PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

ROBERT F. WILLIAMS

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Publications mathématiques de l'I.H.É.S., tome 50 (1979), p. 73-99
<http://www.numdam.org/item?id=PMIHES_1979__50__73_0>

THE STRUCTURE OF LORENZ ATTRACTORS

by R. F. WILLIAMS (1)

Dedicated to the memory of Rufus Bowen

Introduction.

The system of equations

$$(L) \qquad \begin{cases} \dot{x} = -10x + 10y \\ \dot{y} = 28x - y - xz \\ \dot{z} = -\frac{8}{3}z + xy \end{cases}$$

of E. N. Lorenz [7] has attracted much attention ([3], [10], [12]) lately, in part because of its relation to turbulence. Lorenz obtained this system by " truncating " the Navier-Stokes equation; it offers a striking example of a strange attractor, vis-à-vis Ruelle-Takens [11].

We present the Ruelle-Takens idea briefly. In order that any type of motion be observable, the set of initial conditions leading to this motion must be of positive measure. This essentially says that the motion must be bound to an attractor. Until recently, mathematicians knew of only two types—steady state attractors (or sinks) and periodic attractors. Thus when a *persistent* motion was seen to be neither steady state nor periodic, it was termed " random " or " chaotic ", and stochastic mathematics was invoked. It is just this *non sequitur* that Lorenz was attacking; his article is entitled " Deterministic aperiodic motion " (1963).

Though many scientists, especially experimentalists, knew this article, it is not too surprising that most mathematicians did not, considering for example where it was published. Thus, when Ruelle-Takens proposed (1971) *specifically* that turbulence was likely an instance of a " strange attractor ", they did so without specific solutions of the Navier-Stokes equations, or truncated ones, in mind. This proposal, controversial at first, has gained much favor.

In particular, the paper of Guckenheimer (see below) gives a geometric description

(1) This research partially supported by NSF Grant No. MPS74-06731 AO1.

of what seems to be going on in the system (L) and it is indeed a strange attractor. (To prove this, one would have to make certain estimates; meanwhile computer printouts surely indicate this is about right.) This aids the advocate of strange attractors in two ways: it adds a fairly simple example to our knowledge, and at the same time, one that comes up *naturally*. Meanwhile, the estimates needed to tie the system (L) to the geometric work of Guckenheimer or the present paper have not been made. Though the current work is of independent interest, it would certainly be enhanced by such a direct connection. We begin by summarizing a theorem of [3].

Theorem (Guckenheimer). — *There is an open set* $\mathscr{L}$ *in* $\mathscr{H}$, *the space of all vector fields on* $\mathbf{R}^3$, *such that*:

1) *if* $X \in \mathscr{L}$, *then* X *has a two dimensional attractor* (herein called *Lorenz attractor*) *which contains a singular point*;
2) *there are two dense subsets* $\mathscr{R}, \mathscr{I} \subset \mathscr{L}$ *such that the attractors for* X *in* $\mathscr{R}$ *are topologically distinct from those for* $Y \in \mathscr{I}$.

Here we improve upon Guckenheimer's result by showing that there are uncountably many topologically mutually distinct Lorenz attractors. Therefore this answers in the negative a question asked by R. Thom [13]. In particular, we show that the obvious " kneading sequences " are invariant under homeomorphisms near the identity. Briefly, these sequences tell to which side of the singular point its own unstable manifold passes, in its various " trips " around the attractor.

In the process of proving this, we develop a cell-structure of Lorenz attractors, and a singular fibration into a figure eight space, B_0. We proceed to show that the kneading sequences can be thought of as infinite words in the monoid of positive words of $\pi_1(B_0)$. The second main tool is a kind of pre-zeta function, η, whose arguments x, y are the generators of this monoid. The function η can be computed in the following sense. First there is a (possibly, in fact, *usually* infinite) matrix $B(x,y)$. That is, B is a pairing on certain symbols Σ, with values either x, y, or 0. Then

$$\eta = \sum_i \frac{\operatorname{tr} B^i}{i}$$

where one must take care, as $\pi_1(B_0)$ is not abelian. Finally we show that η is a topological invariant and that the correspondence between the kneading sequences and η is one-to-one. This proves our basic proposition, that the kneading sequences are topological invariants.

More precisely:

Theorem. — *There is a positive number* Δ *such that, if the attractors* A_X *and* A_Y, *for* $X, Y \in \mathscr{L}$, *are homeomorphic* via *a homeomorphism within* Δ *of the identity* (C^0-*sense*), *then* X *and* Y *have the same kneading sequences.*

The number Δ is the " diameter " of the hole (see Figure 1) or about 30 for the equations of Lorenz.

From here it is just set-theoretic topology to show that the $\omega\Sigma$-conjecture of René Thom [1] — long thought to be false — is indeed false. That is, the Lorenz attractors are *not* generically countable up to topological type.

This differs significantly from Guckenheimer's result inasmuch as one of his two dense subsets is not a Baire set, and hence has no existence, generically.

Another basic geometric fact about Lorenz attractors is brought out, and used as a strong tool. This is the fact that these attractors are real objects, in ordinary euclidian 3-space, and that they consist of many-many two-dimensional layers, stretching from front to back in our line of sight. It follows that these layers are *linearly ordered*, by this front to back-ness. For example, see the stereoscopic computer printouts of Rössler [10].

We conclude the introduction with two types of comments. First, we use branch manifolds ([17], [18]) in our proofs, and would like to call the reader's attention to the sketches in Lorenz's original (1963) paper [7]. Also his comments, particularly about his Figure 3, correspond quite well to the author's theorem C [18]. Secondly, we emphasize below certain nice aspects of Lorenz attractors. They have a relative 2-manifold structure, are *orientable*, have a smooth line as boundary, form a singular fiber bundle, and have a rich cell-complex structure; in a sense, all of this depends continuously on the original equation.

It is a pleasure to thank Dennis Pixton for his helpful conversations. Also, J. Milnor for his conversations about work on kneading sequences he and W. Thurston have done recently, in another, basically more difficult connection. Michael Kervaire for his hospitality and encouragement at the third cycle in Geneva. Finally, and most important, the long conversation with W. Parry, in part about his early papers on maps like our Poincaré map f; in particular he seems to have singled out the property we call l.e.o, locally eventually onto (Prop. 1, § 2).

1. Use of branched manifolds.

Our point of departure is to describe a type of semi-flow, φ_t, $t \in \mathbf{R}^+$, on a certain smooth branched manifold L of dimension 2. Then $\{L, \varphi_t, t \in \mathbf{R}^+\}$ forms an inverse system, and its inverse limit

$$\hat{L} = \varprojlim \{L, \varphi_t, t \in \mathbf{R}^+\}$$

inherits a flow $\hat{\varphi}_t$, $t \in \mathbf{R}$. These $\hat{L}$, $\hat{\varphi}_t$ are the Lorenz attractors.

There are several additional steps, required to show that these $\hat{L}$, $\hat{\varphi}_t$ are indeed attached to the differential equations of Lorenz. First, there are *analytic estimates* to be made on the stable and unstable manifolds of the singular point. This task has

[1] First proposed by Thom in about 1967 and restated in the volume on Hilbert's problems [20], p. 59.

been carried out by various researchers, on computers ([7], [10], [12]), the only way now known.

The missing step involves a novel and fascinating problem, which I state here as a conjecture.

Conjecture. — There is a vector field X_x, transversal to the flow Φ_t of the Lorenz equation, such that for each t and each x near the attractor,

$$d\Phi_t . X_x = c\lambda^t X_{\Phi_t x}, \qquad c = c(x, t) \in (0, m)$$

where $0 < \lambda < 1$ and $m > 0$ are independent of x and t.

Thus X determines a strong stable (oriented) line bundle. Next, one needs to prove a strong stable manifold theorem for the Lorenz attractors, along the lines of the Hirsch-Pugh [5] version of the Smale formulation [13] for hyperbolic systems, and related to the Hirsch-Pugh-Shub paper [6]. However, one familiar with these techniques will have little trouble making this step; admittedly, this should be done in print, but should probably await a more general description of Lorenz structures.

Finally, one needs:

a) to proceed from the actual attractors to the artifact, L, φ_t, $t \geq 0$;

b) to proceed from $\hat{L}$, $\hat{\varphi}_t$, $t \in \mathbf{R}$ to a vector field (= differential equation) in some neighborhood of $\mathbf{R}^3$.

These two steps were treated in great detail in the author's papers ([17], [18]) for the case of diffeomorphisms. Admittedly, this too should be done in print; meanwhile, those familiar with this earlier work will have no trouble in these last two steps.

As a final remark, note that we do not use the assumption that the equations (and hence the attractors) of Lorenz are symmetric (see, *e.g.* [12]). This generality seems natural to us. On the other hand, all our work is (or can be) done symmetrically, so that the theorems apply in the symmetric case as well.

2. The branched manifold L.

Let L be the branched manifold of Figure 1

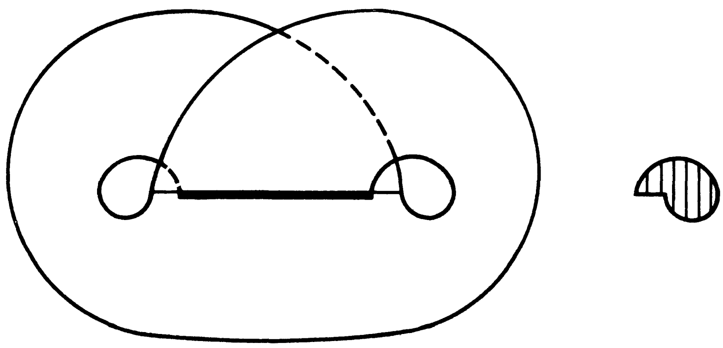


FIG. 1

Note that it has the homotopy type of a figure eight, that these two holes could be filled in by inserting two disks, or plugs, like the one to the right of figure one. The branch points are indicated by a heavy line, in the middle. Note that we have indicated an immersion into the plane, from which it inherits a counterclockwise orientation. Its boundary ∂L is an open line interval terminating in the end points of the branch line; $L-(\partial L \cup \text{branch set, extended})$ is an open disk.

