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INVARIANT MEASURES FOR A DIFFEOMORPHISM WHICH EXPANDS THE LEAVES OF A FOLIATION

by David RUELLE

The present note is a companion to [1]. It studies certain measures on a foliated manifold with a leaf-expanding diffeomorphism.

Let M be a compact C^r manifold, and $\mathscr{F}$ a C^r foliation, preserved by the C^r diffeomorphism g, with finite $r \geq 2$. We assume that g is expanding on leaves. We proceed to define a set $\mathscr{K}_+$ of probability measures on M such that their conditional probabilities on leaves have certain smoothness properties.

Choose a covering of M by a finite number of charts $D^{m-k} \times D^k$ such that the leaves of $\mathscr{F}$ are of the form $D^{m-k} \times \{v\}$. Let a probability measure ρ on M have, to each of these charts, a restriction of the form:

$$\rho^0(u, v)\, du \otimes \mu(dv)$$

where du is Lebesgue measure on D^{m-k} and $\mu(dv)$ some positive measure on D^k. Assume that, for μ-almost all v, the function $u \mapsto \rho^0(u, v)$ is strictly positive, and its logarithm has derivatives up to order $r-2$ which are Lipschitz with Lipschitz constant $\leq \ell$. Let $\mathscr{K}(\ell)$ be the set of such measures ρ, where ℓ is fixed, but the $\mu(dv)$ are allowed to vary. Define:

$$\mathscr{K}_+ = \bigcup_{\ell > 0} \bigcap_{n > 0} g^n \mathscr{K}(\ell).$$

It is easily seen that $\mathscr{K}_+$ does not depend on the choice of the charts $D^{m-k} \times D^k$ used to define $\mathscr{K}(\ell)$.

Theorem. — Let M, $\mathscr{F}$, g, *and* $\mathscr{K}_+$ *be as above.*

(a) $\mathscr{K}_+$ *is vaguely compact, it is a Choquet simplex, and the conditional measure* $\rho^0(u, v)\, du$ *of* $\rho \in \mathscr{K}_+$ *on a leaf of* $\mathscr{F}$ *is independent of* ρ *(up to normalization). Any two distinct extremal points of* $\mathscr{K}_+$ *are disjoint measures.*

(b) *If* σ *is any probability measure absolutely continuous with respect to Lebesgue measure on* M, *any vague limit of* $g^n \sigma$ *when* $n \to \infty$ *is in* $\mathscr{K}_+$.

(c) *Let* $\mathscr{K}_g$ *be the set of g-invariant elements of* $\mathscr{K}_+$ *(or* $\bigcup_{\ell > 0} \mathscr{K}(\ell)$*); then* $\mathscr{K}_g$ *is a simplex, and its extremal points are g-ergodic measures.*

At each $x \in \mathrm{M}$, there is a C^r chart $\mathrm{D}^{m-k} \times \mathrm{D}^k$ such that the leaves of $\mathscr{F}$ are of the form $\mathrm{D}^{m-1} \times \{v\}$. With respect to such charts g^{-1} takes the form:

$$(u, v) \rightarrow (f_1(u, v), f_2(v))$$

where f_1, f_2 are C^r. Possibly replacing g by g^n we may assume that:

$$\left\| \frac{\partial f_1(u, v)}{\partial u} \right\| \leq \alpha < 1.$$

We may also assume that there is a finite set of charts covering M such that for each piece of leaf $\mathrm{D}^{m-k} \times \{v\}$ in one chart, its image by g^{-1} is entirely in some chart of the set.

