{4\beta\gamma(x),h\beta\varepsilon(k,\chi,\ell),h}$*) then:*

$$N\cap C(x, h, k, \chi, \ell)$$

is an admissible (u, h)*-manifold (resp.* (s, h)*-manifold) near* x.

Proposition **(2.5)**. — *Let* $x \in \Lambda^k_{\chi,\ell}$, $0<h\leq 1$. *Then any admissible* (s, h)*-manifold near the point* x *intersects any admissible* (u, h)*-manifold near that point at exactly one point and the intersection is transversal.*

Proof. — (i) Existence. Let $K=\Phi_x(\text{graph } \varphi)$, $L=\Phi_x(\text{graph } \psi)$ be an admissible (s, h)-manifold and an admissible (u, h)-manifold near x, respectively, with:

$$\varphi \in C^1(B^k_{h\varepsilon/2}, B^{s-k}_{h\varepsilon/2}) \quad \text{and} \quad \psi \in C^1(B^{s-k}_{h\varepsilon/2}, B^k_{h\varepsilon/2}).$$

Let us consider the map:

$$\psi \circ \varphi : B^k_{h\varepsilon/2} \to B^k_{h\varepsilon/2}.$$

Since this map is continuous it has a fixed point u_0 (by the Brower fixed point theorem). Thus, $\psi(\varphi(u_0))=u_0$ or:

(2.14) $$(u_0, \varphi(u_0))=(\psi(\varphi(u_0)), \varphi(u_0)).$$

But $(u_0, \varphi(u_0)) \in \text{graph } \varphi$, $(\psi(\varphi(u_0)), \varphi(u_0)) \in \text{graph } \psi$ so that (2.14) implies that:

$$\Phi_x(u_0, \varphi(u_0)) \in K \cap L.$$

(ii) Uniqueness. Let $(u_0, v_0) \in \text{graph } \varphi \cap \text{graph } \psi$. Then if $(u, v) \in \text{graph } \varphi$ the following inequality is true:

(2.15) $$\|v-v_0\| \leq \gamma \|u-u_0\|$$

and similarly for $(u, v) \in \text{graph } \psi$:

(2.16) $$\|v-v_0\| \geq \gamma^{-1} \|u-u_0\|.$$

Since $\gamma<1$ inequalities (2.15) and (2.16) are satisfied simultaneously only for $u=u_0$, $v=v_0$.

(iii) Transversality. Once more let:

$$(u_0, v_0) \in \text{graph } \varphi \cap \text{graph } \psi.$$

If $\xi=(\eta, \zeta) \in T_{(u_0, v_0)} \text{graph } \varphi$ then:

(2.17) $$\|\zeta\| \leq \gamma \|\eta\|.$$

If $\xi=(\eta, \zeta) \in T_{(u_0, v_0)} \text{graph } \psi$ then:

$$\|\zeta\| \geq \gamma^{-1} \|\eta\|.$$

Thus if $\xi \in T_{(u_0, v_0)} \text{graph } \varphi \cap T_{(u_0, v_0)} \text{graph } \psi$ then by (2.16) and (2.17) we have $\xi=0$. This means that the intersection is transversal. ■

3. Approximation of recurrent regular point by periodic point.

Main Lemma. — *Let* f *be a* $C^{1+\alpha}$ $(\alpha>0)$ *diffeomorphism of a compact* s*-dimensional Riemannian manifold* M. *Then for any* $k=0, \dots, s$ *and any positive numbers* χ, ℓ, δ *there* *exists a number* $\psi=\psi(k, \chi, \ell, \delta)>0$ *such that if for some point* $x\in\Lambda^k_{\chi,\ell}$ *and for some integer n one has*

$$f^n x\in\Lambda^k_{\chi,\ell} \tag{3.1}$$

and

$$d(x,f^n x)<\psi \tag{3.2}$$

then there exists a point $z=z(x)$ *such that:*

(i) $f^n z=z$;

(ii) $d^f_n(x, z)<\delta$;

(iii) *the point z is a hyperbolic periodic point for f and its local stable and unstable manifolds are the admissible* $(s, 1)$*-manifold and admissible* $(u, 1)$*-manifold near the point x, respectively.*

Let us fix some numbers $\beta>\dfrac{1+\lambda(\gamma)}{8}$ and $h:0<h\leq 1$ and assume that:

$$d(x,f^n x)<\min(\varkappa(k, \chi, \ell, \beta, h), \varkappa(k, \chi, \ell, \beta, 1)) \tag{3.3}$$

where the number $\varkappa$ is found from Corollary (2.2).

Moreover, by Proposition (2.3) (iii) we can find for every $0<\tau<1$ the number $\psi=\psi(\chi, \ell, \tau)$ such that if $x\in\Lambda^k_{\chi,\ell}$ and conditions (3.1) and (3.2) are satisfied then for every $y_1,y_2\in\hat{C}(x, 1)$:

$$\tau<\frac{d'_x(y_1,y_2)}{d'_{f^n x}(y_1,y_2)}<\tau^{-1}. \tag{3.4}$$

We assume that ψ in (3.2) is chosen to satisfy (3.4) with τ sufficiently close to 1; particular choice of τ will be specified below (cf. (3.9), remark after (3.20), (3.25)). Other conditions on ψ will occur explicitly in the course of the proof (cf. (3.6), remark after (3.12)).

During the proof of the Main Lemma we shall use the following simplified notations: ε instead of $\varepsilon(k, \chi, \ell)$ and $\hat{C}(x, h)$ instead of $C(x, h, k, \chi, \ell)$. Also, we shall omit the dependence of γ and λ on χ.

All other constructions here and below will be effected for the chosen number h (which may be very small if δ is small, cf. (3.36)) and for $h=1$. Obviously for $h=1$ some of the notations become simpler.

We shall use this second version of the constructions only in the final step of the proof dealing with assertion (iii).

Step 1. — Let us denote by A_0 and B_0 the following manifolds:

$$A_0=(\Phi_{f^n x}(B^k_{h\varepsilon(f^n x)}\times\{0\}))\cap\hat{C}(x, h)$$

$$B_0=\Phi_x(\{0\}\times B^{s-k}_{h\frac{\varepsilon}{2}}).$$

Obviously, B_0 is an admissible (u, h)-manifold near the point x. Since

$$\Phi_{f^n x}(B^k_{h\varepsilon(f^n x)}\times\{0\})\in S^{0,0,h}_{f^n x}$$

then by Corollary (2.2) the manifold A_0 is an admissible (s, h)-manifold near x.

Let us define the manifolds A^i_0, B^i_0, $i=1, \dots, n-1$ in the following way:

$$A^1_0=f^{-1}A_0\cap C(f^{n-1}x, h)$$
$$A^i_0=f^{-1}A^{i-1}_0\cap C(f^{n-i}x, h) \quad i=2, \dots, n-1$$
$$B^1_0=fB_0$$
$$B^i_0=f(B^{i-1}_0\cap C(f^{i-1}x, h)) \quad i=2, \dots, n-1.$$

We can conclude from Proposition (2.4) (i) that each manifold B^i_0 is a part of a manifold from the class $U^{\lambda^i\gamma,0,h}_{f^i(x)}$ (or maybe the whole such manifold). Similarly, A^i_0 is a part of a manifold from $S^{\lambda^i\gamma,0,h}_{f^{n-i}x}$. Now let:

$$A_1=f^{-1}A^{n-1}_0\cap\hat{C}(x, h)$$
$$B_1=fB^{n-1}_0\cap\hat{C}(x, h).$$

Applying once more Proposition (2.4) (i) we can conclude that the manifold A_1 is a part of an admissible (s, h)-manifold near the point x and B_1 is a part of a manifold from the class $U^{\lambda^n\gamma,0,h}_{f^n x}$. Thus Corollary (2.2) and (3.3) guarantee that B_1 is part of an admissible (u, h)-manifold near x.

