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THE DENSITY AT INFINITY OF A DISCRETE GROUP OF HYPERBOLIC MOTIONS

by DENNIS SULLIVAN

Dedicated to the memory of Rufus Bowen

Imagine in the hyperbolic space $\mathbf{H}^{d+1}$ an infinite completely symmetrical array of points. We study here the distribution of these points at large distances from an observation point p. One can define the density at ∞ for the array viewed from p. For each p this density is a finite measure on S^d which is proportional to a certain power of the metric on S^d associated to p by radial projection. Thus denoting the measure by μ_p,

$$\frac{d\mu_p}{d\mu_q} = \left(\frac{\theta(p)}{\theta(q)}\right)^\delta, \quad \text{for } p,\ q \text{ in } \mathbf{H}^{d+1},$$

where $\theta(p)$ is the spherical metric associated to p. So μ behaves like a Lebesgue or Hausdorff measure of dimension δ.

The construction of the density, which was made by Patterson [4] for orbits of Fuchsian groups and is extended here to arrays in $\mathbf{H}^{d+1}$, makes sense for any discrete array of points and is invariant by the symmetry group Γ of the array in the sense that $\gamma_* \mu_p = \mu_{\gamma p}$, for γ in Γ.

It is a tautology that a density μ with these invariance properties is unique if and only if the action of Γ on S^d is ergodic relative to the measure class defined by μ.

In [5] we derived an ergodicity criterion for Lebesgue measure in terms of the divergence of the absolute Poincaré series

$$g_s(x, y) = \sum_\Gamma e^{-s(x, \gamma y)},$$

where $(x, \gamma y)$ is hyperbolic distance, at $s = d$. In fact this divergence was seen to be equivalent to the much stronger ergodicity of the action of Γ on $(S^d \times S^d) - \text{diagonal}$.

In the construction of the density μ above, the critical exponent of this series, $\delta(\Gamma)$, is the dimension δ for the density μ. (The series diverges for $s < \delta(\Gamma)$ and converges for $s > \delta(\Gamma)$.) The last theorem of this paper (§ 7, theorem 32) is that *divergence at* $\delta(\Gamma)$ *implies ergodicity of the action of* Γ *relative to* $\mu \times \mu$ *on* $(S^d \times S^d)$ — *diagonal for* $\delta > \frac{d}{2}$. In particular, if the series diverges at the critical exponent $\delta(\Gamma) > \frac{d}{2}$, then the Γ-invariant density at ∞ is canonical. The proof makes use of a Markov process constructed with the positive eigenfunction Φ (for the eigenvalue $-\lambda = -\delta(d-\delta)$) associated to the density μ, with $\Phi(p) =$ total mass of μ_p. The transition probabilities for the process are $e^{\lambda t} \frac{\Phi(y)}{\Phi(x)} p_t(x, y)$, where $p_t(x, y)$ are the transition probabilities for the usual Brownian motion on $\mathbf{H}^{d+1}$. The paths of the process starting at p hit ∞ with probability distribution $\frac{1}{\Phi(p)} \cdot \mu_p$, and the absolute Poincaré series of weight δ estimates the Green's measure of the process.

In an earlier section (§ 4) we construct from μ an invariant measure for the geodesic flow acting on (unit tangent spaces of $\mathbf{H}^{d+1}$)$/\Gamma$. *For this measure dm the geodesic flow is either ergodic or completely dissipative* (Theorem 14). This generalizes the case of Lebesgue measure, which is E. Hopf's theorem [8]. The proof is exactly the same as Hopf's once a simple estimate is verified.

This dichotomy " ergodic or dissipative " for the geodesic flow is equivalent to a geometric dichotomy concerning the support of the density μ. By construction the measures of μ live on the *topological limit set* of Γ, Λ_t. The geometric dichotomy is whether or not μ gives positive measure to the *radial limit set* $\Lambda_r \subset \Lambda_t$. (A point ξ of S^d belongs to Λ_r if a ray ending at ξ comes within a bounded distance of infinitely many points of the array (which is by the symmetry assumption an orbit of Γ).)

In fact consider the conditions:

(i) *the radial limit set* $\Lambda_r \subset \Lambda_t$ *has positive* μ*-measure*;
(ii) *the action of* Γ *on* $(S^d \times S^d)$ — *diagonal is ergodic*;
(iii) *the geodesic flow on the quotient* $\mathbf{H}^{d+1}/\Gamma$ *is ergodic*;
(iv) *the Poincaré series diverges at the critical exponent* $\delta(\Gamma) = \delta$;
(v) *the Markov process on the quotient* $\mathbf{H}^{d+1}/\Gamma$ *is recurrent.*

Then the first three are equivalent and imply the fourth. If $\delta > d/2$, *the fourth implies the fifth which implies the first three* (§ 4, § 5, § 7). *So if* $\delta > \frac{d}{2}$ *all five are equivalent.*

In § 2, § 3, § 5, § 6 we study the local properties of such densities (which are canonical in the above mentioned situation). One finds easily that there are many balls on S^d such that the ratio $\mu(\text{ball of radius } r)/r^\delta$ is bounded above and below. The best case (§ 3) is when Γ acts with a compact fundamental domain on the convex hull of the limit set. We say " Γ *is convex cocompact* ". This is equivalent to $\Lambda_{\text{radial}} = \Lambda_{\text{topological}}$. *Then the density* μ *is canonical and is just the* δ*-Hausdorff density on* Λ_t *which is then a positive finite measure in every metric.* This is Theorems 7 and 8, whose proof only depends on the existence of μ and § 2.

Such a theorem was proved first for quasi-Fuchsian surface groups by Bowen [1] using Markov partitions and equilibrium states in Gibbs formalism to construct the relevant measure. Sections § 1 and § 7 here have a certain rapport with his methods, and it would be interesting to understand more about this.

In the convex cocompact case one can also derive estimates on the array of points such as the following one: *If n_r is the number of orbit points in a ball of radius r about a fixed center,*

$$ce^{\delta r} \leq n_r \leq Ce^{\delta r}, \quad \text{for} \quad 0 < c, \ C < \infty.$$

These suggest that *the convex hull is like a hyperbolic space of real dimension* $\delta(\Gamma)+1$.

The relation between δ and the Hausdorff dimension can be further studied (§ 6) in the $\delta(\Gamma)$-*finite volume case*, *i.e.* the invariant measure for the geodesic flow has finite total mass. Using the Birkhoff ergodic theorem one finds that in this case $\delta(\Gamma)$ *is the Hausdorff dimension of the radial limit set.*

Such groups include finitely generated Fuchsian groups, a non-trivial fact depending on Patterson [4]. We find that:

(i) *for a finitely generated Fuchsian group the Hausdorff dimension of the entire limit set is* $\delta(\Gamma)$ (Corollary 26);

(ii) *for an arbitrary Fuchsian group the Hausdorff dimension of the radial limit set is* $\delta(\Gamma)$ (Corollary 27).

Part (i) was done by Patterson [4] assuming either no cusps or $\delta(\Gamma)$ is not in the interval $(1/2, 2/3)$. (Actually by Beardon [7] one knew cusps imply $\frac{1}{2} < \delta(\Gamma) \leq 1$).

We close by acknowledging our great debt to the two papers by Bowen and Patterson. The Patterson measure which is rather remarkable allows one to achieve a general version of Bowen's beautiful result.

1. Conformal densities on the limit set.

A *conformal density of dimension* δ on a manifold V is a function which assigns a positive finite measure $\mu(\rho)$ to each element ρ in a non empty collection of Riemann metrics on V. It is assumed that if ρ and ρ' are conformally the same, *i.e.* $\rho = \varphi\rho'$ where φ is a positive function, then $\mu = \mu(\rho)$ and $\mu' = \mu(\rho')$ belong to the same measure class (1) and the Radon-Nikodym ratio $\frac{d\mu}{d\mu'}$ is φ^δ. Thus $\frac{d\mu}{d\mu'} = \left(\frac{\rho}{\rho'}\right)^\delta$. Note that given a measure μ and a metric ρ on V we can use the formula to define a conformal density on the set of metrics conformally the same as ρ.

(1) The measure class of a measure picks out the sets of positive measure.

One example of a conformal density defined on the class of all Riemann metrics is the following. Suppose a Borel subset C of a compact manifold V has finite positive Hausdorff δ-measure in one Riemann metric (and therefore all). Then

$$\text{(Riemann metric)} \mapsto \text{(Hausdorff } \delta\text{-measure on C)}$$

is a conformal density of dimension δ [1].

This is clear because given a metric, Hausdorff measure (with gauge function r^δ) is constructed from the numbers $\sum_i r_i^\delta$, where the r_i are the radii of tiny metric balls covering subsets $A \subset C$. If we change the metric by a continuous function φ, the factor $\varphi(x)^\delta$ comes out for those terms near x. Actually, this conformal variance of dimension δ is true even if we use a more general gauge function such as $r^\delta(\log 1/r)^{-\delta'}(\log\log 1/r)^{-\delta''}\ldots$.

Now we will employ an ingenious construction of Patterson [4] to obtain a conformal density of a certain dimension δ on the topological limit set Λ_t of an arbitrary infinite discrete group Γ of hyperbolic motions in $\mathbf{H}^{d+1}$. The conformal density μ will assign a finite measure to each of the metrics on S^d obtained by radially projecting the unit tangent sphere from the various points x of $\mathbf{H}^{d+1}$. This conformal density μ will be invariant under Γ in the sense that $\gamma_*\mu(\rho)=\mu(\gamma_\rho)$, for all γ in Γ (γ_ρ is the metric such that γ is an isometry between ρ and γ_ρ).

For each x in $\mathbf{H}^{d+1}$ we now construct, following Patterson [4], a measure μ_x by looking from x at the orbit under Γ of some point y. We are interested in how the orbit looks near ∞ (as viewed from x). A unit object at the point γy appears from x to have size $e^{-(x,\gamma y)}$ where $(x, \gamma y)$ is the hyperbolic distance. Thus in dimension δ we want to associate the scale factor $e^{-\delta(x,\gamma y)}$ to the point γy. To construct something really at ∞ we proceed as follows.

For s a positive real number consider the infinite [2] series

$$(1) \qquad g_s(x,y)=\sum_{\gamma\in\Gamma} e^{-s(x,\gamma y)}, \qquad (x,\gamma y) \text{ the distance in } \mathbf{H}^{d+1}.$$

For x and y fixed this series is proportional to [3]

$$(2) \qquad \sum_{k=0}^{\infty} s_k e^{-ks}$$

where s_k is the number of orbit points in a half-open shell of radii in $\left(k-\frac{1}{2}, k+\frac{1}{2}\right]$ centered about x.

(1) The function on the space of all Riemannian metrics

$$\text{metric} \mapsto \text{Hausdorff } \delta\text{-mass of C}$$

seems an interesting one (compare § 7).

(2) Since the log of the derivative of γ in the Euclidean metric of the unit ball model of $\mathbf{H}^{d+1}$ is proportional to $(x, \gamma x)$ this series is proportional to the absolute Poincaré series of weight s associated to Γ.

(3) Two functions are « proportional » if their ratio is bounded above and below by finite, positive constants.

The series (2) converges for $s>\delta$ and diverges for $s<\delta$, where

$$\delta=\overline{\lim_{k\to\infty}}\frac{1}{k}\log s_k. \tag{3}$$

Since Γ is discrete, $s_k\leq ce^{dk}$ for some constant depending on the minimal separation of the orbit points Γy. Thus $\delta\leq d$.

If we define n_k to be the number of orbit points in the closed ball of radius $k+\frac{1}{2}$ about x, then $n_k=\sum_{i=0}^{k}s_i$, so that we may also write

$$\delta=\overline{\lim_{k\to\infty}}\frac{1}{k}\log n_k. \tag{4}$$

Using the triangle inequalities $(x,\gamma y)\leq(x,y)+(y,\gamma y)$ and $(x,\gamma y)\geq(y,\gamma y)-(x,y)$ yields

$$e^{-s(x,y)}g_s(y,y)\leq g_s(x,y)\leq e^{s(x,y)}g_s(y,y). \tag{5}$$

In particular δ depends not on x or y but only on the discrete group Γ. We call δ the *critical exponent* of the group Γ.

For simplicity of the discussion now (and later for mathematical reasons) let us *assume* that the series (1) diverges at the critical value δ. Thus since all terms are positive

$$\lim_{s\to\delta}\sum_{\Gamma}e^{-s(x,\gamma y)}=\infty, \qquad \text{for } s>\delta,$$

and this is true for all x, y using (5).