A smooth semi-flow is sketched in Figure 2; we also sketch to the right in Figure 2, the *first return*, or Poincaré map, f.

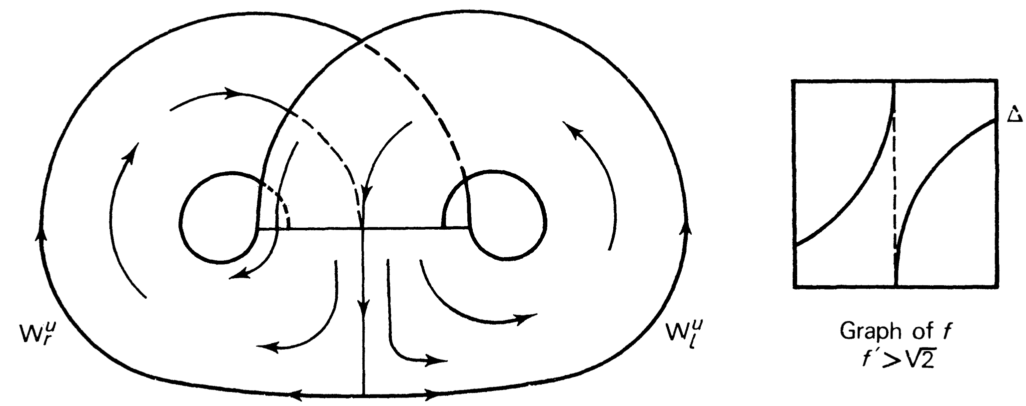


FIG. 2

Note the singular point O, where the linearized equation has the form

$$\begin{cases} \dot{y} = -\mu y \\ \dot{x} = \lambda x \end{cases} \qquad 0 < \mu < \lambda.$$

Note that if the plugs described above were inserted, they could carry flows with singular points (sources) where the eigenvalues are complex with positive real part. Note that as L is embedded in $\mathbf{R}^3$, it can be thickened in $\mathbf{R}^3$ by adding a tubular neighborhood; this can clearly be done, so that the flow lines can be put in. The semi-flow φ_t is defined only for $t \geq 0$, because at each point of the branch line, *two* trajectories enter while only one leaves. But in the thickened version in $\mathbf{R}^3$, such trajectories just come closer together, without touching.

It is of considerable importance that the unstable manifold at O (which fills ∂L, then goes on into the interior) is *not* thickened in this process of " exfoliating " ∂L into the attractor Λ in $\mathbf{R}^3$. This is automatically handled by the process of taking inverse limits and is described in detail, below.

The branch line is extended to the right and left as indicated by the dotted lines, to form I, our section. The Poincaré map $f: I \to I$ is indicated to the right in figure 2. Note that f is undefined at a central point O′, corresponding to the fact that this point on I is on the stable manifolds of O and hence never returns. Next, that with our choice $\mu<\lambda$, f has infinite derivatives on both sides of this point. One can adjust, with some liberty, the remainder of the graph of f, and we do so to arrange that $f'>\sqrt{2}$ at all points of I. This is a simplifying assumption, adequate for our purposes here. In fact, a similar analysis can be carried out for slopes $2^{1/(n+1)}<f'<2^{1/n}$, $(f^{2^n})'>\sqrt{2}$ (see [9]), and thus for all *expanding* f. Note that this range is compatible with the computer machine studies which indicate Δ (the upper right in the graph of f) to be about 12 % of the length of I. It follows that the "slope" of f is somewhere near $2(.88)=1.76>\sqrt{2}$.

We close this section with a basic Proposition, which gives the motivation of our choice of $\sqrt{2}$ as the lower level for f.

Proposition 1. — If $J\subset I$ *is a subinterval, then there is an integer n such that* $f^n(J)=I$. *That is, f is locally eventually onto.*

Proof. — Let $I_0=I$, if $O\notin I_0$; otherwise let I_0 be the bigger of the two intervals O splits I into. Similarly, for each i such that I_i is defined, set

$$I_{i+1}=\begin{cases} f(I_i), & \text{if} \quad O\notin f(I_i) \\ \text{bigger of two parts O splits } f(I_i) \text{ into, if } O\in f(I_i). \end{cases}$$

Now $\text{length}\, f(I_{i+1})>\lambda\, \text{length}\, I_{i+1}$, where $\lambda=\min f'(x)>\sqrt{2}$.

Thus unless O is in both $f(I_i)$ and $f(I_{i+1})$ we have

$$\text{length}\, I_{i+2}\geq\frac{\lambda^2}{2}\text{length}\, I_i.$$

But as $\lambda^2>2$, this last cannot always hold, say

$$O\in f(I_{n-2}) \quad \text{and} \quad O\in f(I_{n-1}).$$

Then $f(I_{n-1})$ contains O and one end point of I, so that I_n is one "half" of I. Note $f(I_n)$ contains the other half, and finally $f^3(I_n)>I$.

Basic assumption. — The Poincaré map f satisfies $f'>\sqrt{2}$ and the kneading sequences (see § 3, below) begin $yx^3\ldots$ and $xy^3\ldots$

Neither of these assumptions is necessary, but doing without them would be a further complication, whereas this is already complicated enough. In particular, the example of Guckenheimer, and some of the other illustrative examples of this paper *do not* satisfy the assumptions on the kneading sequences. However, they are illustrative, and are comparatively simple.

3. The orthogonal trajectory space B_0 and kneading sequences.

Consider the unstable manifold $W_u(O)\subset L$. It has two sides; we label the one that leaves O to the right, W^u_l, and the other W^u_r. This seeming perversion of labeling (the right hand one is called the left, and vice-versa) is a compromise which makes notation simpler, below. This is because W_r^u first enters I at its right-most point, and similarly W_ℓ^u enters first at the left-most point.

Next, let $p_0 : L \to B_0$ be the quotient map of the orthogonal trajectories of φ. In detail, note that the orthogonal trajectories of L form smooth line intervals except that F, the one through O, is the union of two intervals, intersecting at an angle at O. Then these intervals foliate L, and the leaf space formed by collapsing each one to a point is B_0; p_0 is the collapsing map. Then $F = p_0^{-1}(\mathcal{O})$, where

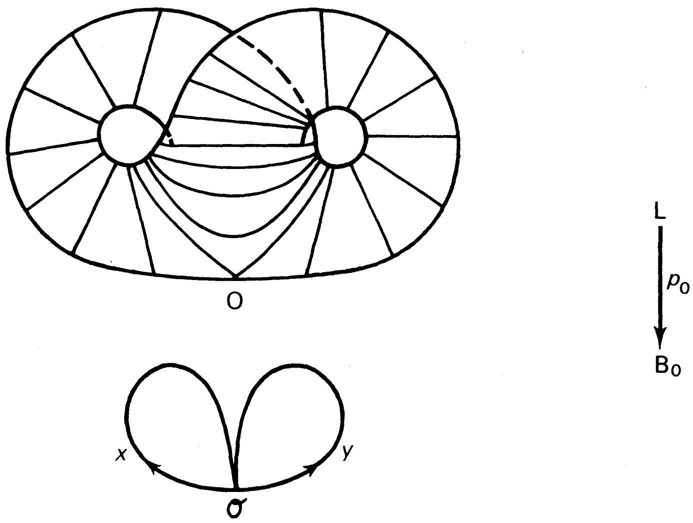


FIG. 3

we use $\mathcal{O} \in B_0$ to denote $p_0(O)$. Then B_0 has the homotopy type of a figure eight, and we label two generators x and y of $\pi_1(B_0, \mathcal{O})$. We also think of x and y as oriented paths; in this sense, each orbit beginning on F, forms a *positive* word in x and y, where positive means that no negative exponents are involved.

Kneading sequences have been considered by many researchers who have studied endomorphisms of a line interval. The phrase is due to B. Thurston [8]. They are sometimes sequences of +'s and −'s and possibly a o; for our purposes they are sequences of x's and y's.

Definition. — Given the branched manifold L and a semi-flow φ_t, $t \geq 0$ on L, we define the kneading sequences k_ℓ, k_r by

$$k_\ell = y^{-1} \circ p_0(W_\ell^u), \qquad k_r = x^{-1} \circ p_0(W_r^u).$$

The y^{-1} and x^{-1} are to simplify the ordering, below. Then $p_0(W_\ell^u)$ is a path in B_0 and can be written uniquely in the monoid of positive powers of the elementary paths x and y, so that $p_0(W_\ell^u) = yxxx\ldots$ Hence $k_\ell = xxx\ldots$ and similarly $k_r = yyy\ldots$

We emphasize here that we are making the following assumptions throughout the paper:

Basic Assumptions. — k_r begins with yyy and k_ℓ begins with xxx. The Poincaré return map f satisfies $f' > \sqrt{2}$ and hence is locally eventually onto (l.e.o).

We close this section with a quick indication of how Guckenheimer's result is proved. We introduce two sequences r_i, ℓ_i which we use below.

Define $r_i = i$-th point in which W^u_r hits F and $\ell_i = i$-th point in which W^u_ℓ hits F. Note $r_0 = \ell_0 = O \in F$. Note the sequences $\{\ell_i\}$, $\{r_i\}$ can be finite in case k_r or k_ℓ is finite. The two cases, both finite and both infinite, correspond to the two topologically distinct examples of Guckenheimer:

Theorem (Guckenheimer). — *If the flow φ_t on L yields finite sequences $\{r_i\}$, $\{\ell_i\}$ and φ'_t on L′ yields infinite ones, then the inverse limits $\hat{L}$, $\hat{L}'$ are not homeomorphic.*

Proof. — A point $\hat{x} \in \varprojlim\{L, \varphi_t\}$ consists of a point $x_0 \in L$ together with a choice $\hat{x}(s)$, $s \leq 0$, of its " prehistory ". That is, $\hat{x} = \{\hat{x}(s)\}^0_{s=-\infty}$ such that $\varphi_t \hat{x}(s) = \hat{x}(s+t)$, $s+t \leq 0$. We can distinguish the points $\hat{x} \in \hat{L}$ which are in the unstable manifold of O, as follows.