We shall show that *if $d(x,f^n x)$ is small enough then B_1 actually does coincide with some admissible (u, h)-manifold near x.* This statement is a particular case of a statement that will be proved in Step 2. So the reader can either omit the subsequent proof and proceed directly to Step 2 or try to understand the idea of both proofs (which is basically the same) on the particular case which is technically easier.

Suppose that B_1 is the proper part of an admissible (u, h)-manifold near x. Then we can extend B_0 to a manifold $\widetilde{B}_0\subset U^{\gamma,0,h}_x$, apply inductively Proposition (2.4) (ii), then Corollary (2.2) and construct manifolds $N_i\in U^{\lambda^i\gamma,0,h}_{f^i x}$, $i=0, \dots, n-1$, and an admissible (u, h)-manifold $\widetilde{B}_1$ near x such that for some point $y\in\widetilde{B}_0\setminus B_0$:

$$f^i x\in N_i, \quad i=1, \dots, n-1 \quad \text{and} \quad f^n y\in\widetilde{B}_1\setminus B_1.$$

Since $y\in\widetilde{B}_0\setminus B_0$ we have:

$$d'_x(x,y)>\frac{h\varepsilon}{2}. \tag{3.5}$$

Let:

$$f^n x=\Phi_x(u_1, v_1)$$
$$f^n y=\Phi_x(u_2, v_2)$$

and assume that:

$$d'_x(x,f^n x)<\frac{h\varepsilon\gamma}{4}. \tag{3.6}$$

Then:

$$(3.7)\qquad d'_x(f^n x, f^n y) \leq \|u_1 - u_2\| + \|v_1 - v_2\| \leq \|v_1\| + \|v_2\| + \gamma h.\varepsilon \leq d'_x(x, f^n x) + \frac{h\varepsilon}{2} + \gamma h.\varepsilon \leq \frac{h\varepsilon}{2}(1 + 3\gamma).$$

It follows from Proposition (2.3) (i) that (3.6) can be fulfilled if ψ is chosen small enough.

On the other hand it follows from Proposition (2.4) (ii) that:

$$(3.8)\qquad d'_{f^n x}(f^n x, f^n y) > \left(\frac{1}{2} + \frac{1}{2\lambda}\right)^n d'_x(x, y).$$

Suppose that in (3.4) τ is chosen so that:

$$(3.9)\qquad \tau > \frac{1 + \frac{1-\lambda}{2}}{1 + \frac{1-\lambda}{2\lambda}}.$$

Then we have combining (3.4)-(3.9):

$$\frac{h\varepsilon}{2}(1 + 3\gamma) \geq d'_x(f^n x, f^n y) > \frac{1 + \frac{1-\lambda}{2}}{1 + \frac{1-\lambda}{2\lambda}} d'_{f^n x}(f^n x, f^n y) \geq \left(1 + \frac{1-\lambda}{2}\right) d'_x(x, y) > \frac{h\varepsilon}{2}\left(1 + \frac{1-\lambda}{2}\right)$$

or:

$$\frac{3}{20}(1-\lambda) = 3\gamma(\chi) > \frac{1}{2}(1-\lambda)$$

which is a contradiction.

Thus we have proved that B_1 is an admissible (u, h)-manifold near x. Similar arguments show that A_1 is an admissible (s, h)-manifold near the same point.

Step 2. — Let us define by induction the manifolds:

$$A_m, A_m^1, \ldots, A_m^{n-1}, B_m, B_m^1, \ldots, B_m^{n-1}, \quad m = 1, 2, \ldots$$

in the following way

$$(3.10)\qquad \begin{cases} A_m^1 = f^{-1} A_m \cap C(f^{n-1} x, h) \\ A_m^i = f^{-1} A_m^{i-1} \cap C(f^{n-i} x, h) \quad i = 2, \ldots, n-1 \\ A_{m+1} = f^{-1} A_m^{n-1} \cap \hat{C}(x, h) \end{cases}$$

$$(3.11)\qquad \begin{cases} B_m^1 = f B_m \\ B_m^i = f(B_m^{i-1} \cap C(f^{i-1} x, h)), \quad i = 2, \ldots, n-1 \\ B_{m+1} = f B_m^{n-1} \cap \hat{C}(x, h). \end{cases}$$

We shall apply arguments similar to those that were used in Step 1 to prove that:

For every $m=1, 2, \ldots, B_m$ *is an admissible* (u, h)*-manifold near* x *and* A_m *is an admissible* (s, h)*-manifold near* x.

We shall use induction in m. Let us assume that B_m is an admissible (u, h)-manifold near x. Then, by (3.11) and Proposition (2.4) (i), B_{m+1} is a part of a manifold from the class $U_{f^n x}^{\lambda^n\gamma, \left(\frac{1+\lambda}{2}\right)^n \cdot \frac{h\varepsilon}{4}, h}$ and, consequently by Corollary (2.2), B_{m+1} is a part of (u, h)-admissible manifolds near the point x. Let $B_m = \Phi_x(\text{graph}\,\varphi)$ where

$$\varphi \in C^1(B_{h\varepsilon/2}^{s-k}, B_{h\varepsilon/2}^k).$$

Let us extend B_m to a manifold:

$$\widetilde{B}_m \in U_x^{\gamma, \frac{h\varepsilon}{4}, h}, \qquad \widetilde{B}_m = \text{graph}\,\widetilde{\varphi}, \qquad \widetilde{\varphi} \in C^1(B_{h\varepsilon(x)}^{s-k}, B_{h\varepsilon(x)}^k),$$
$$\|\widetilde{\varphi}(0)\| \leq \frac{h\varepsilon}{4}, \qquad \|d\widetilde{\varphi}\| \leq \gamma.$$

In other words, $\varphi = \widetilde{\varphi}\Big|_{B_{h\varepsilon/2}^{s-k}}$.

Let us define the manifolds $\widetilde{B}_m^i$, $i=1, \ldots, n-1$ and $\widetilde{B}_{m+1}$ by formulas similar to (3.11):

$$\begin{aligned} &\widetilde{B}_m^1 = f\widetilde{B}_m \\ (3.12) \qquad &\widetilde{B}_m^i = f(\widetilde{B}_m^{i-1} \cap C(f^{i-1}x, h)), \quad i=2, \ldots, n-1 \\ &\widetilde{B}_{m+1} = f(\widetilde{B}_m^{n-1} \cap C(f^{n-1}x, h)) \cap C(f^n x, h). \end{aligned}$$

Obviously, if $d(f^n x, x)$ is small enough then $\widetilde{B}_{m+1} \supset B_{m+1}$. Applying Proposition (2.4) (i) inductively we conclude that for $i=1, \ldots, n-1$:

$$\widetilde{B}_m^i \cap C(f^i x, h) \subset U_{f^i x}^{\lambda^i\gamma, \frac{h\varepsilon}{4}\cdot\left(\frac{1+\lambda}{2}\right)^i, h}$$

and:

$$\widetilde{B}_{n+1} \subset U_{f^n x}^{\lambda^n\gamma, \frac{h\varepsilon}{4}\cdot\left(\frac{1+\lambda}{2}\right)^n, h}.$$