Consider the family of measures

$$\mu_s(x)=\frac{1}{g_s(y,y)}\sum_{\Gamma}e^{-s(x,\gamma y)}\delta(\gamma y) \tag{6}$$

where $\delta(\gamma y)$ is the unit Dirac mass at γy. The total mass of these measures is bounded above and below independently of s, using (5).

Let $\mu(x)=\lim_{s_i\to\delta}\mu_{s_i}(x)$ denote a weak limit of these in the space of measures on $\mathbf{H}^{d+1}$ compactified by S^d, the sphere at ∞. Since $g_s(y,y)\to\infty$ as $s\to\delta$, $\mu(x)$ is concentrated on the cluster points of the orbit $\Gamma(y)$. Thus $\mu(x)$ is a measure on the topological limit set $\Lambda_t(\Gamma)$.

If x' is another point we claim that $\lim_{s_i\to\delta}\mu_{s_i}(x')$ converges to $\mu(x')$, an equivalent measure, and the ratio $\frac{d\mu(x')}{d\mu(x)}$ is just $e^{s(x,x')_\xi}$ where ξ belongs to S^d, and $(x,x')_\xi$ is the signed distance between the horospheres based at ξ and passing through x and x', respectively.

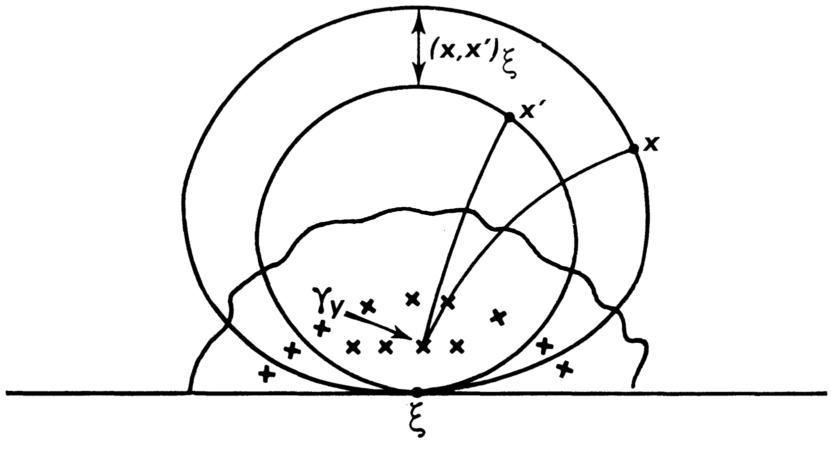


FIG. 1

The point is that for orbit points γy near to ξ (in the compactified space) the difference $(x, \gamma y) - (x', \gamma y)$ is approximately the horospherical distance $(x, s')_\xi$. Thus for those terms near ξ for either $\mu_s(x)$ or $\mu_s(x')$ the ratio of the coefficients of $\delta(\gamma y)$ are all nearly $e^{-s(x,x')_\xi}$. As $s_i \to \delta$ only terms near S^d count and this proves the claim.

Now if x and x' are related by an isometry η in Γ, $x' = \eta x$, then η on S^d is an isometry between the metric associated to x and the one associated to x'. Thus we have:

Theorem 1. — *There is a conformal density of dimension* $\delta(\Gamma)$ *on the topological limit set which is invariant by* Γ.

This result for Fuchsian groups is due to Patterson [4]. To complete the proof here we should remark why the ratio at ξ of the metrics on S^d corresponding to x and x' is just $e^{(x,x')_\xi}$. In the upper half-space model with ξ at 0 move x and x' together using parabolic transformations to put them on the vertical ray from the origin.

Then use a homothety γ to bring these points ρ and ρ' together. The parabolic transformations have derivative 1 at $\xi = 0$ (in any metric) and the linear derivative of the homothety is $\gamma = e^{(\rho,\rho')}$ using the infinitesimal formula:

$$(\text{hyperbolic metric}) . (\text{vertical coordinate}) = (\text{euclidean metric}).$$

To completely finish the proof we also need to eliminate the assumption that the series (1) diverges at $s = \delta$. Following Patterson [4], we do this by increasing the weights of the Dirac masses $\delta(\gamma y)$ by factors $h(\text{distance}(x, \gamma y))$. We choose $h : \mathbf{R}_+ \to \mathbf{R}_+$ continuous, non decreasing, so that the series $\sum_k h(k) s_k e^{-ks}$ diverges at δ, and $h(k)$ has slow growth, namely, for $\varepsilon > 0$ and d bounded, $|(h(r+d)/h(r)) - 1| < \varepsilon$ for r sufficiently large.

Then in the previous calculation the new factor $h(d+r)/h(r)$ which emerges (d is approximately $(x, x')_\xi$ and r is $(x, \gamma y)$) becomes 1 as $s_i \to \delta$.

Note that if δ were zero, the various measures of one conformal density would all be equal. We would thus have a Γ-invariant measure on the limit set. It is easy to see that up to finite groups such a group is either elementary parabolic or elementary hyperbolic.

Corollary 2. — For any infinite discrete group Γ, $\delta(\Gamma)>0$ *unless* Γ *is a finite extension of the elementary hyperbolic group.*

This may be proved in other ways (Schottky subgroups) and was done so by Beardon [7] for Fuchsian and Kleinian groups. We note that, for a parabolic group of rank k, $\delta=\frac{1}{2}k$. This calculation follows since a piece of horocycle connecting two points at distance r has length proportional to $e^{\frac{1}{2}r}$, for r large.

2. The local properties of an invariant conformal density.

If μ is a finite measure on S^d with or without atoms and a metric is given, one can (1) choose $\nu<\text{mass}\,\mu$ and $\varepsilon>0$ so that every ball on S^d of radius $\leq\varepsilon$ has μ-measure $\leq\nu$. If μ has no atoms ν can be chosen arbitrarily small, otherwise it is chosen to be a little larger than the largest mass of an atom of μ. One can take ε to be less than the Lebesgue number of a finite subcover of the cover $\{U_\xi\}$, where U_ξ is a disk about ξ in S^d so that $\mu(U_\xi)\leq\nu$.

Now consider an invariant conformal density μ of dimension α for the group Γ. By Theorem 1 these exist for $\alpha=\delta$. Relative to the metric ρ_x on S^d associated to x in $\mathbf{H}^{d+1}$ we want to estimate $\mu_x(\text{ball of radius } r)/r^\alpha$. Note that an asymptotic property of these radii as $r\to 0$ will be unchanged if we change the metric conformally.

The geometric point of what follows is that a hyperbolic translation t preserving a geodesic ℓ through x in $\mathbf{H}^{d+1}$ uniformly expands a ball B on S^d centered about the expanded endpoint of ℓ and having radius $ce^{-(x,tx)}$. The non uniformity depends only on c and not on the distance (x, tx). By choosing (for given $\varepsilon>0$) $c=c(\varepsilon)$ large enough, the image of B will omit a ball on S^d of radius $\leq\varepsilon$. This picture follows easily by conjugating the translation acting on S^d to the affine expansion of Euclidean space $\mathbf{R}^d$ *via* stereographic projection.

We apply this statement to all the geodesics ℓ connecting x and $\gamma^{-1}x$, for γ in Γ. We can factor such a γ into a translation followed by a rotation about x. We conclude that the ball $B_\gamma(x)$ of radius $c(\varepsilon)e^{-(x,\gamma^{-1}x)}$ about the point ξ on S^d (which is the endpoint of the directed ray $x\to\gamma^{-1}x$) is uniformly expanded to engulf all of S^d except a ball of radius $\leq\varepsilon$. If ε is chosen relative to the ν above this image contains a definite proportion $\geq\dfrac{\text{mass}\,\mu_x-\nu}{\text{mass}\,\mu_x}$ of the mass of μ_x. Since the expansion rate is essentially

(1) Except for the case when μ consists of a single atom. This case can happen in our context only for elementary groups.

$e^{(x, \gamma^{-1}x)}$ we conclude the ratios (mass of μ in B_γ)/(radius B_γ)$^\alpha$, for γ in Γ, are bounded above and below independent of γ.

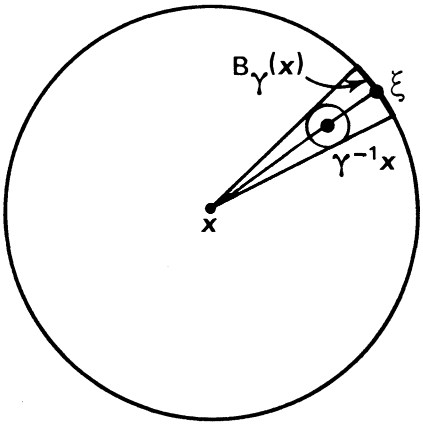


FIG. 2

So if μ is any Γ-invariant conformal density of dimension α which is not a single atom, we have:

Proposition 3. — *Given x in $\mathbf{H}^{d+1}$, write $r_\gamma = e^{-(x, \gamma^{-1}x)}$. There are balls in S^d, $\{B_\gamma\}$, of radius (constant) $. r_\gamma$ centered at the ends of the rays $x \to \gamma_x^{-1}$, for γ in Γ, such that $\mu_x(B_\gamma)/r_\gamma^\alpha$ is bounded above and below. Moreover, γ has an (essentially) uniform expansion on B_γ and the proportion of the mass of μ in γB_γ is as close as we like to* 1—((largest mass of atom of μ_x)/(mass μ_x)).

The balls B_γ on S^d for $(x, \gamma^{-1}x)$ in the interval $\left(k - \frac{1}{2}, k + \frac{1}{2}\right)$, for k large, cover the points ξ of S^d with a multiplicity m_ξ satisfying

$$0 \leq m_\xi \leq C \quad \text{(constant of proposition, minimal separation of points of orbit } x\text{)}.$$

This is so because the B_γ are the radial projections from x of balls of a fixed size placed along the orbit of x.

The sum of the μ-measures of these balls is proportional to the number s_k of orbit points in the shell, times the mass in each, $e^{-k\alpha}$. But this total is no more than the maximum multiplicity C times the mass of μ_x. Thus $s_k e^{-k\alpha} \leq c'$, or $s_k \leq c' e^{k\alpha}$. Summing this from 1 up to k yields, for $\alpha > 0$, $n_k \leq c e^{k\alpha}$. In particular $\alpha \geq \delta$ and we have:

Corollary 4. — *The critical exponent $\overline{\lim_{k \to \infty}} \frac{1}{k} \log n_k = \delta$ is the infimum (which is achieved) of the set of α for which there is a conformal density of dimension α invariant by Γ (the elementary parabolic groups excepted).*

Corollary 5. — *If $n_k(x)$ is the number of points in the orbit of x at distance at most k, then, for some constant c_x, $n_k(x) \leq c_x e^{\delta k}$ for $\delta = \delta(\Gamma) > 0$. In particular (see (9), § 4) the Poincaré series satisfies the upper bound*

$$\sum_{\gamma \in \Gamma} e^{-s(x, \gamma y)} \leq \frac{c_x}{s - \delta}, \quad \textit{for } s > \delta.$$

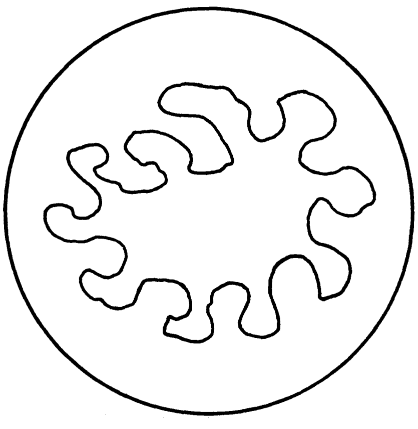
Corollary 6. — *If a discrete group* Γ *is written as a union of subgroups* Γ_α, *then* $\delta(\Gamma) = \sup_\alpha \delta(\Gamma_\alpha)$. *In particular* $\delta(\Gamma) = \sup \delta(\text{finitely generated subgroups})$.

To prove cor. 6, choose invariant conformal densities $\mu(\alpha)$ for Γ_α of dimension $\delta(\Gamma_\alpha)$. Choose a sequence of indices so that $\lim_{i\to\infty} \delta_i = \sup_\alpha \delta_\alpha = \delta'$ and $\lim_{i\to\infty} \mu_x(i)$ converges weakly to μ'_x. Any element γ in Γ is eventually in Γ_α and $\gamma_* \mu_x(i) = \frac{1}{|\gamma'|} \delta_i \mu_x(i)$ since $\mu(i)$ is an invariant conformal density. Letting $i \to \infty$ yields $\gamma_* \mu'_x = \frac{1}{|\gamma'|} \delta' \mu'_x$. Since this is true for each γ in Γ, μ'_x determines an invariant conformal density of dimension δ'. Thus $\delta' \geq \delta$. The reverse inequality $\delta' \leq \delta$ is obvious.

