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GROUPS OF SIMPLE ALGEBRAS

by Moss E. SWEEDLER [1]

This paper is dedicated to George Rinehart.
I often feel he is still next door.

Foreword

This paper may be read from three different points of view. The first point of view is that we are presenting a generalization of the relative Brauer Group and associated theory.

The second point of view is that we are studying and constructing simple algebras. The third point of view is homological. The introduction is divided into three parts. One for each point of view.

Introduction

Relative Brauer Group

Here is the realization of the relative Brauer Group generalized in this paper.

k is a field and A is a finite degree field extension of k. One can consider A as being contained in $\mathrm{End}_k A$ since A acts on itself by translation. Suppose U and V are k algebras each of which contains a copy of A. Write $V \sim U$ if there is an algebra isomorphism $V \cong U$ which is the identity on the copy of A. Let $\langle V \rangle$ denote the " $\sim$ " equivalence class of V.

Form $U \otimes_A V$ with respect to A acting on the left of both U and V (so that $au \otimes v = u \otimes av$). Let $U \times_A V$ denote the k-subspace of $U \otimes_A V$ consisting of

$$\{\sum_i u_i \otimes v_i \in U \otimes_A V \mid \sum_i u_i a \otimes v_i = \sum_i u_i \otimes v_i a,\ a \in A\}.$$

$U \times_A V$ has an algebra structure with unit $1 \otimes 1$ and with product determined by

$$(\sum_i u_i \otimes v_i)(\sum_j u'_j \otimes v'_j) = \sum_{i,j} u_i u'_j \otimes v_i v'_j .$$

Let $\mathscr{E}$ denote the set of " $\sim$ " equivalence classes of algebras U where $U \cong \mathrm{End}_k A$ *as an* A-*bimodule*. For $\langle U \rangle$, $\langle V \rangle \in \mathscr{E}$ one has $\langle U \times_A V \rangle \in \mathscr{E}$ and " $\times_A$ " defines a

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commutative associative product on $\mathscr{E}$ with unit $\langle \mathrm{End}_k A \rangle$. For $\langle U \rangle \in \mathscr{E}$, U is a central simple k-algebra if and only if $\langle U \rangle$ is invertible in the monoid $\mathscr{E}$. The group of units in $\mathscr{E}$ is naturally isomorphic to $\mathrm{Br}(A/k)$, the subgroup of the Brauer group of k consisting of classes split by A.

Let U^{op} denote the opposite algebra to U. Another realization of $U^{op} \times_A U$ is the following:

Form $U \otimes_A U$ with the "slip by" indicated by $ua \otimes u' = u \otimes au'$.

Take the supspace

$$\{\sum_i u_i \otimes u_i' \in U \otimes_A U \mid \sum_i au_i \otimes u_i' = \sum_i u_i \otimes u_i' a,\ a \in A\}.$$

The product is determined by $(\sum_i u_i \otimes u_i')(\sum_j v_j \otimes v_j') = \sum_{i,j} v_j u_i \otimes u_i' v_j'$.

Suppose A is its own centralizer in U, as is the case if $\langle U \rangle \in \mathscr{E}$. Then A is naturally a right $U^{op} \times_A U$-module. In terms of the second realization of $U^{op} \times_A U$, for $\sum_i u_i \otimes u'_i \in U^{op} \times_A U$ and $a \in A$ one can form

$$\sum_i u_i a u_i' \text{ (product in U).}$$

The element is in the centralizer of A, i.e. A itself, and so the right module structure is defined.

For $\langle U \rangle \in \mathscr{E}$ the following three statements are equivalent:

(i) A is a faithful right $U^{op} \times_A U$-module;

(ii) $\langle U \rangle$ is invertible in $\mathscr{E}$;

(iii) U is a central simple k-algebra.

For invertible $\langle U \rangle \in \mathscr{E}$ the right $U^{op} \times_A U$-module structure provides an algebra isomorphism $U^{op} \times_A U \cong (\mathrm{End}_k A)^{op}$ which is the identity on A; in other words

$$(*) \qquad \langle U^{op} \rangle \langle U \rangle = \langle (\mathrm{End}_k A)^{op} \rangle.$$

It is true that $\langle (\mathrm{End}_k A)^{op} \rangle = \langle \mathrm{End}_k A \rangle$ and so $(*)$ gives the classical result that the equivalence class of an algebra and the equivalence class of its opposite algebra are inverse. However notice that $(*)$ is a natural equivalence, but the equivalence $\langle (\mathrm{End}_k A)^{op} \rangle = \langle \mathrm{End}_k A \rangle$ is *not* natural. This latter equivalence depends upon A being Frobenius over k. Therefore when we start "generalizing" and A is no longer Frobenius over k the equivalence $\langle (\mathrm{End}_k A)^{op} \rangle = \langle \mathrm{End}_k A \rangle$ no longer holds.

Since A is finite dimensional over k there is a natural isomorphism $\mathrm{End}_k A \cong A \otimes_k A^*$, where $A^* = \mathrm{Hom}_k(A, k)$. Since A^* is the dual to a finite dimensional k-algebra it is a k-coalgebra. Thus $\mathrm{End}_k A = A \otimes_k A^*$ is naturally an A-coalgebra. The coalgebra diagonalization $\Delta : \mathrm{End}_k A \to (\mathrm{End}_k A) \otimes_A (\mathrm{End}_k A)$ has image in $(\mathrm{End}_k A) \times_A (\mathrm{End}_k A)$ and provides the natural equivalence $\mathrm{End}_k A \sim (\mathrm{End}_k A) \times_A (\mathrm{End}_k A)$ or

$$\langle \mathrm{End}_k A \rangle \langle \mathrm{End}_k A \rangle = \langle \mathrm{End}_k A \rangle.$$

In the above $\mathrm{End}_k A$ is the model for the identity class $\langle \mathrm{End}_k A \rangle$ in $\mathscr{E}$. In the generalization developed herein we deal with a commutative k-algebra A over a commutative ring k. The generalization of $\mathrm{End}_k A$ is a " $\times_A$-bialgebra " E where $\langle E \rangle$ is the model of the identity class in $\mathscr{E}\langle E \rangle$ which is the generalization of $\mathscr{E}$ above.

Homology Theory

Suppose A is a ring and M and N are A-bimodules. The " product " $\widetilde{M} \times_A N$ is defined as the additive subgroup of $M \otimes_A N$ consisting of

$$\{\sum_i m_i \otimes n_i \in M \otimes_A N \mid \sum_i a m_i \otimes n_i = \sum_i m_i \otimes n_i a, \ a \in A\}.$$

Here the tensor product $M \otimes_A N$ is with respect to M_A and ${}_A N$. Some properties of the functor $\widetilde{M} \times_A N$ are derived, properties which are needed in studying $\widetilde{M} \times_A N$ when M and N are rings. Suppose M and N are rings and $i : A \rightarrow M$, $j : A \rightarrow N$ are ring maps. These maps give M and N A-bimodule structures, permitting the formation of $\widetilde{M} \times_A N$. However, now $\widetilde{M} \times_A N$ has a ring structure with unit $1 \otimes 1$ and with product

$$(\sum_i m_i \otimes n_i)(\sum_j m'_j \otimes n'_j) = \sum_{i,j} m'_j m_i \otimes n_i n'_j$$

for $\sum_i m_i \otimes n_i$, $\sum_j m'_j \otimes n'_j \in \widetilde{M} \times_A N \subset M \otimes_A N$. With this ring structure $\widetilde{M} \times_A N$ is naturally isomorphic to $\mathrm{End}_{M \otimes_Z \overline{N}}(M \otimes_A N)$ where $\overline{N}$ is the *opposite ring* to N.

If M is an A-bimodule or ring over A then the symbol " $\widetilde{M}$ " is not defined when A is not commutative. It needs the rest, the " $\times_A N$ ". When A is commutative and M is an A-bimodule, then $\widetilde{M}$ is defined as the opposite A-bimodule, where $a\widetilde{m}b = \widetilde{bma}$. If A is commutative and $i : A \rightarrow M$ a ring map, then $\widetilde{M}$ is the opposite ring to M and $\widetilde{i} : A \rightarrow \widetilde{M}$, $a \mapsto \widetilde{i(a)}$. We identify $\widetilde{\widetilde{M}}$ with M. We define $M \times_A N$ as

$$\widetilde{(\widetilde{M})} \times_A N$$

when A *is commutative.*

In this case $M \times_A N$ may be thought of as being contained in ${}_{\bullet}M \otimes_A {}_{\bullet}N$, the tensor product with respect to ${}_A M$ and ${}_A N$. (See the definition of $U \times_A V$ in the beginning of the previous section, Relative Brauer Group.) If M and N are simply A-bimodules, then $M \times_A N$ is an A-bimodule, where

$$a(\sum_i m_i \otimes n_i)\, b = \sum_i (a m_i) \otimes (n_i b)$$

for $a, b \in A$, $\sum_i m_i \otimes n_i \in M \times_A N \subset {}_{\bullet}M \otimes_A {}_{\bullet}N$. If M and N are rings over A with respect to maps i, j as before then $M \times_A N$ is a ring over A with respect to

$$A \rightarrow M \times_A N, \quad a \mapsto i(a) \otimes 1 = 1 \otimes j(a),$$

$a \in A$. Thus, when A is commutative, " $\times_A$ " gives a product on the category of A-bimodules and a product on the category of rings over A.

Even when A is commutative the " $\times_A$ " product of rings over A is defined more generally than the tensor product. If $i : A \to M$ and $j : A \to N$ do not have images in the centers of the respective rings then $M \otimes_A N$ is not a well defined ring. Nevertheless $M \times_A N$ is a well defined ring. If i and j have central images then $M \times_A N$ is naturally isomorphic to $M \otimes_A N$ as a ring.

Two *rings over* A are considered equivalent if they are isomorphic by an isomorphism preserving the maps from A to each of them. When A is commutative " $\times_A$ " induces a product on equivalence classes of rings over A. The product is commutative. Certain equivalence classes are idempotent and hence are candidates for playing the role of identity element in a group. For a given A there may be several groups built around different identity elements. Each of these groups is essentially the H^2 in a cohomology theory. The cohomology theory is determined by the identity element or a representative of it.

Among the main difficulties that arise with the " $\times_A$ " product is lack of associativity. Suppose M, N and P are A-bimodules. There are natural maps from $M \times_A (N \times_A P)$ and $(M \times_A N) \times_A P$ to a third A-bimodule, Y. When the maps to Y are injective and have the same image they induce a natural isomorphism

$$(M \times_A N) \times_A P \cong M \times_A (N \times_A P).$$

This isomorphism is automatically an isomorphism of rings over A if M, N and P happen to be rings over A. A fair amount of technical detail is developed to establish when $M \times_A (N \times_A P)$ is naturally isomorphic to $(M \times_A N) \times_A P$ as above. For example the natural isomorphism holds if both M and P are the directed union of subbimodules which are projective as left A-modules. Other conditions are presented.

The notion of $\times_A$-bialgebra is introduced. These are rings over A which are like Hopf algebras but with respect to the product " $\times_A$ ", rather than tensor product. $\times_A$-bialgebras or rather their equivalence classes are good candidates for being the identity of a group as mentioned two paragraphs above. $\times_A$-bialgebras also determine a cohomology theory which is akin to the Hopf algebra cohomology of [1].

However, in particular cases, it is shown that the $\times_A$-bialgebra cohomology is naturally isomorphic to some other cohomology. For example if A is a commutative R-algebra and is a finite projective R-module, then $\mathrm{End}_R A$ is a $\times_A$-bialgebra and the $\times_A$-bialgebra cohomology is isomorphic to Amitsur cohomology. Another important example is rings of differential operators.

Let $\mathfrak{M} = \mathrm{Ker}(A \otimes A \xrightarrow{\text{mult}} A)$. We say A has *almost finite projective differentials* if there is a collection of ideals of $A \otimes A$ which is cofinal with $\{\mathfrak{M}^n\}_{n=0}^{\infty}$ and where for each ideal I in the collection $(A \otimes A)/I$ is a finite projective *left* A-module. When A has almost finite projective differentials, then D_A, the ring of differential operators on A, is a $\times_A$-bialgebra. A is said to have *finite projective differentials* when $(A \otimes A)/\mathfrak{M}^n$ is a finite projective A-module for each n. In this case, when the ground ring contains $\mathbf{Q}$, the $\times_A$-bialgebra cohomology is naturally isomorphic to the algebraic De Rham cohomology of A from degree two onward. This leads to an interpretation of $H^2_{\text{DeRham}}(A)$ as classifying a certain Brauer-type group.

Some examples of rings with almost finite projective differentials. Suppose A is an algebra which is a finite projective module over the ground ring. A is called purely inseparable over the ground ring if $\operatorname{Ker}(A\otimes A \xrightarrow{\text{mult}} A)$ consists of nilpotent elements. In this case A has almost finite projective differentials although A is not necessarily differentially smooth [8, (16.10.1)]. Suppose A is a localization of a finitely generated algebra over a field. If A is regular then A has finite projective differentials. The tensor product of two algebras with almost finite projective differentials again has almost finite projective differentials.

Our investigations lead us to consider the following type of cohomology theory. Say A is a commutative algebra and $\{L_\alpha\}$ is a collection of ideals in $A\otimes A$. Let $e : A\otimes A \to A\otimes A\otimes A$, $a\otimes b \mapsto a\otimes 1\otimes b$. Assume that $\{L_\alpha\}$ has the following properties:

1) Given L_α and L_β there is L_γ with $L_\gamma \subset L_\alpha \cap L_\beta$.
2) Given L_α and L_β there is L_γ with $e(L_\gamma) \subset A\otimes L_\alpha + L_\beta\otimes A$.

In the n-fold tensor product $A\otimes\ldots\otimes A$ form the collection of ideals of the form

$$L_{\alpha_1}\otimes A\otimes\ldots\otimes A + A\otimes L_{\alpha_2}\otimes A\otimes\ldots\otimes A + \ldots + A\otimes\ldots\otimes A\otimes L_{\alpha_{n-1}}.$$

Let $\widehat{A\otimes\ldots\otimes A}$ be the completion of the n-fold tensor product with respect to the family of ideals. (In degree 0, $\hat{A} = A$.) The second condition, $e(L_\gamma) \subset A\otimes L_\alpha + L_\beta\otimes A$, insures that the Amitsur complex maps

$$e_i : \overbrace{A\otimes\ldots\otimes A}^{n} \to \overbrace{A\otimes\ldots\otimes A}^{n+1}$$

$$a_1\otimes\ldots\otimes a_n \mapsto a_1\otimes\ldots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\ldots\otimes a_n$$

are continuous and induce maps $\hat{e}_i : \widehat{A\otimes\ldots\otimes A} \to \widehat{A\otimes\ldots\otimes A}$, raising degree by one. The $\hat{e}_i$ are algebra maps. There are two natural cohomologies to consider at this point: with respect to the functor " underlying additive group " and with respect to the functor " multiplicative group of invertible elements ". (In some cases there is an exponential map relating the two. This happens in the theory about De Rham cohomology.)

Suppose that $\{L_\alpha\}$ has the additional property:

3) $(A\otimes A)/L_\alpha$ is a finite projective *left* A-module for each L_α and there is an L_α contained in $\operatorname{Ker}(A\otimes A \xrightarrow{\text{mult}} A)$. In this case $\{L_\alpha\}$ gives rise to a $\times_A$-bialgebra which lies in End A. End A is naturally an $A\otimes A$-module where

$$((a\otimes b).f)(c) = af(bc)$$

$a, b, c \in A$, $f\in \operatorname{End} A$. The $\times_A$-bialgebra C arising from $\{L_\alpha\}$ is

$$\{f\in \operatorname{End} A \mid L_\alpha.f = 0 \text{ for some } L_\alpha\}.$$

The $\times_A$-bialgebra cohomology is isomorphic to the cohomology of the complex $\{\widehat{A\otimes\ldots\otimes A}, \{\hat{e}_j\}\}$ with respect to the functor " multiplicative group of invertible elements ". The cohomology of the complex $\{\widehat{A\otimes\ldots\otimes A}, \{\hat{e}_j\}\}$ with respect to the functor " underlying additive group " is naturally isomorphic to $\mathrm{Ext}^*_C(A, A)$ when A is projective over the ground ring.

Simple Algebras

Let twist: $A\otimes A \to A\otimes A$, $a\otimes b \mapsto b\otimes a$. $\{L_\alpha\}$ may have the property:

4) Given L_α there is L_β with $\mathrm{twist}(L_\beta)\subset L_\alpha$.

Suppose $\{L_\alpha\}$ satisfies properties 1), 2), 3) and 4) and C is the associated $\times_A$-bialgebra. We identify A with $A^\ell\subset \mathrm{End}\,A$ where A^ℓ is A acting on itself by left translation. Since some $L_\alpha\subset \mathrm{Ker}(A\otimes A \xrightarrow{\text{mult}} A)$ it follows that $A^\ell\subset C$.

Theorem. — The following statements are equivalent:

1. C *is a simple ring.*
2. A *is a simple* C*-module.*
3. *For each ideal* $I\subset A$ *there is an* L_α *where* $A\otimes I \not\subset I\otimes A+L_\alpha$.
4. *Suppose* U *and* V *are rings over* A *where* $U\cong V\cong C$ *as* A*-bimodules and* $U\times_A V\cong C$ *as a ring over* A. *Then* U *and* V *are simple rings.*

This is one of the main simplicity theorems. It is used to establish the simplicity of C as well as the simplicity of the rings in the equivalence classes which form the Brauer-type group determined by C. One of the main applications is to establish simplicity of rings of differential operators. In a moment we state our main theorem concerning simplicity of rings of differential operators.

Definition. — An element $0\neq a\in A$ has the *strong intersection property* if for each commutative algebra B the elements $x\in B\otimes J_a$ are such that $1+x$ is invertible in $B\otimes A$. Here $J_a=\{y\in A\,|\,ya=0\}$. The algebra A has the strong intersection property if each $0\neq a\in A$ has the strong intersection property.

A has the strong intersection property for example if J_a consists of nilpotent elements for each $0\neq a\in A$. In the section on homology some examples of algebras with almost finite projective differentials were given. These examples also have the strong intersection property. Hence the following theorem applies:

Theorem. — Suppose A *has the strong intersection property and almost finite projective differentials. Furthermore suppose that for each ideal* $0\neq I\subsetneq A$ *both* I *and* A/I *are flat over the ground ring and* $(A/I)\otimes A$ *is a Noetherian ring. Then the ring of differential operators on* A *is a simple ring.*

The center of the ring of differential operators is characterized by:

Theorem. — *Suppose the ground ring is a field with algebraic closure* S. *If* $S\otimes A\otimes A$ *is Noetherian, then the center of the ring of differential operators on* A *is* $(\text{Sep } A)^{\ell}$; i.e. Sep A *acting on* A *as left translation operators.* (Sep A is the subalgebra of A consisting of elements satisfying non-zero separable polynomials over the ground ring.) *Moreover* Sep A *is finite dimensional.*

In the beginning of the section on Homology the product $\widetilde{M}\times_A N$ is described. This product without much other theory is used to give a criterion for simplicity of a ring. Here we are no longer assuming that A is commutative. Let $i: A \to M$ be a map of rings and let L denote the centralizer of $i(A)$ in M. Then L is a right $\widetilde{M}\times_A M$-module where

$$\ell.(\sum_i m_i\otimes m_i') = \sum_i m_i \ell m_i',$$

$\ell\in L$, $\sum_i m_i\otimes m_i' \in \widetilde{M}\times_A M \subset M\otimes_A M$.