In other words:

$$\widetilde{B}_m^i \cap C(f^i x, h) = \Phi_{f^i x}(\text{graph}\,\widetilde{\varphi}_i)$$
$$\widetilde{B}_m = \Phi_{f^n x}(\text{graph}\,\widetilde{\varphi}_n)$$

where for $i=1, \ldots, n$:

$$\begin{aligned} &\widetilde{\varphi}_i \in C^1(B_{h\varepsilon(f^i x)}^{s-k}, B_{h\varepsilon(f^i x)}^k), \\ (3.13) \qquad &\|\widetilde{\varphi}_i(0)\| \leq \frac{h\varepsilon}{4}\left(\frac{1+\lambda}{2}\right)^i \\ &\|d\widetilde{\varphi}_i\| \leq \lambda^i\gamma. \end{aligned}$$

Let us consider the point:

$$z_n = \Phi_{f^n x}(\widetilde{\varphi}_n(0), 0)$$

and set:

$$z_i = f^{i-n} z_n, \quad i = 0, 1, \ldots, n-1.$$

Since $z_i \in \widetilde{B}^i_m$ we can represent these points in the form:

$$z_i = \Phi_{f^i x}(\widetilde{\varphi}_i(v_i), v_i)$$

where by (3.13):

$$\|\widetilde{\varphi}_i(v_i)\| \leq \|\widetilde{\varphi}_i(0)\| + \gamma\|v_i\| \leq \left(\frac{1+\lambda}{2}\right)^i \frac{h\varepsilon}{4} + \gamma\|v_i\|. \tag{3.14}$$

Now let us consider a k-dimensional manifold:

$$N = \Phi_{f^n x}(B_{h\varepsilon(f^n x)} \times \{0\}) \in S^{\gamma, 0, h}_{f^n x}$$

which contains the points $f^n x$ and z_n and apply to that manifold inductively the statement similar to Proposition (2.4). We can conclude from that statement that $z_i \in N_i$ for some $N_i \in S^{\gamma, 0, h}_{f^i x}$ and, consequently:

$$\|v_i\| \leq \gamma \|\widetilde{\varphi}_i(v_i)\| \tag{3.15}$$

Combining (3.14) and (3.15) we obtain:

$$\|\widetilde{\varphi}_i(v_i)\| \leq \frac{h\varepsilon\left(\frac{1+\lambda}{2}\right)^i}{4(1+\gamma^2)}, \qquad \|v_i\| \leq \frac{\gamma h\varepsilon}{4(1+\gamma^2)}\left(\frac{1+\lambda}{2}\right)^i. \tag{3.16}$$

Since $B^i_m \cap C(f^i x, h) \subset \widetilde{B}^i_m \cap C(f^i x, h)$ we can represent the first manifold in the form:

$$B^i_m \cap C(f^i x, h) = \Phi_{f^i x}(\text{graph } \widetilde{\varphi}_i\big|_{D_i}).$$

It follows from (3.16) that $z_0 \in B_m$ and consequently $z_i \in B^i_m \cap C(f^i x, h)$.

In other words, $v_i \in D_i$. The arguments from the proof of Proposition (2.4) (esp. (2.11) and (2.12)) show that:

$$D_{i+1} = \pi_{\widetilde{\varphi}_i} D_i \cap B^{s-k}_{h\varepsilon(f^{i+1}(x))} \tag{3.17}$$

where:

$$\pi_{\widetilde{\varphi}_i}(v) = B_{f^i x} v + h_{2 f^i x}(\widetilde{\varphi}_i(v), v). \tag{3.18}$$

Obviously:

$$D_0 = B^{s-k}_{h\varepsilon/2}.$$

Every map $\pi_{\widetilde{\varphi}_i}$ is expanding on $B^{s-k}_{h\varepsilon(f^i x)}$ (and consequently on D_i) and by (2.12) the coefficient of expansion is bigger than $\frac{1}{2} + \frac{1}{2\lambda}$. Thus (3.16), (3.17) and (3.18) imply the following statement:

If D_i *contains a ball around* v_i *of radius* r *then* D_{i+1} *contains a ball around* v_{i+1} *of radius:*

$$r' = \min\left(\left(\frac{1}{2}+\frac{1}{2\lambda}\right)r,\ h\varepsilon(f^{i+1}x) - \frac{\gamma h\varepsilon}{4(1+\gamma^2)}\left(\frac{1+\lambda}{2}\right)^{i+1}\right).$$

Since $\varepsilon(x)\geq\varepsilon$ and, by (2.6), $\dfrac{\varepsilon(f(x))}{\varepsilon(x)} \geq \max\left(\dfrac{2}{3}, \lambda^{\frac{1}{495}}\right) > \dfrac{1+\lambda}{2}$, we have:

$$\varepsilon(f^{i+1}x) > \left(\frac{1+\lambda}{2}\right)^{i+1}\varepsilon(x)$$

whence:

(3.19) $$r' \geq \min\left(\left(\frac{1}{2}+\frac{1}{2\lambda}\right)r,\ h(1-\gamma)\varepsilon(f^{i+1}x)\right).$$

Applying inequality (3.19) inductively and using the fact that D_0 contains a ball around v_0 of radius $(1-\gamma)\dfrac{h\varepsilon}{2}$ (what follows from (3.16) for $i=0$) we obtain that D_n contains a ball around the origin (recall that $v_n=0$) of radius:

$$\min\left((1-\gamma)\cdot\left(\frac{1}{2}+\frac{1}{2\lambda}\right)^n\frac{h\varepsilon}{2},\ h(1-\gamma)\varepsilon(f^n x)\right).$$

This number is bigger than:

(3.20) $$\frac{h\varepsilon}{2}\left(\frac{1}{2}+\frac{1}{2\lambda}\right)^{1/2}.$$

This means that if τ in (3.4) is close enough to 1 then:

$$\Phi_x^{-1}\Phi_{f^n x}(\text{graph}\ \widetilde{\varphi}_n\Big|_{D_n}) = \text{graph}\ \varphi_n$$

where the domain of φ_n covers the ball $B^{s-k}_{h\varepsilon/2}$.

In other words, since:

$$B_{m+1} = \Phi_{f^n x}(\text{graph}\ \widetilde{\varphi}_n\Big|_{D_n}) \cap \hat{C}(x, h)$$

this manifold is an (u, h)-admissible manifold near the point x.

Step 3. — By Proposition (2.5) every manifold A_k intersects every B_ℓ, $k, \ell = 0, 1, \ldots$ at exactly one point. We denote this point of intersection by $z_{k,\ell}$. Obviously:

$$x = z_{1,0} \quad f^n x = z_{0,1}.$$

Let us prove that if $k\geq 1$, $\ell\geq 0$ *then:*

(3.21) $$f^n z_{k,\ell} = z_{k-1,\ell+1}.$$

In other words, we are to prove that:

$$f^n z_{k,\ell}\in A_{k-1} \quad\text{and}\quad f^n z_{k,\ell}\in B_{\ell+1}.$$

The first inclusion follows directly from the definition of $z_{k,\ell}$ and (3.10) because:

$$f^n z_{k,\ell}\in f^n A_k \subset f^{n-1}A^{n-1}_{k-1} \subset \ldots \subset A_{k-1}.$$

To prove the second inclusion it is enough to show that for $i=1, \ldots, n-1$:

(3.22) $$f^i z_{k,\ell} \subset B_\ell^i$$

because in this case:

(3.23) $$f^n z_{k,\ell} = f(f^{n-1} z_{k,\ell}) \subset f B_\ell^{n-1}$$

and since:

(3.24) $$f^n z_{k,\ell} \subset A_{k-1} \subset \hat{C}(x, h)$$

we have from (3.23), (3.24) and (3.11) that:

$$f^n z_{k,\ell} \in B_{\ell+1}.$$

Now we proceed to the proof of (3.22) by induction in i. Suppose that:

$$f^{i-1} z_{k,\ell} \in B_\ell^{i-1}.$$

Then:

$$f^i z_{k,\ell} \subset f B_\ell^{i-1}$$

and by (3.10):

$$f^i z_{k,\ell} \subset f A_{k-1}^{n-i+1} = A_{k-1}^{n-i} \cap f C(f^{i-1} x, h)$$

i.e., by (3.11):

$$f^i z_{k,\ell} \subset f B_\ell^{i-1} \cap f C(f^{i-1} x, h) = f(B_\ell^{i-1} \cap C(f^{i-1} x, h)) = B_\ell^i.$$