3. Convex cocompact groups and Hausdorff measure.

If $\Lambda \subset S^d$ is a closed set we can form the convex hull of Λ, $C(\Lambda)$, in hyperbolic space $\mathbf{H}^{d+1}$. In the projective model $C(\Lambda)$ is identical to the usual convex hull. If Λ is the limit set of a discrete group Γ, then $C(\Lambda)$ is invariant by Γ. We say that Γ is *convex cocompact* if this action of Γ on the convex hull $C(\Lambda)$ has a compact fundamental domain.

Finitely generated Fuchsian groups without cusps have this property. In $\mathbf{H}^{d+1}$ for all dimensions the condition " convex cocompact " is equivalent to the condition that the fundamental domain has finitely many sides and doesn't meet the limit set. Such groups arise from compact convex [1] hyperbolic manifolds with boundary. These form a rich class in dimension 3 and are the building blocks in Thurston's theory [9]. In a typical example of one of these the limit set is obtained by removing countably many open 2-disks (with non-rectifiable boundaries) from S^2.



quasi-fuchsian

typical

FIG. 3

[1] Every homotopy class of paths between two points in the manifold is represented by a geodesic in the manifold.

Now choose a Γ-invariant conformal density μ in Λ of dimension δ, the critical exponent of a convex cocompact group Γ. From the previous proposition 3 we can compute the ratios μ_x (ball of radius r)$/r^\delta$ for all balls centered at ξ on the limit set. To do this consider the ray from x (chosen in the convex hull $C(\Lambda)$) to ξ. There are points of the orbit of x within a bounded distance from every point of this ray because Γ has a compact fundamental domain in its action on $C(\Lambda)$. If we adjust the constant appropriately in the definition of the balls B_γ of § 2, then those B_γ for γ near the ray will contain the ball of radius εr_γ centered at ξ ($r_\gamma = \text{radius } B_\gamma$). As we go out along the ray these B_γ such that $B(\xi, \varepsilon r_\gamma) \subset B_\gamma$ shrink geometrically in size (on S^d) and for these $\mu(B_\gamma)/r^\delta$ is bounded above and below by Proposition 3. It follows from the inclusions $B(\xi, \varepsilon r_\gamma) \subset B_\gamma$ that, for all r, $\mu_x(B(\xi, r))/r^\delta$ is bounded above and below.

Since this is true for all ξ in the limit set we can evaluate the Hausdorff δ-measure on Λ. If $\bigcup_i B_i$ is a covering of a Borel subset $A \subset \Lambda$ by balls centered on Λ of radius r_i, then $\sum_i r_i^\delta \geq \text{constant} \sum_i \mu_x(B_i) \geq \text{constant} . \mu_x(\bigcup_i B_i) \geq \text{constant} . \mu_x(A)$. So the Hausdorff δ-measure of A, defined as the limit for $\varepsilon \to 0$ of the infimum of the sums $\sum_i r_i^\delta$ for all covers of A with $r_i \leq \varepsilon$, is certainly at least a constant times $\mu_x(A)$.

To get the reverse inequality, consider $\varepsilon > 0$ and construct a cover of Λ by balls B_1, $B_2, \ldots$ centered on Λ, such that (radius B_i) $\geq$ (radius B_{i+1}), (radius B_1) $\leq \varepsilon$, and the center of B_{i+1} is outside $B_1 \cup \ldots \cup B_i$.

It is clear that this can be done. Now the balls of $\frac{1}{2}$ the radii and the same centers are disjoint. Denote this disjoint union by B. So

$$\sum_i r_i^\delta = 2^\delta \sum_i \left(\frac{1}{2} r_i\right)^\delta \leq \text{constant} . \mu_x(B) \leq \text{constant} . \mu_x(\Lambda).$$

So Hausdorff δ-measure of Λ is $\leq$ constant times $\mu_x(\Lambda)$.

We can extend this to any subset $A \subset \Lambda$ of positive measure using density points. Namely almost all points of A satisfy $\lim_{r \to 0} \mu_x(B(a, r) \cap A)/\mu_x(B(a, r)) = 1$ (Federer [3], (2.9.11)). Thus there is a subset $A' \subset A$ with $\mu(A - A') < \varepsilon$ and an $r_0 > 0$ so that the above ratio is at least $1 - \varepsilon$ for all $r < r_0$ and a in A'.

Now the above argument for balls of radii $< r_0$ yields

$$\text{Hausdorff } \delta\text{-measure of } A' \leq \text{constant} . \mu_x(A).$$

Letting $\varepsilon \to 0$ yields

$$\text{Hausdorff } \delta\text{-measure of } A \leq \text{constant} . \mu_x(A).$$

Thus we have [1]:

Theorem 7. — For a convex cocompact group Γ, the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ is the critical exponent $\delta(\Gamma)$. The Hausdorff δ-measure of the limit set is positive and finite.

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[1] Theorem 7 for quasi-Fuchsian surface groups was done in Bowen [1] except for the identification of the Hausdorff dimension. The latter was done for finitely generated Schottky Fuchsian groups in Patterson [4] and in Bowen [1] for Kleinian Schottky groups.

Moreover, the Hausdorff δ-measure of the part of $\Lambda(\Gamma)$ in a ball of radius r centered on $\Lambda(\Gamma)$ is proportional to r^δ.

We recall the Hausdorff dimension of a set S is the infimum of the set of α where the Hausdorff α-measure of S is zero. This equals the supremum of the set of α where the Hausdorff α-measure is ∞. In particular if the Hausdorff α-measure is finite and positive for some α, then this α is unique and equals the Hausdorff dimension of the set S.

The proof showed that if μ was a Γ-invariant α-conformal density on Λ, then α was the Hausdorff dimension of Λ and the measure class of μ was that of Hausdorff measure. Thus $\alpha=\delta(\Gamma)$ and the measure class of μ is ergodic under the action of Γ. For if some subset of positive (but not full) measure were invariant we could restrict μ to this subset and obtain an inequivalent conformal density invariant by Γ.

This ergodicity implies that the confoımal density is unique. For if two such densities, μ and μ', are given, $\nu=\frac{1}{2}(\mu+\mu')$ is also one and the ratios $d\mu/d\nu$, $d\mu'/d\nu$ are Γ-invariant functions. Thus they are constant.

In particular any two limits, as $s_i\to\delta$, of the family $\mu_s(x)=\dfrac{1}{g_s(y,y)}\sum\limits_\Gamma e^{-s(x,\gamma y)}\delta(\gamma y)$ discussed in § 1 are equal. Recall (x,y) is the hyperbolic distance and $\delta(x)$ is the Dirac mass at x.

Theorem 8. — For a convex cocompact group Γ there is on $\Lambda(\Gamma)$ one and only one Γ-invariant conformal density of any dimension and this is Hausdorff density of dimension $\delta(\Gamma)$ on the limit set. In terms of the metric associated to x in $\mathbf{H}^{d+1}$ the δ-Hausdorff measure satisfies

$$\mu_x=\lim_{s\to\delta}\Big(\sum_\Gamma e^{-s(x,\gamma y)}\delta(\gamma y)\Big)\Big/\Big(\sum_\Gamma e^{-s(y,\gamma y)}\Big)$$

for any y in $\mathbf{H}^{d+1}$. In particular, the Poincaré series $\sum\limits_\Gamma e^{-s(y,\gamma y)}$ diverges at the critical value $s=\delta$.

The series $\sum\limits_{\gamma\in\Gamma}\exp(-s(y,\gamma y))$ diverges for s equal to the critical value $\delta=\delta(\Gamma)$ (in the convex cocompact case). This follows immediately because the balls B_γ contain coverings of Λ of arbitrarily small diameter. The amount of μ_x in each one is proportional (by proposition 3) to the corresponding term of the series for $s=\delta$. Thus arbitrarily far out in the series for $s=\delta$ we have a definite sum and thus divergence.

We note here that this divergence argument and the one on ergodicity (slightly reinterpreted) only uses that there are infinitely many orbit points at bounded distance from a ray connecting x to almost any point ξ in Λ. This will be formalized in § 5 on the radial limit set. But now we want to derive a more precise divergence estimate and a lower bound on the number of orbit points that really uses our convex cocompact hypothesis.

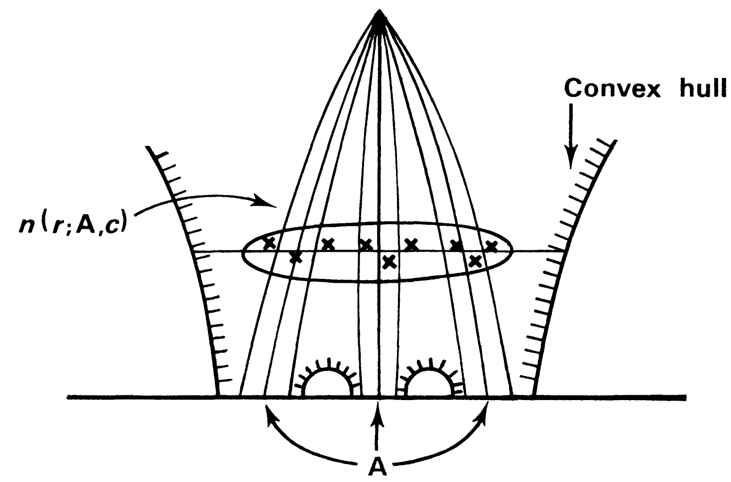

FIG. 4

Let A be some subset of the limit set $\Lambda(\Gamma)$ and let $n(r, A, C)$ be the number of points in the orbit of x so that the distance is at most C to the set A_r obtained from A by projecting radially onto the sphere of radius r about x. Since x was chosen to lie in the convex hull of the limit set and Γ is convex cocompact the balls about the orbit of x of radius D (sufficiently large) cover A_r. The shadows of these balls in S^d form a cover of A with bounded multiplicity. The μ_x-mass of one of these balls is proportional to $e^{-\delta r}$ by Proposition 3.

Thus for convex cocompact groups we have the estimates:

Theorem 9. — There are constants c and C *so that, for all subsets* A *of the limit set,*

$$n(r, A, C) \geq c(\text{Hausdorff } \delta\text{-measure } A) \cdot e^{\delta r},$$

where δ *is the Hausdorff dimension of* $\Lambda(\Gamma)$.

Taking A to be the entire limit set and using Corollary 1 we have:

Corollary 10. — There are constants c and C *so that* n_r, *the number of orbit points in the ball of radius* r, *satisfies*

$$ce^{\delta r} \leq n_r \leq Ce^{\delta r}.$$

These estimates suggest the following interpretation. Since Γ is acting with compact fundamental domain on the convex hull, the number of orbit points in a certain region of the convex hull can be compared to the volume of that region. In hyperbolic space $\mathbf{H}^{d+1}$ the volume of a ball of large radius r is about e^{dr} and the volume of the part of that ball in a cone from the center of spherical measure θ is about θe^{dr}.

Thus we should think of the convex hull as a " hyperbolic space " in its own right with the limit set playing the role of sphere at ∞, the Hausdorff measure playing the role of spherical measure.

In summary. — Volume computations in the convex hull of the limit set for a convex cocompact group Γ *behave like those in a hyperbolic space which has* real *dimension* $\delta(\Gamma)+1$.

For Kleinian groups Γ one can show in the convex cocompact case that the Hausdorff dimension of Λ, $D(\Lambda)$, satisfies $1<D(\Lambda)<2$ unless Γ is Schottky or Fuchsian or Λ is all of S^2.

This follows using Bowen [1] for quasi-circles, and the fact that if one removes a collection of circles of total finite length from a region, a positive area remains.

4. The associated invariant measure for the geodesic flow.

We start with the interesting identity expressing the geometric mean value theorem for a conformal transformation of Euclidean space $\mathbf{R}^{d+1}$:

$$|Ax-Ay|^2=(A'x)(A'y)|x-y|^2. \tag{7}$$

Here A is a conformal transformation, $A'x$ is the magnitude of the conformal distortion at x, and $|x-y|$ is the Euclidean distance. This relation can be checked for $d=1$ by writing A in the form $z\mapsto\dfrac{az+b}{cz+d}$. It then follows (David Freid) for inversions in higher dimensions since the plane of $(x, y,$ inversion point) is carried to itself by a linear fractional transformation. The general case now follows from the chain rule.