Theorem. — *Suppose* L *is a faithful right* $\widetilde{M}\times_A M$-*module,* M *is flat as a left* A-*module and* $0 \neq \widetilde{I}\times_A M$ *for non-zero two-sided ideals* $I\subset M$. *Then* M *is a simple ring if* L *is a simple* $\widetilde{M}\times_A M$-*module.*

This is result (3.7). All the other results on simplicity eventually come down to this theorem

0. Conventions

Throughout we are working over a commutative ring R with identity.

We use unadorned $\otimes$, Hom and End to denote $\otimes_R$, Hom_R and End_R. All algebras are R-algebras. They have unit and subalgebras have the same unit. All modules are unitary. Our typical algebra A is assumed to be commutative in all sections except 0, 1 and 3.

For an algebra A let $\overline{A}$ denote the " opposite " algebra where

$$A \xrightarrow{(a\mapsto\bar{a})} \overline{A}$$

is an algebra *anti*-isomorphism.

If M is a left A-module, also consider M as a right $\overline{A}$-module by setting $m\bar{a}\equiv am$, $a\in A$, $m\in M$. Similarly, right A-modules are made into left $\overline{A}$-modules.

If M is simultaneously a right and left A-module and satisfies $(am)b = a(mb)$ $a, b\in A$, $m\in M$ and the right and left R-module actions on M are the same then M is called an A-bimodule. In this case M is also an $\overline{A}$-bimodule " switching both sides ". A bimodule map is one which is both a left and right module map. An A-bimodule M can be viewed as a left $A\otimes\bar{A}$-module where $(a\otimes\bar{b}).m=amb$, $a, b\in A$, $m\in M$. This gives an equivalence between the category of A-bimodules and the category of left $A\otimes\bar{A}$-modules.

If we write $M\otimes_A N$ this indicates the tensor product with respect to the right A-module structure of M and left A-module structure of N even if M or N are A-bimodules.

If we write $\mathrm{Hom}_A(M, N)$ this indicates the " hom " with respect to the left A-module structure of M and left A-module structure of N even if M and N are A-bimodules. Thus if M and N are A-bimodules the set of bimodule maps from M to N is the same as $\mathrm{Hom}_{A\otimes\bar{A}}(M, N)$.

Let M be an R-module. Giving an A-module structure of M is the same as giving a representation $\rho : A\to \mathrm{End}\, M$ where ρ is an algebra homomorphism if M is a left A-module, and ρ is an algebra anti-homomorphism if M is a right A-module. (Of course ρ is determined by $a.m=\rho(a)(m)$, $a\in A$, $m\in M$.) When discussing several A-module structures on M it will sometimes be convenient to use the associated representations. For example if M has two A-module structures with representations ρ_1 and ρ_2 we say that the A-module structures (or actions) *commute* if for all $a, b\in A$

$$\rho_1(a)\rho_2(b)=\rho_2(b)\rho_1(a)\in \mathrm{End}\ M.$$

Suppose M has several A-module structures with representations $\{\rho_i\}_{i\in I}$. The R-*module equalizer* of the A-module structures denotes

$$\{m\in M \mid \rho_i(a)(m)=\rho_j(a)(m),\ i, j\in I,\ a\in A\}.$$

This is only an R-submodule of M in general. However, if M has an A-module structure " * " which commutes with all the A-module structures used in forming the R-module equalizer, then the R-module equalizer is a sub *-A-module of M.

The R-*module coequalizer* of the A-module structures (with representations $\{\rho_i\}_{i\in I}$) denotes M/N where N is the R-submodule of N generated by

$$\{\rho_i(a)(m)-\rho_j(a)(m)\in M \mid i, j\in I,\ a\in A,\ m\in M\}.$$

This is only an R-quotient module of M in general. However, if M has an A-module structure " * " which commutes with all the A-module structures used in forming the R-module coequalizer, then the R-module coequalizer is a quotient *-A-module of M.

We only deal with R-module equalizers and R-module coequalizers to define the symbols " $\int^x$ " and " $\int_x$ ".

Many R-modules have several A-module structures indicated by " position ". For example if M and N are A-bimodules then ${}_\ell M\otimes_A N_r$ has ℓ and r A-module structures. A more complicated example: Suppose F is an n variable additive functor from the category of R-modules to the category of R-modules and $M_1, \ldots, M_n$ are A-bimodules. The A-module structure ${}_A M_i$ induces an A-module structure on $F(M_1, \ldots, M_n)$ indicated by the symbol $F(M_1, \ldots, M_{i-1}, {}_x M_i, M_{i+1}, \ldots, M_n)$ where x is an indeterminate. Similarly the A-module structure M_{iA} induces an A-module structure on $F(M_1, \ldots, M_n)$ indicated by the symbol $F(M_1, \ldots, M_{i-1}, M_{ix}, M_{i+1}, \ldots, M_n)$.

Now say that M is an R-module which has several A-module structures indicated by positions. Following Mac Lane we denote the R-module equalizer of those A-module structures by the symbol

$$\int^x \text{(M with } x \text{ placed in the appropriate positions).}$$

The R-module coequalizer of those A-module structures is denoted by the symbol

$$\int_x \text{(M with } x \text{ placed in the appropriate positions).}$$

For example, if M and N are A-bimodules then

$$M \otimes_A N = \int_x M_x \otimes {}_x N$$

$$\mathrm{Hom}_A(M, N) = \int^x \mathrm{Hom}({}_x M, {}_x N)$$

$$\{m \in M \mid am = ma, \ a \in A\} = \int^x {}_x M_x.$$

The x is merely a place holder and may be replaced by other letters, especially in iterated integrals. For example, if M, N and P are A-bimodules then

$$M \otimes_A N \otimes_A P = \int_y \int_x M_x \otimes {}_x N_y \otimes {}_y P = \int_x \int_y M_x \otimes {}_x N_y \otimes {}_y P.$$

One of the main concerns of this paper is studying

$$\int^y \int_x {}_y M_x \otimes {}_x N_y = \int^y {}_y M \otimes_A N_y.$$

As another example the set of A-bimodule maps from M to N may be described as

$$\int^y \int^x \mathrm{Hom}({}_x M_y, {}_x N_y) = \int^x \int^y \mathrm{Hom}({}_x M_y, {}_x N_y).$$

When A is commutative we shall have to consider

$$\int_x {}_x M \otimes {}_x N \otimes {}_x P$$

which is the triple tensor product over A of M, N and P with respect to A acting on the *left*. In general

$$\int_x {}_x M \otimes {}_x N \otimes {}_x P \neq M \otimes_A N \otimes_A P = \int_y \int_x M_x \otimes {}_x N_y \otimes {}_y P.$$

One of the reasons for introducing the " $\int_x \ldots$ " notation is to easily distinguish different tensor products of bimodules.

Suppose A is a commutative algebra and M is an A-bimodule. Let $\widetilde{M}$ denote the A-bimodule where

$$M \xrightarrow{(m \mapsto \tilde{m})} \widetilde{M}$$

is an R-module isomorphism and $a\tilde{m}b = \widetilde{bma}$ $\quad a, b \in A, \quad m \in M$.

If A is an algebra then an *algebra over* A is a pair (U, i) where U is an algebra and $i: A \to U$ is an algebra map. *Notice that this does not make* U *into an* A*-algebra.* For U to be an A-algebra, A would have to be commutative and $\operatorname{Im} i$ would have to be in the *center* of U.

If i is injective we may then identify A with its image in U so that i is the inclusion map. If (U, i) and (U', i') are algebras over A, then $f: U \to U'$ is a map of algebras over A if f is an algebra map and $fi = i'$. If f is bijective it is called an isomorphism of algebras over A. In this case f^{-1} is also an isomorphism of algebras over A and (U, i) and (U', i') are called isomorphic algebras over A.

If (U, i) is an algebra over A, the canonical A-bimodule structure on U is given by $aub \equiv i(a)ui(b)$, $a, b \in A$, $u \in U$. A map of algebras over A is an A-bimodule map.

If (U, i) is an algebra over A, then a subalgebra $V \subset U$ is called a subalgebra over A if $\operatorname{Im} i \subset V$. In this case (V, i with its range restricted to V) is an algebra over A. Usually it will be written (V, i).

Let $\ell : A \to \operatorname{End} A$ be the injective algebra homomorphism determined by $a^{\ell}(b) = ab$, $a, b \in A$. For $a \in A$ the element a^{ℓ} is sometimes called a as a *left translation operator*. The pair $(\operatorname{End} A, \ell)$ is an algebra over A and defines the canonical A-bimodule structure of End A. Thus $(afb)(c) = af(bc)$, $a, b, c \in A$, $f \in \operatorname{End} A$.

If (U, i) is an algebra over A then $\bar{i}$ denotes the map $\bar{A} \xrightarrow{(\bar{a} \mapsto \overline{i(a)})} \bar{U}$, making $(\bar{U}, \bar{i})$ an algebra over $\bar{A}$. If A is commutative and (U, i) is an algebra over A, let $\tilde{U}$ denote the opposite algebra to U considered as an algebra *over* A. Thus $U \xrightarrow{(u \mapsto \tilde{u})} \tilde{U}$ is an algebra *anti*-isomorphism and $\tilde{i} : A \xrightarrow{(a \mapsto \widetilde{i(a)})} \tilde{U}$ is an algebra map giving $(\tilde{U}, \tilde{i})$ the structure of algebra over A.

If M is a module with a family of submodules $\{M_\alpha\}$ then M is the *directed union* of $\{M_\alpha\}$ if each finite subset of M is contained in an M_α.

The term " finite projective module " is used interchangeably with the term " finitely generated projective module ".

1. $M \times_A N$ as a module

M and N are bimodules for the algebra A.

Definition (**1.1**). — $\tilde{M} \times_A N$ denotes the R-submodule $\int^x {}_x M \otimes_A N_x$ of $M \otimes_A N$.

Since A may not be commutative " $\tilde{M}$ " is not defined, it needs the rest of the symbol " $\times_A N$ ". The natural equivalences $M \otimes_A A = M = A \otimes_A M$ induce

$$\tilde{M} \times_A A = \int^x {}_x M_x = \tilde{A} \times_A M.$$

If $f: M\to M'$, $g: N\to N'$ are maps of A-bimodules then $f\otimes g : M\otimes_A N \to M'\otimes_A N'$ satisfies $(f\otimes g)\left(\int^x {}_xM\otimes_A N_x\right) \subset \int^x {}_xM'\otimes_A N'_x$.

Definition (1.2). — $\tilde{f}\times g : \widetilde{M}\times_A N \to \widetilde{M}'\times_A N'$ is the R-module map induced by $f\otimes g$.

The following properties hold:

(1.3)
1. If f and g are A-bimodule isomorphisms then $\tilde{f}\times g$ is an R-module isomorphism with inverse $\widetilde{f^{-1}}\times g^{-1}$.
2. If f is injective and N is flat as a left A-module then $\tilde{f}\times I : \widetilde{M}\times_A N \to \widetilde{M}'\times_A N$ is injective.
3. If M is flat as a *right* A-module and g is injective then $\tilde{I}\times g : \widetilde{M}\times_A N \to \widetilde{M}\times_A N'$ is injective.

Suppose X is a right A-module and C is a left A-submodule of End A. There is the map

$$(1.4) \qquad \Lambda : X\otimes_A C \to \mathrm{Hom}(A, X)$$
$$\Lambda(x\otimes c)(a) = xc(a), \quad x\in X, \quad c\in C, \quad a\in A.$$

Proposition (1.5):

1. *If* X *is a flat right* A*-module and is the directed union of finitely presented submodules then* $\Lambda : X\otimes_A C \to \mathrm{Hom}(A, X)$ *is injective.*

2. *If* X *is the directed union of submodules* $\{X_\alpha\}$ *and each* $X_\alpha\otimes_A C \xrightarrow{\Lambda} \mathrm{Hom}(A, X_\alpha)$ *is injective then* $\Lambda : X\otimes_A C \to \mathrm{Hom}(A, X)$ *is injective.*

3. *If* $X\otimes_A C \xrightarrow{\Lambda} \mathrm{Hom}(A, X)$ *is injective then* $Y\otimes_A C \xrightarrow{\Lambda} \mathrm{Hom}(A, Y)$ *is injective if* Y *is an* A*-submodule of* X *which is an* A*-direct summand.*

4. *If* X *is a projective right* A*-module then* $\Lambda : X\otimes_A C \to \mathrm{Hom}(A, X)$ *is injective.*

Proof:

1. Let $F\to A$ be a surjective R-module map where F is a free R-module. This induces injections

$$\mathrm{Hom}(A, X) \xrightarrow{\beta} \mathrm{Hom}(F, X) \qquad \text{and} \qquad \mathrm{Hom}(A, A) \xrightarrow{\gamma} \mathrm{Hom}(F, A).$$

If $\mathrm{Hom}(F, {}_xA)$ has the left x A-module structure, then γ is an A-module map. The diagram

$$\begin{array}{ccccc} X\otimes_A C & \xrightarrow{I\otimes\alpha} & X\otimes_A \mathrm{Hom}(A, A) & \xrightarrow{I\otimes\gamma} & X\otimes_A \mathrm{Hom}(F, A) \\ \downarrow{\scriptstyle\Lambda} & & & & \downarrow{\scriptstyle\rho} \\ \mathrm{Hom}(A, X) & & \xrightarrow{\quad\beta\quad} & & \mathrm{Hom}(F, X) \end{array}$$

commutes, where α is the injection $C \to \mathrm{Hom}(A, A)$ and ρ is determined by $\rho(x \otimes g)(f) \equiv xg(f)$, $x \in X$, $f \in F$, $g \in \mathrm{Hom}(F, A)$. By flatness of X the top row consists of injections. Thus it suffices to prove that ρ is injective.

If Y is a submodule of X the diagram

$$\begin{array}{ccc} Y \otimes_A \mathrm{Hom}(F, A) & \xrightarrow{\rho} & \mathrm{Hom}(F, Y) \\ \downarrow & & \downarrow \\ X \otimes_A \mathrm{Hom}(F, A) & \xrightarrow{\rho} & \mathrm{Hom}(F, X) \end{array}$$

commutes, where the vertical arrows are induced by $Y \to X$. By left exactness of Hom the right vertical arrow is injective. By the directed union hypothesis each element of $X \otimes_A \mathrm{Hom}(F, A)$ is in the image of the left vertical arrow for some finitely presented submodule Y of X. Thus it suffices to prove that ρ is injective when X is finitely presented.

Let $0 \to K \to L \to X \to 0$ be an exact sequence of A-modules where K is finitely generated and L is free and finitely generated. The diagram

$$\begin{array}{ccccccccc} & & K \otimes_A \mathrm{Hom}(F, A) & \longrightarrow & L \otimes_A \mathrm{Hom}(F, A) & \longrightarrow & X \otimes_A \mathrm{Hom}(F, A) & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho & & \\ 0 & \longrightarrow & \mathrm{Hom}(F, K) & \longrightarrow & \mathrm{Hom}(F, L) & \longrightarrow & \mathrm{Hom}(F, X) & \longrightarrow & 0 \end{array}$$

commutes. The top row is exact by right exactness of "$\otimes$". The bottom row is exact because F is a free R-module. The left ρ is surjective because K is a finitely generated A-module and F is a free R-module. The center ρ is bijective because L is a finitely generated free A-module. Thus by the 5-lemma the right ρ is injective and Part 1 is proved.

2. The diagram

$$\begin{array}{ccc} X_\alpha \otimes_A C & \xrightarrow{\Lambda} & \mathrm{Hom}(A, X_\alpha) \\ \downarrow & & \downarrow \\ X \otimes_A C & \xrightarrow{\Lambda} & \mathrm{Hom}(A, X) \end{array}$$

commutes, where the vertical arrows are induced by $X_\alpha \to X$. By left exactness of Hom the right vertical arrow is injective. By the directed union hypothesis each element of $X \otimes_A C$ is in the image of the left vertical arrow for some α. This proves part 2.

3. The diagram

$$\begin{array}{ccc} Y\otimes_A C & \longrightarrow & X\otimes_A C \\ \downarrow & & \downarrow \\ \mathrm{Hom}(A, Y) & \longrightarrow & \mathrm{Hom}(A, X) \end{array}$$

commutes. By the direct summand hypothesis the top horizontal arrow is injective. Also by hypothesis the right vertical arrow is injective. This proves the left vertical arrow is injective and Part 3.

4. A free A-module is the directed union of finitely generated free submodules. Hence, Part 4 follows from Part 1 and 3. Q.E.D.

Proposition **(1.6).** — *Let* M *be an* A-*bimodule and* C *a sub*-A-*bimodule of* End A.

1. *There is an* R-*module map* $\sim\theta : \widetilde{M}\times_A C \rightarrow M$ *determined by*

$$\sum_i m_i\otimes c_i \xrightarrow{\sim\theta} \sum_i m_i c_i(1), \qquad \sum_i m_i\otimes c_i \in \int^x {}_xM\otimes_A C_x = \widetilde{M}\times_A C.$$

2. $\sim\theta$ *is injective if* $\Lambda : M\otimes_A C \rightarrow \mathrm{Hom}(A, M)$ *is injective.*

Proof. — $\Lambda\left(\int^x {}_xM\otimes_A C_x\right) \subset \mathrm{Hom}_A(A, M)$ which as usual is identified with M. The diagram

$$\begin{array}{ccc} \widetilde{M}\times_A C = \int^x {}_xM\otimes_A C_x & \xrightarrow{\Lambda \mid \int^x {}_xM\otimes_A C_x} & \mathrm{Hom}_A(A, M) = M \\ \downarrow & & \downarrow \\ M\otimes_A C & \xrightarrow{\Lambda} & \mathrm{Hom}(A, M) \end{array}$$

commutes, where the vertical arrows are natural inclusions. The top horizontal map from $\widetilde{M}\times_A C$ to M is $\sim\theta$. Q.E.D.

2. $M\times_A N$ for Commutative A

Throughout this section A is a commutative algebra. Thus if M is an A-bimodule $\widetilde{M}$ is defined as the " opposite " A-bimodule.