Step 4. — Let us assume that in (3.4) we have chosen:

(3.25) $$\tau \geq \left(\frac{1}{2} + \frac{1}{2\lambda(\chi)}\right)^{-1/100}.$$

We shall prove that for every $k_1, k_2 \geq 1$, $\ell \geq 0$:

(3.26) $$d_x'(z_{k_1,\ell}, z_{k_2,\ell}) \leq \lambda' d_x'(z_{k_1-1,\ell+1}, z_{k_2-1,\ell+1})$$

where $\lambda' = \lambda'(\chi, n) = \left(\frac{1}{2} + \frac{1}{2\lambda(\chi)}\right)^{-n+1/100} < 1$. The following inclusion follows from (3.22) and (3.11):

(3.27) $$f^i z_{k,\ell} = f^{-1}(f^{i+1} z_{k,\ell}) \in f^{-1} B_\ell^{i+1} = B_\ell^i \cap C(f^i x, h).$$

Since B_ℓ is an admissible (u, h)-manifold near the point x it follows from Proposition (2.4) (i) that the manifold $B_\ell^i \cap C(f^i x, h)$ is a part of a manifold from the class:

$$U_{f^i x}^{\lambda^i \gamma, \frac{\varepsilon h}{4}\left(\frac{1+\lambda}{2}\right)^i, h}.$$

Therefore, we have from Proposition (2.4) (ii) and (3.27) using (3.25) and (3.4):

$$\begin{aligned} d_x'(z_{k_1-1,\ell+1}, z_{k_2-1,\ell+1}) &\geq \tau d_{f^n x}'(z_{k_1-1,\ell+1}, z_{k_2-1,\ell+1}) \\ &\geq \tau \left(\frac{1}{2} + \frac{1}{2\lambda}\right)^n d_x'(z_{k_1,\ell}, z_{k_2,\ell}) \\ &\geq \lambda' d_x'(z_{k_1,\ell}, z_{k_2,\ell}). \end{aligned}$$

(Recall that $\varepsilon \cdot \left(\frac{1+\lambda}{2}\right)^i < \varepsilon(f^i x)$.)

Similarly, we have for $k \geq 0$, $\ell_1, \ell_2 \geq 1$:

$$d'_x(z_{k,\ell_1}, z_{k,\ell_2}) \leq \lambda' d'_x(z_{k+1,\ell_1-1}, z_{k+1,\ell_2-1}) \tag{3.28}$$

Step 5. — Now we are going to prove that:

$$\lim_{k\to\infty} d'_x(z_{k,k-1}, z_{k-1,k}) = 0 \tag{3.29}$$

and:

$$\sum_{k=1}^{\infty} d'_x(z_{k+1,k}, z_{k,k-1}) < \infty. \tag{3.30}$$

We have:

$$d'_x(z_{k,k-1}, z_{k-1,k}) \leq d'_x(z_{k,k-1}, z_{k-1,k-1}) + d'_x(z_{k-1,k-1}, z_{k-1,k}).$$

We shall estimate each term in the right-hand part of this inequality.

The points $z_{k,k-1}$ and $z_{k-1,k-1}$ belong to the manifold B_{k-1}. From (3.21) we have:

$$f^{n(k-1)} z_{k-1,k-1} = z_{0,2k-2} \in B_{2k-2}$$

$$f^{n(k-1)} z_{k,k-1} = z_{1,2k-2} \in B_{2k-2}.$$

For every $i = 0, \ldots, k-2$ inequality (3.26) gives:

$$d'_x(z_{k-1-i,k-1+i}, z_{k-i,k-1+i}) \leq \lambda' d'_x(z_{k-2-i,k+i}, z_{k-1-i,k+i})$$

and consequently:

$$d'_x(z_{k,k-1}, z_{k-1,k-1}) \leq (\lambda')^{k-1} d'_x(z_{1,2k-2}, z_{0,2k-2}) \leq 2\varepsilon h(\lambda')^{k-1}. \tag{3.31}$$

Similarly, using (3.28) instead of (3.26) we obtain:

$$\begin{aligned} d'_x(z_{k-1,k-1}, z_{k-1,k}) &\leq (\lambda')^{k-1} d'_x(z_{2k-2,1}, z_{2k-2,0}) \\ &\leq 2\varepsilon h(\lambda')^{k-1}. \end{aligned} \tag{3.32}$$

Since $\lambda' < 1$ (3.29) follows immediately from (3.31) and (3.32). The same two inequalities imply (3.30) because:

$$\begin{aligned} d'_x(z_{k+1,k}, z_{k,k-1}) &\leq d'_x(z_{k+1,k}, z_{k,k}) + d'_x(z_{k,k}, z_{k,k-1}) \\ &\leq 4\varepsilon h(\lambda')^k. \end{aligned}$$

It follows from (3.30) that the sequence $z_{k,k-1}$, $k = 1, 2, \ldots$ converges as $k \to \infty$ to some point $z \in \hat{C}(x, h)$.

Since by (3.21) $z_{k-1,k} = f^n z_{k,k-1}$ we have from (3.29):

$$f^n z = \lim_{k\to\infty} f^n z_{k,k-1} = \lim_{k\to\infty} z_{k-1,k} = \lim_{k\to\infty} z_{k,k-1} = z.$$

Thus, we have proved (i).

Since f^i is a continuous map then by (3.27):

$$f^i z = \lim f^i z_{k,k-1} \in C(f^i x, h) \tag{3.33}$$

and consequently:

(3.34) $d'_{f^i x}(f^i x, f^i z) \leq 2h\varepsilon(f^i x) \leq 2hr_0$.

It follows from the compactness of M and Proposition (2.3) (i) that there exists a constant $K>0$ such that for every point $y \in \Lambda = \bigcup_{k,\chi,\varepsilon} \Lambda^k_{\chi,\ell}$ and every two points $w_1 w_2 \in C(y, 1)$:

(3.35) $d'_y(w_1 w_2) > Kd(w_1 w_2)$.

Therefore, from (3.34) and (3.35):

$$d'_n(x, z) \leq \max_{0 \leq i \leq n-1} d(f^i x, f^i z) \leq 2hr_0 K^{-1}.$$

Taking:

(3.36) $h < \dfrac{\delta K}{2r_0}$

we obtain the statement (ii).

Step 6. — In this section we shall prove that df^n_z is a hyperbolic linear operator. For $0<\beta<1$ let us denote by K_β and L_β the following cones in $\mathbf{R}^s$:

$$K_\beta = \{(w_1, w_2) \in \mathbf{R}^k \times \mathbf{R}^{s-k} : \|w_1\| \leq \beta \|w_2\|\}$$
$$L_\beta = \{(w_1, w_2) \in \mathbf{R}^k \times \mathbf{R}^{s-k} : \|w_2\| \leq \beta \|w_1\|\}.$$

It follows from the proof of Proposition (2.4) (the section starting from (2.9) through (2.10) and below) that for $x \in \Lambda^k_{\chi,\ell}$, $(u, v) \in B^k_{\varepsilon(x)} \times B^{s-k}_{\varepsilon(x)}$:

(3.37) $(df_x)_{(u,v)} K_\gamma \subset K_{\lambda\gamma}$.