We apply this to conformal transformations preserving the unit sphere in $\mathbf{R}^{d+1}$. We can then write the same formula on S^d using the chordal metric on S^d.

Write $|\xi-\eta|$ for the chordal distance between two points on S^d and let μ be the measure associated by a given Γ-invariant conformal density to the round Riemann metric on S^d. If μ is not a single atom we can form the non-trivial Radon measure (1) $\nu=(\mu\times\mu)/|\xi-\eta|^{2\delta}$ on $(S^d\times S^d)$—diagonal.

Proposition 11. — The measure $\nu=(\mu\times\mu)/|\xi-\eta|^{2\delta}$ *on* $(S^d\times S^d)$*—diagonal is invariant under the product action of* Γ, $(\xi,\eta)\mapsto(\gamma\xi,\gamma\eta)$, *for* γ *in* Γ.

This follows immediately from (7) and $\gamma^*\mu=|\gamma'\xi|^\delta\mu$, which is true since μ arises from a Γ-invariant conformal density.

Now think of the manifold T of unit tangent vectors to $\mathbf{H}^{d+1}$ fibred by the geodesic flow lines over $(S^d\times S^d)$—diagonal, geodesic $\mapsto$ ($+\infty$ endpoint, $-\infty$ endpoint). We combine ν with arc length along geodesics to obtain a measure $\widetilde{dm}$ invariant under Γ and the geodesic flow g_t ($g_t v=$ the tangent vector to the oriented geodesic determined by v a distance $+t$ away from v). Since Γ acts discontinuously on T we can form a quotient space T/Γ with a quotient measure dm which is kept invariant by the quotient geodesic flow.

(1) A Radon measure gives finite mass to compact sets.

Corollary 12. — *A Γ-invariant conformal density on S^d determines an invariant measure for the geodesic flow on* {tangent vectors to $\mathbf{H}^{d+1}$}$/\Gamma$.

Let

$$\sigma : T/\Gamma \to \mathbf{R}^+ \tag{8}$$

be the distance from the base point of a tangent vector v to a fixed orbit in $\mathbf{H}^{d+1}$, say that of the center of the unit ball in $\mathbf{R}^{d+1}$. We want to show that $\overline{\lim\limits_{r\to\infty}} \frac{1}{r} \log dm(B_r) < \infty$ where $B_r = \sigma^{-1}[0, r]$.

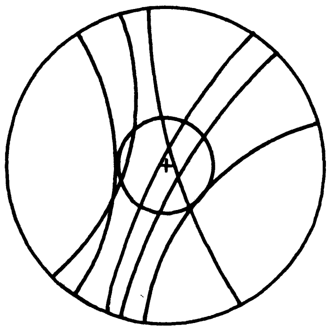

Fig. 5

Consider all geodesics passing through a ball of hyperbolic radius r about the center point. The hyperbolic length of each intersection of a geodesic with the ball is at most $2r$ and the chordal distance between the endpoints is at least (a constant)$.e^{-r}$. Thus the total $\widetilde{dm}$-mass of all these tangent vectors is at most a constant times $re^{2\delta r}$. On T/Γ this set covers B_r and this proves what we want.

Now consider the function $\rho = e^{-(2\delta+\varepsilon)\sigma}$ for $\varepsilon > 0$. Then in terms of the natural quotient metric on T/Γ we have that $\left|\frac{d\rho}{\rho}\right|$ is bounded and ρ belongs to $L^1(dm)$. For calculating the differential in $\mathbf{H}^{d+1}$ yields $\left|\frac{d\rho}{\rho}\right| \leq |2\delta+\varepsilon| . |d\sigma|$, but $|d\sigma| \leq 1$. And by construction, on $B_{n+1} - B_n$, ρ is at most $e^{-(2\delta+\varepsilon)n}$ and $B_{n+1} - B_n$ has dm-measure less than (constant)$.ne^{2\delta n}$. Thus the dm-integral of ρ on $B_{n+1} - B_n$ is at most $ne^{-\varepsilon n}$. Summing over Λ yields that ρ belongs to $L^1(dm)$. We record this as

Proposition 13. — *There is a positive dm-integrable function ρ on T/Γ which satisfies*

$$(\rho(x) - \rho(y))/\rho(y) \leq c.\text{distance } (x, y).$$

The function ρ allows us to prove the analog of E. Hopf's ergodic theorem for the geodesic flow:

Theorem 14. — *Assume the geodesic flow g_t on T/Γ is conservative for the measure dm associated to the Γ-invariant conformal density on S^d. Then g_t is ergodic for the measure dm.*

One may follow Hopf's clear exposition ([8], pp. 873-876) sentence by sentence. Conservative means that if $\rho>0$, then $\int_0^\infty \rho(g_t v)\,dt=\infty$ almost everywhere. The Hopf form of the Birkhoff ergodic theorem for a conservative flow with invariant measure dm (possibly infinite) states that

$$\lim_{\tau\to\infty}\frac{\int_0^\tau f(g_t v)\,dt}{\int_0^\tau \rho(g_t v)\,dt}=f_\rho(v),\qquad \text{for } f,\ \rho>0 \text{ in } L^1(dm),$$

exists almost everywhere, is constant on flow lines, and is constant for all f in $L^1(dm)$, ρ fixed, if and only if the flow is ergodic. The continuous functions with compact support are dense in $L^1(dm)$ and one computes (using Proposition 13) directly for one of these that $f_\rho(v)=f_\rho(v')$ if $g_t v$ and $g_t v'$ are asymptotic as $t\to+\infty$. The same is true for negatively asymptotic trajectories. The function f_ρ lifted to T determines a Γ-invariant function on $(S^d\times S^d)$ — diagonal which is almost everywhere constant on each factor $\mathrm{pt}\times S^d$ and $S^d\times \mathrm{pt}$. Thus it is constant almost everywhere by Fubini. By density f_ρ is constant for all f in $L^1(dm)$ and the generalized Hopf Ergodic Theorem is proved. For more details see [8].

Corollary 15. — *For the measure class on S^d determined by a Γ-invariant conformal density, the product action of Γ on $(S^d\times S^d)$ — diagonal is ergodic for the product measure class if and only if it is conservative.*

For the remainder of this paper we assume μ is of pure type ([1]). We note that g_t is conservative if almost all trajectories return for arbitrarily large times to a fixed bounded neighborhood B. For then there will be an induced measurable flow on B preserving the finite measure $dm|B$. This induced flow is conservative by the Poincaré recurrence theorem, and this implies g_t is conservative. In § 5 we show that the existence of such a bounded neighborhood is equivalent to the radial limit set having positive mass for the conformal density.

Conversely, we show there that if the radial limit set has measure zero then the product action of Γ on $(S^d\times S^d)$ — diagonal is completely dissipative. Equivalently the flow g_t on T/Γ is completely dissipative. Thus we have *in general* that the geodesic flow g_t is either *completely dissipative* or *ergodic*, relative to *dm*.

Now let us study the average time a trajectory spends in a given bounded set $\mathscr{B}$. Since the *dm*-measure of a bounded set $\mathscr{B}$ is finite and g_t is measure preserving, in the dissipative case almost all trajectories spend a finite amount of time in $\mathscr{B}$. Thus almost everywhere, $\lim \frac{1}{t}(\text{time spent in } \mathscr{B})=0$.

([1]) Pure type means either the measure has no atoms or is entirely comprised of atoms.

In the ergodic case, this limit exists almost everywhere and equals the constant $dm(\mathscr{B})/dm(\mathrm{T}/\Gamma)$ (by the ergodic theorem).

Corollary 16. — *The expected time a positive trajectory $g_t v$ of the geodesic flow on* T/Γ *spends in a bounded set $\mathscr{B}$, $\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \chi_{\mathscr{B}}(g_t v)\,dt$, exists for dm-almost all v in* T/Γ. *The value is zero unless dm is a finite measure, in which case the limit is* $dm(\mathscr{B})/dm(\mathrm{T}/\Gamma)$.

Let us take $\mathscr{B}$ to be all the tangent vectors in a ball B about x in $\mathbf{H}^{d+1}$ projected to T/Γ. A trajectory $g_t(v)$ for v in $\mathscr{B}$ lies again in $\mathscr{B}$ in T/Γ iff the ray in $\mathbf{H}^{d+1}$ of length t expanding v lies in a ball γB for some γ in Γ. Clearly the average time for a slightly larger ball is at least a constant times the expected number of visits to B which in turn is larger than a constant times the expected time in a slightly smaller ball.

Thus the average time spent in $\mathscr{B}$ up to time τ for a probability space of starting vectors v in $\mathscr{B}$ is proportional to $\frac{1}{\tau}\sum_{(x,\gamma x)\leq\tau} a_\gamma$ where a_γ is the proportion of starting vectors whose extended rays pass through the ball γB for $(x, \gamma x)\leq\tau$.

Furthermore, since the averages are bounded, we can interchange the orders of averaging over a probability space of starting vectors for which the expected time exists, and let $\tau\to\infty$.

Now the vectors in $\mathscr{B}$ are nicely sliced into families of those $\mathscr{B}_\xi$ with given $(-\infty)$-end point ξ in S^d, for the representative of v in the ball B about x. (In what follows we may have to choose a ball B so large that there are finitely many representatives of v. This complication is easily dealt with.)

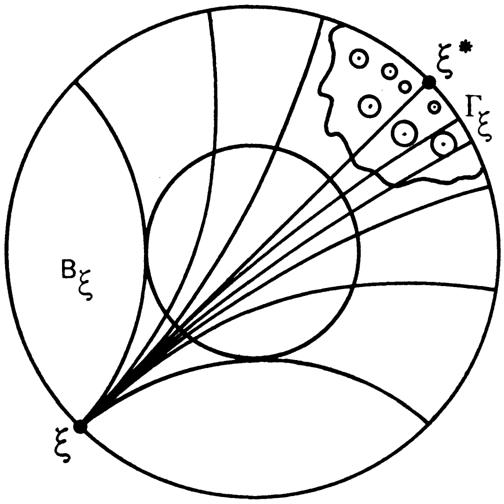

Fig. 6

Thus $\mathscr{B}=$(union of $\mathscr{B}_\xi$) with ξ in S^d. Since by definition dm is

$$(\text{arc length}) \mathbin{\widetilde{\times}} \frac{1}{|\xi-\eta|}\,2\delta(\mu_\xi\times\mu_\eta)$$

we see $dm|B$ is made of the measure μ_ξ on S^d for the parameter ξ, and of measures (arc length) $\widetilde{\times}\ \dfrac{1}{|\xi-\eta|} 2\delta\, d\mu_\eta$ constructed on each $\mathscr{B}_\xi$.

Let us consider the proportion a_γ of v in $\mathscr{B}_\xi$ (with the above measure) so that the ray extending v passes through γB. We will only consider a subset Γ_ξ of Γ consisting of γ such that γx is near enough the point ξ^* antipodal about x to ξ.

Then for example the arc length contribution to the measure is bounded from above and below for the proportion a_γ, for γ in Γ_ξ. The factor $\dfrac{1}{|\xi-\eta|} 2\delta$ is also bounded above and below for η the $(+\infty)$-endpoint of v in $\mathscr{B}_\xi$. So to compute the dm measure of the proportion a_γ we have to compute the ratio of $\mu(\gamma B)_\xi/\mu(B)_\xi$ where $(B')_\xi$ means the image by projection from ξ into S^d of the ball B'.

Now we will choose B large enough so that the denominator is a definite positive amount independent of ξ. Next, by considering only γx near enough to the point antipodal to ξ about x, the balls $(\gamma B)_\xi$ will be approximately the same as $(\gamma B)_x$, where $(B')_x$ means the ball obtained by projecting B' onto S^d from x.

By proposition 3, if B is chosen with radius in an interval $[R, R']$ large enough, the μ-mass of $(\gamma B)_x$ is proportional to $e^{-\delta(x,\gamma x)}$.

Thus choosing first R compatible with this condition and the one previous we then define " near to the antipodal point of ξ " so that $(\gamma B)_\xi$ will be trapped between the smallest and largest of the $(\gamma B)_x$.

Namely, $(\gamma B_R)_x \subset (\gamma B)_\xi \subset (\gamma B_{R'})_x$. Then the μ-mass of $(\gamma B)_\xi$ is proportional to $e^{-\delta(x,\gamma x)}$ for γ in the subset $\Gamma_\xi = \{\gamma : \gamma x$ is near to the antipode of $\xi\}$.