For A-bimodules M and N the R-module $\int_x {}_xM_y\otimes_x M_z$ has the set $\{x, y, z\}$ of A-module structures. $\int^y\int_x {}_xM_y\otimes_x N_y$ is an x, y and z A-submodule of $\int_x {}_xM_y\otimes_x N_z$ and the y and z A-module structures on $\int^y\int_x {}_xM_y\otimes_x N_{yz}$ are the same.

Definition **(2.1)**. — $M\times_A N$ is the R-module $\int^y \int_x {}_x M_y \otimes {}_x N_y$. As an A-bimodule the *left* A-module structure is the x A-module structure and the *right* A-module structure is the y A-module structure.

Since A is commutative, if M is an A-bimodule, $\widetilde{M}$ is defined. Thus, if N is another A-bimodule, the symbol $\widetilde{M}\times_A N$ has meaning in terms of (1.1) and (2.1). The two definitions are related by the commutative diagram

$$\text{(2.2)}\qquad \begin{array}{ccc} \int^y {}_y M\otimes_A N_y \overset{(1.1)}{=} \widetilde{M}\times_A N \overset{(2.1)}{=} \int^y \int_x {}_x\widetilde{M}_y \otimes {}_x N_y \\ \cap \qquad\qquad\qquad \cap \\ M\otimes_A N \xleftarrow{\;m\otimes n = \tilde{m}\otimes n\;} \int_x {}_x\widetilde{M}\otimes_x N \end{array}$$

Notice that $\widetilde{M\times_A N}$ is also naturally identified with $\int^y \int_x {}_x M_y \otimes_x N_y$, with the left A-module structure on $\widetilde{M\times_A N}$ being the y A-module structure, and the right A-module structure on $\widetilde{M\times_A N}$ being the x A-module structure. In a later section we shall deal with $\widetilde{\widetilde{M}\times_A N}$. By (1.1) $\widetilde{M}\times_A N$ is $\int^y {}_y M\otimes_A N_y$. $\widetilde{\widetilde{M}\times_A N}$ may also be identified with $\int^y {}_y M_x \otimes_A N_y$, where the right A-module structure on $\widetilde{\widetilde{M}\times_A N}$ is the x A-module structure and the left A-module structure on $\widetilde{\widetilde{M}\times_A N}$ is the y A-module structure. Thus the natural inclusion

$$\widetilde{\widetilde{M}\times_A N} \hookrightarrow {}_y M_x \otimes_A N_z$$

is a right A-module map if the right A-module structure on $M_x\otimes_A N$ is the x A-module structure. The natural inclusion is a left A-module map if the left A-module structure on ${}_y M\otimes_A N_z$ is either the y or z A-module structure.

If $f: M\to M'$ and $g: N\to N'$ are maps of A-bimodules, then

$$f\otimes g : \int_x {}_x M\otimes_x N \to \int_x {}_x M'\otimes_x N'$$

carries $M\times_A N$ to $M'\times_A N'$.

Definition **(2.3)**. — $f\times g : M\times_A N \to M'\times_A N'$ is the R-module map induced by $f\otimes g$.

$f\times g$ is an A-bimodule map. If $\widetilde{f}$ is the A-bimodule map defined by $\widetilde{M} \xrightarrow{(\tilde{m}\mapsto \widetilde{f(m)})} \widetilde{M}'$ then $\widetilde{f}\times g$ makes sense in terms of (1.2) or (2.3). That the two definitions agree follows from (2.2). Thus (1.3) gives the following properties for $f\times g$:

(2.4)
1. If f and g are A-bimodule isomorphisms then $f \times g$ is an A-bimodule isomorphism with inverse $f^{-1} \times g^{-1}$.
2. If f is injective and N is flat as a *left* A-module then $f \times I : M \times_A N \to M' \times_A N$ is injective.
3. If M is flat as a *left* A-module and g is injective then $I \times g : M \times_A N \to M \times_A N'$ is injective.

Proposition **(2.5)**. — M, N *and* P *are* A-*bimodules*.

1. *The natural isomorphism* $\int_x {}_x M \otimes_x N \xrightarrow{(m \otimes n \mapsto n \otimes m)} \int_x {}_x N \otimes_x M$ *induces an* A-*bimodule isomorphism* $M \times_A N \to N \times_A M$ *which is denoted* " *twist* ".

2. *There is an* R-*module map* $(M \times_A N) \times_A P \xrightarrow{\alpha} \int^y \int_x {}_x M_y \otimes_x N_y \otimes_x P_y$ *induced by the composite* $(M \times_A N) \times_A P \xhookrightarrow{\iota} \int_x {}_x(M \times_A N) \otimes_x P \xrightarrow{\iota \otimes I} \int_x {}_x M \otimes_x N \otimes_x P$.

3. *The map* α *is injective when* P *is flat as a left* A-*module. If in addition* A *is of finite presentation as an* $A \otimes A$-*module then* α *is bijective.*

4. *The map* α *is bijective when* P *is projective as a left* A-*module.*

5. *The map* α *is bijective if* $M \times_A N$ *is flat as a left* A-*module and* P *is the directed union of projective* left *sub*-A-*modules.*

Proof. — Parts 1 and 2 are left to the reader. (The ι maps in part 2 are the natural inclusions.)

It is clear in part 3 that α is injective if P is flat as a left A-module.

Let L be an A-bimodule. There are identifications

$$(*)\quad \begin{cases} \int^x {}_x L_x = \mathrm{Hom}_{A \otimes A}(A, L), \quad \ell \in \int^x {}_x L_x \text{ corresponds to } (a \mapsto a\ell = \ell a), \\ \int^x ({}_x L_x \otimes_A P) = \mathrm{Hom}_{A \otimes A}(A, L \otimes_A P) \quad \text{where } L \otimes_A P \text{ is an } A \otimes A\text{-module by} \\ \quad (a \otimes b).(\ell \otimes x) = a\ell b \otimes x \text{ for } a, b \in A,\ \ell \in L,\ x \in P. \end{cases}$$

The map $\left(\int^x {}_x L_x\right) \otimes_A P \to L \otimes_A P$ has image in $\int^x ({}_x L_x \otimes_A P)$ and induces a map $\left(\int^x {}_x L_x\right) \otimes_A P \xrightarrow{z} \int^x ({}_x L_x \otimes_A P)$. In terms of the identifications $(*)$

$$z : \mathrm{Hom}_{A \otimes A}(A, L) \otimes_A P \to \mathrm{Hom}_{A \otimes A}(A, L \otimes_A P)$$

is given by $(z(f \otimes x))(a) = f(a) \otimes x$, $f \in \mathrm{Hom}_{A \otimes A}(A, L)$, $x \in P$, $a \in A$.

If U and V are rings and there are modules ${}_U X$, ${}_U Y_V$, ${}_V Z$ then there is a natural map

$$\mathrm{Hom}_U(X, Y) \otimes_V Z \xrightarrow{z} \mathrm{Hom}_U(X, Y \otimes_V Z)$$

[3, Prop. 10, p. 38]. This is our z map in the case $U = A \otimes A$, $V = A$, $X = A$, $Y = L$, $Z = P$. By [3, Prop. 10, p. 38] z is a bijection if X is a finitely presented U-module and Z is a flat left V-module. It is easily checked that z is bijective if X is a finitely generated left U-module and Z is a projective left V-module. Thus the map z is bijective under the hypothesis of parts 3 or 4.

If L is flat as a right A-module and P is the directed union of left sub A-modules $\{P_\gamma\}$ then $L\otimes_A P = L\otimes_A(\varinjlim P_\gamma) = \varinjlim (L\otimes_A P_\gamma)$ and the right hand limit is a directed union. If for each γ the map $\left(\int^x {}_xL_x\right)\otimes_A P_\gamma \to \int^x ({}_xL_x\otimes_A P_\gamma)$ is an isomorphism then

$$\begin{aligned}\left(\int^x {}_xL_x\right)\otimes_A P &= \left(\int^x {}_xL_x\right)\otimes_A (\varinjlim P_\gamma) = \varinjlim \left(\left(\int^x {}_xL_x\right)\otimes_A P_\gamma\right)\\ &\cong \varinjlim \int^x ({}_xL_x\otimes_A P_\gamma) = \varinjlim \mathrm{Hom}_{A\otimes A}(A,\ L\otimes_A P_\gamma)\\ &= = \mathrm{Hom}_{A\otimes A}(A,\ \varinjlim L\otimes_A P_\gamma) = \mathrm{Hom}_{A\otimes A}(A,\ L\otimes_A \varinjlim P_\gamma)\\ &= \mathrm{Hom}_{A\otimes A}(A,\ L\otimes_A P) = \int^x ({}_xL_x\otimes_A P)\end{aligned}$$

where the doubled equality $(= =)$ follows from the facts that the direct limit is monomorphic, i.e. a directed union, and A is a finitely generated $A\otimes A$-module.

(✱) Thus the map $\left(\int^x {}_xL_x\right)\otimes_A P \to L\otimes_A P$ maps $\left(\int^x {}_xL_x\right)\otimes_A P$ isomorphically to $\int^x ({}_xL_x\otimes_A P)$ when

(i) P is flat as a left A-module and A is a finitely presented $A\otimes A$-module,

or

(ii) P is projective as a left A-module,

or

(iii) L is a flat right A-module and P is the directed union of left submodules $\{P_\gamma\}$ where for each γ, $\left(\int^x {}_xL_x\right)\otimes_A P_\gamma \to L\otimes_A P_\gamma$ maps $\left(\int^x {}_xL_x\right)\otimes_A P_\gamma$ isomorphically to $\int^x ({}_xL_x\otimes_A P_\gamma)$.

Thus if the hypothesis of parts 3 or 4 or 5 hold, then, by (✱), the $\iota\otimes I$ map in part 2 maps $\int_x {}_x(M\times_A N)\otimes_x P$ isomorphically to $\int^y\int_x {}_xM_y\otimes_x N_y\otimes_x P$. Thus $\iota\otimes I$ maps $(M\times_A N)\times_A P$ isomorphically to

$$\int^z\int^y\int_x {}_xM_y\otimes_x N_{y,z}\otimes_x P_z = \int^y\int_x {}_xM_y\otimes_x N_y\otimes_x P_y. \qquad \text{Q.E.D.}$$

Similarly to α in (2.5, 2) there is a map

$$\alpha' : M\times_A(N\times_A P) \to \int^y\int_x {}_xM_y\otimes_x N_y\otimes_x P_y.$$

And (2.5, parts 3, 4, 5) with suitable modifications gives conditions for α' to be injective and bijective.

Definition **(2.6)**. — An ordered triple (M, N, P) of A-bimodules is said to *associate* if the maps α and α' are injective and have the same image.

$$(M \times_A N) \times_A P \xrightarrow{\ \alpha\ } \int^y \int_x {}_xM_y \otimes {}_xN_y \otimes {}_xP_y \xleftarrow{\ \alpha'\ } M \times_A (N \times_A P)$$

In this case the induced isomorphism $(M \times_A N) \times_A P \cong M \times_A (N \times_A P)$ is called the *association isomorphism.*

The association isomorphism is an A-bimodule isomorphism. Sometimes (M, N, P) associate because α and α' are isomorphisms. For example by (2.5), (M, N, P) associate when M and P are projective left A-modules or when M and P are flat left A-modules and A is of finite presentation as on $A \otimes A$-module.

Definition **(2.7)**. — An A-bimodule M is an *associative bimodule* if (M, M, M) associates.

By symmetry M is an associative bimodule if

$$\alpha : (M \times_A M) \times_A M \to \int^y \int_x {}_xM_y \otimes {}_xM_y \otimes {}_xM_y$$

is an isomorphism.

Let M be an A-bimodule and C a sub A-bimodule of End A.

Definition **(2.8)**. — θ is the composite $M \times_A C \cong \tilde{\tilde{M}} \times_A C \xrightarrow{\sim\theta} \tilde{M} \xrightarrow{(\tilde{m} \mapsto m)} M$, where $\sim\theta$ is defined in (1.6).

For $\sum_i m_i \otimes c_i \in M \times_A C \subset \int_x {}_xM \otimes {}_xC$, $\theta(\sum_i m_i \otimes c_i) = \sum_i c_i(1) m_i$. The map θ is an A-bimodule map. Conditions for θ to be injective are provided by (1.6) and (1.5).

Suppose M and M′ are A-bimodules and N and N′ are left A-modules. φ denotes the map

(2.9) $$\int_x (M \times_A M')_x \otimes_x N \otimes_x N' \xrightarrow{\varphi} \int_x {}_xM \otimes_A N \otimes_x M' \otimes_A N'$$

determined by $(\sum_i m_i \otimes m'_i) \otimes n \otimes n' \mapsto \sum_i (m_i \otimes n) \otimes (m'_i \otimes n')$ for $\sum_i m_i \otimes m'_i \in M \times_A M'$, $n \in N$, $n' \in N'$.

If N and N′ are also A-bimodules then the composite

$$(M \times_A M') \otimes_A (N \times_A N') \xrightarrow{I \otimes \iota} \int_x (M \times_A M')_x \otimes_x N \otimes_x N' \xrightarrow{\varphi} \int_x {}_xM \otimes_A N \otimes_x M' \otimes_A N'$$

has image in $(M \otimes_A N) \times_A (M' \otimes_A N') \subset \int_x {}_xM \otimes_A N \otimes_x M' \otimes_A N'$. Let ξ denote the induced map

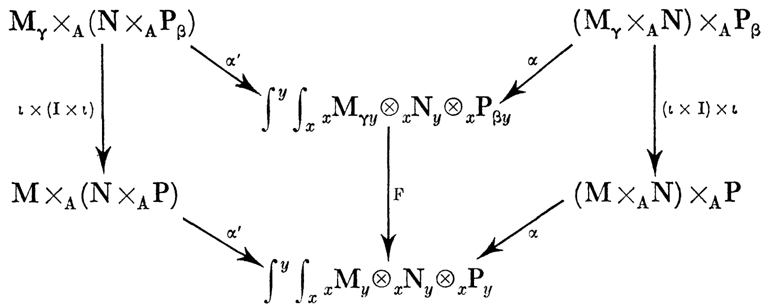
(2.10) $$(M \times_A M') \otimes_A (N \times_A N') \xrightarrow{\xi} (M \otimes_A N) \times_A (M' \otimes_A N').$$

ξ is an A-bimodule map. The various module structures preserved by φ and ξ will be mentioned as needed.

We conclude the section with final results on A-bimodules:

Proposition **(2.11)**. — *Let* M, N, P *be* A-*bimodules where* M *is the directed union of subbimodules* $\{M_\gamma\}$ *and* P *is the directed union of subbimodules* $\{P_\beta\}$. *Moreover assume each* M_γ *and* P_β *is projective as a left* A-*module. Assume* N *is flat as a left* A-*module. Then* (M, N, P) *associates as* A-*bimodules. Moreover the* α *and* α' *maps are isomorphisms.*

Proof. — For each γ and β there is a commutative diagram



where F is induced by

$$\int^y \int_x {}_xM_{\gamma y} \otimes {}_xN_y \otimes {}_xP_{\beta y} \longrightarrow \int_x {}_xM_\gamma \otimes {}_xN \otimes {}_xP_\beta \xrightarrow{\iota\otimes I\otimes\iota} \int_x {}_xM \otimes {}_xN \otimes {}_xP.$$

Since M and P are the directed union of projective left A-modules they are flat. Therefore all the vertical maps in the diagram are injections. Moreover the range modules of the vertical maps are the directed union of the images of the vertical maps as γ and β vary. By (2.5, 4) the upper α is an isomorphism and similarly the upper α' is an isomorphism. Hence the lower α and α' are isomorphisms. Q.E.D.

Since End A is an A-bimodule it is an $A\otimes A$-module where $(a\otimes b)f = afb$, $a, b \in A$, $f \in \text{End A}$. Consider ${}_xA\otimes A$ as a left A-module by the x A-module structure. Let $\{L_\alpha\}$ be a set of left sub A-modules of $A\otimes A$ with the properties:

(i) For each L_α, $(A\otimes A)/L_\alpha$ is a finite projective (left) A-module.

(ii) Given L_α and L_β there is L_γ with $L_\alpha \supset L_\gamma \subset L_\beta$.

Let $C_\alpha = \{f \in \text{End A} \mid x.f = 0,\ x \in L_\alpha\}$. Let $C = \bigcup_\alpha C_\alpha$. By property (ii) above it easily follows that given C_α and C_β there is C_γ with $C_\alpha \subset C_\gamma \supset C_\beta$. Hence C is the directed union of $\{C_\alpha\}$.

Theorem **(2.12)**. — a) C_α *is naturally isomorphic to* $\text{Hom}_A((A\otimes A)/L_\alpha, A)$. *The isomorphism is given as follows: for* $f \in C_\alpha \subset \text{End A}$ *and* $z \in (A\otimes A)/L_\alpha$ *let* $\sum_i a_i \otimes b_i \in A\otimes A$ *lie in the coset of* z. *Then if* F *is the element of* $\text{Hom}_A((A\otimes A)/L_\alpha, A)$ *corresponding to* f

$$F(z) = \sum_i a_i f(b_i).$$

b) *Each* C_α *is a finite projective* A*-module and* C *is flat as a left* A*-module.*

c) *If* M *is any right* A*-module then* $M \otimes_A C \xrightarrow{\Lambda} \mathrm{Hom}(A, M)$ *is injective.* I.e. « *all* Λ*-maps are injective for* C ».

d) *If* L_α *is an ideal then* C_α *is a sub-*A*-bimodule of* End A. *If all* (L_α)*'s are ideals of* $A \otimes A$ *then* C *is a sub-*A*-bimodule of* End A. (This result does not use the finite projectivity of $(A \otimes A)/L_\alpha$.)

e) *If* C *is a sub-*A*-bimodule of* End A *and* M *is any* A*-bimodule, then* $M \times_A C \xrightarrow{\theta} M$ *is injective.* I.e. « *all* θ*-maps are injective for* C ».

f) *If each* L_α *is an ideal in* $A \otimes A$, *then* C *is associative as an* A*-bimodule, in fact the* α *and* α' *maps are isomorphisms.*

Proof. — *a)* There is a natural identification $\mathrm{Hom}_A(A \otimes A, A) = \mathrm{Hom}(A, A) = \mathrm{End}\, A$. Under the identification C_α corresponds to the image of

$$\mathrm{Hom}_A((A \otimes A)/L_\alpha, A) \hookrightarrow \mathrm{Hom}_A(A \otimes A, A).$$

This is the duality given in *a)*.

b) Since $(A \otimes A)/L_\alpha$ is finite projective so is its A-dual C_α. And C the directed union of (C_α)'s is flat.

c) For each C_α there is the commutative diagram

$$\begin{array}{ccc} M \otimes_A C & \xrightarrow{\Lambda} & \mathrm{Hom}(A, M) \\ \uparrow & \nearrow & \\ M \otimes_A C_\alpha & & \end{array}$$

Since C is the directed union of the (C_α)'s it suffices to prove that each

$$M \otimes_A C_\alpha \to \mathrm{Hom}(A, M)$$

is injective.

$$\begin{aligned} M \otimes_A C_\alpha &\cong M \otimes_A \mathrm{Hom}_A((A \otimes A)/L_\alpha, A) \\ &\cong \mathrm{Hom}_A((A \otimes A)/L_\alpha, M \otimes_A A) \\ &= \mathrm{Hom}_A((A \otimes A)/L_\alpha, M) \to \mathrm{Hom}_A(A \otimes A, M) = \mathrm{Hom}(A, M). \end{aligned}$$

The first isomorphism follows from the identification of C_α with the A-module dual of $(A \otimes A)/L_\alpha$. The second isomorphism results from the fact that $(A \otimes A)/L_\alpha$ is a finite projective left A-module. It is left to the reader to show that the resulting injection $M \otimes C_\alpha \to \mathrm{Hom}(A, M)$ coincides with Λ.

d) is an easy calculation and is left to the reader.

e) follows from (1.6).

f) follows from parts *b)* and *d)* and (2.11). Q.E.D.