Moreover, for $w \in K_\gamma$:

(3.38) $\|(df_x)_{(u,v)} w\| \geq \left(\dfrac{1}{2} + \dfrac{1}{2\lambda}\right)\|w\|$.

The proof of (3.38) is similar to the proof of part (ii) of Proposition (2.4). Since $w \in K_\gamma$ we have $\|w\| < (1-\gamma)^{-1}\|w_2\|$. Furthermore, we have from (2.4) and (2.7):

$$\begin{aligned}
\|(df_x)_{(u,v)}(w_1, w_2)\| &= \|(A_x w_1, B_x w_2) + (dh_x)_{(u,v)}(w_1, w_2)\| \\
&\geq \|B_x w_2\| - \|A_x w_1\| - \|(dh_x)_{(u,v)}\| \cdot \|w\| \\
&\geq \lambda^{-1}(1-\gamma) \cdot \|w\| - \lambda\gamma\|w\| - \frac{(1-\lambda)^2}{100}\|w\| \\
&= \left(\lambda^{-1} - \left(\frac{\lambda^{-1}(1-\lambda)}{20} - \frac{\lambda(1-\lambda)}{20} - \frac{(1-\lambda)^2}{100}\right)\right)\|w\| \\
&\leq \left(\frac{1}{2} + \frac{1}{2\lambda}\right)\|w\|.
\end{aligned}$$

similarly, we have for f^{-1}:

(3.39) $$(df_x^{-1})_{(u,v)}L_\gamma \subset L_{\lambda\gamma}$$

and for $w \in L_\gamma$:

(3.40) $$\|(df_x^{-1})_{(u,v)}w\| > \left(\frac{1}{2}+\frac{1}{2\lambda}\right)\|w\|.$$

Let us now consider the periodic point z constructed in the previous section. Inclusion (3.33) shows that, for $i=0, \ldots, n-1$, we have $f^i z = \Phi_{f^i x}(u_i, v_i)$ for some $(u_i, v_i) \in B^k_{\varepsilon(f^i x)} \times B^{s-k}_{\varepsilon(f^i x)}$. Let us set:

$$F^{(n)}_{x,z} = (df_{f^{n-1}x})_{(u_{n-1}, v_{n-1})} \circ \ldots \circ (df_x)_{(u_0, v_0)}.$$

Applying (3.37) inductively for (u_0, v_0), $(u_1, v_1) = f_x(u_0, v_0)$, etc., we obtain:

(3.41) $$F^{(n)}_{x,z} K_\gamma \subset K_{\lambda^n \gamma}$$

and from (3.37) and (3.38) we can see that for $w \in K_\gamma$:

(3.42) $$\|F^{(n)}_{x,z} w\| > \left(\frac{1}{2}+\frac{1}{2\lambda}\right)^n \|w\|.$$

Similarly, from (3.39) and (3.40) we have:

(3.43) $$(F^{(n)}_{x,z})^{-1} L_\gamma \subset L_{\lambda^n \gamma}$$

and for $w \in L_\gamma$:

(3.44) $$\|(F^{(n)}_{x,z})^{-1} w\| > \left(\frac{1}{2}+\frac{1}{2\lambda}\right)^n \|w\|.$$

The following equalities follow directly from the definition of Φ_x and f_x (cf. Proposition (2.3) (ii)):

$$df^n_z = (d\Phi_{f^n x})_{(u_n, v_n)} \circ F^{(n)}_{x,z} \circ (d\Phi_x^{-1})_z$$
$$= (d\Phi_{f^n x})_{(u_n, v_n)} \circ (d\Phi_x^{-1})_z \circ (d\Phi_x)_{(u_0, v_0)} \circ F^{(n)}_{x,z} (d\Phi_x^{-1})_z.$$

Let us denote for $0<\beta<1$:

$$\widetilde{K}_\beta = (d\Phi_x)_{(u_0, v_0)} K_\beta$$
$$\widetilde{L}_\beta = (d\Phi_x)_{(u_0, v_0)} L_\beta.$$

Since $d\Phi_x$ transforms the Euclidean norm in $\mathbf{R}^s$ into the norm $\|\cdot\|'_x$ the properties similar to (3.41)-(3.44) with the cones $\widetilde{K}_\gamma$ and $\widetilde{L}_\gamma$ instead of K_γ and L_γ and with the norm $\| \ \|'_x$ instead of the Euclidean norm in $\mathbf{R}^n$ take place for the operator:

$$(d\Phi_x)_{(u_0, v_0)} \circ F^{(n)}_{x,z} \circ (d\Phi_x^{-1})_z : T_z M \to T_z M.$$

The operator

$$(d\Phi_{f^n x})_{(u_0, v_0)} \circ (d\Phi_x^{-1})_z : T_z M \to T_z M$$

transforms the norm $||\cdot||'_x$ into $||\cdot||'_{f^n x}$ and the decomposition

$$(d\Phi_x)_{(u_0,\, v_0)}\mathbf{R}^k \times (d\Phi_x)_{(u_0,\, v_0)}\mathbf{R}^{s-k}$$

into
$$(d\Phi_{f^n x})_{(u_n,\, v_n)}\mathbf{R}^k \times (d\Phi_{f^n x})_{(u_n,\, v_n)}\mathbf{R}^{s-k}.$$

It follows from Proposition (2.3), parts (iii) and (iv) that if the number ψ in (3.2) is chosen sufficiently small then:

$$(d\Phi_{f^n x})_{(u_n,\, v_n)}\mathrm{K}_{\lambda\gamma} \subset \widetilde{\mathrm{K}}_{\lambda^{1/2}\gamma} \tag{3.45}$$

$$(d\Phi_{f^n x})_{(u_n,\, v_n)}\mathrm{L}_{\lambda\gamma} \subset \widetilde{\mathrm{L}}_{\lambda^{1/2}\gamma} \tag{3.46}$$

and consequently:

$$df_z^n \widetilde{\mathrm{K}}_\gamma \subset \widetilde{\mathrm{K}}_{\lambda^{1/2}\gamma} \tag{3.47}$$

$$df_z^{-n} \widetilde{\mathrm{L}}_\gamma \subset \widetilde{\mathrm{L}}_{\lambda^{1/2}\gamma}. \tag{3.48}$$

Moreover, (3.9) together with (3.42) and (3.44) guarantee that for $w \in \widetilde{\mathrm{K}}_\gamma$:

$$||df_z^n w||'_x > \left(1 + \frac{1-\lambda}{2}\right)||w||'_x \tag{3.49}$$

and for $w \in \widetilde{\mathrm{L}}_\gamma$:

$$||df_z^{-n} w||'_x > \left(1 + \frac{1-\lambda}{2}\right)||w||'_x. \tag{3.50}$$

Standard arguments (which we do not reproduce) show that:

$$\mathrm{H}_1 = \bigcap_{k=0}^{\infty} df_z^{kn} \widetilde{\mathrm{K}}_{\gamma(x)}$$

and:

$$\mathrm{H}_2 = \bigcap_{k=0}^{\infty} df_z^{-kn} \widetilde{\mathrm{L}}_{\gamma(x)}$$

are respectively an $(s$-$k)$-dimensional subspace and a k-dimensional subspace of $\mathrm{T}_z\mathrm{M}$ invariant with respect to df_z^n. Since $\mathrm{H}_1 \cap \mathrm{H}_2 = \{0\}$ we have:

$$\mathrm{T}_z\mathrm{M} = \mathrm{H}_1 \oplus \mathrm{H}_2.$$

Obviously $\mathrm{H}_1 \subset \widetilde{\mathrm{K}}_{\gamma(x)}$, $\mathrm{H}_2 \subset \widetilde{\mathrm{L}}_{\gamma(x)}$ so we can apply (3.49) and (3.50) and conclude that the spectrum of $df_z^n\big|_{\mathrm{H}_1}$ lies outside the unit circle Δ and the spectrum of $df_z^n\big|_{\mathrm{H}_2}$ lies inside Δ. Therefore, df_z^n is a hyperbolic linear operator.