A finite number of the near regions covers infinity near the limit set. Thus we consider the probability space C consisting of a finite number of $\mathscr{B}_\xi$ and conclude that except for finitely many γ the proportion a_γ of the vectors in C whose rays pass through γB is at least a constant times $e^{-\delta(x,\gamma x)}$.

Applying the previous remark about interchanging the limit over τ and integration over C and assuming the finite number of $\mathscr{B}_\xi$ are chosen generically, so that the ergodic limits exist, we have (elementary groups included):

Theorem 17. — If dm on T/Γ *has infinite total mass, then, for* γ *in* Γ,

$$\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{(x,\gamma x)\leq\tau} e^{-\delta(x,\gamma x)} = 0.$$

Theorem 17 and Corollary 5 imply:

Corollary 18. — If the Poincaré series satisfies an inequality

$$\sum_\Gamma e^{-s(x,\gamma x)} \geq \frac{\text{constant}}{s-\delta}, \quad \textit{for } s>\delta,$$

then the measure dm on T/Γ *has finite total mass.*

From Corollary 5, $s_k = \text{card}\left\{\gamma : (x, \gamma x) \in \left(k - \frac{1}{2}, k + \frac{1}{2}\right)\right\}$ is at most constant$.e^{\delta k}$. This implies the series $\sum_{k=0}^{\infty} s_k e^{-sk}$ is term by term majorized by the geometric series constant$.\sum_{k=0}^{\infty} e^{(\delta - s)k}$ which sums to $\dfrac{\text{constant}}{s - \delta}$.

In particular, the Poincaré series, which is proportional to $\sum_{k=0}^{\infty} s_k e^{-sk}$, always satisfies the upper bound given in Corollary 5:

$$\sum_{\Gamma} e^{-\delta(x, \gamma x)} \leq \frac{\text{constant}}{s - \delta}. \tag{9}$$

Now estimate the tail $\sum_{k = c_\tau}^{\infty} s_k e^{-sk}$ (for c_τ conveniently chosen) from above to see that a lower bound of the form

$$\sum_{\Gamma} e^{-\delta(x, \gamma x)} \geq \frac{\text{constant}}{s - \delta}$$

implies $\dfrac{1}{\tau} \sum_{(x, \delta x) \leq \tau} e^{-\delta(x, \gamma x)} \geq \varepsilon > 0$. The latter inequality can only happen, according to Theorem 17, when T/Γ has finite dm-measure. This proves Corollary 18.

Now we want to estimate how far away a trajectory can get in time t. Using the function σ above (8), define a function ψ on T/Γ by $\psi(v) = (d\sigma, v)$, the directional derivative of σ in the direction v.

On T/Γ there is the canonical involution A defined on T by sending v to $-v$. The involution A sends trajectories to trajectories reversing direction and preserving arc length. Passing to the quotient of T by flow lines in $(S^d \times S^d)$—diagonal, A becomes the interchange of coordinates; A commutes with the action of Γ, so it passes to the quotient T/Γ. From the above description, A has the following properties:

(i) A preserves the measure dm;
(ii) A conjugates g_t to g_{-t}, $Ag_t = g_{-t}A$;
(iii) A reverses ψ, $\psi \circ A = -\psi$.

Now assume $dm(T/\Gamma) < \infty$. Then Properties (i) and (iii) imply $\int \psi \, dm = 0$ since ψ is bounded. By the ergodic theorem ($dm(T/\Gamma) < \infty$ implies conservative hence ergodic) for almost all v, $\lim_{\tau \to \infty} \dfrac{1}{\tau} \int_0^\tau \psi(g_t v)\, dt = \int_{T/\Gamma} \psi \, dm = 0$. But $\dfrac{1}{\tau} \int_0^\tau \psi(g_t v)\, dt = \dfrac{1}{\tau}(\sigma(g_\tau v) - \sigma(v))$, so we have:

Corollary 19. — If $dm(T/\Gamma) < \infty$, for almost all starting points v the distance of the point $g_t(v)$ from a fixed point satisfies $\lim_{t \to \infty} \dfrac{1}{t}$ (distance) $= 0$.

We conjecture that using property (ii) allows one to replace in Corollary 19 the assumption that $dm(T/\Gamma) < \infty$ by only the ergodicity of the geodesic flow.

5. Conformal densities supported on the radial limit set.

The set of cluster points in S^d of an orbit of Γ in $\mathbf{H}^{d+1}$, *i.e.* the topological limit set Λ_t, has the interesting topological property that every orbit of Γ in Λ_t is dense. For metrical properties such as ergodicity, or Hausdorff dimension and measure, the subset $\Lambda_r \subset \Lambda_t$ of radial limit points plays an important role.

A point ξ in S^d is a *radial limit* of Γ if some geodesic ray ending at ξ is within a bounded distance of infinitely many points of an orbit of Γ in $\mathbf{H}^{d+1}$. The set of these radial limit points Λ_r is a Γ-invariant residual subset of Λ_t [1]. Fixing the orbit Γx for the moment, the infimum b_ξ of all these bounds at ξ is a finite Γ-invariant Borel function on Λ_r (depending on the choice of orbit Γx).

We choose a Γ-invariant conformal density μ on Λ_t and we make the (strong) assumption that the radial limit set Λ_r has positive measure, relative to μ. For the moment the dimension α of the density is not specified.

Then the function b_ξ just defined is less than some constant c on an invariant set of positive measure $A \subset \Lambda_r$. Now choose R, R′ large enough and balls B centered at x with radii in [R, R′] so that:

(i) the balls $(\gamma B)_x$ in S^d (obtained by radially projecting the γB from x) work in proposition 3;

(ii) the balls of radius R resp. R′ for γx within $c+\varepsilon$ of the ray connecting x to ξ trap a ball $(\gamma B)_\xi$ centered at ξ, $(\gamma B_R)_x \subset (\gamma B)_\xi \subset (\gamma B_{R'})_x$. (See figure 7.)

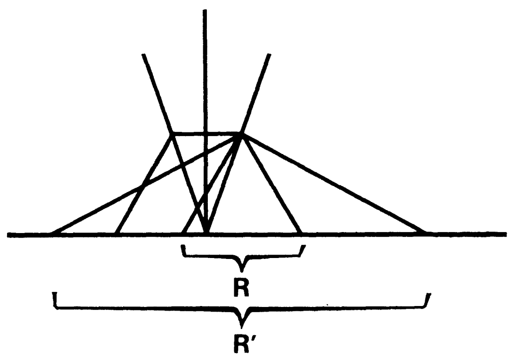

FIG. 7

By construction and Proposition 3 (first part) the balls $(B\gamma)_x$ have μ_x-measure proportional to $e^{-\alpha(x,\gamma x)}$. The $(B\gamma)_x$ contain coverings of A of arbitrarily small diameter since they contain for each ξ the balls $(B\gamma)_\xi$. For such a tiny covering of A by $\bigcup_i (B\gamma_i)_x$ we have

$$\mu_x(A) \leq \mu_x(\bigcup_i (B\gamma_i)_x) \leq \text{constant} . \sum_i e^{-\delta(x,\gamma_i x)}.$$

(1) The convex cocompact groups of § 3 are characterized by the property that $\Lambda_{\text{radial}} = \Lambda_{\text{topological}}$, and we wrote Λ there.

Corollary 20. — *If a Γ-invariant conformal density of dimension α gives positive measure to the radial limit set $\Lambda_r \subset \Lambda_l$, then the Poincaré series diverges: $\sum_{\Gamma} e^{-s(x,\gamma x)}$ diverges at $s=\alpha$. In particular, the dimension α must be the critical exponent of the group, $\alpha = \delta(\Gamma)$.*

The divergence is clear from the above because there are terms far out in the series corresponding to small coverings of A. We know by Corollary 4 that $\delta \leq \alpha$ and by the definition (3) of δ the series converges for $s > \delta$. Thus $\alpha = \delta$. Q.E.D.

If $\gamma_i^{-1}x$ approaches ξ in a sector the derivatives $|\gamma_i'\xi|$ are unbounded, so μ_x can deposit no atom at ξ. (We note as an aside that the divergence above implies μ may only have atoms at parabolic fixed points.) We may then discard any atoms from μ_x and still have $\mu_x(\Lambda_r) > 0$.

We now can apply the second part of Proposition 3 at a density point ξ of A ($\lim_{r\to 0} \mu(B(\xi, r) \cap A)/\mu(B(\xi, r)) \to 1$) using the balls $(\gamma B)_\xi$ to see that A has full μ-measure. For if $\gamma_i^{-1}x$ approaches ξ in a sector, $\gamma_i((\gamma_i^{-1}B)_\xi)$ is almost the entire S^d with almost all the mass of μ (Proposition 3).

The argument here only uses that A is an invariant set of radial limit points with positive μ_x-measure and we conclude that the action of Γ on the radial limit points is ergodic for the measure class of μ [1].

A corollary of ergodicity is that the function b_ξ defined above must be constant almost everywhere. This has the following interpretation: there is a constant C so that for μ_x-almost all ξ on S^d all geodesic rays ending at ξ pass within a distance C of the orbit of x infinitely often. (We take C a little larger than the constant function b_ξ and use the fact that two geodesic rays ending at ξ come arbitrarily close together.)

As we remarked in the previous section this property implies that the geodesic flow on T/Γ is conservative with respect to the associated measure dm. Thus by Theorem 14 and Corollary 15 the action of Γ on $(S^d \times S^d)$—diagonal is ergodic.

Assume Γ is a non-elementary discrete group. Then:

Theorem 21. — *The atomic part μ_a of a Γ-invariant density μ gives measure zero to the radial limit set Λ_r. The non atomic part μ_c gives either zero or full measure to Λ_r.*

In the case of full measure the dimension of μ_c is the critical exponent $\delta(\Gamma)$, and μ_c is the only Γ-invariant density giving full measure to Λ_r. Also the action of Γ on $(S^d \times S^d)$—diagonal is ergodic for the measure class of $\mu_c \times \mu_c$.

Conversely, the ergodicity of the product action on $(S^d \times S^d)$—diagonal for $\mu \times \mu$ implies μ has no atoms and gives full measure to Λ_r.

We prove, for the converse, that if a Γ-invariant conformal density μ gives zero measure to Λ_r, then the product action of Γ on $(S^d \times S^d)$—diagonal is completely dissipative. To see this, associate (($\mu \times \mu$)-almost all) geodesics to finite subsets of the orbit by

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[1] This argument assumes $\delta > 0$, *i.e.* Γ is not the elementary hyperbolic group, a trivial counterexample to the ergodicity of the action of Γ on Λ_r.

taking closest points. This partitions $(S^d \times S^d)$ — diagonal into countably many subsets so that each intersects a Γ-orbit in finitely many points. This proves dissipativity.

The uniqueness of μ_c follows from ergodicity just as in § 3. The argument there about Hausdorff measure also has a generalization here.

Namely consider the collection $\mathscr{F}$ of balls $(\gamma B)_\xi$ defined above. Each point of Λ_r is contained in arbitrarily small such balls centered at the point. One may define a Hausdorff-Caratheodory measure $H_\delta(\ ,\mathscr{F})$ relative to this collection $\mathscr{F}$ of balls and the gauge function r^δ. If A is a subset of Λ_r, the inequality $H_\delta(A, \mathscr{F}) \geq \text{constant} . \mu_x(A)$ follows exactly as in § 3. The other inequality requires a strong covering theorem (Federer [3], Theorem (2.8.14)) which implies there is a fixed k so that for each $\varepsilon > 0$ there is an ε-covering of A by balls $(\gamma B)_\xi$ which is the union of k disjoint collections. Given this the argument for the second inequality proceeds just as in § 3.

Theorem 22. — In the round metric on S^d *there is a collection* $\mathscr{F}$ *of balls so that each radial limit point is the center of arbitrarily small balls of* $\mathscr{F}$, *and the Hausdorff* δ*-measure (relative to* $\mathscr{F}$ *and the gauge function* r^δ*) of* Λ_r *is a positive finite measure* $(\delta = \delta(\Gamma))$. *This measure is proportional to the measure associated to the round metric by the canonical* Γ*-invariant conformal density positive on* Λ_r *(which we assume exists).*

In the next section we discuss the harder problem of the (usual) Hausdorff geometry of Λ_r relative to the collection of all balls centered on Λ_r. We note here the information already following from § 2, § 3, Theorem 21, and Theorem 22:

Corollary 23. — The Hausdorff dimension of the radial limit set is positive. If δ *is the Hausdorff dimension of any* Γ*-invariant subset* $A \subset \Lambda_t$, *then if* $\delta < \delta(\Gamma)$ *the Hausdorff* δ*-measure of* A *is zero or infinity. If* $A \subset \Lambda_r$ *and* $\delta = \delta(\Gamma)$ *the Hausdorff* δ*-measure of* A *is either zero or " one ".*

6. Hausdorff Dimension of the Radial Limit Set.

The radial limit set Λ_r of a discrete group Γ has a Hausdorff dimension $D(\Lambda_r)$ (1) and we will study now the relation between $D(\Lambda_r)$ and $\delta(\Gamma)$. One has directly the upper bound $D(\Lambda_r) \leq \delta(\Gamma)$ for all groups Γ (see below). The lower bound is more difficult, $D(\Lambda_r) \geq \delta(\Gamma)$. This was done by a vigorous argument for finitely generated Fuchsian groups (Patterson [4]) with cusps, assuming δ belonged to the interval [2/3, 1]. In § 3 we have shown $D(\Lambda_r) = \delta(\Gamma)$ for convex cocompact groups.