3. $\widetilde{U} \times_A V$ as an Algebra, Simplicity

Suppose U and V are algebras over the algebra A. $U \otimes_A V$ has the left $U \otimes_R \overline{V}$-module structure determined by

$$(u \otimes \bar{v})\,(u' \otimes v') = (uu') \otimes (v'v), \qquad u, u' \in U, \qquad v, v' \in V.$$

Proposition (3.1). — 1. *There is an* R-*module isomorphism*

$$N : \widetilde{U} \times_A V \to \mathrm{End}_{U \otimes_R \overline{V}}(U \otimes_A V)$$

determined by $N(\sum_i u_i \otimes v_i)\,(u \otimes v) = \sum_i uu_i \otimes v_i v$, $\sum_i u_i \otimes v_i \in \widetilde{U} \times_A V \subset U \otimes_A V$, $u \in U$, $v \in V$.

2. $\widetilde{U} \times_A V$ *has an* R-*algebra structure determined by*

$$(\sum_i u_i \otimes v_i)\,(\sum_j w_j \otimes x_j) = \sum_{i,j} w_j u_i \otimes v_i x_j, \qquad \sum_i u_i \otimes v_i, \qquad \sum_j w_j \otimes x_j \in \widetilde{U} \times_A V.$$

3. N *is an algebra isomorphism.*

4. *Let* $U \overset{i}{\leftarrow} A \overset{j}{\rightarrow} V$ *be the maps making* U *and* V *into algebras over* A *and let* Z *be the center of* A. *Then* $\widetilde{U} \times_A V$ *is an algebra over* Z *with respect to the map* $Z \to \widetilde{U} \times_A V$, $z \mapsto i(z) \otimes 1 = 1 \otimes j(z)$.

Proof. — $U \otimes_R V$ is a left $U \otimes_R \overline{V}$-module, the structure defined by letting $A = R$ in the paragraph above (3.1). As a $U \otimes_R \overline{V}$-module, $U \otimes_R V$ is free with basis $\{1 \otimes 1\}$. For each left $U \otimes_R \overline{V}$-module M identify $\mathrm{Hom}_{U \otimes_R \overline{V}}(U \otimes_R V, M)$ with M by $f \leftrightarrow f(1 \otimes 1)$, $f \in \mathrm{Hom}_{U \otimes_R \overline{V}}(U \otimes_R V, M)$. Thus $U \otimes_A V$ and $\mathrm{Hom}_{U \otimes_R \overline{V}}(U \otimes_R V, U \otimes_A V)$ are identified. The natural map $U \otimes_R V \to U \otimes_A V$ is a surjective $U \otimes_R \overline{V}$-module map and induces the injection $\mathrm{End}_{U \otimes_R \overline{V}}(U \otimes_A V) \to \mathrm{Hom}_{U \otimes_R \overline{V}}(U \otimes_R V, U \otimes_A V) = U \otimes_A V$. The image of this injection in $U \otimes_A V$ is $\int^x {}_x U \otimes_A V_x = \widetilde{U} \times_A V$. This proves Part 1.

Parts 2, 3 and 4 are left to the reader. Q.E.D.

The canonical R-algebra structure on $\widetilde{U} \times_A V$ is that given in the proposition.

Example (3.2). — Suppose R is a field and A is a field extension where $[A : R] = n < \infty$. In the Brauer group over R let x and y be elements split by A. Let U be a central simple R-algebra of dimension n^2 which contains A and is a representative of x. Similarly V for y. By (3.1) and [10, bottom p. 486, top p. 487] it follows that $\widetilde{U} \times_A V$ is a central simple R-algebra of dimension n^2 which contains A and represents $x^{-1}y$.

Previous theory applies to $\widetilde{U} \times_A V$ as follows:

(3.3)

1. If $f : U \to U'$ and $g : V \to V'$ are maps of algebras over A then
$$\widetilde{f \times g} : \widetilde{U} \times_A V \to \widetilde{U}' \times_A V'$$
is an algebra map.
2. If E is a subalgebra *over* A of End A then $\sim\theta : \widetilde{U} \times_A E \to U$ (defined in (1.6)) is an algebra *anti*-homomorphism.

Throughout the rest of this section A is an algebra and (U, i) an algebra over A. Let L denote the centralizer of $i(A)$ in U, $(L=\int^x {}_xU_x)$ and let k denote the center of U.

Proposition (3.4). — $\xi : \tilde{U}\times_A U \to \mathrm{End}_k L$ *determined by*

$$\xi(\sum_i u_i\otimes v_i)(\ell)=\sum_i u_i\ell v_i, \quad \sum_i u_i\otimes v_i\in\tilde{U}\times_A U\subset U\otimes_A U, \quad \ell\in L$$

is an R-*algebra* anti-*homomorphism. If* Z *is the center of* A *then* $i(Z)\subset L$ *and* $\mathrm{End}_k L$ *is an algebra over* Z. ξ *is an anti-homomorphism of algebras over* Z.

Proof. — For $\ell\in L$ the map $f_\ell : U\otimes_A U\to U$ is determined by $f_\ell(u\otimes v)=u\ell v$, $u, v\in U$. Since f_ℓ is an A-bimodule map it carries $\tilde{U}\times_A U=\int^x {}_xU\otimes_A U_x$ to $\int^x {}_xU_x=L$. For $y\in\tilde{U}\times_A U$, $\zeta(y)(\ell)=f_\ell(y)\in L$. The rest is left to the reader. Q.E.D.

ζ gives L a *right* $\tilde{U}\times_A U$-module structure.

Definition (3.5). — The pair, (U, i) is called *Jake* if ζ is injective, i.e. L is a faithful right $\tilde{U}\times_A U$-module.

Lemma (3.6). — *Assume* (U, i) *is Jake and* M *is a sub*-A-*bimodule of* U.

1. *If* U *is flat as a* left A-*module and* $0\neq\tilde{M}\times_A U$, *then* $0\neq MU\cap L$.
2. *If* U *is flat as a* right A-*module and* $0\neq\tilde{U}\times_A M$, *then* $0\neq UM\cap L$.
3. *If* I *is a 2-sided ideal in* U, *then* $I\cap L$ *is a* $\tilde{U}\times_A U$-*submodule of* L.

Proof. — Let ι denote the inclusion $M\to U$. By (1.3), $\tilde{\iota}\times I : \tilde{M}\times_A U\to\tilde{U}\times_A U$ is injective. Since $0\neq\tilde{M}\times_A U$ there is $0\neq\sum_i m_i\otimes u_i\in\mathrm{Im}(\tilde{\iota}\times I)$, $\{m_i\}\subset M$, $\{u_i\}\subset U$. Since (U, i) is Jake there is $\ell\in L$ with $0\neq\sum_i m_i\ell u_i\in L$. This proves Part 1. Part 2 is proved similarly.

If $\sum_i u_i\otimes v_i\in\tilde{U}\times_A U$ then $\sum_i u_i I v_i\subset I$ for a 2-sided ideal I. This proves Part 3. Q.E.D.

Theorem (3.7). — *Suppose* U *is Jake, flat as a left (right)* A-*module and* $0\neq\tilde{I}\times_A U$ $(\tilde{U}\times_A I)$ *for non-zero 2-sided ideals* $I\subset U$. *Then* U *is a simple algebra if* L *is a simple* $\tilde{U}\times_A U$-*module.*

Proof. — If I is a non-zero 2-sided ideal in U then, $0\neq I\cap L$ by Part 1, 2) of (3.6). By part 3 of (3.6) $I\cap L$ is a $\tilde{U}\times_A U$ submodule of L. If L is a simple $\tilde{U}\times_A U$-module then $L=I\cap L$, which implies that $1\in I$ and $I=U$. Thus U is a simple algebra. Q.E.D.

$(\mathrm{End}\,A, \ell)$ is an algebra over A. Let $r : A\to\mathrm{End}\,A$ be the injective algebra anti-homomorphism determined by $a^r(b)=ba$, $a, b\in A$. The element a^r is called " a as a right translation operator ". The centralizer of A^ℓ in End A is A^r. Suppose U is a subalgebra of End A and $A^\ell\subset U\supset A^r$. Thus (U, ℓ) is an algebra over A and A^r is the centralizer of A^ℓ in U. Furthermore, if $k=\{a\in A\,|\,u(a)=au(1),\ u\in U\}$ then k lies in the center of A, so $x^\ell=x^r$ for $x\in k$. Furthermore $k^\ell=k^r$ is the center of U.

Lemma **(3.8)**. — *The diagram*

$$\begin{array}{ccc} \widetilde{U}\times_A U & \xrightarrow{\ \zeta\ } & \operatorname{End}_{k^r}A^r=\operatorname{End}_k A \\ {\scriptstyle \sim\theta}\downarrow & & \downarrow \\ U & \longrightarrow & \operatorname{End} A \end{array}$$

commutes, where the two maps to End A *are the natural inclusions and the equality* $\operatorname{End}_{k^r}A^r=\operatorname{End}_k A$ *is induced by r.*

Proof. — Let $\sum_i u_i\otimes v_i\in\widetilde{U}\times_A U\subset U\otimes_A U$. For $a\in A$

$\sim\theta(\sum_i u_i\otimes v_i)(a)=\sum_i u_i(v_i(1)a)=\sum_i u_i a^r v_i(1)=(\sum_i u_i a^r v_i)(1)=(\zeta(\sum_i u_i\otimes v_i)(a^r))(1)$. Q.E.D.

Thus $\sim\theta$ being injective is equivalent to ζ being injective. Hence (1.5) and (1.6) provide conditions for U to be Jake. The lemma also shows that the image of ζ is in U.

In the next lemma, we still assume that U is a subalgebra of End A and $A^\ell\subset U\supset A^r$.

Lemma **(3.9)**. — *If* U *is a simple algebra, then* A *is a simple* U-*module.*

Proof. — Assume that U is a simple algebra. Then for $0\neq a\in A$, $U=Ua^rU$ and so the identity I of End A can be written $I=\sum_i u_i a^r v_i$, $\{u_i\}\cup\{v_i\}\subset U$. Thus for $b\in A$, $b=(\sum_i u_i a^r v_i)(b)=\sum_i u_i(v_i(b)a)=(\sum_i u_i(v_i(b)^\ell))(a)$. This proves that $A=U(a)$. Q.E.D.

When A is commutative $A^\ell=A^r$ and the condition on U is simply that U is a subalgebra over A of End A.

Lemma **(3.10)**. — *Assume that* A *is a field (and still an* R-*algebra),* M *is an* A-*bimodule and* C *a* left A-*submodule of* End A.

1. *If* $\{c_i\}\subset C$ *is a finite* A-*linearly independent set of s elements, then there exists* $\{a_{ij}\}\cup\{b_{ij}\}\subset A$ *satisfying* $\sum_j a_{ij}c_k(b_{ij})=\delta_{ik}$ *with* $i,k=1,\ldots,s$.

2. *If* C *is a sub-*A-*bimodule of* End A (*so that* $\sim\theta:\widetilde{M}\times_A C\to M$ *is defined*), *then* $m\in\sim\theta(\widetilde{(AmA)}\times_A C)$ *if* $m\in\sim\theta(\widetilde{M}\times_A C)$.

Proof. — 1. Let $N=\sum_i Ac_i$ the span of $\{c_i\}$, a finite dimensional vector space over A. For $a\in A$ let $\gamma_a\in\operatorname{Hom}_A(N,A)$ be determined by $\gamma_a(n)=n(a)$, $n\in N$. If $0\neq n\in N$ there is $a\in A$ with $0\neq n(a)=\gamma_a(n)$. Thus Im γ spans $\operatorname{Hom}_A(N,A)$ and, given the basis $\{c_i\}$ for N, there is $\{a_{ij}\}\cup\{b_{ij}\}\subset A$ where $\{\sum_j a_{ij}\gamma_{b_{ij}}\}_{1\leq i\leq s}$ is a dual basis in $\operatorname{Hom}_A(N,A)$. Thus $\{a_{ij}\}\cup\{b_{ij}\}$ is the desired set.

2. Suppose $m = \sim\theta(\sum_i m_i \otimes c_i)$, $\sum_i m_i \otimes c_i \in \widetilde{M} \times_A C \subset M \otimes_A C$, assuming that $\{c_i\} \subset_x C$ is an x A-linearly independent set. Let $\{a_{ij}\} \cup \{b_{ij}\} \subset A$ be as in Part 1.

$$\begin{aligned}\sum_j b_{ij} m a_{ij} &= \sum_{j,k} b_{ij} m_k c_k(1) a_{ij} \\ &= \sum_{j,k} m_k [c_k(b_{ij}) a_{ij}] \\ &= m_i.\end{aligned}$$

Thus $\{m_i\} \subset AmA$. Since A is a field the inclusion $AmA \subset M$ induces an inclusion $\widetilde{(AmA)} \times_A C \subset \widetilde{M} \times_A C$. Then $\sum_i m_i \otimes c_i \in \widetilde{(AmA)} \times_A C$. Q.E.D.

The following corollary is immediate:

Corollary (**3.11**). — *Let* A *be a field,* M *an* A*-bimodule and* C *a sub-*A*-bimodule of* End A. *The map* $\sim\theta : \widetilde{M} \times_A C \to M$ *is surjective if and only if* $\sim\theta : \widetilde{N} \times_A C \to N$ *is surjective for all sub-*A*-bimodules* $N \subset M$. *In this case* $0 \neq \widetilde{N} \times_A C$ *for a non-zero sub-*A*-bimodule* N.

Theorem (**3.12**). — *Let* A *be a field and* E *a subalgebra over* A *of* End A. *If* $\sim\theta : \widetilde{E} \times_A E \to E$ *is surjective, then* E *is a simple algebra. The center of* E *is*

$$\{a^\ell \in E \mid a \in A \text{ and } e(a) = ae(1),\ e \in E\}.$$

Proof. — By (1.5) and (1.6), $\sim\theta$ is injective so that by (3.8) E is Jake. Since A is a field E is a flat left A-module. Since $A^\ell \subset E$, A is a simple E-module. The remaining hypothesis of (3.7) is satisfied by (3.11). Q.E.D.

4. $U \times_A V$ for commutative A

Throughout this section A is a commutative algebra. Thus if (U, i) is an algebra over A so is the opposite algebra $(\widetilde{U}, \widetilde{i})$.

If U and V are algebras over A then $U \times_A V \cong \widetilde{\widetilde{U}} \times_A V$ is an algebra. In terms of the realization $U \times_A V \subset \int_x {}_x U \otimes_x V$ the product is given by

$$(\sum_i u_i \otimes v_i)(\sum_j w_j \otimes x_j) = \sum_{i,j} u_i w_j \otimes v_i x_j, \quad \sum_i u_i \otimes v_i, \quad \sum_j w_j \otimes x_j \in U \times_A V.$$

If (U, i) and (V, j) are algebras over A there is an algebra map $\beta : A \to U \times_A V$ determined by

$$a \mapsto 1 \otimes j(a) = i(a) \otimes 1 \in \mathbf{U} \times_A V \subset \int_x {}_x \mathbf{U} \otimes_x V.$$

Definition **(4.1)**. — $(U \times_A V, \beta)$ is the *canonical structure* on $U \times_A V$ as an algebra over A.

(4.2)

1. If U or V is flat as a left A-module and both i and j are injective, then so is β.
2. If $f: U \to U'$ and $g: V \to V'$ are maps of algebras over A, then
$$f \times g : U \times_A V \to U' \times_A V'$$
is a map of algebras over A.
3. The A-bimodule structure on $U \times_A V$ defined in (2.1) is the same as the A-bimodule structure on $U \times_A V$ as an algebra over A. The above algebra over A structure on $U \times_A V$ agrees with the algebra over $Z = A$ structure on $\approx{U} \times_A V$ in (3.1, 4).
4. The isomorphism twist : $U \times_A V \to V \times_A U$, defined in (2.5), part 1, is an isomorphism of algebras over A.
5. If U, V and W are algebras over A, where (U, V, W) associates in the sense of (2.6), then the association isomorphism $(U \times_A V) \times_A W \cong U \times_A (V \times_A W)$ is an isomorphism of algebras over A.
6. If E is a subalgebra over A of End A and U is an algebra over A, then $\theta : U \times_A E \to U$ is a map of algebras over A, where θ is defined in (2.8).

With the exception of number 5 these results are easily verified. Number 5 is proved in the following manner. $\int^y \int_x {}_x U_y \otimes {}_x V_y \otimes {}_x W_y$ has an algebra structure where $(\sum_i u_i \otimes v_i \otimes w_i)(\sum_j u'_j \otimes v'_j \otimes w'_j) = \sum_{i,j} u_i u'_j \otimes v_i v'_j \otimes w_i w'_j$. In fact, with this algebra structure $\int^y \int_x {}_x U_y \otimes {}_x V_y \otimes {}_x W_y$ is isomorphic to $\mathrm{End}_{\bar{U} \otimes \bar{V} \otimes \bar{W}} \int_x {}_x U \otimes {}_x V \otimes {}_x W$ and the details are similar to the proof of (3.1). It is easily verified that the maps α and α' in (2.6) are maps of algebras over A for (U, V, W). This gives number 5 above.

We may put a product structure on isomorphism classes of algebras over A by means of " $\times_A$ ".

Remarks **(4.3)**. — If (U, i) is an algebra over A then $\langle U \rangle$ denotes the class of algebras over A which are isomorphic to U as *algebras over* A. From (4.2), if U and V are algebras over A, then the product $\langle U \rangle \langle V \rangle$ is well-defined as $\langle U \times_A V \rangle$. This is the canonical product on isomorphism classes of algebras over A. By (4.2) the product is commutative, and if U, V and W are algebras over A, where (U, V, W) associates, then $(\langle U \rangle \langle V \rangle) \langle W \rangle = \langle U \rangle (\langle V \rangle \langle W \rangle)$.