Let $\sigma \in \mathscr{K}(\ell)$. The density of the conditional measure corresponding to $g\sigma$ on $\mathrm{D}^{m-k} \times \{v\}$ is:

$$\sigma^1(u, v) = \mathrm{K}\sigma^0(f_1(u, v), f_2(v)) \,.\, \left| \det \frac{\partial f_1(u, v)}{\partial u} \right| \tag{1}$$

where σ^0 is a density for σ and K an arbitrary constant >0. (Notice that σ^0 and σ^1 are only fixed up to a multiplicative constant.) Writing ∂_1 for the derivative with respect to the first argument, we have:

$$\partial_1 \log \sigma^1(u, v) = \frac{\partial}{\partial u}\left(\log \sigma^0(f_1(u, v), f_2(v)) + \log \left| \det \frac{\partial f_1(u, v)}{\partial u} \right| \right)$$

$$= (\partial_1 \log \sigma^0(f_1(u, v), f_2(v))) \,.\, \frac{\partial f_1(u, v)}{\partial u} + \frac{\partial}{\partial u} \log \left| \det \frac{\partial f_1(u, v)}{\partial u} \right|$$

hence:

$$|\partial_1 \log \sigma^1| \leq \alpha |\partial_1 \log \sigma^0| + \mathrm{C} \tag{2}$$

where C is a constant.

There is thus a constant $\mathrm{C}_1 = 1 + \mathrm{C}/(1-\alpha)$ such that the densities σ^n associated with $g^n\sigma$ satisfy:

$$|\partial_1 \sigma^n / \sigma^n| = |\partial_1 \log \sigma^n| < \mathrm{C}_1 \tag{3}$$

for sufficiently large n. Inequalities similar to (2) yield bounds similar to (3) for the derivatives of order $\ell \leq r-1$, namely:

$$|\partial_1^\ell \sigma_n / \sigma_n| < \mathrm{C}_\ell \tag{4}$$

From (4) it is clear that a w^*-limit of points in $\mathscr{K}_+$ is again in $\mathscr{K}_+$, and therefore $\mathscr{K}_+$ is vaguely compact. If $\rho \in \mathscr{K}_+$, the formula (1) permits the calculation of a density ρ^0 from the knowledge of a density ρ^{-1} on a piece of leaf of smaller diameter. By iteration one can express ρ^0 in terms of ρ^{-n}, which for large n is defined on a very small piece of leaf, and therefore almost constant because $|\partial_1 \rho^{-n} / \rho^{-n}| < \mathrm{C}_1$. Since ρ^{-n} can be approximated in norm by a constant, it is immediate that ρ^0 is unique (up to a multiplicative constant), *i.e.* independent of the choice of ρ in $\mathscr{K}_+$.

Let $\mathscr{K}$ be the linear space of real measures generated by $\mathscr{K}_+$. Then, because of the uniqueness of conditional measures, each $\rho \in \mathscr{K}$ is of the form $\rho = \alpha\rho_1 - \beta\rho_2$ with $\rho_1, \rho_2 \in \mathscr{K}_+$, $\alpha, \beta \geq 0$, and $||\rho|| = \alpha + \beta$. Thus, if $\rho \in \mathscr{K}$, then $|\rho| \in \mathscr{K}$. This implies that the cone of positive measures in $\mathscr{K}$ is simplicial; therefore $\mathscr{K}_+$ is a Choquet simplex and any two distinct extremal points of $\mathscr{K}_+$ are disjoint measures (see [1], Lemma 2). This concludes the proof of (a).

To prove (b), we may approximate in norm σ by a measure which has a C^{r-1} density with respect to Lebesgue measure on M, and is therefore in $\bigcup_{\ell > 0} \mathscr{K}(\ell)$. It follows then from (4) that any vague limit of $g^n\sigma$ when $n \to \infty$ is in $\mathscr{K}_+$.

To prove (c), we note that the same argument used to show that $\mathscr{K}_+$ is a simplex also shows that $\mathscr{K}_g$ is a simplex. Suppose $\rho \in \mathscr{K}_g$ is not ergodic. One can then write $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$ with $0 < \alpha < 1$ and disjoint g-invariant probability measures ρ_1, ρ_2. But then ρ_1, ρ_2 are in $\mathscr{K}_g$ and therefore g is not extremal in $\mathscr{K}_g$.

Remark. — The above theorem is close to results of Sinai on Anosov diffeomorphisms [2], [3] (these results extend readily to Axiom A attractors), but the setting and the method of proof are different. The measures considered in the theorem are similar to the Gibbs states of statistical mechanics in being defined by conditions on their expectation values.

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