$\hat{W}^u_r = \{\hat{x} \in \hat{L} : \hat{x}_s \to O$ from the left as $s \to -\infty\}$. Similarly for $\hat{W}^u_\ell$, whereas $\hat{O} = \{O\}$ as $\varphi_t O = O$ all t. Roughly speaking, $\hat{W}^u$ is distinguished in that it comprises the only two semi-orbits with a unique (unbranched) past history. We show below that each point $\hat{x} \in \hat{W}^u_\ell \cup \hat{W}^u_r$ lies in the interior of an interval $I' \subset \hat{W}^u_\ell \cup \hat{W}^u_r$, so that I′ in turn lies in a set $I' \times C \subset \hat{L}$, where C is the cone over a Cantor set,

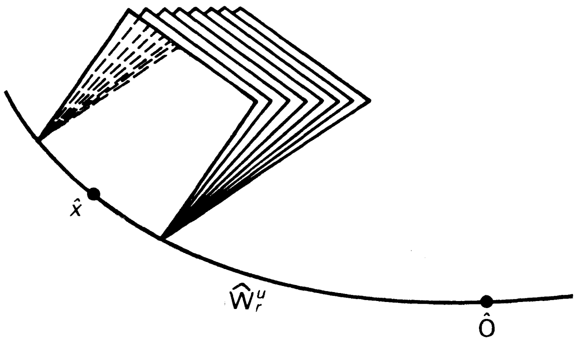


FIG. 4

whereas no other point of L with the possible exception of O lies in such a set.

Then Guckenheimer's theorem follows, as $\hat{W}^u_r \cup O \cup \hat{W}^u_\ell$ is a distinguished line in $\hat{L}'$ in one case, and a distinguished figure eight in the other case, $\hat{L}$.

4. $\hat{F}$ as cell complex.

Recall that we have chosen F (the " fiber ") to be made up of two line intervals, joined at O. Then $\hat{F} \subset \hat{L}$ consists of all $\hat{x} \in \hat{L}$ with $\hat{x}(O) \in F$. In particular $r_i, \ell_i \in F$, defined above (§ 3), yield the vertices $\hat{\ell}$, $\hat{r}_i$ of $\hat{F}$, where

$$\hat{r}_0 = \hat{\ell}_0 = \hat{O} \text{ is the rest point,}$$

$$\hat{r}_i = O^- \to r_1 \to r_2 \to \ldots \to r_i,$$

and $$\hat{\ell}_j = O^+ \to \ell_1 \to \ell_2 \to \ldots \to \ell_j.$$

More formally, note that there are positive numbers $\Delta_1, \Delta_2, \ldots, \Delta_{i-1}$ such that $\varphi_{\Delta_{i-1}}(r_{i-1}) = r_i$. In case of choice, choose the Δ's to be the smallest possible. Now define

$$\hat{r}_i(s) = \varphi_{\Delta+s}(r_1)$$

where $\Delta = \Delta_1 + \Delta_2 + \ldots + \Delta_{i-1}$. This last is okay because φ_t is unambiguously defined on r_1 for all negative as well as positive values of t. Note that as $t \to -\infty$, $\varphi_t(r_1) \to \mathcal{O}^-$. Similarly for a fixed $j \geq 1$, there are $\delta_1, \delta_2, \ldots, \delta_{j-1} > 0$ such that

$$\varphi_{\delta_1}(\ell_1) = \ell_2, \ldots, \varphi_{\delta_{j-1}}(\ell_{j-1}) = \ell_j.$$

These are chosen as small as possible, and we define $\delta = \delta_1 + \ldots + \delta_{j-1}$ and

$$\hat{\ell}_j(s) = \varphi_{\delta+s}(\ell_1).$$

There is a tricky point here: the vertices of $\hat{F}$ are finite *only* if there is a saddle connection on both sides of $W^u(O)$, and not when the sequences $\{\ell_i\}$, $\{r_i\}$ are periodic. We illustrate this by an example.

Example **(4.1)**. — Consider the case $r_3 = \ell_2$, $\ell_3 = r_2$. Then the vertices of $\hat{F}$ are infinite.

Proof. — The prehistories for r_i, $i = 0, 1$ are unique. But for $r_2 = r_{2i} = \ell_{2i+1}$, $i = 1, 2, \ldots$ there are infinitely many prehistories, as follows:

$$\hat{r}_2 : O^- = r_0^- \to r_1 \to r_2. \text{ From } O^+ \text{ to } r_1 \text{ is infinite.}$$

$$\hat{\ell}_3 : O^+ = \ell_0^+ \to \ell_1 \to \ell_2 \to \ell_3 = r_2.$$

$$\hat{r}_4 : O^- = r_0^- \to r_1 \to r_2 \to r_3 \to r_4 = r_2.$$

E.g., $\hat{r}_2(s) = \hat{r}_4(s + s_0)$, when $s_0 = \min\{t > 0 : \varphi_t r_2 = r_4\}$. Hence $\hat{r}_2 \neq \hat{r}_4$.

(4.2) The one cells of $\hat{F}$ are in a one-to-one correspondence with a one-sided shift space on a set Σ of certain (usually infinitely many) symbols. We proceed to define these symbols inductively in such a way that if $[i, j]$ is defined, then ℓ_i and r_j are points on the same side of O. (For this purpose, O itself is on the same side of O as any point of F.)

1) [0, 1] and [1, 0] are symbols.
2) If $[i, j]$ is a symbol and if
   a) $\ell_{i+1} = 0$, then $[0, j+1]$ is a symbol;
   b) $r_{j+1} = 0$, then $[i+1, 0]$ is a symbol;

c) $\ell_{i+1}<0<r_{j+1}$, then both $[i+1, 0]$ and $[0, j+1]$ are symbols;

d) neither *a*, *b*, nor *c*, then $[i+1, j+1]$ is a symbol.

Next, define the (possibly infinite) matrix $B(x,y)$ as a pairing as follows: if $\sigma, \tau \in \Sigma$,

$$B(x,y)_{\sigma\tau} \in \begin{Bmatrix} x \\ y \\ 0 \end{Bmatrix} = p_0(\text{orbit from } \sigma \text{ to } \tau)$$

where the answer is 0 unless

$$\sigma, \tau = [i, *], [i+1, *]$$
$$\text{or } [*, j], [*, j+1]$$

where $*$ means this term is unimportant. In the latter cases, there is an orbit which proceeds from ℓ_i to ℓ_{i+1} (resp. r_j to r_{j+1}); its projection onto B_0 traces out either x or y and this is the value of $B(x,y)_{\sigma\tau}$.

Examples.

$$\begin{bmatrix} 0 & x & x \\ 0 & 0 & x \\ y & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & x & x \\ 0 & 0 & x & 0 \\ y & y & 0 & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \text{ are the simplest such matrices.}$$

The second corresponds to the original example [3] of Guckenheimer.

Definition (**4.2.1**). — If $B_{\sigma\tau} \neq 0$, we say $\sigma \rightarrow \tau$ (σ maps to τ) *via x*, if $B_{\sigma\tau} = x$ and *via y*, if y; in either case $\sigma \rightarrow \tau$, or σ *precedes* τ.

We next prove that our symbolic system has a property like " *indecomposability* ".

Lemma (**4.2.2**). — *Given any two symbols* σ, τ, *there is a finite sequence*

$$\sigma_0 = \sigma, \sigma_1, \sigma_2, \ldots, \sigma_n = \tau$$

such that σ_{i-1} *precedes* σ_i *for each* $i = 1, \ldots, n$.

Proof. — Note that it suffices to prove this for $\tau = [0, 1]$ and $[1, 0]$ as all other symbols follow these by our inductive definition. And this in turn is essentially a special case of our earlier lemma about " locally eventually onto ". Recall that there we find a sequence $I_0 = I, I_1, I_2, \ldots, I_n$, such that for each i, either

$$f(I_i) = I_{i+1}$$

or

$$f(I_i) = I_{i+1} \cup J$$

when J is another interval, intersecting I_{i+1} in only the point O. It thus follows that I_{i+1} follows I_i. Finally I_{n-1} and I_n are the two intervals into which O divides I, *i.e.* one corresponds to $[\ell_0, r_1]$, the other to $[\ell_1, r_0]$.

Definition **(4.2.3).** — Let C(B) be the sub-shift space of all sequences $\{\sigma_i\}_{i=0}^{\infty}=\sigma$ such that $B_{\sigma_{i+1}\sigma_i}\neq 0$, *i.e.* $\sigma_{i+1}\to\sigma_i$. We write such a sequence ending with σ_0:

$$\sigma=\ldots\sigma_2\sigma_1\sigma_0.$$

The cylinder set $C[i,j]=\{\sigma\in C(B):\ \sigma_0=[i,j]\}$. For $\sigma\in C(B)$ define the *prekneading sequence* $k^*(\sigma)$ by

$$k^*(\sigma)=\ldots z_2 z_1 z_0,$$

where $z_i=x$ or y according as to whether $\sigma_{i+1}\to\sigma_i$ via x or via y. We lexicographically order the $k^*(\sigma)$'s with $x<y$. This induces a linear ordering $\sigma<\tau$ on C(B). *E.g.*, [1, 0], [0, 1] ... < [0, 1], [1, 0] ... as their kneading sequences begin with x, y respectively.

(4.2.4) *The following sequences are allowable by* B:

$$[1,0]\to[2,0]\to[0,1]$$
$$[0,1]\to[1,0]\to[0,1].$$

Proof. — The basic assumption (§ 3) about k_ℓ means $\ell_1<\ell_2<\ell_3<O$. Thus

$$[1,0]\to[2,0],[0,1];\qquad [2,0]\to[3,0],[0,1];$$

and dually

$$[0,1]\to[1,0],[0,2];\qquad [0,2]\to[1,0],[0,3].$$

The lemma follows.