Example **(4.4)**. — Suppose that R is a field and A is an overfield of R. Let k be an intermediate field, $A \supset k \supset R$, where $[A : k] < \infty$. Both $(\mathrm{End}_k A, \ell)$ and $(\widetilde{\mathrm{End}_k A}, \tilde{\ell})$ are algebras over A. As k-algebras $\mathrm{End}_k A \cong \widetilde{\mathrm{End}_k A}$, by the transpose map for example.

Thus by Skolem-Nœther $\langle \mathrm{End}_k A \rangle = \langle \widetilde{\mathrm{End}_k A} \rangle$. Let $n = [A : k]$. $\mathrm{End}_k A$ is the unique n^2-dimensional (over k) representative containing A of the identity class of the Brauer group of K. By (3.2) it follows that $\langle \mathrm{End}_k A \rangle \langle \mathrm{End}_k A \rangle = \langle \mathrm{End}_k A \rangle$. Thus $\langle \mathrm{End}_k A \rangle$ is idempotent with respect to the product on isomorphism classes. Suppose k' is another field intermediate between A and R and $\langle \mathrm{End}_k A \rangle = \langle \mathrm{End}_{k'} A \rangle$. Then there is an algebra isomorphism $f : \mathrm{End}_k A \to \mathrm{End}_{k'} A$ which is the identity on A (actually A^l). Since f must carry k, the center of $\mathrm{End}_k A$, to k', the center of $\mathrm{End}_{k'} A$, and $k \subset A \supset k'$, it follows that $k = k'$. Thus if $k \neq k'$ we have that $\langle \mathrm{End}_k A \rangle \neq \langle \mathrm{End}_{k'} A \rangle$ are both idempotent classes.

Since the only idempotent in a group is the identity, the example demonstrates one problem in using " $\times_A$ " to put a group structure on isomorphism classes of algebras over A. Lack of associativity is a second problem and a third problem is that the equivalence classes of algebras over A *don't form a set.*

For an A-bimodule M, define $\mathscr{E}_M$ by

(4.5) $\mathscr{E}_M = \{$isomorphism classes $\langle U \rangle$ of algebras over A where $U \cong M$ as an A-bimodule.$\}$

If M and N are A-bimodules and $e \in \mathscr{E}_M$, $f \in \mathscr{E}_N$ then $ef \in \mathscr{E}_{M \times_A N}$.

Definition **(4.6)**. — An A-bimodule M is *idempotent as a bimodule* if $M \cong M \times_A M$ as A-bimodules. An algebra (U, i) over A is *idempotent as an algebra over* A if $U \cong U \times_A U$ is an algebra over A, i.e. $\langle U \rangle = \langle U \rangle \langle U \rangle$.

Remark. — Suppose M is idempotent as an A-bimodule and M is the directed union of projective left sub-A-modules. Then M is flat as a left A-module, being the direct limit of flat left A-modules. Thus $M \times_A M \cong M$ is flat as a left A-module. By (2.5), part 5 and the lines following (2.7) it follows that M is associative as an A-bimodule.

If M is an idempotent A-bimodule, then, for $e, f \in \mathscr{E}_M$, the product $ef \in \mathscr{E}_M$. Thus $\mathscr{E}_M$ has a commutative product. If M is also an associative bimodule, then the commutative product in $\mathscr{E}_M$ is associative by (4.2), part 5.

Lemma **(4.7)**. — *Let* S *be a set with associative product. For each idempotent* $e \in S$ *let* $S(e) = \{s \in S \mid se = s = es\}$. *Then* $S(e)$ *is the unique maximal " submonoid " of* S *with identity* e *and the group of invertible elements in* $S(e)$ *is the unique maximal " subgroup " of* S *with identity* e.

Proof. — Left to reader.

Definition **(4.8)**. — If (U, i) is an algebra over A which is idempotent as an algebra over A (4.6), and associative as an A-bimodule (2.7), let $\mathscr{E}\langle U \rangle$ denote the monoid of equivalence classes $C \in \mathscr{E}_U$ where $C\langle U \rangle = C$. Let $\mathscr{G}\langle U \rangle$ denote the group of invertible elements in $\mathscr{E}\langle U \rangle$.

It will be shown that for certain U the group $\mathscr{G}\langle U\rangle$ is (isomorphic to) a relative Brauer group. We shall also consider such matters as:

a) If V is an algebra over A with $\langle V\rangle\in\mathscr{G}\langle U\rangle$, is V a simple algebra?

b) Is $\langle V\rangle^{-1}$ equal to $\langle\tilde{V}\rangle$? (Recall that $\tilde{V}$ is the opposite algebra to U still considered as an algebra over A.)

c) How are $\mathscr{E}_U$, $\mathscr{E}\langle U\rangle$ and $\mathscr{G}\langle U\rangle$ classified by cohomology?

Proposition **(4.9)**. — *Suppose* E *is a subalgebra over* A *of* End A *where* $E\times_A E\xrightarrow{\theta}E$ *is an isomorphism of algebras over* A *(so that* E *is idempotent as an algebra over* A*) and assume* E *is associative as an* A*-bimodule. Then, if* U *is an algebra over* A *with* $U\cong E$ *as an* A*-bimodule,* i.e. $\langle U\rangle\in\mathscr{E}_E$, *it follows that*

$$U\times_A E\xrightarrow{\theta}U$$

is an isomorphism of algebras over A, i.e. $\langle U\rangle\in\mathscr{E}\langle E\rangle$. *Thus* $\mathscr{E}\langle E\rangle=\mathscr{E}_E$.

Proof. — If M is any A-bimodule $M\times_A E\xrightarrow{\theta}M$ is defined (2.8) and depends upon the bimodule structure of M. Since by hypothesis $E\times_A E\xrightarrow{\theta}E$ is bijective it follows that for any A-bimodule M which is bimodule isomorphic to E the map $M\times_A E\xrightarrow{\theta}M$ is bijective. In case M happens to be an algebra over A the map θ is also a map of algebras over A. Q.E.D.

5. $\times_A$-Coalgebras and Bialgebras

Throughout this section A is a commutative algebra.

Definition **(5.1)**. — Let C be an associative A-bimodule (2.7). Let $\Delta: C\to C\times_A C$ be an A-bimodule map and let $\mathscr{I}: C\to \text{End } A$ be an A-bimodule map. Then $(C, \Delta, \mathscr{I})$ is a $\times_A$-*coalgebra* if the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\quad\Delta\quad} & C\times_A C \\ \Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle I\times\Delta} \\ & & C\times_A(C\times_A C) \\ & & \wr\Vert \\ C\times_A C & \xrightarrow{\Delta\times I} & (C\times_A C)\times_A C \end{array}$$

where the isomorphism is the association isomorphism (2.6)

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\times_A C \\ \Big\downarrow{\scriptstyle I} & & \Big\downarrow{\scriptstyle I\times\mathscr{I}} \\ C & \xleftarrow{\theta} & C\times_A \operatorname{End} A \end{array}$$

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\times_A C \\ & & \Big\downarrow{\scriptstyle \mathscr{I}\times I} \\ \Big\downarrow{\scriptstyle I} & & \operatorname{End} A\times_A C \\ & & \Big\downarrow{\scriptstyle \text{twist}} \\ C & \xleftarrow{\theta} & C\times_A \operatorname{End} A \end{array}$$

Definition (5.2). — For a right A-module C and an R-module map $f: C\to A$, let $f^t: C\to \operatorname{End} A$ be defined by $f^t(c)(a)=f(ca)$ $c\in C$, $a\in A$. Let $\epsilon: \operatorname{End} A\to A$, $f\mapsto f(1)$.

It is easily verified that f^t is a right A-module map, ϵ is a left A-module map, $\epsilon(f^t)=f$ and if $g: C\to \operatorname{End} A$ is a right A-module map then $(\epsilon g)^t=g$. Moreover, if C happens to be an A-bimodule and $f: C\to A$ a left A-module map then $f^t: C\to\operatorname{End} A$ is an A-bimodule map. If $g: C\to\operatorname{End} A$ is an A-bimodule map then $\epsilon g: C\to A$ is a left A-module map.

We may use $(\ )^t$ and ϵ to show the following diagram commutes

$$\begin{array}{ccc} \operatorname{End} A\times_A\operatorname{End} A & \xrightarrow{\theta} & \operatorname{End} A \\ \Big\downarrow{\scriptstyle \text{twist}} & \nearrow{\scriptstyle \theta} & \\ \operatorname{End} A\times_A\operatorname{End} A & & \end{array}$$

It suffices to show that $\epsilon\theta=\epsilon\theta(\text{twist})$ since then

$$\theta=(\epsilon\theta)^t=(\epsilon\theta(\text{twist}))^t=\theta(\text{twist}).$$

Viewing $\operatorname{End} A\times_A\operatorname{End} A\subset\int_x {}_x\operatorname{End} A\otimes_x\operatorname{End} A$ for an element

$$z=\sum_i e_i\otimes e_i'\in\operatorname{End} A\times_A\operatorname{End} A$$

it is easily verified that

$$\epsilon\theta(z)=\sum_i e_i'(1)e_i(1)=\sum_i e_i(1)e_1'(1)=\epsilon\theta(\text{twist})(z).$$

Proposition (5.3). — *Let* C *be an* A-*bimodule,* $\Delta: C\to C\times_A C$ *an* A-*bimodule map,* $\mathscr{I}: C\to\operatorname{End} A$ *an* A-*bimodule map and let* $\varepsilon=\epsilon\mathscr{I}$ *so that* $\mathscr{I}=\varepsilon^t$.

a) *The diagram*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\times_A C \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I\times\mathscr{S}} \\ C & \xleftarrow{\theta} & C\times_A \mathrm{End}\, A \end{array}$$

commutes if and only if the diagram

$$\begin{array}{ccc} C & \xrightarrow{\iota\Delta} & \int_x {}_xC\otimes_x C \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I\otimes\varepsilon} \\ C & = & \int_x {}_xC\otimes_x A \end{array}$$

commutes. ι *is the natural inclusion* $C\times_A C \xrightarrow{\iota} \int_x {}_xC\otimes_x C$.

b) *The diagram*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\times_A C \\ \Big\downarrow & & \downarrow{\scriptstyle \mathscr{S}\times I} \\ & & \mathrm{End}\, A\times_A C \\ \Big\downarrow & & \downarrow{\scriptstyle \text{twist}} \\ C & \xleftarrow{\theta} & C\times_A \mathrm{End}\, A \end{array}$$

commutes if and only if the diagram

$$\begin{array}{ccc} C & \xrightarrow{\iota\Delta} & \int_x {}_xC\otimes_x C \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle \varepsilon\otimes I} \\ C & = & \int_x {}_xA\otimes_x C \end{array}$$

commutes.

Proof. — Left to the reader.

Corollary (5.4). — a) *If* $(C, \Delta, \mathscr{S})$ *is a* $\times_A$*-coalgebra then* $(C, \iota\Delta, \varepsilon = \epsilon\mathscr{S})$ *gives* C *the structure of an* A*-coalgebra* [17, Definition p. 4, where " vector space " should be read as " module "].

b) *Conversely, if* C *is an associative* A*-bimodule,* $\varepsilon : C \to A$ *a left* A*-module map and*

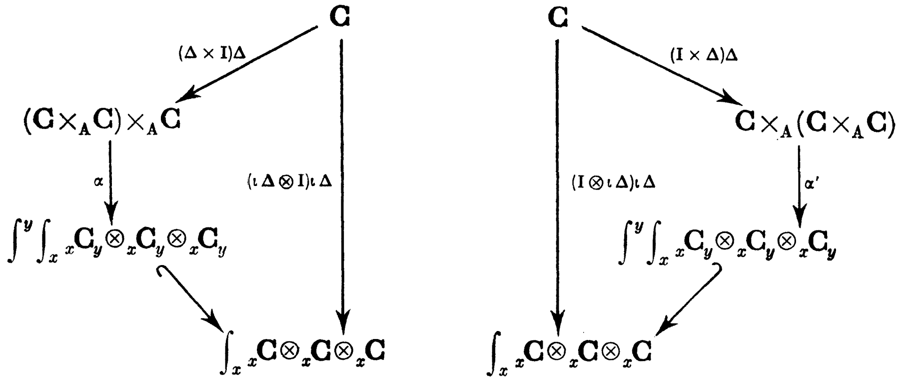
$\Delta : C \to C \times_A C$ *an* A-*bimodule map, where* $(C, \iota\Delta, \varepsilon)$ *gives* C *the structure of an* A-*coalgebra, then* $(C, \Delta, \mathscr{I} = \varepsilon^t)$ *is a* $\times_A$-*coalgebra.*

Proof. — Left to the reader.

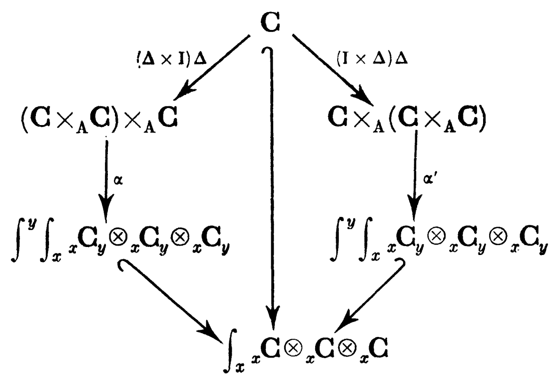
If $(C, \Delta, \mathscr{I})$ is a $\times_A$-coalgebra then the *underlying coalgebra structure on* C refers to the A-coalgebra $(C, \iota\Delta, \varepsilon = \boldsymbol{\varepsilon}\mathscr{I})$.

Proposition (5.5). — *Suppose* C *is an* A-*bimodule,* $\varepsilon : C \to A$ *a left* A-*module map and* $\Delta : C \to C \times_A C$ *an* A-*bimodule map where* $(C, \iota\Delta, \varepsilon)$ *gives* C *the structure of an* A-*coalgebra. If* Δ *is an isomorphism then* C *is associative as an* A-*bimodule and by* (5.4), b), $(C, \Delta, \mathscr{I} = \varepsilon^t)$ *is a* $\times_A$-*coalgebra.*

Proof. — It is easily verified that the following two diagrams commute:



where α is defined in (2.5), 2) and α' is defined just above (2.6). By coassociativity of coalgebras $(\iota\Delta \otimes I)\iota\Delta = (I \otimes \iota\Delta)\iota\Delta$. By the counit condition of coalgebras it is easily shown that $(\iota\Delta \otimes I)\iota\Delta$ is injective. Thus the two diagrams may be pushed together as a commutative diagram:



If Δ is an A-bimodule isomorphism then so are $(\Delta \times I)\Delta$ and $(I \times \Delta)\Delta$. When $(I \times \Delta)\Delta$ and $(\Delta \times I)\Delta$ are isomorphisms the above diagram shows that α and α' are both injective and have the same image. In other words C is associative as an A-bimodule. Q.E.D.

Suppose $(C, \Delta, \mathscr{I})$ is a $\times_A$-coalgebra. The map Δ is called the diagonalization of C and $\mathscr{I}$ is called the co-unit of C. $(C, \Delta, \mathscr{I})$ is cocommutative if $(\text{twist})\Delta=\Delta$ where twist is defined in (2.5), part 1. This is the same as $(C, \iota\Delta, \varepsilon)$ being a cocommutative coalgebra [17, Def. p. 63].

Definition **(5.6)**. — Let B be an algebra over A. The triple $(B, \Delta, \mathscr{I})$ is a $\times_A$-*bialgebra* if $(B, \Delta, \mathscr{I})$ is a $\times_A$-coalgebra (where the bimodule structure on B is determined by B being an algebra over A), and the maps $\Delta : B \to B \times_A B$, $\mathscr{I} : B \to \text{End } A$ are maps of algebras over A.

A $\times_A$-bialgebra is *cocommutative* if the underlying $\times_A$-coalgebra structure is.

Definition **(5.7)**. — If B is a $\times_A$-bialgebra the *natural* B-*module structure* on A is that given by $\mathscr{I}$. Thus $b.a=\mathscr{I}(b)(a)$, $b \in B$, $a \in A$.

If B is an algebra over A and $\mathscr{I} : B \to \text{End } A$ an A-bimodule map and $\varepsilon=\boldsymbol{\epsilon}\mathscr{I}$, then $\mathscr{I}$ is a map of algebras over A if and only if $\varepsilon(bc)=\varepsilon(b\varepsilon(c))$, $\varepsilon(1)=1$ for $1, b, c \in B$. In this case the B-module structure on A with representation $\mathscr{I}$ is given by $b.a=\varepsilon(ba)$ for $b \in B$, $a \in A$.

By definition a $\times_A$-bialgebra is associative as an A-bimodule. If B is a $\times_A$-bialgebra and $\Delta : B \to B \times_A B$ is an isomorphism, then B is idempotent as an algebra over A (4.6). In this case $\mathscr{E}\langle B \rangle$ and $\mathscr{G}\langle B \rangle$ are defined. In a later section we give a cohomology theory arising from B where the H^2 is naturally isomorphic to $\mathscr{G}\langle B \rangle$.

Proposition **(5.8)**. — *Suppose* $C \subset \text{End } A$ *is a sub*-A-*bimodule and* (C, Δ, ι) *gives* C *a* $\times_A$-*coalgebra structure, where* $\iota : C \to \text{End } A$ *is the natural inclusion.*

a) *The composites* $C \xrightarrow{\Delta} C \times_A C \xrightarrow{\theta} C$ *and* $C \xrightarrow{\Delta} C \times_A C \xrightarrow{(\text{twist})} C \times_A C \xrightarrow{\theta} C$ *are the identity.*

b) *If* Δ *is surjective or* θ *is injective, then both are isomorphisms and* $\Delta=\theta^{-1}$.

c) *Suppose that* D *is a* $\times_A$-*coalgebra except that coassociativity of* $\Delta : D \to D \times_A D$ *is not assumed. If* $d \in D$ *and* $\Delta d=\sum_i d_i \otimes d_i' \in D \times_A D \subset \int_x {}_x D \otimes_x D$ *then, for* $a, b \in A$,

i) $\sum_i (\mathscr{I}(d_i)(a))d_i'=da=\sum_i (\mathscr{I}(d_i')(a))d_i$

ii) $\sum_i (\mathscr{I}(d_i)(a))(\mathscr{I}(d_i')(b))=(\mathscr{I}(d)(ab))$.