We will first prove the conjecture that $\delta(\Gamma) = D(\Lambda_r)$ for groups Γ with $\delta(\Gamma)$*-finite volume*. By this we mean there is on S^d a Γ-invariant conformal density μ such that the associated measure dm on T/Γ invariant under the geodesic flow has *finite total mass.*

(1) This is the notation of Mandelbrot [6], which has beautiful pictures of sets like limit sets of Kleinian groups and discussions of their Hausdorff geometry.

By the results of § 4 and § 5 the dimension of μ is necessarily $\delta(\Gamma)$, μ is unique, and μ gives full measure to the radial limit set. We begin with the upper bound,

(10) $\qquad D(\Lambda_r) \leq \delta(\Gamma)$, for any Γ.

To see (10) directly note that the shadows of large enough balls along the orbit of x in $\mathbf{H}^{d+1}$ give fine coverings of Λ_r. Thus if $\sum_\Gamma e^{-\alpha(x,\gamma x)} < \infty$ all terms far out are $< \varepsilon$ and we can construct a small diameter covering of Λ_r such that $\sum_i r_i^\alpha < \varepsilon$. So we have the more precise statement:

Theorem 24. — If the Poincaré series $\sum_\Gamma e^{-s(x,\gamma x)}$ converges at $s=\alpha$, then the Hausdorff α-measure of Λ_r is zero. In particular the Hausdorff dimension of Λ_r is at most $\delta(\Gamma)$.

In one sense Theorem 24 is merely a generalization of part of an argument in Beardon and Maskit [11].

We turn to the question of the lower bound $D(\Lambda_r) \geq \delta(\Gamma)$ for the case in which Γ has $\delta(\Gamma)$-finite volume. To motivate what follows we note (but do not use logically) a well-known construction of Frostman (see [10] or Frostman's beautifully written thesis). Suppose C is a compact set whose Hausdorff α-measure is >0, then there must exist on C (Frostman's lemma) a finite measure ν so that $\nu(\text{ball of radius } r)/r^\alpha$ is bounded from above for all r. To see what this means if our desired lower bound is true, (Hausdorff dimension $\Lambda_r) \geq \delta(\Gamma)$, note that by definition (Hausdorff α-measure $(\Lambda_r)) > 0$ for $\alpha = \delta - \varepsilon$, $\varepsilon > 0$. Now Λ_r is not compact but some compact subset of Λ_r must also have positive Hausdorff α-measure. Then by Frostman's construction the nice diffuse measure ν exists.

In our case we have a canonical measure on Λ_r provided by the conformal density μ. So we naturally try to prove it has the nice property of Frostman; namely,

(11) $\qquad \mu_x(B(\xi, r))/r^{\delta-\varepsilon} \leq C(\varepsilon)$, for $\varepsilon > 0$,

where μ_x is the measure associated to the metric on S^d obtained by radial projection from x in $\mathbf{H}^{d+1}$, ξ ranges over some compact subset K of Λ_r of positive μ_x-measure, and $B(\xi, r)$ is the intersection of the ball of radius r centered at ξ with K.

Actually, it is trivial that if we prove (11) we are finished. For then if K is covered by a union of balls centered on K, $K \subset \bigcup_i B_i$ where $r_i = \text{radius } B_i$, then

$$\sum_i r_i^{\delta-\varepsilon} \geq \text{constant} . \sum_i \mu_x B_i \geq \text{constant} . \mu_x(\bigcup_i B_i) \geq \mu_x K > 0.$$

So K has positive Hausdorff $(\delta-\varepsilon)$-measure, which means

$$(\text{Hausdorff dimension of K}) \geq \delta - \varepsilon.$$

Since $K \subset \Lambda_r$ and ε was an arbitrary positive number, we get the inequality

$$(\text{Hausdorff dimension of } \Lambda_r) \geq \delta(\Gamma),$$

following from (11).

To derive (11) for μ_x, consider a geodesic ray $g_t(v)$ starting from x in $\mathbf{H}^{d+1}$ and heading toward the point ξ in S^d. The projection from x onto S^d of a ball of fixed size centered at $g_t(v)$ is a ball B_t of radius constant$\cdot e^{-t}$ centered at ξ. Let $\sigma(t)$ as in § 4 denote the distance from $g_t(v)$ to the nearest point in the orbit of x. Let R_t be the region of $\mathbf{H}^{d+1}$ in the cone from B_t to x consisting of points at large ($\gg t$) distance from x.

If we construct μ_x by the procedure of § 1, then $\mu_x(B_t)$ will be approximately the ratio

$$\sum_{\gamma x \in R_t} e^{-s(x,\gamma x)} \Big/ \sum_{\Gamma} e^{-s(x,\gamma x)}$$

for some s close to δ. (We are in the divergence case by Corollary 20.)

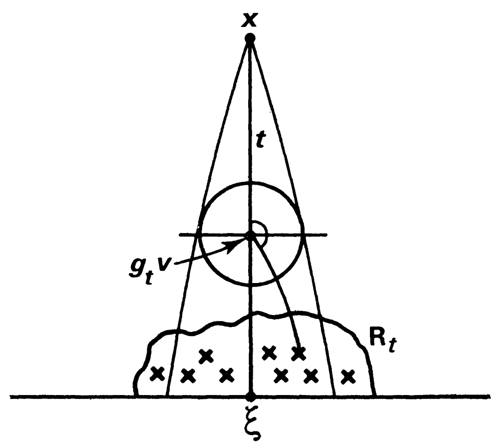

FIG. 8

Since t is large and the angle between g_t and geodesics connecting $g_t(x)$ to γx in R_t cannot be too small we can write, using non-Euclidean plane geometry, $(x, \gamma x) \geq t + (g_t x, \gamma x) -$ constant.

Thus $\sum_{\gamma x \in R_t} e^{-s(x,\gamma x)} \leq c e^{-st} \sum_{\gamma x \in R_t} e^{-s(g_t x, \gamma x)}$.

Then, using the triangle inequality $(g_t x, \gamma x) \geq (\gamma_0 x, \gamma x) - \sigma(t)$, where $\gamma_0 x$ is the closest orbit point to $g_t x$, we have $(\sigma(t) = (g_t x, \gamma_0 x))$

$$\begin{aligned} e^{st} \sum_{\gamma x \in R_t} e^{-s(x,\gamma x)} &\leq c e^{\sigma(t)} \sum_{\gamma x \in R_t} e^{-s(\gamma_0 x, \gamma x)} \\ &\leq c e^{\sigma(t)} \sum_{\Gamma} e^{-s(\gamma_0 x, \gamma x)} \\ &= c e^{\sigma(t)} \sum_{\Gamma} e^{-s(x,\gamma x)}. \end{aligned}$$

Thus letting s approach δ to construct μ we have $e^{\delta t} \mu(B_t) \leq c e^{\sigma(t)}$. In particular, if $\sigma(t)/t \to 0$ as $t \to \infty$, for $\varepsilon > 0$ there is a constant $c(\xi, \varepsilon)$ such that, writing $r = e^{-t}$,

$$\mu(\text{ball of radius } r \text{ about } \xi)/r^{\delta-\varepsilon} \leq c(\xi, \varepsilon).$$

Now in the finite volume case we have shown $\sigma(t)/t \to 0$ as $t \to \infty$ for almost all trajectories (Corollary 19). Thus if $\varepsilon > 0$ is fixed, $c(\xi, \varepsilon)$ will be less than $c(\varepsilon)$ on a compact set K of ξ in Λ_r with positive measure. For this set K we have established (11). So:

Theorem 25. — If a group Γ *has* $\delta(\Gamma)$*-finite volume, then the Hausdorff dimension of the radial limit set is* $\delta(\Gamma)$.

We apply this to the finitely generated Fuchsian groups. These have finite sided fundamental domain, using which the strong inequality

$$\sum_{\Gamma} e^{-s(x,\gamma x)} \geq \frac{c}{s-\delta}, \quad \text{for } s > \delta,$$

was derived by Patterson at the end of his paper [4].

By Corollary 18 the inequality implies Γ has $\delta(\Gamma)$-finite volume. Thus for finitely generated Fuchsian groups the Hausdorff dimension of Λ_r equals $\delta(\Gamma)$. This has two corollaries. Consider only non-elementary groups:

Corollary 26. — For a finitely generated Fuchsian group the Hausdorff dimension of the topological limit set is $\delta(\Gamma)$.

Corollary 27. — For an arbitrary Fuchsian group the Hausdorff dimension of the radial limit set is $\delta(\Gamma)$.

The first corollary follows since Beardon-Maskit [11] show that for groups (in $\mathbf{H}^2$ or $\mathbf{H}^3$) with a finite-sided fundamental domain the topological limit set is the union of the radial limit set and a countable set of points.

The second follows since we may write any Fuchsian group Γ as a union of finitely generated groups Γ_α. Then $\Lambda_r^\alpha \subset \Lambda_r$ and writing D for Hausdorff dimension, $D(\Lambda_r) \geq \sup_\alpha D(\Lambda_r^\alpha) = \sup_\alpha \delta^\alpha = \delta(\Gamma)$ using the above and Corollary 5. Combining this lower bound with (10) yields Corollary 27.

7. The Markov process associated to a conformal density.

We will now construct from a Γ-invariant conformal density μ on S^d a Γ-invariant Markov process P_t^μ in $\mathbf{H}^{d+1}$. Intuitively, paths of this process starting from x in $\mathbf{H}^{d+1}$ hit ∞ at time ∞ with probability measure μ_x normalized to mass 1.

Let $\Phi(x)$ on $\mathbf{H}^{d+1}$ be the function which assigns to x the total mass of the measure μ_x, *i.e.* the mass of μ as viewed from x. Let $p_t(x,y)$ denote the transition probabilities for the random motion on $\mathbf{H}^{d+1}$ associated to the heat equation $\left(\Delta u - \frac{\partial u}{\partial t} = 0\right)$. Thus if dy denotes the Riemannian volume in $\mathbf{H}^{d+1}$, then the probability of going from x to

$E \subset \mathbf{H}^{d+1}$ in time t is the amount of mass deposited by the probability measure $p_t(x,y)dy$ in E.

For the μ-associated process the natural measure is $dy^\mu = \Phi(y)^2 dy$ and the transition probabilities (relative to this measure) are $p_t^\mu(x,y) = e^{\lambda t}\dfrac{p_t(x,y)}{\Phi(x)\Phi(y)}$ where $\lambda = \delta(d-\delta)$, $\delta =$ dimension μ. Thus the probability of going from x to E in time t for the process P_t^μ will be $\int_E e^{\lambda t}\dfrac{p_t(x,y)}{\Phi(x)\Phi(y)}dy^\mu = \int_E e^{\lambda t}\dfrac{\Phi(y)}{\Phi(x)}p_t(x,y)dy$. The last equation shows the process only depends on the scalar multiples of Φ.

Proposition 28. — The function Φ is an eigenfunction of the Laplacian Δ on $\mathbf{H}^{d+1}$ with eigenvalue $-\lambda$, where $\lambda = \delta(d-\delta)$, $\delta =$ dimension μ. Thus:

(i) $\int_{\mathbf{H}^{d+1}} p_t(x,y)\Phi(y)dy = e^{-\lambda t}\Phi(x)$.

(ii) *The operator on functions,* $P_t^\mu f = \int_{\mathbf{H}^{d+1}} p_t^\mu(x,y)f(y)dy^\mu$, *fixes the function* 1.

(iii) *The dual operator on measures,* $\nu \mapsto \nu P_t^\mu$, *defined by* $\nu P_t^\mu(E) = \int_{\mathbf{H}^{d+1}}\left(\int_E p_s^\mu(x,y)dy^\mu\right)d\nu$ *fixes the measure* $dy^\mu = \Phi(y)^2 dy$.