Step 7. — Finally we shall prove the statement about local stable and unstable manifolds. We explain in detail the case of stable manifolds; unstable manifolds are treated similarly. Let us construct the manifolds A_k and B_k, $k = 0, 1, \ldots$ for $h = 1$. Obviously, they are extensions of corresponding manifolds constructed for smaller h.

The set S_x of all $(s, 1)$-admissible manifolds near x can be provided a C^0-topology in the following way. Let $\mathrm{W}_1, \mathrm{W}_2 \in \mathrm{S}_x$:

$$\mathrm{W}_i = \Phi_x(\text{graph } \varphi_i), \quad i = 1, 2.$$

Then the distance:

$$\rho_0(W_1, W_2) = \max_{u \in B^k_{\varepsilon/2}} \|\varphi_1(u) - \varphi_2(u)\|.$$

Obviously, the closure $\overline{S}_x$ in this topology consists of all C^0-manifolds W of the form:

$$W = \{\Phi_x \text{ graph } \varphi : \varphi \in C^0(B^k_{\varepsilon/2}, B^{s-k}_{\varepsilon/2}), \|\varphi(0)\| \leq \frac{\varepsilon}{4},$$
$$\|\varphi(u_1) - \varphi(u_2)\| \leq \gamma \|u - u_2\|, \forall\, u_1, u_2 \in B^k_{\varepsilon/2}\}.$$

This closure is a compact set. Consequently, the sequence $\{A_m\}$, $m = 1, 2, \ldots$ contains a subsequence $\{A_{m_\ell}\}$, $\ell = 1, 2, \ldots$ $m_\ell \to \infty$, such that A_{m_ℓ} converges in the C^0-topology to some manifold $A \subset \overline{S}_x$.

Let $w \in A$. I shall prove that

(*) $f^{nm} w \in \hat{C}(x, 1)$ for $m = 1, 2, \ldots$;

(**) for some constants $K > 0$, $\tilde{\lambda} < 1$:

$$d(f^{mn} w, z) < K(\tilde{\lambda})^{mn} d(w, z).$$

Let us fix m and find a sequence of points $w_\ell \in A_{m_\ell}$ such that $w = \lim_{\ell \to \infty} w_\ell$.

If $m_\ell \geq m$ then by (3.10):

$$f^{mn} w \in A_{m_\ell - m} \subset \hat{C}(x, 1) \tag{3.51}$$

which implies (*).

We have from the statement similar to Proposition (2.4) (ii), Proposition (2.3) (i) and (3.9):

$$\begin{aligned} d(f^{mn} w, z) &= d(f^{mn} w, f^{mn} z) \leq K_1^{-1} d'_x(f^{mn} w, f^{mn} z) \\ &\leq K_1^{-1} \left(1 + \frac{1-\lambda}{2}\right)^{-m} d'_x(w, z). \\ &\leq K_1^{-1} K_2 A(x) \left(1 + \frac{1-\lambda}{2}\right)^{-m} d(x, z). \end{aligned} \tag{3.52}$$

To verify (**) it is enough to set in (3.52) $K = K_1^{-1} K_2 A(x)$ $\tilde{\lambda} = \left(1 + \frac{1-\lambda}{2}\right)^{-1}$. It follows from (*) and (**) that A is contained in the local stable manifold $V^s(z)$ of the point z for f^n.

Since $T_z V^s(z) = H_2$ and $(d\Phi_x^{-1})_z H_2 \subset K_\gamma$ we can conclude that locally near z the manifold $V^s(z)$ has the following form:

$$V^s(z) = \Phi_x(\text{graph } \varphi)$$

where φ is a C^1 function defined in a neighborhood of u_0 with the values from $\mathbf{R}^{s-k}$ and $\|d\varphi_{u_0}\| \leq \gamma$.

Since the manifold $\Phi_x^{-1}A$ has the same form we conclude that, locally, A coincides with $V^s(z)$. Since the extension of any arbitrarily small piece of the local stable manifold is unique we come to the conclusion that A is a local stable manifold. ■

Remark. — The above arguments actually show that the sequence $\{A_m\}$, $m=1, 2, \ldots$ converges in the C^0-topology because any limit point of that sequence may serve as the manifold A in that argument and the local stable manifold is unique.

4. Proof of the main results.

Theorem **(4.1)**. — *Let f be a $C^{1+\alpha}$ ($\alpha>0$) diffeomorphism of a compact manifold* M, *and μ a Borel probability f-invariant measure with non-zero Lyapunov exponents. Then* $\overline{\operatorname{Per} f} \supset \operatorname{supp} \mu$.

Proof. — Let us fix some smooth Riemannian metric on M with distance function $d(\ ,\ )$ and denote by $B(x, r)$ the r-ball around the point $x\in M$. To prove the theorem we shall show how to find a periodic point in the ball $B(x_0, \varepsilon)$ for a given point $x_0\in \operatorname{supp} \mu$ and a number $\varepsilon>0$.

First, we can find numbers k, χ, ℓ such that:

$$\mu\left(B\left(x_0, \frac{\varepsilon}{4}\right)\cap \Lambda^k_{\chi,\ell}\right)>0$$

(cf. Corollary (2.1)) and define the number:

$$\psi=\psi\left(k, \chi, \ell, \frac{\varepsilon}{4}\right)>0$$

satisfying the assertion of the Main Lemma. Let B be a subset of the intersection $B\left(x_0, \frac{\varepsilon}{4}\right)\cap \Lambda^k_{\chi,\ell}$ such that $\mu(B)>0$ and the diameter of B is less than ψ. By the Poincaré recurrence theorem, for almost every point $x\in B$ there exists a positive integer $n(x)$ such that $f^{n(x)}x\in B$ and consequently $d(x, f^{n(x)}x)<\psi$. Since $B\subset \Lambda^k_{\chi,\ell}$ we can apply the Main Lemma and find a point z of period $n(x)$ such that $d(x, z)<\frac{\varepsilon}{4}$. Obviously:

$$d(x_0, z)<d(x_0, x)+d(x, z)<\frac{\varepsilon}{2}. \quad ■$$

Let us set $\operatorname{Per}_h(f)=\{x\in \operatorname{Per} f : x$ is hyperbolic and has a transversal homoclinic point$\}$.

Theorem **(4.2)**. — *If in addition to the assumptions of Theorem* (4.1) *the measure μ is ergodic and not concentrated on a single periodic trajectory then* $\overline{\operatorname{Per}_h f}\supset \operatorname{supp} \mu$.

Proof. — First, let us show that the Lyapunov exponents of μ cannot be of the same sign. If so we can suppose (taking f^{-1} instead of f, if necessary) that all exponents are negative. Suppose that x is a recurrent point of f (*i.e.* $f^{n_k}x\to x$ for some sequence $n_k\to\infty$)

and that $x \in \Lambda^k_{\chi,\ell}$ for k, χ from (2.1) and some ℓ. Then there exists a positive integer m and $\varepsilon_1 > 0$ such that f^m maps the disc $B(x, \varepsilon_1)$ into itself and $f^m\big|_{B(x,\varepsilon_1)}$ is a contracting map. Consequently the points $f^{km}x$ tend to some point y as k tends to infinity. Obviously $f^m y = y$. If $y \neq f^s x$ for some integer s then x is not a recurrent point. Consequently in this case almost all recurrent points are periodic. But since almost all points are recurrent and μ is an ergodic measure it has to be concentrated on a single periodic trajectory.