Proof. — *a)* follows from the second and third commutative diagrams in (5.1).

b) follows from *a)*.

c) Let d and $\sum_i d_i \otimes d_i'$ be as in part *c)*. Since Δ is an A-bimodule map, $\Delta(da)=\sum_i d_i \otimes d_i' a$. Applying $\theta(I \times \mathscr{I})$ to both sides (and using the second diagram in (5.1) and the fact that $\mathscr{I}$ is a right A-module map) gives $da = \sum_i (\mathscr{I}(d'_i)(a))d_i$. Similarly applying $\theta(\text{twist})(\mathscr{I} \times \mathrm{I})$ to both sides of $\Delta(da) = \sum_i d_i a \otimes d'_i$ gives

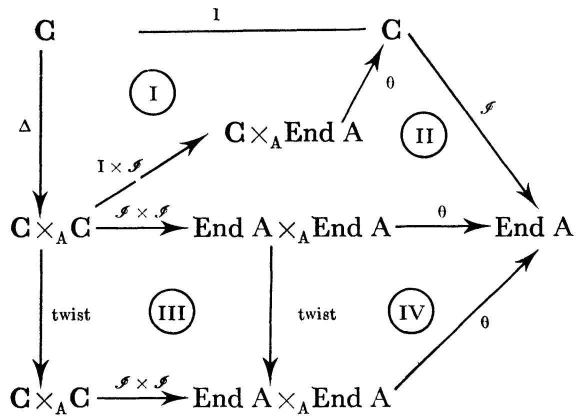
$$da = \sum_i (\mathscr{I}(d_i)(a))d'_i.$$

For (ii) $\mathscr{I}(d)(ab) = \mathscr{I}(da)(b) = \mathscr{I}(\sum_i (\mathscr{I}(d_i)(a))d'_i)(b) = \sum_i (\mathscr{I}(d_i)(a))(\mathscr{I}(d'_i)(b))$.

Q.E.D.

Proposition **(5.9)**. — *Suppose* (C, Δ, $\mathscr{I}$) *is a* $\times_A$*-coalgebra where* $\mathscr{I}$ *is injective and* Δ *is surjective. Then* C *is cocommutative. In fact twist:* $C \times_A C \to C \times_A C$ *is the identity map.*

Proof. — In the diagram below region I commutes by the co-unit condition for $\times_A$-coalgebras. Regions II and III are directly verified to commute. Region IV commutes by the remarks between (5.2) and (5.3).



Thus the composites $\theta(\mathscr{I} \times \mathscr{I})\Delta$ and $\theta(\mathscr{I} \times \mathscr{I})(\text{twist})\Delta$ both equal $\mathscr{I}$. Since $\mathscr{I}$ is injective and Δ surjective it follows that $\theta(\mathscr{I} \times \mathscr{I})$ is injective. Since Δ is surjective it follows that $\theta(\mathscr{I} \times \mathscr{I}) = \theta(\mathscr{I} \times \mathscr{I})(\text{twist})$. This with the injectivity of $\theta(\mathscr{I} \times \mathscr{I})$ shows that twist is the identity. Q.E.D.

Suppose M is a left A-module and D is a left sub-A-module of End A. Let Λ' denote the map $\Lambda' : \int_x {}_x M \otimes_x D \to \mathrm{Hom}(A, M)$ where $\Lambda'(m \otimes d)(a) = d(a)m$, $m \in M$, $d \in D$, $a \in A$. It is easily shown that $\Lambda' : \int_x {}_x M \otimes_x D \to \mathrm{Hom}(A, M)$ is injective if and only if $\Lambda : \widetilde{M} \otimes_A D \to \mathrm{Hom}(A, \widetilde{M})$ is injective. Thus all Λ'-maps for D are injective if and only if all Λ-maps are injective for D.

If D is actually a sub-A-bimodule of End A and $\Lambda' : \int_x {}_x M \otimes_x D \to \mathrm{Hom}(A, M)$ is injective then $M \times_A D \xrightarrow{\theta} M$ is injective. This follows from (1.6).

Proposition **(5.10)**. — *Let* $C \subset \operatorname{End} A$ *be a sub-A-bimodule, where* (C, Δ, ι) *is a* $\times_A$*-coalgebra.*

a) *If* $\Lambda' : \int_x {}_x C \otimes_x C \to \operatorname{Hom}(A, C)$ *is injective, then* $\theta : C \times_A C \to C$ *is injective.*

b) *If* $C \times_A C \xrightarrow{\theta} C$ *is injective, then* C *is cocommutative. In fact twist*: $C \times_A C \to C \times_A C$ *is the identity map.*

c) *If* C *is actually a subalgebra over* A *of* End A *and* $C \times_A C \xrightarrow{\theta} C$ *is injective, then* (C, Δ, ι) *makes* C *into a* $\times_A$*-bialgebra.*

Proof. — *a)* follows from the remark just before (5.10).

b) By (5.8), b) Δ must be surjective. Hence, (5.9) gives part *b)*.

c) By (5.8), b) $\Delta = \theta^{-1}$. The map $C \times_A C \xrightarrow{\theta} C$ is a map (isomorphism) of algebras over A. Hence, Δ is also. Clearly $\iota : C \to \operatorname{End} A$ is a map of algebras over A. Q.E.D.

In the next section we study when θ (actually θ^{-1}) can be used to induce a $\times_A$-coalgebra structure on a sub-A-bimodule of End A.

6. $\times_A$-Coalgebras

In a later example we show that End A is a $\times_A$-bialgebra when A is a finite projective R-module. To present this and other examples we must first develop some $\times_A$-coalgebra theory.

Definition **(6.1)**. — For a $\times_A$-coalgebra $(C, \Delta, \mathscr{I})$ let E_C (or E) denote $\operatorname{Im} \mathscr{I}$ a sub-A-bimodule of End A.

(6.2) For $D \subset \operatorname{End} A$ a sub-R-module let $\epsilon : D \to A$ denote the R-module map determined by $\epsilon(d) = d(1)$. This ϵ is really the ϵ in (5.2) restricted to D.

Suppose C and D are $\times_A$-coalgebras and $f : C \to D$ is an A-bimodule map. f is called a map of $\times_A$-coalgebras if $(f \times f)\Delta_C = \Delta_D f$ and $\mathscr{I}_D f = \mathscr{I}_C$. It is easily shown that f is a $\times_A$-coalgebra map if and only if f is a coalgebra map of the underlying coalgebras.

If C and D are $\times_A$-bialgebras then $f : C \to D$ is a map of $\times_A$-bialgebras if f is a $\times_A$-coalgebra map and a map of algebras over A.

Proposition **(6.3)**. — *Let* C *be a* $\times_A$*-coalgebra and assume that* $\theta : E \times_A E \to E$ *is injective* (θ *is defined in* (2.8)). *Then* θ *is an* A*-bimodule isomorphism. Moreover if* $\iota : E \to \operatorname{End} A$ *is the natural inclusion, then* (E, θ^{-1}, ι) *gives* E *the structure of a* $\times_A$*-coalgebra and* $\mathscr{I}_C : C \twoheadrightarrow E$ *is a* $\times_A$*-coalgebra map.*

Proof. — From the second $\times_A$-coalgebra diagram $\theta(I\times\mathscr{I})=I$ and so

$$\theta(\mathscr{I}\times\mathscr{I})\Delta=\mathscr{I}.$$

Since $\mathscr{I}:C\to E$ is surjective it follows that $\theta:E\times_A E\to E$ is surjective and hence an A-bimodule isomorphism. Applying θ^{-1} to both sides of $\theta(\mathscr{I}\times\mathscr{I})\Delta=\mathscr{I}$ yields $(\mathscr{I}\times\mathscr{I})\Delta=\theta^{-1}\mathscr{I}$.

It is left to the reader to show that $(E, \iota\theta^{-1}, \epsilon)$ gives E the structure of an A-coalgebra and $\mathscr{I}$ is a coalgebra map. Then by (5.5) E is associative as an A-bimodule and (E, θ^{-1}, ι) is a $\times_A$-coalgebra. Since $\mathscr{I}$ is a coalgebra map it is a map of $\times_A$-coalgebras.

Q.E.D.

Lemma **(6.4)**. — *Let* C *and* D *be sub*-A-*bimodules of* End A.

1. *The diagram*

$$\begin{array}{ccccc} C\times_A D & & \xrightarrow{\text{(twist)}} & & D\times_A C \\ \downarrow{\scriptstyle\theta} & & & & \downarrow{\scriptstyle\theta} \\ C & \xrightarrow{\iota} & \text{End } A & \xleftarrow{\iota} & D \end{array}$$

commutes, where the (ι)'*s are the natural inclusions. In particular* $\operatorname{Im}\theta\subset C\cap D$.

2. *Suppose* $D\times_A D\overset{\theta}{\to}D$ *is an isomorphism and*

$$\int_x\Big(\int_y {}_{x,y}D\otimes_y D\Big)\otimes_x D\overset{\Lambda'}{\to}\operatorname{Hom}\Big(A, \int_y {}_yD\theta_y D\Big)$$

is injective. Then $\iota\theta^{-1}$ *is coassociative.* I.e. $(I\otimes\iota\theta^{-1})\iota\theta^{-1}=(\iota\theta^{-1}\otimes I)\iota\theta^{-1}$.

3. *Suppose* $\int_x {}_xD\otimes_x D\overset{\Lambda'}{\to}\operatorname{Hom}(A, D)$ *is injective and* $u=\sum_i d_i\otimes d_i'\in\int_x {}_xD\otimes_x D$. *Then* $u\in D\times_A D$ *if and only if* $\sum_i d_i(a)d_i'(bc)=\sum_i d_i(ab)d_i'(c)$, $a, b, c\in A$.

Proof. — Let $z=\sum_i c_i\otimes d_i\in C\times_A D\subset\int_x {}_xC\otimes_x D$. Then $\iota\theta(z)=\sum_i d_i(1)c_i\in\text{End } A$ and $\iota\theta(\text{twist})(z)=\sum_i c_i(1)d_i\in\text{End } A$. For $a\in A$

$$\begin{aligned}\sum_i d_i(1)c_i(a)&=\sum_i(d_i(1))(c_i a^{\ell}(1))\\ &=\sum_i(d_i a^{\ell}(1))(c_i(1))=\sum_i d_i(a)c_i(1).\end{aligned}$$

This proves Part 1.

By injectivity of Λ' in Part 2 it suffices to show that for $d\in D$, $a\in A$

$$(*)\qquad \Lambda'((I\otimes\iota\theta^{-1})\iota\theta^{-1}(d))(a)=\Lambda'((\iota\theta^{-1}\otimes I)\iota\theta^{-1}))(a)\in\int_x {}_xD\otimes_x D.$$

Clearly $\theta\theta^{-1}=I$ and by Part 1 $\theta(\text{twist})\theta^{-1}=I$. Thus we may use the formulae in (5.8), c) to evaluate $(*)$. The reader may verify that the left hand side is $\sum_i d_i\otimes d_i' a$

and the right hand side is $\iota\theta^{-1}(da)$ where $\sum_i d_i \otimes d'_i = \iota\theta^{-1}(a)$. Since θ^{-1} is an A-bimodule map both sides are equal.

By injectivity of Λ' in part 3 it follows that $u \in D \times_A D$ if and only if

$$\Lambda'(\sum_i d_i b \otimes d_i) = \Lambda'(\sum_i d_i \otimes d'_i b), \qquad b \in A.$$

This exactly reduces to the condition given in part 3. Q.E.D.

Theorem **(6.5)**. — *Suppose* $D \subset \text{End } A$ *is a sub-A-bimodule where* $D \times_A D \xrightarrow{\theta} D$ *is an isomorphism and* $\Lambda' : \int_x \left(\int_y {}_{x,y}D \otimes {}_yD\right) \otimes {}_xD \to \text{Hom}\left(A, \int_y {}_yD \otimes {}_yD\right)$ *is injective. Then* D *is associative as an* A*-bimodule and* (D, θ^{-1}, ι) *gives* D *the structure of cocommutative* $\times_A$*-coalgebra.*

If $\Delta : D \to D \times_A D$ *is an* A*-bimodule map where* (D, Δ, ι) *gives* D *the structure of* $\times_A$*-coalgebra, then* $\Delta = \theta^{-1}$. *If in addition* D *is a subalgebra over* A *of* End A *then* (D, θ^{-1}, ι) *is a* $\times_A$*-bialgebra.*

Proof. — By (6.4), 2) $\iota\theta^{-1}$ is coassociative. Clearly $\theta\theta^{-1} = I$ and by (6.4), 1) $\theta(\text{twist})\theta^{-1} = I$. Thus the formulae in (5.8), *c)* hold and it easily follows that $(D, \iota\theta^{-1}, \epsilon)$ is an A-coalgebra. By (5.5) D is associative as an A-bimodule and $(D, \theta^{-1}, \iota = \epsilon^t)$ is a $\times_A$-coalgebra. Cocommutativity follows from (5.10), *b)* and (5.10), *c)* shows that D is a $\times_A$-bialgebra when it is a subalgebra over A of End A.

By (5.8), *b)* it follows that $\Delta = \theta^{-1}$ if (D, Δ, ι) gives D the structure of $\times_A$-coalgebra. Q.E.D.

Consider End A as an A⊗A-module where $(a \otimes b).f = a^l f b^l$, $a, b \in A$, $f \in \text{End } A$. Let $\{L_\alpha\}$ be a collection of *ideals* in A⊗A with the properties:

(i) $({}_x A \otimes A)/L_\alpha$ is a finite projective x A-module for each α.

(ii) Given L_α and L_β there is an L_γ with $L_\gamma \subset L_\alpha \cap L_\beta$.

Let $C_\alpha = \{f \in \text{End } A \mid z.f = 0,\ z \in L_\alpha\}$, and let $C = \bigcup_\alpha C_\alpha$. Some results about C can be found in (2.12) such as each C_α is an A-bimodule and projective as a left A-module.

Let $e : A \otimes A \to A \otimes A \otimes A$, $a \otimes b \to a \otimes 1 \otimes b$.

Theorem **(6.6)**. — a) *If* $e(L_\gamma) \subset L_\alpha \otimes A + A \otimes L_\beta$ *then* $C_\alpha C_\beta \subset C_\gamma$. *If for each* L_α *and* L_β *there is* L_γ *with* $e(L_\gamma) \subset L_\alpha \otimes A + A \otimes L_\beta$ *and there is an* L_τ *contained in the kernel of the map* $A \otimes A \xrightarrow{\text{mult}} A$, *then* C *is a subalgebra over* A *of* End A. (This result does not require the condition that $(A \otimes A)/L_\alpha$ be a finite projective A-module for the (L_α)'s.)

b) *Let* N *be an* A*-bimodule and hence an* A⊗A*-module. Suppose there is an* L_α *with* $L_\alpha.N = 0$. *Then the maps* $N \times_A C_\alpha \xrightarrow{\theta} N$ *and* $N \times_A C \xrightarrow{\theta} N$ *are isomorphisms.*

c) *The map* $C \times_A C \xrightarrow{\theta} C$ *is an isomorphism,* C *is associative as an* A*-bimodule, in fact* α *and* α' *are isomorphisms, and* (C, θ^{-1}, ι) *makes* C *into a* $\times_A$*-coalgebra which is cocommutative.*

d) *If* C *is a subalgebra over* A *of* End A, *then* (C, θ^{-1}, ι) *makes* C *into a* $\times_A$*-bialgebra.*

Proof. — *a)* Let comp: $C_\alpha \otimes_A C_\beta \to \text{End } A$, $c_1 \otimes c_2 \to c_1 c_2$. Consider $C_\alpha \otimes_A C_\beta$ as an A⊗A⊗A-module where $(a_1 \otimes a_2 \otimes a_3).(c_1 \otimes c_2) \equiv a_1 c_1 \otimes a_2 c_2 a_3 = a_1 c_1 a_2 \otimes c_2 a_3$. It is easily shown that for $z \in L_\alpha \otimes A + A \otimes L_\beta \subset A \otimes A \otimes A$, $z.(c_1 \otimes c_2) = 0$, $c_1 \in C_\alpha$, $c_2 \in C_\beta$. It is also easily shown that for $y \in A \otimes A$

$$y.(\text{comp}(c_1 \otimes c_2)) = \text{comp}(e(y).(c_1 \otimes c_2)).$$

Thus if $y \in L_\gamma$ and $e(y) \in L_\alpha \otimes A + A \otimes L_\beta$ it follows that $y.(C_\alpha C_\beta) = 0$. This gives the assertion about $C_\alpha C_\beta \subset C_\gamma$.

If for each L_α and L_β there is an L_γ with $e(L_\gamma) \subset L_\alpha \otimes A + A \otimes L_\beta$ it follows that C is closed under product. If there is $L_\gamma \subset \text{Ker}(A \otimes A \xrightarrow{\text{mult}} A)$ then $A = A^\ell \subset C_\tau$ and C is a subalgebra over A of End A.

b) Assume $L_\alpha . N = 0$.

$$\begin{aligned} N \times_A C_\alpha &\overset{1}{=} \int^y \int_x {}_x N_y \otimes {}_x C_{\alpha y} \\ &\overset{2}{=} \int^y \int_x {}_x N_y \otimes \text{Hom}_A(({}_x A \otimes A_y)/L_\alpha, A) \\ &\cong \int^y \text{Hom}_A((A \otimes A_y)/L_\alpha, N_y) \\ &\overset{3}{=} \text{Hom}_{A \otimes A}((A \otimes A)/L_\alpha, N) \\ &\overset{4}{=} \text{Hom}_{(A \otimes A)/L_\alpha}((A \otimes A)/L_\alpha, N) \\ &\overset{5}{=} N. \end{aligned}$$

The first equality follows from the definition of $\times_A$. The second equality follows from the identification of C_α with the dual of $(A \otimes A)/L_\alpha$ in (2.12), *a)*. The isomorphism is the natural isomorphism which exists because $(A \otimes A)/L_\alpha$ is a finite projective left A-module. The third equality follows from the definition of " $\int^y$ ". Since $L_\alpha . N = 0$ it follows that N is naturally an $(A \otimes A)/L_\alpha$-module and the fourth equality is immediate. Equality number five is the usual identification. It is left to the reader to verify that the resulting isomorphism $N \times_A C_\alpha \cong N$ is given by θ.

Consider the commutative diagram:

$$\begin{array}{ccc} N \times_A C_\alpha & \xrightarrow{\theta} & N \\ \downarrow & \nearrow_{\theta} & \\ N \times_A C & & \end{array}$$

By (2.12), *e)* the bottom θ is injective. The top θ we have just shown to be an isomorphism. Hence the bottom θ is surjective. This proves Part *b)*, *c)* and *d)*. By (2.12), *f)* the α and α' maps for C are isomorphisms. By (2.12), *c)* all Λ-maps for C are injective and hence all Λ'-maps are injective. Hence by (6.5) we will have proved *c)* and *d)* once we have shown $C \times_A C \xrightarrow{\theta} C$ is an isomorphism.