(iv) $p_t^\mu(x,y) = p_t^\mu(y,x)$ *(symmetry property).*

(v) $\int_{\mathbf{H}^{d+1}} p_t^\mu(x,y)p_s^\mu(y,z)dy^\mu = p_{t+s}^\mu(x,z)$ *(semi-group property).*

Proof. — To see the first part fix an x_0 in $\mathbf{H}^{d+1}$ and write y_ξ for the function in $\mathbf{H}^{d+1}$ which in upper half-space coordinates (with ξ at ∞ and x_0 at height one) is just the vertical coordinate (the y-coordinate). In terms of the horocycle distance $(x, x_0)_\xi$, $y_\xi(x) = e^{-(x,x_0)_\xi}$.

Since in the upper half-space coordinates the hyperbolic metric is $\dfrac{ds}{y}$, where ds is the Euclidean metric, one computes the hyperbolic Laplacian $\Delta = *'d*'d$, where $*'$, sending i-forms to $((d+1)-i)$-forms, is just $\left(\dfrac{1}{y}\right)^{d+1-2i}$ times the Euclidean star operator, $*$. Thus the hyperbolic laplacian is

$$\Delta = (y)^{d+1} * d\left(\left(\frac{1}{y}\right)^{d-1} * d\right) = (y^2 \text{ Euclidean } \Delta) + (1-d)y*(dy \wedge * d(\ )).$$

If we apply the hyperbolic laplacian to the function y^α, we get

$$\begin{aligned}\Delta y^\alpha &= \alpha(\alpha-1)y^\alpha + (1-d)y*(dy \wedge \alpha y^{\alpha-1} * dy)\\ &= (\alpha(\alpha-1) + (1-d)\alpha)y^\alpha \quad (\text{since } *(dy\wedge * dy) \equiv 1)\\ &= \alpha(\alpha-d)y^\alpha.\end{aligned}$$

Thus $\Delta(y_\xi)^\alpha = \alpha(\alpha-d)(y_\xi)^\alpha$. Now the function Φ is a convex combination of the functions $(y_\xi)^\delta$, namely $\Phi(x') = \int (y_\xi(x'))^\delta d\mu_{x_0}(\xi)$ because

$$\left(\frac{d\mu_{x'}}{d\mu_{x_0}}\right)^\delta = \left(\frac{\text{metric associated to } x'}{\text{metric associated to } x_0}\right)^\delta = e^{-\delta(x_0,x')_\xi} = (y_\xi(x'))^\delta.$$

Thus Φ is also an eigenfunction of Δ with eigenvalue $-\lambda$, where $\lambda=\delta(d-\delta)>0$. (Compare Patterson [4].) Then, since $p_t(x,y)dy$ is by definition the kernel for the operator $e^{\Delta t}$, we have $e^{\Delta t}\Phi=e^{-\lambda t}\Phi$ or $\int_{\mathbf{H}^{d+1}} p_t(x,y)\Phi(y)dy=e^{-\lambda t}\Phi(x)$, which is (i).

We then calculate that $\int_{\mathbf{H}^{d+1}} e^{\lambda t}\frac{p_t(x,y)}{\Phi(x)\Phi(y)}\Phi^2(y)dy=e^{\lambda t}\frac{1}{\Phi(x)}\int_{\mathbf{H}^{d+1}} p_t(x,y)\Phi(y)dy=1$, which is (ii).

Since $p_t(x,y)=p_t(y,x)$ for the heat kernel, $p_t^\mu(x,y)=p_t^\mu(y,x)$, and (iv) is clear. But then since in (iii) we want to show for all $E\subset\mathbf{H}^{d+1}$, $\int_{\mathbf{H}^{d+1}}\left(\int_E p_t^\mu(x,y)dy^\mu\right)d\nu=\int_E dy^\mu$, this becomes $\int_{\mathbf{H}^{d+1}} p_t^\mu(x,y)dx^\mu=1$, which follows from (ii) using (iv).

The semi-group property (v) follows from the similar property for $p_t(x,y)$ by direct calculation. Q.E.D.

Now we consider the positive harmonic functions f for the process $p_t^\mu(x,y)$, namely such that $P_t^\mu f=f$, $f>0$. Write $g=f.\Phi$, then $f(x)=\int_{\mathbf{H}^{d+1}} p_t^\mu(x,y)f(y)dy^\mu$ is true if and only if $\int_{\mathbf{H}^{d+1}} e^{\lambda t}\frac{p_t(x,y)}{\Phi(x)\Phi(y)}\frac{g(y)}{\Phi(y)}\Phi^2(y)dy=f(x)$, or equivalently $\int_{\mathbf{H}^{d+1}} p_t(x,y)g(y)dy=e^{-\lambda t}g(x)$.

Corollary 29. — The positive P_t^μ*-harmonic functions are just the functions* g/Φ, *where* g *is a positive eigenfunction of* $e^{\Delta t}$ *with eigenvalue* $e^{-\lambda t}$, *which* $\lambda=\delta(d-\delta)$.

In particular positive P_t^μ-harmonic functions bounded by 1 correspond to positive eigenfunctions g satisfying $g\leq\Phi$. Each subset A of S^d of positive μ-measure determines one of these by the formula $\Phi_A(x)=\int\chi_A(\xi)y_\xi(x)d\mu_{x_0}(\xi)$ where χ_A is the characteristic function of A; $\Phi_A(x)$ is also the μ_x-mass of A, *i.e.* the μ-mass of A as viewed from x [1].

Let us estimate $\Phi_A(x)$ for x near ∞ in the compactified space $\mathbf{H}^{d+1}\cup S^d$. Draw a ray from x_0 through x to ξ on S^d. As x moves toward ξ the ratio of the measures $\left(\frac{d\mu_x}{d\mu_{x_0}}\right)^\delta$ is a function which is uniformly small outside smaller and smaller balls around ξ.

Then, writing $\mu_A(\xi,r)$ for the μ_{x_0}-mass of A in a ball of radius r, there is the formula $\Phi_A(x)=\int_0^1 t(r)^\delta d\mu_A(\xi,r)$ where t is the translation taking x back to x_0, and $t(r)$ is the conformal derivative of t on S^d at distance r from ξ in the metric from x_0.

Suppose ξ is such that $\lim_{r\to 0}\frac{\mu_A(\xi,r)}{\mu(\xi,r)}$ exists and equals $a\mu(\xi,r)=\mu_A(\xi,r)$ for $A=S^d$.

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(1) The integral formula for Φ was suggested by Patterson [4], who treated $d=1$. The μ-mass description was suggested by Thurston (verbal communication) in a discussion of δ-Hausdorff measure.

We take ε small enough for the limit to be good, then (x, x_0) large enough to ignore the contribution of the integral outside ε. Partial integration yields

$$\int_0^\varepsilon t(r)^\delta d\mu_A(\xi, r) = -\int_0^\varepsilon \frac{d}{dr} t(r)^\delta \mu_A(\xi, r) dr + t(\varepsilon)^\delta \mu_A(\xi, \varepsilon).$$

Thus we see $\Phi_A(x)$ and $\Phi(x)$ are sums of terms in approximate ratio a. Thus (radial limit for $x \to \xi$ of $\Phi_A(x)/\Phi(x)) = \lim_{r \to 0} \frac{\mu_A(\xi, r)}{\mu(\xi, r)} = a$. By the density theorem (Federer [3], (2.9.11)), for μ-almost all ξ, $\lim_{r\to 0} \frac{\mu_A(\xi, r)}{\mu(\xi, r)} = \chi_A(\xi)$. A minor extension allows $x \to \xi$ in a sector, the set of points at a bounded distance from the ray.

Corollary 30. — The bounded P_t^μ*-harmonic function* $\Phi_A(x)/\Phi(x)$ *has sectorial limits equal to the characteristic function of* A *at* μ*-almost all points of* S^d.

To go further we need to discuss the measures on the paths of the process. The space of one-sided paths is at first the uncountable product $(\mathbf{H}^{d+1})^{\mathbf{R}^+}$ provided with a natural measure associated to a process P_t such as P_t^μ above. On the starting point we use some appropriate measure on $\mathbf{H}^{d+1}$. For the paths starting at x we think of the Dirac mass flowing out (as mass) from x by applying the operator P_t. One defines the measure of a cylinder set:

at time t_i the path lies in $A_i \subset \mathbf{H}^{d+1}$, for $i = 1, \ldots, n$,

as the amount of this flowing mass which lies in A_1 at t_1, A_2 at t_2, ..., A_n at t_n.

The semi-group or Markov property for P_t implies that compatibilities with respect to dropping conditions are satisfied. By the Kolmogoroff extension theorem one then generates a countably additive probability measure on the σ-algebra generated by the above cylinder sets of paths starting at x in $\mathbf{H}^{d+1}$.

If a measure dm on the state space (in our case $\mathbf{H}^{d+1}$) is invariant under the process, then one can construct a measure on the two-sided infinite paths. The space is the uncountable product $(\mathbf{H}^{d+1})^{\mathbf{R}}$, times runs from $-\infty$ to $+\infty$, the cylinders sets are again of the form

$$(A_1, t_1; A_2, t_2; \ldots; A_n, t_n), \quad t_i \text{ in } (-\infty, \infty).$$

Now one imagines the mass of dm in A_1 flowing under the process started at time t_1. Then the measure of the cylinder set is the amount of mass starting in A_1, which lies in A_2 at t_2, ..., and in A_n at t_n. Now the invariance of the measure means the amount of mass flowing into any set from the entire space equals the amount of mass originally in the set. Thus dropping the A_1 condition is compatible. Dropping the other conditions is compatible as above. Thus again by the Kolmogoroff extension theorem there is generated a countably additive measure on the space of all biinfinite paths defined on the σ-algebra generated by the cylinders. This measure is σ-finite because the measure of each (A, t) is the dm-measure of A, and if we write $\mathbf{H}^{d+1} = \bigcup_i A_i$ the entire space of biinfinite paths is the union of (A_i, t) for t fixed.

In our case one can measure by sets defined by conditions at a countable dense set of times, using the continuity properties of the process. This means we have a separable process and we can work with a countably generated σ-algebra.

We will use the biinfinite paths to prove that a certain analytical condition implies that the action of Γ on $(S^d \times S^d)$—diagonal is ergodic. The hypothesis emerges later. If W is a subset of $(S^d \times S^d)$—diagonal and χ_W is the characteristic function, then form a function of two points x, x', in $\mathbf{H}^{d+1}$,

$$h(x, x') = \frac{1}{\Phi(x)\Phi(x')} \int_{(S^d \times S^d)-\text{diagonal}} \chi_W(\xi, \xi') y_\xi(x) y_{\xi'}(x') \, d\mu_{x_0}(\xi) \, d\mu_{x_0}(\xi').$$

Then h is P_t^μ-harmonic in x and x' separately, Γ-invariant if W is, and not constant if W has positive but not full measure. The latter point follows from Fubini and the corollary 30.

Now we construct from h a function on the biinfinite paths. Let $P_{a,t}$ be all the biinfinite paths at a in $\mathbf{H}^{d+1}$ at time t.

If we condition our measure ν constructed on biinfinite paths to P_{a,t_0} we see we have a product situation. We have as probability spaces $P_{a,t_0} = P_- \times P_+$, where P_+ is the measure on one-sided paths starting at a and P_- is the measure on one-sided paths ending at a. Since our process P_t^μ is symmetric, $P_- = P_+$. (In general P_- would be the measure constructed on one-sided paths for the dual process associated to dy^μ.)

Now the abstract theory for a process (Martingale convergence theorem) tells us a bounded harmonic function for a process has limits along almost all paths. Moreover, the function can be reconstructed from these limits using the mean value property for harmonic functions and dominated convergence.

Define a function on the biinfinite paths $\{w\}$ by fixing a time s and then forming $\lim_{t\to\infty} h(w(s), w(t))$. This limit only depends on any part of the path w near $+\infty$ and the coordinate $w(s)$ in $\mathbf{H}^{d+1}$. Fixing the former, the function is P_t^μ-harmonic in the latter, using dominated convergence and symmetry of P_t^μ. Now take a limit over $w(s)$ to construct a function of biinfinite paths.

This produces a non trivial function of biinfinite paths which is invariant under the time shift and the action of Γ.

Corollary 31. — If the combined action of Γ and time shift on the biinfinite paths of P_t^μ is ergodic, then the action of Γ on $(S^d \times S^d)$—diagonal is ergodic.