Now we are able to follow the line of the proof of theorem (4.1) but instead of a single set B we take two different points $x_1, x_2 \in M$ with the following properties:

(i) $x_1, x_2 \in B\left(x_0, \frac{\varepsilon}{4}\right)$;

(ii) there exists ℓ such that for any $\delta > 0$ $\mu(\Lambda^k_{\chi,\ell} \cap B(x_i, \delta)) > 0$, $i = 1, 2$,

(iii) $d(x_1, x_2) < \frac{1}{2} \min\left(\frac{\varepsilon}{4}, \varkappa\left(k, \chi, \ell, \frac{1+\lambda(\chi)}{8}, h\right)\right)$ where the number $\varkappa$ is found from Corollary (2.2).

Such two points exist because μ is a continuous measure.

Let us now take subsets:

$$B_1 \subset \Lambda^k_{\chi,\ell} \cap B\left(x_1, \frac{d(x_1, x_2)}{10}\right), \quad B_2 \subset \Lambda^k_{\chi,\ell} \cap B\left(x_2, \frac{d(x_1, x_2)}{10}\right)$$

such that for $i = 1, 2$, $\mu(B_i) > 0$ and:

$$\operatorname{diam} B_i < \psi\left(k, \chi, \ell, \frac{d(x_1, x_2)}{100}\right).$$

Using the Poincaré recurrence theorem we can find points $y_1 \in B_1$, $y_2 \in B_2$ and positive integers $n(y_1)$, $n(y_2)$ such that $f^{n(y_1)} y_1 \in B_1$, $f^{n(y_2)} y_2 \in B_2$. Therefore, we can apply the Main Lemma and find periodic points z_1, z_2 such that:

$$d(z_i, y_i) < \frac{d(x_1, x_2)}{100}, \quad i = 1, 2.$$

Let us estimate the distance between z_1 and z_2. Evidently:

$$\begin{aligned} d(x_1, x_2) - d(y_1, x_1) - d(z_1, y_1) - d(y_2, x_2) - d(z_2, y_2) \\ \leq d(z_1, z_2) \\ \leq d(x_1, x_2) + d(y_1, x_1) + d(z_1, y_1) + d(y_2, x_2) + d(z_2, y_2) \end{aligned}$$

whence:

$$\frac{1}{2} d(x_1, x_2) < d(z_1, z_2) < \frac{3}{2} d(x_1, x_2).$$

The inequality on the left shows that $z_1 \neq z_2$, the one on the right together with (iii) and the last statement of the Main Lemma guarantee that the stable manifold $W^s(z_1)$

has a point of transversal intersection with $W^u(z_2)$ and the unstable manifold $W^u(z_1)$ with $W^s(z_2)$. Indeed, local pieces of $W^s(z_1)$ and $W^s(z_2)$ are $(s, 1)$-admissible manifolds near the point y_1, and local pieces of $W^u(z_1)$ and $W^u(z_2)$ are $(u, 1)$-admissible manifolds near the same point. By Proposition (2.5), $W^u(z_1)$ has a point of transversal intersection with $W^s(z_2)$, and $W^s(z_1)$ with $W^u(z_2)$. It is well known that the existence of such two points of transversal intersection guarantees the existence of transversal homoclinic points for z_1 and z_2. ■

Corollary **(4.1)**. — *Under the assumptions of Theorem* (4.2) *the diffeomorphism f has a closed invariant hyperbolic set Γ such that the restriction of f to Γ is topologically conjugate to a topological Markov chain (subshift of finite type) and* $h(f|_\Gamma)>0$.

This fact follows immediately from Theorem (4.2) and the existence of such a set Γ in any neighborhood of the trajectory of any transversal homoclinic point [20].

As an immediate consequence of this fact we obtain something like an estimation of the topological entropy from below:

Corollary **(4.2)**. — *If a $C^{1+\alpha}$ diffeomorphism f of a compact manifold has a Borel probability invariant continuous non-atomic ergodic measure with non-zero Lyapunov exponents then* $h(f)>0$.

If $\dim M=2$ then the converse of Corollary (4.2) is true. Combining this remark with Corollary (4.1) we obtain the following result:

Corollary **(4.3)**. — *Any $C^{1+\alpha}$ diffeomorphism of a two-dimensional manifold with positive topological entropy has an invariant set as described in Corollary* (4.1).

Proof. — Since $h(f)=\sup h_\mu(f)$ where sup is taken over the set of all Borel probability f-invariant measures (or only over the set of ergodic measures) we can find an ergodic invariant measure μ with positive entropy. Such a measure is obviously continuous. Let $\chi_1\geq\chi_2$ be the Lyapunov exponents of μ. Since the entropy is less than or equal to the sum of positive Lyapunov exponents then $\chi_1>0$. Since $h_\mu(f^{-1})=h_\mu(f)>0$ then $-\chi_2>0$, *i.e.*, $\chi_2<0$. Consequently we can apply Corollary (4.1). ■

The presented results give some rather qualitative information about the set of periodic points. The next theorem gives an estimation of the asymptotic growth of the numbers of periodic points.

Theorem **(4.3)**. — *With the assumptions of Theorem* (4.1):

$$\max\left(0, \overline{\lim_{n\to\infty}} \frac{\ln P_n(f)}{n}\right)\geq h_\mu(f).$$

Proof. — We can assume that μ is an ergodic measure. In this case we shall construct for every positive numbers ε, ℓ and every positive integer n a finite set

$K_n = K_n(\varepsilon, \ell)$ satisfying the following four properties (for some n the set $K_n(\varepsilon, \ell)$ may be empty):

1. $K_n \in \Lambda^k_{\chi,\ell}$, where χ and k are defined by (2.1).
2. If $x, y \in K_n$ and $x \neq y$ then $d^\ell_n(x, y) > \varepsilon(k, \chi, \ell) \cdot \ell^{-1}$ (cf. (2.8)).
3. For every $x \in K_n$ there exists a number $m(x)$: $n \leq m(x) \leq (1+\varepsilon)n$ such that $f^{m(x)} x \in \Lambda^k_{\chi,\ell}$ and:

$$d(x, f^m x) \leq \psi\left(k, \chi, \ell, \frac{\varepsilon(k, \chi, \ell)}{10\ell}\right).$$

4. For every $\varepsilon > 0$:

$$\varliminf_{\ell\to\infty} \varliminf_{n\to\infty} \frac{\ln \operatorname{Card} K_n(\varepsilon, \ell)}{n} \geq h_\mu(f).$$

To proceed to the construction of the sets K_n let us choose a finite measurable partition ξ such that:

$$\operatorname{diam} \xi < \psi\left(k, \chi, \ell, \frac{\varepsilon(k, \chi, \ell)}{10\ell}\right) \quad \text{and} \quad \xi > \{\Lambda^k_{\chi,\ell}, M \setminus \Lambda^k_{\chi,\ell}\}.$$

The last condition means that every element of ξ either belongs to the set $\Lambda^k_{\chi,\ell}$ or is disjoint from this set. Let us set:

$$\Lambda^{k,n}_{\chi,\ell} = \{x \in \Lambda^k_{\chi,\ell} : \exists\, m : n \leq m \leq (1+\varepsilon)n \text{ such that the points } x \text{ and } f^m x \text{ belong to the same element of } \xi\}.$$

We define the set K_n as a maximal subset of the set $\Lambda^{k,n}_{\chi,\ell}$ satisfying the separation property 2. The properties 1, 2, 3 are true by definition. Let us check 4.

Lemma. — $\lim_{n\to\infty} \mu(\Lambda^{k,n}_{\chi,\ell}) = \mu(\Lambda^k_{\chi,\ell})$.