By (2.12), *e)* $C \times_A C \xrightarrow{\theta} C$ is injective. For each $C_\beta \subset C$ it follows from the definition of C_β that $L_\beta . C_\beta = 0$. Thus by part *b)* the top θ in the diagram below is an isomorphism.

$$\begin{array}{ccc} C_\beta \times_A C & \xrightarrow{\theta} & C_\beta \\ \big\downarrow & & \big\downarrow \iota \\ C \times_A C & \xrightarrow{\theta} & C \end{array}$$

Since C is the union of the (C_β)'s letting β vary shows that the bottom θ is surjective.
Q.E.D.

7. A a field and $A \# H$

Throughout the first part of this section A is assumed to be a field and an algebra over the subfield R.

Suppose E is a left sub-A-module of End A. By (1.5), 4) all Λ-maps for E are injective. If E is a sub-A-bimodule of End A then by (1.6) all θ-maps for E are injective. By (2.5), 4) all triples of A-bimodules associate and thus each A-bimodule is associative.

Suppose $E \subset \text{End } A$ is a sub-A-bimodule and (E, Δ, ι) gives E the structure of $\times_A$-coalgebra. By (5.8), *b)* $\theta : E \times_A E \to E$ is an isomorphism and $\Delta = \theta^{-1}$. By (5.9) E is cocommutative and $(\text{twist}) : E \times_A E \to E \times_A E$ is the identity map. By (5.10), *c)* (E, Δ, ι) makes E into a $\times_A$-bialgebra if E is a subalgebra over A of End A.

Suppose E is a sub-A-bimodule of End A and $\theta : E \times_A E \to E$ is surjective and hence bijective. By (6.5) (E, θ^{-1}, ι) gives E the structure of cocommutative $\times_A$-coalgebra.

Suppose $B \subset \text{End } A$ where (B, Δ, ι) is a $\times_A$-bialgebra and $\theta : \widetilde{B} \times_A B \to \widetilde{B}$ is surjective. In a later section we prove that for $\langle U \rangle \in \mathcal{G}\langle B \rangle$, U is a simple algebra with A as a maximal commutative subring. Moreover U has the same center as B (viewed as a subring of A).

***

A is still assumed to be a field and an algebra over the subfield R.

Let B denote the image of $\theta : \text{End } A \times_A \text{End } A \to \text{End } A$. Since θ is a map of algebras over A, B is a subalgebra over A of End A.

Theorem **(7.1)**. — $\theta : B \times_A B \to B$ *is bijective and* (B, θ^{-1}, ι) *is the unique maximal* $\times_A$*-coalgebra in* End A *with co-unit* ι.

Proof. — Since B is a sub-A-bimodule of End A it follows that $AbA \subset B$ for $b \in B$. Thus by (3.10), part 2, the map $\theta : B \times_A \text{End } A \to B$ is surjective. By (6.4), part 1, $\theta : \text{End } A \times_A B \to \text{End } A$ has the same image B as $\theta : B \times_A \text{End } A \to B$. Thus by (3.10), part 2, the map $\theta : B \times_A B \to B$ is surjective. By the opening remarks of this section θ is bijective and (B, θ^{-1}, ι) is a $\times_A$-coalgebra.

If $C \subset \text{End } A$ and (C, Δ', ι) is a $\times_A$-coalgebra then $C \times_A C \xrightarrow{\theta} C$ is bijective with inverse Δ'. In particular $C \subset \text{Im}(C \times_A \text{End } A \xrightarrow{\theta} C) \subset \text{Im}(\text{End } A \times_A \text{End } A \xrightarrow{\theta} \text{End } A) = B$.

Q.E.D.

By the opening remarks B is actually a cocommutative $\times_A$-bialgebra.

Lemma **(7.2)**. — *Let* M *be an* A-*bimodule and* B *as in* (7.1).

1. *The inclusion* $M \times_A B \xrightarrow{I \times \iota} M \times_A \text{End } A$ *is an isomorphism of* A-*bimodules.*
2. $\theta : M \times_A B \to M$ *and* $\theta : M \times_A \text{End } A \to M$ *have the same image, namely* $\{m \in M \mid {}_x AmA$ has finite x A-dimension$\}$.
3. *If* $N = \text{Im}(M \times_A \text{End } A \xrightarrow{\theta} M)$ *then* N *is a sub*-A-*bimodule of* M *and*

$$N \times_A \text{End } A \xrightarrow{\iota \times I} M \times_A \text{End } A$$

is an A-*bimodule isomorphism. Moreover* $\theta : N \times_A \text{End } A \to N$ *is an* A-*bimodule isomorphism.*

4. *The map* $N \times_A B \xrightarrow{\iota \times I} M \times_A \text{End } A$ *is an* A-*bimodule isomorphism. Moreover*

$$\theta : N \times_A B \to N$$

is an A-*bimodule isomorphism.*

Proof. — $I \times \iota$, $\iota \times I$ and $\iota \times \iota$ are injective A-bimodule maps by (2.3) and (2.4), 1).

Since $\Lambda' : \int_x {}_x M \otimes_x \text{End } A \to \text{Hom}(A, M)$ is injective by (1.6), $\theta : M \times_A \text{End } A \to M$ is injective. Suppose $m = \theta(\sum_i m_i \otimes f_i)$ where $\sum_i m_i \otimes f_i \in M \times_A \text{End } A \subset \int_x {}_x M \otimes_x \text{End } A$. Then $m = \sum_i f_i(1)\, m_i$ and for $a \in A$, $ma = \sum_i f_i(1)\, m_i a = \sum_i f_i a^\ell(1)\, m_i = \sum_i f_i(a)\, m_i$. Thus ${}_x AmA$ has finite x A-dimension. Conversely suppose $m \in M$ and ${}_x AmA$ has finite x A-dimension. Choose a finite x A basis $\{m_i\}$ of ${}_x AmA$. Then there exists $\{g_i\} \subset \text{End } A$ where $ma = \sum_i g_i(a)\, m_i$, $a \in A$. Consider the two elements

$$\sum_i m_i a \otimes g_i, \qquad \sum_i m \otimes g_i a \in \int_x {}_x M \otimes_x \text{End } A$$

for fixed $a \in A$. The map $\Lambda' : \int_x {}_x M \otimes_x \text{End } A \to \text{Hom}(A, M)$ is injective; so to prove the two elements equal, it suffices to apply Λ' and evaluate at $b \in A$. This gives $\sum_i g_i(b)\, m_i a = (mb)\, a = m(ba) = \sum_i g_i(ab)\, m_i$. Thus

$$\sum_i m_i \otimes g_i \in M \times_A \text{End } A \quad \text{and} \quad \theta(\sum_i m_i \otimes g_i) = \sum_i g_i(1)\, m_i = m1 = m.$$

This proves that $\text{Im}(M \times_A \text{End } A \xrightarrow{\theta} M) = \{m \in M \mid {}_x AmA$ has finite x A-dimension$\}$.

Since θ is an A-bimodule map N is a sub-A-bimodule of M. The diagram commutes

$$\begin{array}{ccc} N \times_A \text{End } A & \xrightarrow{\iota \times I} & M \times_A \text{End } A \\ & {\scriptstyle\theta} \searrow \quad \swarrow {\scriptstyle\theta} & \\ & N & \end{array}$$

Both θ maps are injective and by choice of N the right hand θ is bijective. By (3.10), part 2 the left hand θ is bijective. Hence, the top map is bijective and part 3 is proved.

Let $\{n_\alpha\}$ be an x A basis for ${}_x\mathrm{N}$. By the property of $\mathrm{N} = \mathrm{Im}\,\theta$, for each n_α there exists a set $\{f_{\alpha,\beta}\}_\beta \subset \mathrm{End\,A}$ where $\{f_{\alpha,\beta}\}_\beta$ is a finite set; i.e. for fixed α, $f_{\alpha,\beta} = 0$ for all but a finite number of (β)'s and

$$n_\alpha a = \sum_\beta f_{\alpha\beta}(a)\, n_\beta, \qquad a \in \mathrm{A}.$$

Suppose $z \in \mathrm{M} \times_{\mathrm{A}} \mathrm{End\,A} \int_x {}_x\mathrm{M} \otimes_x \mathrm{End\,A}$. By part 3, z can be regarded as lying in $\int_x {}_x\mathrm{N} \otimes_x \mathrm{End\,A}$. So z can be expressed: $z = \sum_\alpha n_\alpha \otimes f_\alpha$ where $\{f_\alpha\} \subset \mathrm{End\,A}$ and $f_\alpha = 0$ for all but a finite number of (α)'s. For $a \in \mathrm{A}$

$$\begin{aligned} \sum_\alpha n_\alpha \otimes f_\alpha a &= \sum_\alpha n_\alpha a \otimes f_\alpha \\ &= \sum_{\alpha,\beta} f_{\alpha,\beta}(a)\, n_\beta \otimes f_\alpha \\ &= \sum_{\alpha,\beta} n_\beta \otimes f_{\alpha,\beta}(a) f_\alpha. \end{aligned}$$

Since the tensor product is over a field and the $\{n_\alpha\}$ is linearly independent it follows that $f_\beta a = \sum_\alpha f_{\alpha,\beta}(a) f_\alpha$ which is a finite sum since almost all $f_\alpha = 0$. Thus ${}_x\mathrm{A} f_\beta \mathrm{A}$ has finite x A-dimension. By the part of part 2 which has been proved it follows that $f_\beta \in \mathrm{Im}(\mathrm{End\,A} \times_{\mathrm{A}} \mathrm{End\,A} \xrightarrow{\theta} \mathrm{End\,A}) = \mathrm{B}$. This proves part 1. The remaining part of part 2 follows from part 1. Part 4 follows from parts 1 and 3. Q.E.D.

Part 3 characterizes B as $\{f \in \mathrm{End\,A} \,|\, {}_x\mathrm{A}f\mathrm{A}$ has finite x A-dimension$\}$.

Let $\mathrm{D} = \{f \in \mathrm{End\,A} \,|\, \mathrm{A}f\mathrm{A}_y$ has finite y A-dimension$\}$. By (7.2), part 2 $\widetilde{\mathrm{D}} \subset \widetilde{\mathrm{End\,A}}$ is the image of $(\widetilde{\mathrm{End\,A}}) \times_{\mathrm{A}} \mathrm{B} \xrightarrow{\theta} \widetilde{\mathrm{End\,A}}$. Since θ is a map of algebras over A it follows that D is a subalgebra over A of $\widetilde{\mathrm{End\,A}}$. By (7.2), part 4 the map $\theta : \widetilde{\mathrm{D}} \times_{\mathrm{A}} \mathrm{B} \to \widetilde{\mathrm{D}}$ is an equivalence of algebras over A. Let $\delta : \widetilde{\mathrm{D}} \to \widetilde{\mathrm{D}} \times_{\mathrm{A}} \mathrm{B}$ denote the inverse to θ. Let $\Delta : \mathrm{B} \to \mathrm{B} \times_{\mathrm{A}} \mathrm{B}$ denote the diagonalization of B (which makes B into a $\times_{\mathrm{A}}$-coalgebra).

Let E denote the sum of all A-bimodules $\mathrm{X} \subset \mathrm{End\,A}$ which satisfy

(i) $\mathrm{X} \subset \mathrm{B} \cap \mathrm{D}$.

(ii) $\Delta\mathrm{X} \subset \mathrm{X} \times_{\mathrm{A}} \mathrm{X} \subset \mathrm{B} \times_{\mathrm{A}} \mathrm{B}$.

(iii) $\delta\widetilde{\mathrm{X}} \subset \widetilde{\mathrm{X}} \times_{\mathrm{A}} \mathrm{X} \subset \widetilde{\mathrm{D}} \times_{\mathrm{A}} \mathrm{B}$.

E again has properties (i), (ii), (iii), and is maximal with respect to these properties. Since B and D are algebras over A, Δ and δ are maps of algebras over A and A^ℓ satisfies properties (i)-(iii), E must be a subalgebra over A of End A. By property (ii), $(\mathrm{E}, \Delta|\mathrm{E}, \ell)$ is a $\times_{\mathrm{A}}$-coalgebra. By the opening remarks E is a $\times_{\mathrm{A}}$-bialgebra. By property (iii), $\widetilde{\mathrm{E}} \times_{\mathrm{A}} \mathrm{E} \xrightarrow{\theta} \widetilde{\mathrm{E}}$ is surjective, hence bijective. This proves that

Theorem **(7.3).** — E *is the unique maximal* $\times_{\mathrm{A}}$*-bialgebra* (*with* $\mathscr{I}_{\mathrm{E}} = \iota$) *in* End A *which satisfies*: $\widetilde{\mathrm{E}} \times_{\mathrm{A}} \mathrm{E} \xrightarrow{\theta} \widetilde{\mathrm{E}}$ *is surjective (or bijective).*

As a consequence of this theorem it will follow that E is a simple algebra. The result on simplicity appears in a later section.

Example **(7.4)**. — Suppose $g: A \to A$ is an R-algebra homomorphism. Then $ga^{\ell} = g(a)g$ and ${}_xAgA$ has x A-dimension 1. Thus $g \in B$. If g is an automorphism then $ag = gg^{-1}(a)^{\ell}$ and $g \in D$. Actually AgA satisfies properties (i)-(iii), since

$$\Delta(g) = g \otimes g \in \int_x {}_xB \otimes {}_xB \quad \text{and} \quad \delta\tilde{g} = g \otimes g^{-1} \in D \otimes_A B.$$

Thus $E \supset \mathrm{Aut}(A/R)$. It can be shown that a sequence of higher derivations [11, p. 195] must lie in E.

Example **(7.5)**. — Suppose $0 \neq f \in D$; i.e. AfA_y has finite y A-dimension. Then there exists a finite y A basis $\{a_i^{\ell} f\}$ for AfA_y and $\{f_i\} \subset \mathrm{End}\, A$ where $af = \sum_i a_i^{\ell} f f_i(a)^{\ell}$. Then $A = A\, \mathrm{Im} f = Af(A) = \sum_i a_i f(f_i(A)A) \subset \sum_i a_i f(A)$. If $\mathrm{Im} f$ is a subfield of A this shows that A is a finite degree extension of $\mathrm{Im} f$. Let $g: A \to A$ be an R-algebra homomorphism where A is *not* a finite extension of the subfield $\mathrm{Im}\, g$. By (7.4), $g \in B$. By what we have just shown $g \notin D$. Thus $g \notin E$ and in general $E \subsetneq B$.

Example **(7.6)**. — Suppose $f \in \mathrm{End}\, A$ where $f(1) = 1$ and $1 = \mathrm{codim}\, \mathrm{Ker} f$; i.e. $\dim_R(A/\mathrm{Ker} f) = 1$. Suppose $f \in B$ and $\{fa_i^{\ell}\}$ is a finite x A basis for ${}_xAfA$. There exists $\{f_i\} \subset \mathrm{End}\, A$ where $fb^{\ell} = \sum_i f_i(b) fa_i^{\ell}$. Since a_i^{ℓ} is an isomorphism of End A, $\mathrm{Ker}(fa_i^{\ell})$ has codimension 1 and $\bigcap_i \mathrm{Ker}(fa_i^{\ell})$ has finite codimension in A. If $\dim_R A$ is not finite then $0 \neq \bigcap_i \mathrm{Ker}(fa_i^{\ell})$ and there exists $0 \neq c \in \bigcap_i \mathrm{Ker}(fa_i^{\ell})$. $fc^{-1\ell} = \sum_i f_i(c^{-1}) fa_i^{\ell}$. Applying both sides to c yields a contradiction. Hence A must be a finite extension of R. This shows that if $[A:R] = \infty$ then $B \subsetneq \mathrm{End}\, A$.

It will follow from our study of End A when A is a finite projective R-module that if A is a finite degree field extension of R then $B = E = \mathrm{End}\, A$.

* * *

We no longer assume that A and R are fields. A is merely a commutative R-algebra.

Example $A \# H$.

Familiarity with standard coalgebra, bialgebra and Hopf algebra theory is assumed in this example. Suppose H is a *cocommutative* bialgebra over R and A is an H-module algebra [17, § 7.2, p. 153]. The smash (semi-direct) product $A \# H$ is $A \otimes H$ as an R-module and left A-module. Define $\Delta: A \# H \to \int_x ({}_xA \# H) \otimes ({}_xA \# H)$ by

$$a \# h \mapsto \sum_{(h)} (a \# h_{(1)}) \otimes (1 \# h_{(2)}).$$

For $b \in A$

$$\sum_{(h)} (a \# h_{(1)})\, b \otimes (1 \# h_{(2)}) = \sum_{(h)} (a(h_{(1)}.b) \# h_{(2)}) \otimes (1 \# h_{(3)})$$

$$= \sum_{(h)} (a \# h_{(1)}) \otimes ((h_{(2)}.b) \# h_{(3)}) = \sum_{(h)} (a \# h_{(1)}) \otimes (1 \# h_{(2)})\, b.$$

Thus $\operatorname{Im} \Delta \subset (A \# H) \times_A (A \# H)$ and we consider Δ as a map from $A \# H$ to $(A \# H) \times_A (A \# H)$. Δ is an A-bimodule map; in fact Δ is a map of algebras over A.

A is naturally an $A \# H$-module where $(a \# h).b = a(h.b)$. Let $\mathscr{I}$ be the associated representation. $\mathscr{I}$ is a map of algebras over A. If $\varepsilon = \boldsymbol{\varepsilon}\mathscr{I}$ then $\varepsilon : A \# H \to A$, $a \# h \to a(h.1)$.

It is easily verified that $(A \# H, \iota\Delta, \varepsilon)$ is a coalgebra over A. Hence by (5.4), *b)* $(A \# H, \Delta, \mathscr{I})$ is a $\times_A$-coalgebra (and so a $\times_A$-bialgebra) if $A \# H$ is associative as an A-bimodule. By (2.5), 4) $A \# H$ is associative as an A-bimodule if $A \# H$ is a projective left A-module. As a left A-module $A \# H \cong A \otimes H$. Thus if H is a projective R-module $A \# H$ is a $\times_A$-bialgebra.

Even if H is not projective as an R-module $A \# H$ may be associative as an A-bimodule. For example by (5.5) if Δ is an isomorphism then $A \# H$ is associative as in A-bimodule.

We shall be interested in when Δ is an isomorphism for other reasons. Among them is that $A \# H$ is idempotent as an algebra over A (4.6) when Δ is an isomorphism.

The question of when $\Delta : A \# H \to (A \# H) \times_A (A \# H)$ is an isomorphism is partially answered by (5.8). If $\mathscr{I} : A \# H \to \operatorname{End} A$ is injective we may identify $A \# H$ with $\operatorname{Im} \mathscr{I}$ and let this be C in (5.8). Then by (5.8) Δ is an isomorphism if $(A \# H) \times_A (A \# H) \xrightarrow{\theta} A \# H$ is injective. By (1.5) and (1.6) it follows that θ is injective if $A \# H$ is projective as a left A-module. As we pointed out before $A \# H$ is projective as a left A-module if H is a projective R-module.