(4.3) *Structure proposition.* — *The fiber* $\hat{F}$:

a) *has vertices* $V=\{\hat{\ell}_i,\hat{r}_j:\ \ell_i$ *and* r_j *are defined*$\}$;

b) *two vertices of* $\hat{F}$ *are joined by a* 1*-cell iff they are* $\hat{\ell}_i$ *and* $\hat{r}_j$ *for some* $[i,j]\in\Sigma$;

c) *a dense subset* E_{ij} *of the* 1*-cells joining* $\hat{\ell}_i$ *to* $\hat{r}_j$ *is in one-to-one correspondence with the cylinder set* $C[i,j]$ (see (4.2.3));

d) $\bigcup_{i,j} E_{ij}$ *is dense in* $\hat{F}$ *and contains all the periodic points of* $\hat{F}$;

e) *the map* $\hat{F}\to F$ *given by* $\hat{x}\mapsto\hat{x}(O)$ *maps each connected component of* $\hat{F}-V$ *homeomorphically onto a (perhaps degenerate) subinterval of* F.

We proceed with the rather lengthy proof, first introducing a sequence $\{F_n\}$ of approximations to $\hat{F}$.

(4.4) *Definition of* F_n, $\varphi_n:\hat{F}\to F_n$. — A prehistory $\hat{x}\in\hat{F}$ is said to *alternate for* $s\leq s^*$, provided adjacent intersections of $\hat{x}(s)$ with F, for $s\leq s^*$, lie on opposite sides of O. A point $\hat{x}\in\hat{F}$ is in F_n provided:

a) $\hat{x}=\hat{O}$, $\hat{\ell}_i$ or $\hat{r}_j$ for some $i,j\leq n$; or

b) $\hat{x}$ alternates for $s\leq s^*$ where s^* is the n-th value of s such that $\hat{x}(s)\in F$. Here the first value is $s=0$, the second is the next value of $s<0$, etc.

Finally, for $n=1, 2, \ldots$ define $\varphi_n : \hat{F} \to F_n$ by

a) $\varphi_n(\hat{x}) = \hat{x}$, for $\hat{x} = \hat{O}$, $\hat{\ell}_i$ or $\hat{r}_j$, $i, j \leq n$; or

b) otherwise $\hat{x}$ crosses F for more than n values of s; let s^* be the n-th one. Then $\varphi_n \hat{x}$ is the unique prehistory which agrees with $\hat{x}$ for $s^* \leq s \leq 0$ and alternates for $s \leq s^*$.

(4.5) $\varphi_n : \hat{F} \to F_n$ *is a continuous retraction within* ε_n *of the identity, where* $\varepsilon_n \to 0$ *as* $n \to \infty$.

Proof. — That φ_n is a retraction is clear, since a point $\hat{x}$ is determined by its initial value $\hat{x}(O)$ and its prekneading sequence. It is easy to prove continuity in each of the three cases of the definition of φ_n.

For example, for $\hat{x} = \hat{\ell}_i$, $i \leq n$, note that for $\hat{y}$ near $\hat{\ell}_i$, $\hat{y} \neq \hat{\ell}_i$, we have $\hat{\ell}_i(O) < \hat{y}(O)$. Furthermore $\hat{y}(s)$ maintains this position to the right of $\hat{\ell}_i(s)$ until $\hat{y}(s)$ passes slowly by O and $\hat{\ell}_i(s) \to O^+$ as $s \to -\infty$. This, because $\hat{\ell}_i(s)$ passes along a boundary of L for $s \leq s^*$, where $\hat{\ell}_i(s^*) = \ell_1$. Then $\varphi_n(\hat{\ell}_i) = \hat{\ell}_i$ and for a large range of s, $(\varphi_n \hat{y})(s) = \hat{y}(s)$. Continuity of φ_n at $\hat{\ell}_i$ follows. Finally, the fact that $\hat{x}(s) = (\varphi_n \hat{x})(s)$ for all s down to the n-th value of s for which $\hat{x}(s)$ is on F, implies the last statement and completes the proof of (4.5).

Lemma **(4.5.1)**. — *For* $\hat{\ell}_i, \hat{r}_j \in F_n$, $\varphi_n^{-1}(\hat{\ell}_i) = \hat{\ell}_i$ *and* $\varphi_n^{-1}(\hat{r}_j) = \hat{r}_j$.

Proof. — As φ_n is a retraction, no point of F_n maps to $\hat{\ell}_i$ except $\hat{\ell}_i$. On the other hand if $\hat{x} \notin F_n$, then $\varphi_n(\hat{x})$ has infinite (alternating) prehistory and hence is not $\hat{\ell}_i$. The other case is similar.

Remark **(4.6)**. — The first return map $\hat{f} : \hat{F} \to \hat{F}$ is given as follows: $\hat{f}(\hat{O}) = \hat{O}$; for $\hat{x} \in \hat{F} - \hat{O}$, let T be the first value of $t > 0$ such that $\varphi_t \hat{x}(O) \in F$. Then

$$\hat{f}(\hat{x})(s) = \begin{cases} \hat{x}(s+T), & s \leq -T. \\ \varphi_{T+s}\hat{x}(O), & -T \leq s \leq 0. \end{cases}$$

Remark **(4.7)**. — $\hat{f}$ maps F_n onto F_{n+1}. We may regard $\hat{L}$ as being "swept out" by the flow lines which determine the first return map. (This "sweeping out" occurs in $\mathbf{R}^3$.)

Proof. — If $\ldots s_i < \ldots < s_1 = 0$ are the values of s for which $\hat{x}$ crosses F, then

$$\ldots s_i - T < \ldots < s_1 - T < -T < 0$$

are the values of s for which $(\hat{f}\hat{x})(s) = 0$. Hence $\hat{f}(F_n) = F_{n+1}$. Finally, $F_n \subset F_{n+1} \subset \hat{F} \subset \mathbf{R}^3$ and the first return $\hat{f}$ is based on the first return map $f : F \to F$, and $f\hat{x}(O) = \varphi_T \hat{x}(O)$.

Definition **(4.8)**. — Let $C_n(B)$ be the set of all $\sigma \in C(B)$ of the form

$$\ldots \to \sigma_n \to \sigma'_n \to \sigma_n \to \sigma_{n-1} \to \ldots \to \sigma_1$$

where $\{\sigma_n, \sigma'_n\} = \{[1, 0], [0, 1]\}$ or $\{[0, 1], [1, 0]\}$. That is, all $\sigma \in C(B)$ such that $\{\sigma_{i+1}, \sigma_i\} = \{[1, 0], [0, 1]\}$ or $\{[0, 1], [1, 0]\}$ for $i \geq n$.

Remark **(4.9)**. — There is the commutative diagram

$$\begin{array}{ccc} F_n & \xrightarrow{\hat{f}} & F_{n+1} \\ \updownarrow & & \updownarrow \\ C_n(B) & \underset{s^{-1}}{\dashrightarrow} & C_{n+1}(B) \end{array}$$

wheret he vertical arrows denote a one-to-one correspondence between the 1-cells of F_n and the points of $C_n(B)$. The map s simply drops the first symbol σ_0 from σ.

Proof. — By induction on n. First, $C_1(B)$ consists of the two sequences which alternate between $[1, 0]$ and $[0, 1]$. Similarly, $\hat{x} \in F_1$ is determined by its initial point $\hat{x}(O) \in F$ as its prekneading sequence is alternating. $\hat{O}$ is in F_1 by choice; $\hat{\ell}_1$ and $\hat{r}_1$ are forced to be in vertices F_1 as we see as follows: $\hat{\ell}_1(O) = \ell_1$ has no point of F to its left. As $s \to -\infty$, $\hat{\ell}_1(s)$ flows along the boundary of L with all of L to its right. Thus there is no $\hat{x}$ to the left of $\hat{\ell}_1$. Similarly for $\hat{r}_1$.

Now suppose we know the lemma for n. Then by (4.7) a 1-simplex e joining $\hat{\ell}_i$ to $\hat{r}_j$ maps to a 1-simplex e' or two 1-simplices e', e'' according to the various cases detailed in the definition (4.2) of B; the symbol $[i, j]$ was defined to map exactly to the corresponding symbol σ', or the two symbols σ', σ'' so that the one-to-one correspondence carries over to $n+1$. The remark follows.

Lemma **(4.10)**. — *If e is a 1-cell joining $\hat{\ell}_i$ to $\hat{r}_j$ in F_n for some n, then the map given by $\hat{x} \mapsto \hat{x}(O)$ is a homeomorphism of e onto $[\ell_i, r_j] \subset F$.*

Proof. — Let $\sigma = \{\sigma_i\}_{i=1}^{\infty}$ be the point of $C_n(B)$ corresponding to e, say $\sigma_\alpha = [i_\alpha, j_\alpha]$. Then $[\ell_{i_\alpha}, r_{j_\alpha}]$ is a subinterval of F, lying on one side of O and mapping onto $[\ell_{i_{\alpha-1}}, r_{i_{\alpha-1}}]$ (and perhaps more). Thus by induction, to each point $x_1 \in [\ell_i, r_j]$ we can complete a " history " $x_\alpha \in [\ell_{i_\alpha}, r_{j_\alpha}]$ so that $fx_\alpha = x_{\alpha-1}$, $\alpha = 1, 2, \ldots, n$. Then there is a unique point $\hat{x} \in F_n$ which passes through the x_α in succession as s decreases, then alternates after the n-th intersection of F. The lemma follows.

Corollary **(4.10.1)**. — *Part e) of the structure proposition* (4.3) *is true.*

Proof. — Note that the map $\varphi : \hat{F} \to F$ given by $\hat{x} \mapsto \hat{x}(O)$, factors as $\varphi = \varphi \circ \varphi_n$ for each n. So let C be a component of $\hat{F} - V$. Then $\varphi_n(C)$ contains no vertex of F_n by (4.5.1), so that it lies in a 1-cell. Since $\varphi | \varphi_n(C)$ is a homeomorphism, it follows that $\varphi \circ \varphi_n | C$ is an ε_n-map. But as this is true for each n and as $\varepsilon_n \to 0$ with n, it follows that $\varphi | C$ is a homeomorphism onto its image, which must be a (perhaps degenerate) subinterval of F.

The following is due to Guckenheimer [3]; we include it here for completeness.

Proposition **(4.11)**. — *In case both k_ℓ and k_r are finite (two "saddle connections") the symbol set Σ is finite and the structure proposition holds with* $\hat{F}=\bigcup_{i,j} E_{ij}$.