Proof of the lemma. — We fix an element $c \in \xi$ belonging to the set $\Lambda^k_{\chi,\ell}$ and set:

$$c_{n,\varepsilon} = \left\{x \in c : \sum_{k=0}^{n-1} \chi_c(f^k x) < n\mu(c)\left(1 + \frac{\varepsilon}{3}\right), \sum_{k=0}^{[n(1+\varepsilon)]} \chi_c(f^k x) > n\mu(c)\left(1 + \frac{2\varepsilon}{3}\right)\right\}$$

where χ_c is a characteristic function of the set c.

Obviously $c_{n,\varepsilon} \subset \Lambda^{k,n}_{\chi,\ell} \cap c$. By the ergodic theorem we have $\mu(c \setminus c_{n,\varepsilon}) \to 0$. Applying these arguments to every element $c \in \xi$ belonging to $\Lambda^k_{\chi,\ell}$ we obtain that

$$\mu(\Lambda^{k,n}_{\chi,\ell}) \to \mu(\Lambda^k_{\chi,\ell}). \quad \blacksquare$$

Since K_n is a maximal subset of $\Lambda^{k,n}_{\chi,\ell}$ having the property 2, the union of $\varepsilon(k, \chi, \ell) . \ell^{-1}$-balls in the d^ℓ_n-metric around points of K_n covers the set $\Lambda^{k,n}_{\chi,\ell}$. Otherwise we could add any uncovered point of $\Lambda^{k,n}_{\chi,\ell}$ to K_n and produce the greater set with the same property. Consequently by the definition of the numbers $N_f(n, \varepsilon, \delta)$ (see § 1) we have:

(4.1) $$\operatorname{Card} K_n \geq N_f(n, \varepsilon(k, \chi, \ell) . \ell^{-1}, 1 - \mu(\Lambda^{k,n}_{\chi,\ell})).$$

Using the lemma we conclude that for every $\delta > 1 - \mu(\Lambda^{k,n}_{\chi,\ell})$:

$$\underline{\lim}_{n\to\infty} \frac{\ln \operatorname{Card} K_n(\varepsilon, \ell)}{n} \geq \underline{\lim}_{n\to\infty} \frac{\ln N_f(n, \varepsilon(k, \chi, \ell).\ell^{-1}, \delta)}{n}.$$

Thus the property 4 follows from Theorem (1.1).

Having the sets $K_n(\varepsilon, \ell)$ we can finish the proof of the theorem. For every point $x \in K_n$ we can by the Main Lemma find a periodic point $z = z(x)$ of period $m(x)$. If $x, y \in K_n$ and $x \neq y$ then:

$$\begin{aligned} d^f_n(z(x), z(y)) &\geq d^f_n(x,y) - d^f_n(x, z(x)) - d^f_n(y, z(y)) \\ &\geq \frac{2}{3}\varepsilon(k, \chi, \ell)\cdot \ell^{-1} \end{aligned} \tag{4.2}$$

so that the points $z(x)$ and $z(y)$ are different. Consequently:

$$\sum_{m=n}^{[(1+\varepsilon)n]} P_m(f) \geq \operatorname{Card} K_n(\varepsilon, \ell)$$

and:

$$\max_{n \leq m \leq (1+\varepsilon)n} P_m(f) \geq \frac{\operatorname{Card} K_n(\varepsilon, \ell)}{\varepsilon n}.$$

Thus, we can find a sequence of integers $m_n : n \leq m_n \leq (1+\varepsilon)n$, such that:

$$\begin{aligned} \underline{\lim}_{n\to\infty} \frac{\ln P_{m_n}(f)}{m_n} &\geq \underline{\lim}_{n\to\infty} \frac{n}{m_n}\left(\frac{\ln \operatorname{Card} K_n(\varepsilon, \ell) - \ln \varepsilon n}{n}\right) \\ &\geq \frac{1}{1+\varepsilon}\left(\underline{\lim}_{n\to\infty} \frac{\ln \operatorname{Card} K_n(\varepsilon, \ell)}{n}\right) = \frac{1}{1+\varepsilon}(h_\mu(f) - \varphi(\ell)) \end{aligned} \tag{4.3}$$

where by the property 4:

$$\overline{\lim}_{\ell\to\infty} \varphi(\ell) \leq 0. \quad \blacksquare$$

Corollary **(4.4)**. — *For every* $C^{1+\alpha}$ $(\alpha > 0)$ *diffeomorphism* f *of a 2-dimensional manifold*:

$$\max\left(0, \overline{\lim}_{n\to\infty} \frac{\ln P_n(f)}{n}\right) \geq h(f).$$

Proof. — We can suppose that $h(f) > 0$ since if $h(f) = 0$ the inequality is obviously true. Then for every $\varepsilon > 0$ we can find a Borel probability f-invariant ergodic measure μ such that $h_\mu(f) > h(f)(1-\varepsilon) > 0$. In the proof of Corollary (4.3) we have shown that one of the Lyapunov exponents of μ is positive and the other is negative. So we can apply Theorem (4.3) and conclude that for every $\varepsilon > 0$:

$$\overline{\lim}_{n\to\infty} \frac{\ln P_n(f)}{n} \geq h_\mu(f) > h(f)(1-\varepsilon). \quad \blacksquare$$

The next fact shows that we can approximate the entropy of any "good" invariant measure by the entropies of limit distributions of periodic points. Let us set:

$$L_n(\varepsilon, \ell)=\{z(x) : x\in K_n(\varepsilon, \ell)\}.$$

Thus, $L_n(\varepsilon, \ell)$ is a set of periodic points with periods between n and $(1+\varepsilon)n$. Let further $\mu_n(\varepsilon, \ell)$ be a uniform measure concentrated on the set $L_n(\varepsilon, \ell)$.

Corollary **(4.5).** — *For any condensation point* $\hat{\mu}$ *of the sequence of measures* $\mu_n(\varepsilon, \ell)$ *in the weak topology we have*:

$$h_{\hat{\mu}}(f)\geq h_\mu(f)-\varphi(\ell).$$

(For the definition of the function $\varphi(\ell)$ *see* (4.3).*)*

Proof. — Let ξ be a finite measurable partition of M such that:

(i) $$\operatorname{diam}\xi<\frac{\varepsilon(k, \chi, \ell)}{2\ell}$$

and

(ii) $$\hat{\mu}(\partial\xi)=0.$$

Condition (i) and inequality (4.2) show that every element of the partition ξ_{-n} contains at most one point of the set $L_n(\varepsilon, \ell)$. Therefore:

(4.4) $$H_{\mu_n}(\xi_{-n})=\ln \operatorname{Card} L_n(\varepsilon, \ell)=\ln \operatorname{Card} K_n(\varepsilon, \ell).$$

Suppose that $\mu_{n_k}\to\hat{\mu}$ in the weak topology. Property (ii) implies that for every positive integer m:

(4.5) $$\lim_{k\to\infty} H_{\mu_{n_k}}(\xi_{-m})=H_{\hat{\mu}}(\xi_{-m}).$$

If $m<n_k$ then:

(4.6) $$\frac{H_{\mu_{n_k}}(\xi_{-m})}{m}\geq\frac{H_{\mu_{n_k}}(\xi_{-n_k})}{n_k}.$$

Combining (4.4), (4.5), (4.6) with the property 4 of the sets $K_n(\varepsilon, \ell)$ we have:

$$\frac{H_{\hat{\mu}}(\xi_{-m})}{m}\geq\lim_{k\to\infty}\frac{\ln \operatorname{Card} K_{n_k}(\varepsilon, \ell)}{n_k}\geq h_\mu(f)-\varphi(\ell). \quad ■$$

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