There is a natural condition concerning the ergodicity of biinfinite paths. It is convenient to take the quotient of the set of biinfinite paths by Γ. We then have a quotient process, quotient path space, etc.

Suppose we have the following *recurrence property* for the quotient process: for any point p in $\mathbf{H}^{d+1}/\Gamma$ and any tiny metric ball B, almost all the paths starting from p enter B.

We will show that this recurrence property implies the ergodicity of the shift map on the biinfinite paths of $\mathbf{H}^{d+1}/\Gamma$.

We work with rectangles $\mathscr{A}=\bigcap_{i=1}^{n}(A_i, t_i)$ $\mathscr{B}=\bigcap_{i=1}^{m}(B_i, t_i)$ where the A_i and B are metric balls in $\mathbf{H}^{d+1}/\Gamma$. Besides recurrence we will use the fact that the probability $p(x, \mathscr{A})$ that a path starting at x in time o lies in a rectangle $\mathscr{A}=\bigcap_{i=1}^{n}(A_i, t_i)$ with $t_{i+1}>t_i>0$ is a positive continuous function of x.

Say that two sets X and Y of paths of positive equal measure are *shift isomorphic* if there is a countable partition of X into pieces so that (different) time shifts applied to the pieces yield a partition of Y (all this modulo sets of measure zero). Ergodicity in the measure preserving context implies that any two sets of equal measure are shift isomorphic. Conversely, if for every pair of rectangles one of minimum mass is shift isomorphic to a subset of the other, then ergodicity follows. For any set of paths X of positive area can be arbitrarily well approximated by disjoint collections of rectangles—namely, for $\varepsilon>0$, the symmetric difference (X, disjoint union of rectangles) has measure less than ε for some choice of rectangles. Then an easy argument consisting of the steps (shift (isomorph) one rectangle into another, partition the complement of the image approximately into disjoint rectangles, and repeat) shows that one can approximately construct a shift isomorphism between two disjoint collections of rectangles of nearly equal mass.

If these collections are approximating two *a priori* given sets of paths of positive measure X and Y, this shows some shift of X intersects Y.

These are the generalities. Now we do the specific step for rectangles, which makes the proof. Start with simple rectangles $\mathscr{A}=(A, t_1)$ and $\mathscr{B}=(B, t_2)$. By *recurrence*, a *countable* collection $(A, t_1)\cap(B, s_i)$ covers (A, t_1) almost everywhere. Similarly (B, t_2) is covered by a countable collection $(B, t_2)\cap(A, t_j)$. Denote the union of these two collections by $\mathscr{C}$. Then by shifting the pieces of $\mathscr{C}$ we obtain (shift) surjections $\mathscr{A}\leftarrow\mathscr{C}\rightarrow\mathscr{B}$ which are countable to one.

Now one can construct piecemeal an injection of the set $\mathscr{A}$ or $\mathscr{B}$ of minimum mass into the other by composing local inverses of one surjection with the other. Thus one of $\mathscr{A}$ or $\mathscr{B}$ is shift isomorphic to a subset of the other.

Now consider more complicated rectangles $\mathscr{A}=\bigcap_{i=1}^{n}(A_i, t_i)$ and $\mathscr{B}=\bigcap_{j=1}^{m}(B_j, s_j)$. Let $b>0$ be the maximum of $p(x, \mathscr{B}')$ where x is in B_1 and $\mathscr{B}'=\bigcap_{j=2}^{m}(B_j, s_j-s_1)$. Choose a countable collection of times r_α so that $\bigcup_\alpha((A_n, t_n)\cap(B_1, r_\alpha))$ covers (A_n, t_n). Let $\mathscr{B}_\alpha=(\mathscr{B}$ shifted by $r_\alpha-s_1)$.

Then the rectangles $\mathscr{A}\cap\mathscr{B}_\alpha$ cover at least the proportion b of $\mathscr{A}$ (by definition of the measure). Remove this proportion from $\mathscr{A}$, approximate the remainder very well by new rectangles $\mathscr{A}'$ and repeat the procedure to similarly cover at least a proportion b of these $\mathscr{A}'$. Again we arrive at a countable collection of subsets of $\mathscr{B}$ whose translates cover $\mathscr{A}$. Similarly cover $\mathscr{B}$ using translates of subsets of $\mathscr{A}$.

Let $\mathscr{C}$ be a countable union of these subsets of $\mathscr{A}$ and $\mathscr{B}$ and construct first the shift surjections $\mathscr{A} \leftarrow \mathscr{C} \rightarrow \mathscr{B}$ and then the shift isomorphism between the one of minimum mass and the other.

Thus the shift map of biinfinite paths is ergodic, assuming the process is recurrent.

This recurrence property is established by an analytical condition. Suppose it were false. Then define $\pi(x)$ to be the probability a path starting at x enters the tiny metric ball $B \subset H/\Gamma$.

Then $\pi(x)$ defines a $\bar{P}^\mu_t$-harmonic function away from B by the Markov property ($\bar{P}^\mu_t$ is the quotient process). Also $\pi(x) \leq 1$ everywhere, $\pi(x) = 1$ on B, $\pi(x) < 1$ off B by the maximum principle, and $\pi(x) \geq \bar{P}^\mu_t \pi(x)$ by the Markov property.

For t small, integrate $\pi - \bar{P}^\mu_t \pi$ against the measure $\int_0^T \bar{p}^\mu_s(x, y)\, dy^\mu\, ds$. Now,

$$\int_0^T \left(\int_{\mathbf{H}^{d+1}} \bar{p}^\mu_s(x, y) (\bar{P}^\mu_t \pi - \pi)\, dy^\mu \right) ds = \left(\int_T^{T+t} - \int_0^t \right) \left(\int_{\mathbf{H}^{d+1}} \bar{p}^\mu_s(x, y) \pi(y)\, dy^\mu \right) ds.$$

The right hand side is at most $2t$. But the left hand side may be rewritten as the measure $\int_0^T \bar{p}^\mu_s(x, y)\, dy^\mu$ evaluated on the non-positive bounded function $\bar{P}^\mu_t \pi - \pi$.

If $\bar{P}^\mu_t \pi - \pi$ is negative on a tiny neighborhood, and the density of the measure $\int_0^T \bar{p}^\mu_s(x, y)\, dy^\mu$ goes to ∞ as $T \to \infty$, then we have a contradiction.

Now $\bar{p}^\mu_s(x, y)$, the transition probability on the quotient, satisfies

$$\bar{p}^\mu_s(x, y) = \sum_\Gamma p^\mu_s(x, \gamma y).$$

Thus we want to consider $\lim\limits_{T \to \infty} \int_0^T \sum\limits_\Gamma p^\mu_s(x, \gamma y)\, ds$ which will diverge if and only if $\sum\limits_\Gamma \left(\int_0^\infty p^\mu_s(x, \gamma y)\, ds \right)$ does.

Now the function $\int_0^\infty p^\mu_s(x, y)\, ds$ is just $(\Phi(y)/\Phi(x)) \int_0^\infty e^{\lambda s} p_s(x, y)\, ds$ where $\Phi(y)/\Phi(x)$ is Γ-invariant. Thus the above diverges for fixed x, y iff $\sum\limits_\Gamma \int_0^\infty e^{\lambda s} p_s(x, \gamma y)\, ds$ does.

The function $g = \int_0^\infty e^{\lambda s} p_s(x, y)\, ds$ satisfies the equation $\Delta g + \lambda g = 0$ in each variable and only depends on the hyperbolic distance (x, y) since $p_t(x, y)$ satisfies the heat equation, $\Delta p - \dfrac{\partial p}{\partial t} = 0$, and only depends on (x, y).

If we write the Laplacian in polar coordinates, we get $\dfrac{\partial^2}{\partial r^2} + \dfrac{A'(r)}{A(r)} \dfrac{\partial}{\partial r}$ ignoring angular derivatives; $A(r)$ is the area of the sphere in $\mathbf{H}^{d+1}$ of radius r, so

$$A(r) = \text{constant} . \sinh dr.$$

The variable change $t = \cosh r$ reduces the equation $\dfrac{\partial^2 g}{\partial r^2} + \dfrac{A'(r)}{A(r)} \dfrac{\partial g}{\partial r} + \lambda g = 0$ to the associated Legendre equation with parameters depending on d and λ. The indicial equation at ∞, $\dfrac{\partial^2 g}{\partial r^2}+d\dfrac{\partial g}{\partial r}+\lambda g=0$, has exponential solutions $e^{-\delta_+ r}$, $e^{-\delta_- r}$ where $\delta_+\delta_-=\lambda$, $\delta_++\delta_-=d$ and $\delta_+\geq\delta_-$.

The smallest solution at ∞ is $e^{-\delta_+ r}$ where $\delta_+=\dfrac{d}{2}+\sqrt{(d^2/4)-\lambda}$. Thus we have the inequality $g(x,y)\geq e^{-\delta_+(x,y)}$.

In particular if $\delta(\Gamma)>d/2$ and $\lambda=\delta(d-\delta)$, then $\delta(\Gamma)=\delta_+$. Then if

$$\sum_\Gamma e^{-\delta(x,\gamma y)}=\infty, \qquad \sum_\Gamma g(x,\gamma y)=\infty$$

and the above chain of reasoning yields:

Theorem 32. — If the Poincaré series $\sum_\Gamma e^{-s(x,\gamma x)}$ *diverges at* $s=\delta(\Gamma)>\dfrac{d}{2}$, *then the action of* Γ *on* $(S^d\times S^d)$*—diagonal is ergodic for the measure class determined by any* Γ*-invariant conformal density* μ *of dimension* $\delta(\Gamma)$.

Then using § 5, theorem 21, we have:

Corollary 33. — If $\delta(\Gamma)>d/2$, *a conformal density* μ *of dimension* $\delta(\Gamma)$ *gives full measure to the radial limit set iff the Poincaré series* $\sum_\Gamma e^{-s(x,\gamma x)}$ *diverges at* $s=\delta(\Gamma)$.

Addendum.

There is an alternative approach to the construction of μ in § 1 and of the eigenfunction Φ (both Γ-invariant) based on positive superharmonic functions in $\mathbf{H}^{d+1}$ invariant by Γ. These functions form a compact convex cone $\mathscr{C}$ invariant by the heat semi-group P_t and satisfy $P_t f\leq f$.

Consider the decreasing sequence of subcones $\mathscr{C}_\lambda$, λ positive and increasing, defined by $P_t f\leq e^{-\lambda t}f$. By compactness we can go to the maximum λ_Γ (easily seen to be finite). Again by the compactness and the fixed point property applied to the rays, there is an element φ such that $P_t\varphi=e^{-at}\varphi$. Now $e^{-a}\leq e^{-\lambda}\Gamma$ because φ belongs to $\mathscr{C}_\lambda$ and strict inequality is impossible because λ_Γ is the maximum. So we have constructed a Γ-invariant positive eigenfunction φ of the Laplacian for the smallest possible eigenvalue $-\lambda_\Gamma$.

If $\delta(\Gamma)>d/2$ and we verify the Green's function for $\lambda=\delta(d-\delta)$ $\int_0^\infty e^{\lambda t}p_t(x,y)\,dt$ is really the small solution near ∞, $g(x,y)\sim e^{-\delta(x,y)}$, then, for $\delta>\delta(\Gamma)$, $\sum_\Gamma g(x,\gamma y)$ converges and is a non-trivial element of $\mathscr{C}_\lambda$ where $\lambda=\delta(d-\delta)$. Thus $\lambda_\Gamma\geq\delta(d-\delta)$. On the other hand, if $\mathscr{C}_\lambda$ is non-void for a λ larger than $\delta(d-\delta)$, $\delta>\delta(\Gamma)$, it must contain either a function satisfying $P_t f=e^{-\lambda t}f$ or $P_t f\leq e^{-\lambda t}f$ with strict inequality at some point.

In the first case we apply the boundary theory for positive harmonic functions relative to the operator $e^{\lambda t}P_t$ to construct an invariant conformal density of weight $<\delta$

which contradicts § 2. In the second case we find a Green's potential for $e^{\lambda t}P_t$ and deduce that the Poincaré series converges at a point $<\delta$, again a contradiction.

Thus $\lambda_\Gamma = \delta(d-\delta)$, we have our eigenfunction φ invariant by Γ, and its representating measure on S^d will be a Γ-invariant conformal density $\left(\delta > \frac{d}{2}\right)$.

It is curious that one does not readily describe the theory for the dimensions $\delta \leq \frac{d}{2}$ by exit boundary potential theory.

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