Proof. — Σ is finite as there are only finitely many ℓ_i and r_j. Note that we have two growing sets

$$F_1 \subset F_2 \subset \ldots \subset \hat{F}$$
$$C_1 \subset C_2 \subset \ldots C(B)$$

the lower corresponding one-to-one with the 1-cells of the upper. Furthermore, $\hat{F}$ is the closure of $\bigcup_i F_i$. Likewise, in this case, C(B) is the closure of $\bigcup_n C_n(B)$. That is, if $\ldots \to \sigma_{\alpha_i} \to \ldots \to \sigma_{\alpha_1}$ is a sequence in C_α, for each α we may choose subsequences $\sigma_{\alpha_i} = \sigma_i$ for $\alpha > \alpha_i$. Thus

$$\lim_\alpha C_\alpha(B) = C(B).$$

We need check only one thing further: that if a sequence e_α of 1-cells, joining $\hat{\ell}_i$ to $\hat{r}_j$, converges to e, then e is actually a 1-cell, *i.e.* a homeomorph of [0, 1]. But by (4.10.1), $\hat{x} \mapsto \hat{x}(O)$ is a homeomorphism of e_α onto $[\ell_i, r_j]$ for all α and the prekneading sequences of all points $\hat{x} \in e_\alpha$ agree. Thus the prekneading sequences of all points on e agree, so that $e \to [\ell_i, r_j]$ is also a homeomorphism.

(4.12.1) If $\sigma_0 \to \sigma_1 \to \ldots \to \sigma_{p-1} \to \sigma_0$ is allowable, then each $\sigma_\alpha = [i_\alpha, j_\alpha]$ satisfies $0 \leq i_\alpha, j_\alpha < p$.

Proof. — Note that for each α one of the following must hold:

a) $i_{\alpha+1} = i_\alpha + 1$, $j_{\alpha+1} = j_\alpha + 1$;
b) $i_{\alpha+1} = 0$, $j_{\alpha+1} = j_\alpha + 1$;
c) $i_{\alpha+1} = i_\alpha + 1$, $j_{\alpha+1} = 0$.

And since $i_p = i_0$ and $j_p = j_0$, both *b)* and *c)* must occur. But beginning at $i_\beta = 0$, it follows that $i_{\beta+\alpha} \leq \alpha$, where $\beta + \alpha$ is computed mod p. Hence all $i_\alpha < p$ as required. Similarly, all $j_\alpha < p$.

Lemma **(4.12.2)**. — *The periodic points of* $\hat{F}$ *lie in* $\bigcup_{i,j} E_{ij}$ *as claimed in part d) of the Structure proposition* (4.3).

Proof. — As case one, we suppose as in (4.11) that both k_ℓ and k_r are finite. Then $\hat{F} = \bigcup_{i,j} E_{ij}$ so that the lemma is clear.

Now the general case. Let $\hat{x} \in \hat{F}$ be a point of period p and let $x = \hat{x}(O)$. Then x is a point of period p of $f: F \to F$ and is stable, since the slope of f is always $> \sqrt{2}$. Thus if we perturb f a small amount to f', there will be a nearby point x' of period p

under f'. Likewise, the relative position of the $2p+1$ points, O', ℓ_1', r_1', $\ldots$, ℓ_p', r_p' for f' and the corresponding points for f will be the same if we perturb only slightly. We suppose this is done and done so that we obtain two saddle connections for f' and hence finite k_ℓ' and k_r' (see § 5, below).

In the perturbed system we can apply (4.11) to the periodic point x' and find a point $\hat{x}'$ such that $\hat{x}'(O)=x'$, $\hat{x}'$ lies in a 1-cell determined by

$$(*) \qquad \ldots \to \sigma_\alpha \to \ldots \to \sigma_0 \text{ in } C(B'),$$

and $\hat{x}'$ is periodic of period p.

It follows that the orbit of the point $\hat{x}'$ under $\hat{f}'$ lies in only p 1-cells of $\hat{F}'$. As these in turn are labelled by the σ_α of $(*)$ it follows that $(*)$ is also periodic of period p. Thus by (4.12.1) all of its entries $\sigma_\alpha=[i_\alpha, j_\alpha]$ satisfy $i_\alpha, j_\alpha < p$.

Hence, by our choice of f', the sequence $(*)$ is also allowable in C(B). Furthermore, such a periodic sequence clearly determines a point $\hat{x}''\in\hat{F}$ which is of period p. We claim, finally, that $\hat{x}''=\hat{x}$. This is because these points have the same prekneading sequences and thus the same kneading sequences as these concepts coincide in the periodic case. But then, by the basic proposition of § 2, $\hat{x}''(O)=\hat{x}(O)$. Therefore $\hat{x}''=\hat{x}$, which completes the proof of (4.12.2).

(4.13) *Completion of the proof of the Structure Proposition.* As $F_n \subset \hat{F}$ for all n, the $\hat{\ell}_i$, $\hat{r}_j$ are vertices in F, which proves *a)*. Similarly, consider a sequence

$$(*) \qquad \ldots \sigma_i \to \ldots \to \sigma_1 \text{ in } C(B).$$

Then, given i, (4.2.2) says there is a finite allowable sequence of the form

$$[1, 0] \to \sigma_i \to \ldots \to \sigma_1.$$

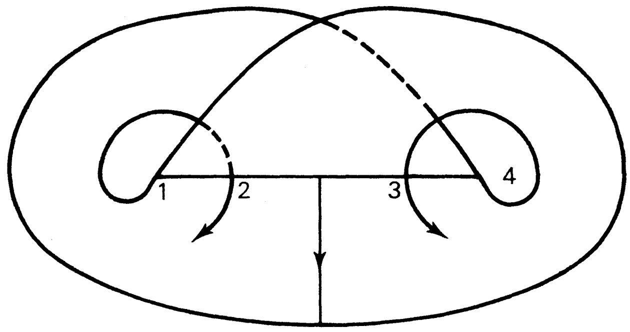
This can in turn be completed by alternating between $[1,0]$ and $[0,1]$. Thus for each α there is an α' and a sequence

$$(**) \qquad \ldots \to \sigma_{\alpha_{i+1}} \to \alpha_{\alpha_i} \to \ldots \to \sigma_{\alpha_1} \text{ in } C_{\alpha'}(B)$$

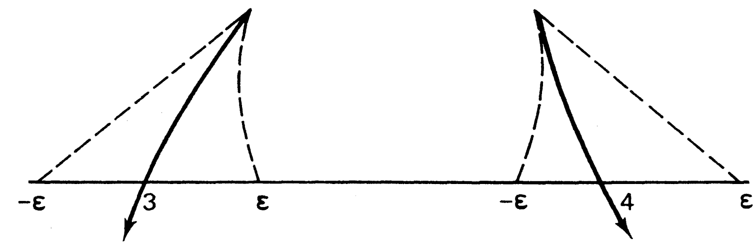
with $\sigma_{\alpha_i}=\sigma_i$ for all $i\leq\alpha$. Then the sequence $(**)$ corresponds to a 1-cell $e_{\alpha'}\subset F_{\alpha'}$; we claim this sequence converges to the sequence $(*)$. This follows quickly from (4.10) as the prekneading sequences of all $\hat{x}\in e_\alpha$ agree and agree with those of $\hat{x}\in e$, for a long way out. Thus $\hat{x}\mapsto\hat{x}(O)$ defines a homeomorphism from $e=\lim_\alpha e_\alpha$ to $[\ell_i, r_j]$, where $\sigma_1=[i,j]$. This is the 1-cell corresponding to the sequence $(*)$, so $e\in E_{ij}$. On the other hand, if a 1-cell e joins two vertices v_1, v_2 of V, then we can choose n large enough so that $v_1, v_2\in F_\alpha$, for $\alpha>n$. It follows that $\varphi_\alpha|e$ is a homeomorphism of e into F_α, which is within ε_n of the identity. Hence v_1, v_2 must be $\hat{\ell}_i$ and $\hat{r}_j$ where $[i,j]\in\Sigma$. This proves *b)* and *c)* of the Structure Proposition. Certainly $\bigcup_{i,j} E \supset \bigcup_\alpha F_\alpha$, so that $\bigcup_{i,j} E_{ij}$ is dense in $\hat{F}$. This completes the proof of the Structure Proposition.

5. Perturbation of the differential equation.

There are four types of perturbations we wish to consider, one each at the four special points of " ∂L ":



We will call these, respectively, left outside, left inside, right inside and right outside. We will discuss formally only the later two, as the others are similar.



At each of these points we make a perturbation by pushing W_r^u to the right or left, corresponding to t positive or negative, for $t \in [-\varepsilon, \varepsilon]$. Each perturbation is to be supported in an interval small enough to miss the other four of the five points (the middle one is unlabeled) indicated in the figure.

Proposition. — *The map* $t \mapsto \{k_t, k_r\}$ *which assigns to* $t \in [-\varepsilon, \varepsilon]$ *kneading sequences of the vector field perturbed by t units is order preserving. The order on* $[-\varepsilon, \varepsilon]$ *is the usual, and the lexicographical ordering on the kneading sequences.*

Proof. — Let f be the unperturbed Poincaré map and $g = g_t$ be the one perturbed by t units. We think of I as being a subset of $\mathbf{R}$ in the natural order. We discuss the inside perturbation first. Then there is the sequence $r'_1, r'_2, \ldots$ (finite or infinite) where r'_i is the i-th point in which the right unstable manifold W_r^u hits I, under the return map f, and similarly $r'_{1t}, r'_{2t}, \ldots$ for g_t. We note that $r'_{1t} = r'_1$ and r'_{2t} is less than, equal to, or greater than r'_2 according as to whether t is negative, o or positive. We suppose $t > 0$ as the other cases are similar. Then as long as the points r'_i and r'_{it} are on the same side of O, for $i=1, \ldots, n$, $r'_i < r'_{it}$, for $i=1, \ldots, n+1$. Furthermore, the distance between them is increasing with i, by more than a factor of $\sqrt{2}$. Hence there is an n so that r'_i and r'_{it} are on the same side of O up to $i=n$ and on opposite sides for $i=n+1$. This latter case is taken to include the possibility that one of them *is* O. That is $r'_{n+1} \leq 0 \leq r'_{n+1,t}$ where only one $=$ can hold. Thus the right sequence for f comes before that for g_t, as they agree up to the $(n+1)$-th place, where there is a change to one of the following cases:

r'_{n+1}	$-$	O	$-$
$r'_{n+1,t}$	$+$	$+$	O

Thus in any case $k_r(f) < k_r(g_t)$.

We next consider $k_\ell(f)$ and $k_\ell(g)$: to this end let $\{\ell'_1, \ell'_2, \ldots\}$ and $\{\ell'_{1t}, \ell'_{2t}, \ldots\}$ be defined as we defined r'_i, r'_{it} above. Now consider the question: is there an integer i so that ℓ'_i is in the support of our perturbation? If not, then $\ell'_i = \ell'_{it}$ for all i and hence $k_\ell(f) = k_\ell(g_t)$. In case there is such an i, let n be the least such and note that $\ell'_i = \ell'_{it}$ for $i=1, \ldots, n$, whereas $\ell'_i < \ell_{it}$ for $i=n+1$. The argument is then completed, just as before.

6. Distinguishing $\hat{\mathrm{W}}$ and $\hat{\mathrm{O}}$.

The unstable manifold W of $\mathrm{O} \in \mathrm{L}$ is clearly well defined, being the union of the left orbit W_r and the right orbit W_ℓ exiting from O. We let $\mathrm{W} = \mathrm{W}_\ell \cup \mathrm{O} \cup \mathrm{W}_r$, and define $\hat{\mathrm{W}}$ by

$$\hat{\mathrm{W}}_\ell = \{\hat{x} \in \hat{\mathrm{L}} : \lim_{s \to -\infty} \hat{x}(s) = \mathrm{O}^+\}$$
$$\hat{\mathrm{W}}_r = \{\hat{x} \in \hat{\mathrm{L}} : \lim_{s \to -\infty} \hat{x}(s) = \mathrm{O}^-\}$$
$$\hat{\mathrm{O}}(s) = \mathrm{O}, \quad \text{all } -\infty < s \leq 0$$
$$\hat{\mathrm{W}} = \hat{\mathrm{W}}_\ell \cup \hat{\mathrm{O}} \cup \hat{\mathrm{W}}_r.$$

In order to distinguish various types of points in $\hat{\mathrm{L}}$ we introduce the following terminology. By a *Cantor-fan* is meant the cone over a Cantor set. By a *Cantor-book* is mean the Cartesian product

$$\mathrm{F} \times \mathrm{I}'$$

where F is a Cantor-fan and I' is a line interval, and the *spine* of a Cantor-book is the obvious arc $= \mathrm{A} \times \mathrm{I}'$, where $\mathrm{A} \in \mathrm{F}$ is the cone-point.

Proposition. — Each point of $\hat{\mathrm{W}}$ *lies on the spine of a Cantor-book lying in* $\hat{\mathrm{K}}$. *No other point in* $\hat{\mathrm{L}}$ *lies on such a spine.*

Proof. — The positive part of this proposition follows from our knowledge of $\hat{\mathrm{F}}$ (§ 4, Lemma 1). This requires a special argument in the case of $\hat{\mathrm{O}}$. But as $f|[1, 0]$

maps this interval to an interval that contains O in its interior, it is easy to argue. Now suppose $\hat{x} \in \hat{L} - \hat{W}$. Then as O is the only singular point of φ_t, $\hat{x}$ has a neighborhood of the form $\hat{N} \times I$, I an interval and $\hat{N} \subset \hat{F}$. Then we may suppose $\hat{x} = \hat{p} \times t$, $\hat{p} \in \hat{F}$ and $t \in I$. Then the points of $\hat{F}$ consist of:

1) vertices;
2) points interior to 1-cells;
3) neither 1) nor 2) but limit points of both 1) and 2).

Of these, only the first type lies on Cantor-fans, so that $\hat{x}$ does not lie on the spine of a Cantor-book.

We next turn to the question as to whether our two sequences $\{r_1, r_2, \ldots\}$, $\{\ell_1, \ell_2, \ldots\}$ can have any behavior other than finite, periodic, and dense in I. They probably can, but for genericity questions this is no problem because of the

Proposition. — *The subset* $\mathscr{D} \subset \mathscr{L}$ *of all vector fields such that the corresponding sets* $A = \{r_0, r_1, r_2, \ldots\}$ *and* $\{\ell_0, \ell_1, \ell_2, \ldots\} = B$ *both have* O *as a limit point is a Baire set* (*= a second category set*).

Proof. — We introduce the set

$$\mathscr{A}_i = \{X \in \mathscr{L} : A_X \cap ((1/i)\text{-neighborhood of } O) = \emptyset\}.$$

Here of course A_X is the set $\{r_0, r_1, r_2, \ldots\}$ resulting from the vector field X, and the $(1/i)$-neighborhood $N_{1/i}(O)$ of O is taken to be open. Then obviously $\mathscr{A}_i$ is closed as its complement is open.

To see that $\mathscr{A}_i$ is nowhere dense, take an instance L, φ_t, and perturb according to a left inside perturbation (§ 5); since we can arrange an arbitrarily small perturbation to yield some $r_j = 0$, we can perturbe a bit less and get $r_j \in N_{1/i}(O)$. Thus $\mathscr{A} = \bigcup_{i=1}^{\infty} \mathscr{A}_i$ is of the first category. Similarly, define $\mathscr{B}$ and note it is of the first category so that $\mathscr{D} = \mathscr{L} - \mathscr{A} \cup \mathscr{B}$ is of the second category.

Corollary. — *For* $X \in \mathscr{D}$, *the corresponding* $\hat{L}$*'s have* $\hat{O}$ *as a distinguished point.*

Proof. — We only need distinguish $\hat{O}$ from the other points of $\hat{W}$. But for $X \in \mathscr{D}$, $\hat{O}$ definitely has no neighborhood of the form $M \times I$, as $\hat{W}$ makes arbitrarily close " passes " at $\hat{O}$, in a hyperbolic manner; $\hat{O}$ is clearly the only such point.

We close with the remark that $\hat{O}$ is *not* distinguished at least in this way, in the periodic, periodic case alluded to above, in Section 4.

7. Annular words and a pre-zeta function.

We show below that the special words

$$\{p_0(\Lambda) : \Lambda \text{ is a closed orbit of } \varphi\} \subset \pi_1(B, \mathcal{O})$$

not only characterize the topological conjugacy class of $\hat{\varphi}$, but even the homeomorphism class of the Lorenz attractor $\hat{L}$. We use the obvious fact that a periodic orbit Λ lies in an annulus A lying in turn in $\hat{L}-\hat{W}$. Conversely:

Proposition. — If A *is an annulus lying in* $\hat{L}-\hat{W}$, *then* A *can be deformed in* $\hat{L}-\hat{W}$ *to an annulus* A′ *whose central circle is a periodic orbit* Λ *or (exceptionally) to an annulus* A′ *where one edge of* A′ *is a saddle connection.*

The proof requires several steps, beginning with the

Definition. — By an *annular word* in $\pi_1(B_0, \mathcal{O})$ is meant a word of the form $p_0(\alpha \circ S^1 \circ \alpha^{-1})$ where S^1 is the central, simple closed curve (no retracing allowed) of an annulus $A \subset \hat{L}-\hat{W}$ and α is an arc lying in $\hat{F}$, joining O to a point of S^1.

Lemma. — Annulus words are monotonic—i.e. *all their exponents are of the same sign.*

Proof. — Let $p_0(\alpha \circ S^1 \circ \alpha^{-1})$ be an annular path in B_0. We think of B_0 as $\mathcal{O}$ together with two directed loops x and y, attached by a slight abuse of notation.

We can obviously deform S^1 up in $\hat{L}-\hat{W}$ so that our path $p_0(\alpha \circ S' \circ \alpha^{-1})$ has no doubling back in the middle of the arcs x and y.

Now let β be the last half of x, ending in $\mathcal{O}$ and γ the last half of y. Then $\beta \circ \gamma^{-1}$ is a path in B_0, but no part of our path $p_0(\alpha \circ S' \circ \alpha^{-1})$ could be like $\beta \circ \gamma^{-1}$, since in $\hat{L}$, O separates $p_0^{-1}(\beta-\mathcal{O})$ from $p_0^{-1}(\gamma-\mathcal{O})$; $p_0^{-1}(\beta-\mathcal{O})$ is the "back half" of $\hat{L}$ and $p_0^{-1}(\gamma-\mathcal{O})$ the "front half". That is, such an S^1 would have to be "tangent" to $\hat{W}$ at $\hat{O}$, which is not allowed in annular words. Similarly $p_0(\alpha \circ S^1 \circ \alpha^{-1})$ cannot contain a segment like $\gamma^{-1} \circ \beta$.

So suppose $p_0(\alpha \circ S' \circ \alpha^{-1})$ contains a bit like β. Then the next portion is either an x or a y and thus ends in β or γ. By induction all parts are positive. The cases where $p_0(\alpha \circ S^1 \circ \alpha^{-1})$ contains a portion like β^{-1} or γ^{-1} are quite similar.

Proof of the proposition. — Thus *via* Lemma 1 we have deformed S^1 and A into S′ and A′ so that the word $p_0(\alpha \circ S' \circ \alpha^{-1})$ is, say, positive. Thus S′ can be taken to be transverse to the orthogonal trajectories (used in the definition of p_0). Let $\sigma_0 \ldots \sigma_{n-1} \in C(B)$ be the 1-cells of $\hat{F}$ that A′ intersects, say $\sigma_i = \{\sigma_{ji}\}_{j=0}^{\infty}$. Then the symbols $\sigma_{00}\sigma_{01} \ldots \sigma_{0,n-1}\sigma_{00}$ form an allowable word in our symbol space. In particular this means that σ_{0i} flows *onto* $\sigma_{0,i+1}$ (and possibly more), $i+1$ taken mod n.

Consider the composite map

$$\sigma_0 \to \sigma_1 \to \ldots \to \sigma_{n-1} \to \sigma_0,$$

where we think of the symbol $[i, j]$ as being the interval $[\ell_i, r_j]$. This map has either

a) a unique fixed point $a \in \operatorname{int} \sigma_0$, or

b) the right end points of the σ_i are permuted cyclically, or

c) the left end points of the σ_i are permuted cyclically.

In case $a)$, one can easily further deform our annulus A′ to A″, an annulus centered about the periodic orbit through a. Note that the minimum period of a is n, just by the geometry: that is, no circle embedded in an annulus goes around it more than once.

For case 2, recall that the only right end points are the points $\hat{r}_j$, and that $\hat{r}_j \to \hat{r}_{j+1}$, unless $r_j = 0$. Thus, in this case, we can renumber the σ_{0i}'s so that $\hat{r}_i$ is the right end point of σ_{0i}, and

$$(*) \qquad \hat{\mathrm{O}} = \hat{r}_0 \to \ldots \to \hat{r}_{n-1} \to \hat{r}_0 = \hat{\mathrm{O}}$$

where the first and last involve an infinite amount of the parameter t.

Thus we can further deform A′ to an annulus A″ intersecting the 1-cell σ_i in its right hand half, so that the right edge of A″ is the saddle connection $(*)$. In particular the annular word of A is the word given by this saddle connection. Case 3 is similar to Case 2, so this completes the proof of the proposition.

For later use, we preserve a bit more of the technical details of our proof. First a

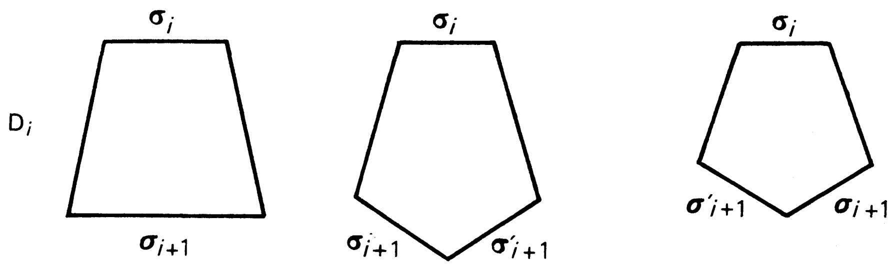
Definition. — Let $\Lambda(w)$ be the periodic orbit in Case 1 and the saddle connection (thought of as a loop based at O) in Case 2 and 3.

Definition. — To each annular word w we have associated a sequence

$$\sigma_0 \to \sigma_1 \to \ldots \to \sigma_{n-1} \to \sigma_0$$

of 1-cells of $\hat{\mathrm{F}}$ so that σ_i maps onto σ_{i+1} (mod n), and perhaps more, under the Poincaré map $\hat{f}$. Then, for each i, let D_i denote the disk formed by moving the closed 1-cell σ_i around to its first return in $\hat{\mathrm{F}}$, under the flow $\hat{\varphi}$.

There are three cases, one in which $\sigma_i \to$ only to σ_{i+1}, and two cases in which $\sigma_i \to \sigma_{i+1}$ and σ'_{i+1}:



We define $\mathrm{A}(w) = \mathrm{D}_0 \cup \mathrm{D}_1 \cup \ldots \cup \mathrm{D}_{n-1}$ and note the interior $\mathrm{A}^0(w)$ if $\mathrm{A}(w)$ is a topological annulus. The boundary consists of $\bigcup_i \sigma'_i$, and portions of $\hat{\mathrm{W}}$—where portions of $\hat{\mathrm{W}}$, including $\hat{\mathrm{O}}$, are perhaps included in several of the disks D_i.

Lemma. — *For each annular word w, there is a path $\Lambda(w)$ determined in $\hat{\mathrm{L}}$, which is either a periodic orbit or a saddle connection; $\Lambda(w)$ lies in a special annulus $\mathrm{A}(w)$ which is the unstable manifold $\mathrm{W}^u(\Lambda(w))$. Moreover $\mathrm{W}^u(\Lambda(w))$ is the arc component of $\hat{\mathrm{L}} - \hat{\mathrm{W}}$ containing $\Lambda(w)$.*

Proof. — We have already proved everything here except the parts about the unstable manifold. But since f is expanding on each 1-cell and since $\Lambda(w)$ is an orbit, it is clear that $A(w) \subset W^u(\Lambda(w))$.

This leaves only the last statement to be proved; but clearly the orbits leaving $A(w)$ exit through and cover the σ_i' mentioned above. This last means that $W^u(\Lambda(w))$ contains the successors of the σ_i', their successors, etc., so that it indeed contains the arc-component of $\hat{L}-\hat{W}$ containing $A(w)$.

Remark. — $W^u(\Lambda(w))$ has exponential growth.

Proof. — This is just a fact (4.2.2) about the matrix B.

We mention this exponential growth because in another paper [14] it was proved that certain types of attractors of dimension u carried u-dimensional homology classes. The technique was to show that the unstable manifolds had less than exponential growth. Note that $\hat{L}$ could carry no 2-dimensional class, as it does not separate $\mathbf{R}^3$. It inherits this last property from L, which obviously has it.

We proceed toward our pre-zeta function.

Definition. — Given Λ, a periodic orbit of φ, its projection $p_0\Lambda$ can be thought of as a positive word $w(\Lambda) \in \pi_1(B_0, \mathcal{O})$, determined up to cyclic permutation. Let

$$\eta(x, y) = \sum_{\Lambda} \sum_{\gamma} \frac{\gamma w(\Lambda)}{\ell(w(\Lambda))},$$

where the sum is over all closed orbits Λ, and for each Λ, all distinct cyclic permutations γ of the word $w(\Lambda)$. Here retracing an orbit Λ is allowed; however, this produces a periodic word, which thus has *fewer* permutations.

Remark. — $\exp \eta(t, t) = \zeta(t)$, the usual ζ-function of the Poincaré map f.

Proof. — Suppose $z \in \mathrm{Fix} f^n$. Say z has minimal period p and $n = pq$. Then the orbit Λ through z determines a word $w(\Lambda)$ of length p. If we retrace it q times we get a contribution of

$$(*) \qquad \sum_{\gamma} \frac{1}{pq} \{\underbrace{\gamma w(\Lambda) . \gamma w(\Lambda) \ldots \gamma w(\Lambda)}_{q\text{-times}}\}$$

to $\eta(x, y)$ where γ is a cyclic permutation of $w(\Lambda)$. The other permutations of the word $\underbrace{w(\Lambda) \ldots w(\Lambda)}_{q\text{-times}}$ are duplicates and don't count. By the usual definition of ζ, the orbit of x counts as p points and contributes p to N_n, $N_n = \mathrm{card}(\mathrm{Fix} f^n)$, which is n times the coefficient of t^n in $\log \zeta(t)$. Evaluating $(*)$ at $(x, y) = (t, t)$ we get

$$\frac{pt^n}{pq} = \frac{pt^n}{n}, \quad \text{as required.}$$

Then $\eta(x,y)$ seems quite natural; we have forced its definition a bit to make it correspond to ζ. In turn, this makes for the formula (where no saddle connections occur)

$$\eta = \sum_i \operatorname{tr} B^i,$$

proved below in § 10, along with various other computations.

Definition. — We write $\eta < \eta'$ if η' has a word w' as summand such that $w < w'$ for all summands w of η.

8. Relations between η and k.

We show here that the correspondence between kneading sequences and η-functions is order preserving and hence one-to-one. A better result would be a formula, giving one in terms of the other. Such a formula has been given elsewhere [21, 22]. It would lead us too far afield to describe it in detail. Briefly, a periodic word w occurs in η iff $k_\ell < uuu \ldots < k_r$, for any cyclic permutation u of w.

Proposition. — *If* L, φ *and* L$'$, φ' *determine two Lorenz attractors, then* $k(\varphi) < k(\varphi')$ *iff* $\eta(\varphi) < \eta(\varphi')$.

Proof. — First consider the case that $k(\varphi) < k(\varphi')$. By symmetry and the facts about saddle connections, we need only deal with the case where $k_r(\varphi)_i = k_r(\varphi')_i$ $i < n$, $k_r(\varphi)_n = x$, $k_r(\varphi')_n = y$.

Then $\hat{W}_r^{u'}$, considered as a path, passes successively through $\hat{r}'_0, \hat{r}'_1, \ldots, \hat{r}'_n$, so that there are allowable symbols

$$[*, 0]' \to [*, 1]' \to \ldots \to [*, n]'$$

in Σ', where the asterisks mean that we are not concerned with this part of the symbols. But by the indecomposability of B$'$ (4.2) we can complete this sequence to a periodic word. This is then an annular word, and taking a cyclic permutation, we may suppose that our annular word w' begins with the first n symbols of $k_r(\varphi')$. Then clearly $w' > \hat{W}_r^u(\varphi)$ considered as a path, and this in turn exceeds or equals any possible path in $\hat{L}$, $\hat{\varphi}$, as other orbits get pushed to the left by $\hat{W}_r^u$.

We conclude the proof by considering the case $k_r(\varphi) = k_r(\varphi')$ and $k_\ell(\varphi) = k_\ell(\varphi')$. This quickly implies that the points $\{r_1, r_2, \ldots, \ell_1, \ell_2, \ldots\}$ are in the same relative positions as are the $\{r'_1, r'_2, \ldots, \ell'_1, \ell'_2, \ldots\}$. For any disparity in order will become greater, until something like $r_i < 0 < r'_i$ occurs, which will contradict the fact that $k_r(\varphi) = k_r(\varphi')$. But this now means that the symbols $\Sigma = \Sigma'$, and the matrix $B = B'$, so that in turn $\eta(\varphi) = \eta(\varphi')$, as required.

9. Homeomorphic Lorenz attractors.

Throughout this section we suppose we have given two systems L, φ and L$'$, φ' and a homeomorphism $h : \hat{L} \to \hat{L}'$ from the attractors they determine. We emphasize that we do *not* assume that h is related to the flows, φ, φ'.

Step 1. — It follows that $h|\hat{W}$ maps $\hat{W}$ homeomorphically onto $\hat{W}'$. Next we can deform h so that $h(\hat{O})=h(\hat{O}')$. This is automatic in case φ (or φ') is in $\mathscr{D}$ (§ 6) or if there is a saddle connection. Then by the barycentric α approximation theorem [IX; 2], we have the factorization $\bar{h}\circ q_t$ of the map $q_0'\circ h$

$$\begin{array}{ccccc} & & (\hat{L},\hat{O}) & \xrightarrow{h} & (\hat{L}',\hat{O}') \\ & \swarrow^{q_0} & \downarrow q_t & & \downarrow q_0' \\ (L,O) & \xleftarrow[\varphi_t]{} & (L,O) & \xrightarrow[\bar{h}]{} & (L',O') \end{array}$$

Here q_t is the projection onto the t-th coordinate—*i.e.* $q_t(\hat{x})=\hat{x}(-t)$. The rectangle is commutative up to homotopy. The triangle to the left is commutative by the definition of inverse limits.

Taking π_1, and adding the projections onto B_0, $\mathcal{O}$ and B_0', $\mathcal{O}'$, we get

$$\begin{array}{ccccc} \pi_1(\hat{L},\hat{O}) & & \xrightarrow{h} & & \pi_1(\hat{L}',\hat{O}') \\ \downarrow q_{0*} & & & & \downarrow q_{0*}' \\ \pi_1(L,O) & \underset{\approx}{\xleftarrow{1}} & \pi_1(L,O) & \xrightarrow[\bar{h}]{} & \pi_1(L',O') \\ \downarrow \approx & & & & \downarrow \approx \\ \pi_1(B_0,\mathcal{O}) & & \xrightarrow[J]{} & & \pi_1(B_0',\mathcal{O}') \end{array}$$

In an earlier version as well as in [21], it was claimed that J is either the identity or interchanges x and y. This is incorrect, though an example would lead us too far afield to reproduce here. This is not needed in our counterexample to $\omega\Sigma$; for the principal theorem we have added an assumption which clearly guarantees that J is the identity:

Remark. — If $h:\hat{L}\to\hat{L}'$ is within Δ (see the introduction) of the identity, then J is the identity.

Proof. — For $x\in L$, x and $\bar{h}x$ are never on opposite sides of a hole in L. Hence there is a deformation of $\bar{h}$ to the identity. Thus $\bar{h}$ is the identity on π_1 and the Remark follows.

Proof of the Main Theorem. — As h is a homeomorphism it sends annuli to annuli. Thus $h(\Lambda(w))=\Lambda'(J(w))=\Lambda'(w)$. Hence $\eta'(x,y)=\eta(x,y)$ so that $k=k'$.

We conclude this section with a remark that is proved just as the lemma in § 7:

Remark. — $J(x)$ is either entirely positive or entirely negative as a word in x, y.

10. $\omega\Sigma$ is false.

Suppose we are given a second category set of vector fields, which we intersect with the set $\mathscr{L}$ of the Introduction, and then call $\mathscr{C}$. By § 5, we have an arc $[-\varepsilon, \varepsilon]$ of vector fields surrounding each point $X \in \mathscr{L}$. Furthermore, this correspondence can be taken to be continuous, with a little care in choosing the perturbation. In fact, for each $X \in \mathscr{L}$ we define $X_u = X + Y_u$, for $u \in [-\varepsilon, \varepsilon]$, where Y_u is independent of X.

Lemma. — *For some* $X \in \mathscr{L}$, $[-\varepsilon, \varepsilon]_X \cap \mathscr{C}$ *is second category in* $[-\varepsilon, \varepsilon]_X$.

Proof. — Say $\mathscr{C} = \bigcap_n \mathscr{C}_n$ where $\mathscr{C}_n$ is open and dense. Choose a countable basis $\{U_n\}$ of open sets in $\mathscr{L}$. Then for each integer i and each pair j, k such that $\overline{U}_k \subset U_j$, the set

$$X_{ijk} = \{X \in \mathscr{L} : [-\varepsilon, \varepsilon]_X \cap \overline{U}_k \neq \emptyset \text{ and } [-\varepsilon, \varepsilon]_X \cap U_j \cap \mathscr{C}_i = \emptyset\}$$

is closed. If the lemma is false, $\bigcup_{i,j,k} X_{ijk} = \mathscr{C}$ so that some X_{ijk} contains an open set, say $\mathscr{V}$. But then

$$\{X + Y_u : X \in \mathscr{V} \text{ and } u \in (-\varepsilon, \varepsilon)\} \cap U_j$$

is a non empty open set which does not intersect $\mathscr{C}_i$. This is a contradiction.

Hence $[-\varepsilon, \varepsilon]_X$ is uncountable, so that, for $\omega\Sigma$ to be true, there must be an uncountable set U such that the Lorenz attractors for all X_u, $u \in U$, are mutually homeomorphic. To simplify the notation, we assume $O \in U$. Then for each $u \in U$, we have the diagram

$$\begin{array}{ccc} \pi_1(\hat{L}(\varphi_u)) & \xrightarrow{h_u} & \pi_1(\hat{L}(\varphi_0)) \\ \Big\downarrow{\scriptstyle p_u} & & \Big\downarrow{\scriptstyle p_0} \\ \pi_1(B_0(\varphi_u)) & \xrightarrow[J_u]{} & \pi_1(B_0(\varphi_0)) \end{array}$$

where we have left out the base point, though it is always our special point O, resp. $\mathcal{O}$. Now the map J_u is determined by two words in (x, y), and thus there are only countably many of them. Hence there are two values $u, u' \in U$ (actually uncountably many) so that $J_u = J_{u'}$. Then the diagram

$$\begin{array}{ccc} \pi_1(\hat{L}(\varphi_u)) & \xrightarrow{h_{u'}^{-1} \circ h_u} & \pi_1(\hat{L}(\varphi_{u'})) \\ \Big\downarrow{\scriptstyle p_u} & & \Big\downarrow{\scriptstyle p_{u'}} \\ \pi_1(B_0(\varphi_u)) & \xrightarrow[J_{u'}^{-1} J_u]{1} & \pi_1(B_0(\varphi_{u'})) \end{array}$$

commutes.

It follows that an annulus $A(w)$, corresponding to the annular word w, maps to an annulus $A'(w)$ corresponding to the *same* word w. Thus $\eta(\varphi_u)=\eta(\varphi_{u'})$, so that $k(\varphi_u)=k(\varphi_{u'})$. But this contradicts the fact that $k(\varphi_u)\neq k(\varphi_{u'})$, by the proposition (§ 5) that says the map $[-\varepsilon, \varepsilon]\to k$ given by $u\mapsto k(\varphi_u)$ is order preserving.

11. Computations of certain η's.

An advantage of the usual ζ-function is its computability in lots of intersecting cases [1, 4, 13, 16, 17]. In some sense, our η is almost as computable, which we illustrate by the following remarks.

Remark 1. — $\eta(x,y)=\sum_i \frac{\operatorname{tr} B^i}{i}\in \mathbf{Z}[[x,y]]$.

Proof. — This is completely formal, see [1], once we get used to multiplying without commutativity. We illustrate with $B(x,y)=\begin{bmatrix} 0 & x & 0\\ 0 & 0 & x\\ y & y & 0\end{bmatrix}$. Then

$$B^2=\begin{bmatrix} 0 & 0 & x^2\\ xy & xy & 0\\ 0 & yx & yx\end{bmatrix}$$

$$B^3=\begin{bmatrix} x^2y & x^2y & 0\\ 0 & xyx & xyx\\ yxy & yxy+y^2x & yx^2\end{bmatrix}.$$

Thus, to 3 terms $\eta=\frac{xy+yx}{2}+\frac{x^2y+xyx+yx^2}{3}+\dots$

It is also possible to retrieve the annular words: just reject any word which is periodic with period ≥ 1. The primitive η-function η_p:

Remark 2. — $\eta_p(x,y)=\sum_i \frac{\operatorname{doc}(\operatorname{tr} B^i)}{i}$ where doc(polynomial in x,y) means:

(1) discard all periodic words (of period >1), and

(2) replace each word by the cyclically equivalent, biggest word.

Remark 3. — If there are two saddle connections, then

$$\eta_H=\frac{(1-W_r^u)(1-W_l^u)}{\det(I-B)}.$$

Here we abelianize to $H_*(B_0, \mathcal{O})$; W_r^u and W_l^u are the words in x, y given by the saddle connection.

Remark 4. — Again in the saddle connection case

$$\zeta(t) = \frac{(1-t^a)(1-t^b)}{\det(I - tB(1, 1))}$$

where the saddle connections are of period a and b, respectively.

Proof. — Just set $x = y = t$ in Remark 3.

We close this section with the

Proposition. — *The periodic orbits are dense in each Lorenz attractor.*

Proof. — Let $\hat{x} \in \hat{L}$ be a point. Follow the orbit of $\hat{x}$ under $\hat{\varphi}$ to the first point $\hat{x}_0 \in \hat{F}$. Continue the orbit to the second $\hat{x}_1$, third $\hat{x}_2, \ldots, \hat{x}_n$ points in $\hat{F}$. Similarly, follow $\hat{x}$ backwards to $\hat{x}_{-1}, \ldots, \hat{x}_{-n} \in \hat{F}$. Then each $\hat{x}_i$ lies in a 1-cell I_i of $\hat{F}$ and each I_i has a first symbol, say σ_i. Then, by the indecomposability property of B (4.2), there are symbols

$$\sigma_n \to \sigma_{n+1} \to \ldots \to \sigma_m \to \sigma_{-n}.$$

Then the infinitely repeated, periodic word

$$\sigma_{-n} \ldots \sigma_m \sigma_{-n} \ldots \sigma_m \ldots \in C(B)$$

determines a periodic orbit Λ. Now Λ, in its passage through the 1-cells, agrees with the orbit of $\hat{x}_0$, as far as the first symbol of these 1-cells are concerned. Since this happens n-times in a row, in both directions, $\hat{x}_0$ must be quite near Λ. This is true as Λ is a hyperbolic orbit, of course.

Northwestern University.

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Manuscrit reçu le 22 février 1977,
Modifié le 7 décembre 1978.