8. Differentials and differential operators and End A

Throughout this section A is a commutative algebra. For left A-modules M and N, $f \in \operatorname{Hom}(M, N)$ and $a \in A$ let $[a, f] \in \operatorname{Hom}(M, N)$ where $[a, f](m) = af(m) - f(am)$.

$\operatorname{Hom}(M, N)$ has a left A-module structure arising from N and a right A-module structure arising from M. This makes $\operatorname{Hom}(M, N)$ into an A-bimodule and $A \otimes A$-module. Then for $f \in \operatorname{Hom}(M, N)$, $[a, f] = (a \otimes 1 - 1 \otimes a).f$, $a \in A$.

Let $\mathfrak{M}$ denote the kernel of $A \otimes A \xrightarrow{\text{mult}} A$. $\mathfrak{M}$ is an x and y A submodule of ${}_xA \otimes A_y$, and is spanned by elements of the form $\{a \otimes 1 - 1 \otimes a\}_{a \in A}$ as an x or y A module.

As in [9, § 2, p. 210] the differential operators from M to N are defined inductively by:

$$\mathrm{Diff}_A^{-1}(M, N) = 0$$

$$\mathrm{Diff}_A^0(M, N) = \{f \in \mathrm{Hom}(M, N) \mid [a, f] = 0,\ a \in A\} = \mathrm{Hom}_A(M, N)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathrm{Diff}_A^n(M, N) = \{f \in \mathrm{Hom}(M, N) \mid [a, f] \in \mathrm{Diff}_A^{n-1}(M, N),\ a \in A\},$$

and

$$\mathrm{Diff}_A(M, N) = \bigcup_n \mathrm{Diff}_A^n(M, N).$$

$\mathrm{Diff}_A^n(M, N) = \{f \in \mathrm{Hom}(M, N) \mid \mathfrak{M}^{n+1}.f = 0\}$ and $\mathrm{Diff}_A^n(M, N)$ is a sub-A-bimodule of $\mathrm{Hom}(M, N)$ for all n. Elements of $\mathrm{Diff}_A^n(M, N)$ are the n^{th} *order differential operators* from M to N. A differential operator from M to N is an element of $\mathrm{Diff}_A(M, N)$.

In case $A = M$ we write $D_A^n(N)$ and $D_A(N)$ in place of $\mathrm{Diff}_A^n(A, N)$ and $\mathrm{Diff}_A(A, N)$. In case both $M = A = N$ we write D_A^n and D_A in place of $D_A^n(A)$ and $D_A(A)$.

For a left A-module M the elements in $D_A^1(M)$ which vanish on 1 are exactly the derivations from A to M. This and other results can be found in [9, § 2, pp. 210-220]. In particular it is shown that $D_A^n D_A^m \subset D_A^{n+m}$. Also, $A^\ell = D_A^0$ since $D_A^0 = \mathrm{Hom}_A(A, A)$. Thus D_A is a subalgebra over A of End A.

Definition **(8.1)**. — An *algebra of differential operators* of A is a subalgebra over A of D_A; i.e. a subalgebra of D_A which contains A^ℓ. The full algebra of differential operators of A is D_A.

Lemma **(8.2)**. — *If* $0 \neq M$ *is a sub*-A-*bimodule of* D_A *then* $M \cap A^\ell = I^\ell$ *for* $0 \neq I$ *an ideal in* A. *Hence* D_A *is an essential extension of* A^ℓ *as an* $A \otimes A$-*module.*

Proof. — $M \cap A^\ell = I^\ell$ for an ideal $I \subset A$ and the problem is to show that $0 \neq M \cap A^\ell$ if $0 \neq M$. Say $0 \neq m \in M$. If $m \in A^\ell$ done. Otherwise there is $1 \leq t \in \mathbf{Z}$ where $\mathfrak{M}^t.m \neq 0$ and $\mathfrak{M}^{t+1}.m = 0$. Choose $y \in \mathfrak{M}^t$ where $y.m \neq 0$. Then $0 \neq y.m \in M$ and $\mathfrak{M}.(y.m) = 0$ so that $y.m \in A^\ell$. Q.E.D.

By [9, p. 215, (2.2.6)] for a left A-module M there is a left A-module $J_n(M)$ and $j_n \in \mathrm{Diff}_A^n(M, J_n(M))$ with the following universal property: If N is a left A-module and $f \in \mathrm{Diff}_A^n(M, N)$, then there is a unique $J(f) \in \mathrm{Hom}_A(J_n(M), N)$ where $f = J(f)j_n$. In other words there is a natural equivalence (adjointness relation)

$$\textbf{(8.3)} \qquad \begin{cases} \mathrm{Diff}_A^n(M, N) \leftrightarrow \mathrm{Hom}_A(J_n(M), N) \\ \qquad\quad g j_n \leftrightarrow g. \end{cases}$$

The explicit construction of $J_n(M)$ appears in [9, p. 214, between (2.2.4) and (2.2.5)]. The construction of $J_n(A)$ is restated here.

${}_x A \otimes A_y$ has an A-bimodule structure, the x A structure being the left and the y A structure being the right. $\mathfrak{M}^{n+1}$ is a sub-A-bimodule; hence, $(A \otimes A)/\mathfrak{M}^{n+1}$ is an

A-bimodule. Let $J_n(A)$ denote $(A\otimes A)/\mathfrak{M}^{n+1}$ which is an A-bimodule and an algebra. Let $j_n : A \to J_n(A)$ be the composite

$$A \xrightarrow{(a \mapsto 1\otimes a)} A\otimes A \longrightarrow (A\otimes A)/\mathfrak{M}^{n+1} = J_n(A).$$

With respect to the left A-module structure of $J_n(A)$, $j_n \in D_A^n(J_n(A))$ and the pair $(J_n(A), j_n)$ has the universal property described in (8.3).

The algebra map $A\otimes A \overset{\text{mult}}{\longrightarrow} A$ induces an algebra map $J_n(A) \to A$ with kernel $\mathfrak{M}/\mathfrak{M}^{n+1}$. This ideal in $J_n(A)$ is denoted $J_n^+(A)$. The composite

$$A \xrightarrow{(a \mapsto a\otimes 1)} A\otimes A \longrightarrow (A\otimes A)/\mathfrak{M}^{n+1} = J_n(A)$$

is a *left* A-module map, an algebra map and a splitting for $J_n(A) \to A$. Thus $A \to J_n(A)$ is given by $a \mapsto a.1$, $a \in A$, 1 the unit of $J_n(A)$, and the image is denoted $A.1$. By the splitting property,

(8.4) $J_n(A) = A.1 \oplus J_n^+(A)$, a direct sum of left A-modules.

Let $j_n^+ : A \to J_n^+(A)$ be the composite

$$A \xrightarrow{j_n} J_n(A) = A.1 \oplus J_n^+(A) \xrightarrow{\text{projection}} J_n^+(A).$$

Then for $a \in A$, $j_n^+(a) = j_n(a) - a.j_n(1)$. $(J_n^+(A), j_n^+)$ has the same universal property for n^{th} order differential operators from A *which vanish at 1* as $(J_n(A), j_n)$ has for all differential operators from A. Since a derivation from A (to M) is the same as a first order differential operator which vanishes at 1 it follows that $(J_1^+(A), j_1^+)$ is the Kaehler module of A (and the universal derivation).

Definition **(8.5)**. — A has *finite projective differentials* if for each $n \in \mathbf{Z}$ there is $m \in \mathbf{Z}$ with $m \geq n$ and $J_m(A)$ is a finitely generated projective left A-module. A has *almost finite projective differentials* if there is a collection $\{L_\alpha\}$ of ideals of $A\otimes A$ which is cofinal with $\{\mathfrak{M}^i\}$ and where $(A\otimes A)/L_\alpha$ is a finite projective left A-module for each L_α.

If A has finite projective differentials, then considering

$$\{\mathfrak{M}^{n+1} \subset A\otimes A \mid J_n(A) \text{ is a finite projective left A-module}\}$$

shows that A has almost finite projective differentials.

By (8.4) it follows that $J_m(A)$ is finitely generated and projective as a left A-module if and only if $J_m^+(A)$ is finitely generated and projective as a left A-module. Thus $J_1(A)$ is finitely generated and projective as a left A-module if and only if the Kaehler module is. The next example shows that A having finite projective differentials does *not* imply that the Kaehler module of A is projective and hence A is not necessarily differentially smooth in the sense of Grothendieck [8, p. 51, (16.10)].

Example **(8.6)**. — Let A be an R-algebra which is finitely generated and projective as an R-module. Furthermore assume that A is purely inseparable over R in the sense of (13.14). By (13.16) there is $N \in \mathbf{Z}$ where $J_m(A) = A \otimes A$ for $m \geq N$. This is a finitely generated projective left A-module since A is a finite projective R-module. Thus A has finite projective differentials. Consider the specific case $A = R[X]/\langle X^2 \rangle$ and let R be a field of characteristic different from 2. A is purely inseparable over R so A has finite projective differentials. Let $\overline{X}$ denote the coset (image) of X in A. In $A \otimes A$, $\mathfrak{M}$ has an R-basis consisting of $\{1 \otimes \overline{X} - \overline{X} \otimes 1, \overline{X} \otimes \overline{X}\}$. Since $(1 \otimes \overline{X} - \overline{X} \otimes 1)^2 = -2\overline{X} \otimes \overline{X}$ and the characteristic is not 2, $\overline{X} \otimes \overline{X} \in \mathfrak{M}^2$. Thus $\mathfrak{M}/\mathfrak{M}^2$ has R-dimension 1 and is " too small " to be a *free* A-module. Since A is local, $\mathfrak{M}/\mathfrak{M}^2 = J_1^+(A)$ is not a projective A-module.

Suppose A has almost finite projective differentials and let $\{L_\alpha\}$ be as in (8.5). Then $\{L_\alpha\}$ satisfy (i) and (ii) above (6.6). The intersection property follows from the fact that $\{L_\alpha\}$ is cofinal with $\{\mathfrak{M}^i\}$. The cofinal property also shows that $C = \bigcup_\alpha C_\alpha = D_A$. The e map in (6.6), *a)* carries $1 \otimes a - a \otimes 1$ to

$$1 \otimes 1 \otimes a - a \otimes 1 \otimes 1 = (1 \otimes 1 \otimes a - 1 \otimes a \otimes 1) + (1 \otimes a \otimes 1 - a \otimes 1 \otimes 1) \in A \otimes \mathfrak{M} + \mathfrak{M} \otimes A.$$

Thus $e(\mathfrak{M}) \subset A \otimes \mathfrak{M} + \mathfrak{M} \otimes A$. Since e is an algebra homomorphism

$$e(\mathfrak{M}^i) \subset (A \otimes \mathfrak{M} + \mathfrak{M} \otimes A)^i \subset A \otimes \mathfrak{M}^r + \mathfrak{M}^s \otimes A$$

where $r + s \leq i + 1$. Thus part *a)* of (6.6) shows that D_A is a subalgebra over A of End A. Theorem (6.6) restated for differential operators becomes:

Theorem **(8.7)**. — *Suppose* A *has almost finite projective differentials*:

a) $D_A \times_A D_A \xrightarrow{\theta} D_A$ *is an isomorphism, the* α *and* α' *maps for* D_A *are isomorphisms so that* D_A *is associative as an* A*-bimodule and* $(D_A, \theta^{-1}, \iota)$ *makes* D_A *into a* $\times_A$*-bialgebra which is cocommutative.*

b) D_A *is flat as a left* A*-module and idempotent as an algebra over* A.

c) *If* M *is any right* A*-module and* N *any left* A*-module then* $M \otimes_A D_A \xrightarrow{\Lambda} \mathrm{Hom}(A, M)$ *and* $\int_x {}_x N \otimes_x D_A \xrightarrow{\Lambda'} \mathrm{Hom}(A, N)$ *are injective.*

Proof. — Part *c)* follows from (2.12), *c)*; (2.12), *b)* gives flatness of D_A.
Part *a)* follows from (6.6).
The isomorphism θ shows that D_A is idempotent as an algebra over A. Q.E.D.

In later sections we study $\mathscr{E}\langle D_A \rangle$ and $\mathscr{G}\langle D_A \rangle$ showing that they are often equal and giving a cohomological interpretation. We also present some answers to the question of when does A have finite projective differentials.

* * *

Suppose A is a finite projective R-module. Let $\{0\}$ be the single element set of ideals in $A \otimes A$. This set has the desired properties stated above (6.6) and the " C " which arises is End A. Thus by (6.6) we have

Theorem **(8.8)**. — *Suppose* A *is a finite projective* R*-module.*

a) $\text{End A} \times_A \text{End A} \xrightarrow{\theta} \text{End A}$ *is an isomorphism, the* α *and* α' *maps for* End A *are isomorphisms so that* End A *is associative as an* A*-bimodule and* $(\text{End A}, \theta^{-1}, \text{I})$ *makes* End A *into a* $\times_A$*-bialgebra which is cocommutative.*

b) *If* M *is any right* A*-module and* N *any left* A*-module, then* $\text{M} \otimes_A \text{End A} \xrightarrow{\Lambda} \text{Hom}(\text{A}, \text{M})$ *and* $\int_x {}_x\text{N} \otimes_x \text{End A} \xrightarrow{\theta} \text{Hom}(\text{A}, \text{N})$ *are injective.*

Proof. — Part *b)* follows from (2.12). Part *a)* follows from (6.6). Q.E.D.

Since End A is a $\times_A$-bialgebra it is associative as an A-bimodule. Since End A is idempotent as an algebra over A the monoid $\mathscr{E}\langle\text{End A}\rangle$ and the group $\mathscr{G}\langle\text{End A}\rangle$ are defined (4.8).

In a later section we prove that $\mathscr{G}\langle\text{End A}\rangle$ is isomorphic to the second Amitsur cohomology group of A over R with respect to the functor " units ".

9. The $\mathscr{S}$ map

Motivation

$\times_A$-coalgebras and $\times_A$-bialgebras have been defined. The next object of interest is derived from the notion of $\times_A$-Hopf algebra.

Suppose B is a cocommutative $\times_A$-bialgebra. An $\times_A$-antipode would be an anti-isomorphism $\text{S} : \text{B} \rightarrow \text{B}$ of algebras over A where $\text{S}^2 = \text{I}$ and S has some additional properties. If $\sim\text{S}$ is the composite $\text{B} \xrightarrow{\text{S}} \text{B} \xrightarrow{\sim} \widetilde{\text{B}}$ then $\sim\text{S}$ is an isomorphism of algebras over A. Using $\sim\text{S}$ one can form the composite $\mathscr{S}$

$$\text{B} \xrightarrow{\sim\text{S}} \widetilde{\text{B}} \xrightarrow{\widetilde{\Delta}} \widetilde{\text{B} \times_A \text{B}} \xrightarrow{\widetilde{(\sim\text{S}) \times \text{I}}} \widetilde{\widetilde{\text{B}} \times_A \text{B}}$$

which would be an isomorphism of algebras over A and have some additional properties.

The map S is not recoverable from the composite $\mathscr{S}$. For our purposes the map $\mathscr{S}$ is all that is needed. Furthermore, for the $\times_A$-bialgebra End A when R is a field and A is a finite dimensional commutative R-algebra there is no $\times_A$-antipode S when A is not a Frobenius R-algebra. But there is always a suitable $\mathscr{S}$ map. (This result will appear in a later section.)

The Ess

Lemma **(9.1)**. — *Suppose* U, V, W *and* X *are* A*-bimodules.*

a) *Consider the composite*

$$\begin{aligned}\widetilde{\widetilde{(U\times_A V)}\times_A(W\times_A X)} &\hookrightarrow (U\times_A V)\otimes_A(W\times_A X)\\ &\xrightarrow{\xi} (U\otimes_A W)\times_A(V\otimes_A X)\\ &\hookrightarrow \int_\ell ({}_\ell U\otimes_A W)\otimes({}_\ell V\otimes_A X)\end{aligned}$$

where the first natural inclusion is defined above (2.3), ξ *is defined in* (2.10) *and the final natural inclusion follows by definition* (2.1). *The image of the composite is in*

$$\int^r\int^y\int_\ell ({}_{y\ell}U_r\otimes_A W_y)\otimes({}_{y\ell}V_r\otimes_A X_y).$$

Let $\mathscr{B}$ *denote the induced map*

$$\widetilde{\widetilde{(U\times_A V)}\times_A(W\times_A X)} \xrightarrow{\mathscr{B}} \int^r\int^y\int_\ell ({}_{y\ell}U_r\otimes_A W_y)\otimes({}_{y\ell}V_r\otimes_A X_y).$$

b) *Consider the composite*

$$\begin{aligned}\widetilde{(\widetilde{U}\times_A W)}\times_A\widetilde{(\widetilde{V}\times_A X)} &\hookrightarrow \int_h {}_h\widetilde{(\widetilde{U}\times_A W)}\otimes_h(\widetilde{V}\times_A X)\\ &\xrightarrow{\sim\otimes\sim} \int_\ell (\widetilde{U}\times_A W)_\ell\otimes(\widetilde{V}\times_A X)_\ell\\ &\xrightarrow{\iota\otimes\iota} \int_\ell ({}_\ell U\otimes_A W)\otimes({}_\ell V\otimes_A X)\end{aligned}$$

where the first inclusion results from the definition (2.1) *and the* (ι)'*s in* $\iota\otimes\iota$ *are each the inclusion above* (2.3). *The image of the composite is in* $\int^r\int^y\int_\ell ({}_{y\ell}U_r\otimes_A W_y)\otimes({}_{y\ell}V_r\otimes_A X_y)$. *Let* $\mathscr{C}$ *denote the induced map*

$$\widetilde{(\widetilde{U}\times_A W)}\times_A\widetilde{(\widetilde{V}\times_A X)} \xrightarrow{\mathscr{C}} \int^r\int^y\int_\ell ({}_{y\ell}U_r\otimes_A W_y)\otimes({}_{y\ell}V_r\otimes_A X_y).$$

Proof. — The proof is straightforward and left to the reader.

Definition **(9.2)**. — Suppose (B, Δ, $\mathscr{I}$) is a $\times_A$-bialgebra, *an* Ess is a map of algebras over A

$$\mathscr{S} : B\to\widetilde{\widetilde{B}\times_A B}$$

which makes the following diagrams commute:

$$\begin{array}{ccc} B\times_A B & \xrightarrow{\mathscr{S}\times\mathscr{S}} & \widetilde{(\widetilde{B}\times_A B)}\times_A\widetilde{(\widetilde{B}\times_A B)}\\ \Delta\uparrow & & \downarrow\mathscr{C}\\ B & & \int^r\int^y\int_\ell ({}_{y\ell}B_r\otimes_A B_y)\otimes({}_{y\ell}B_r\otimes_A B_y)\\ \mathscr{S}\downarrow & & \uparrow\mathscr{B}\\ \widetilde{\widetilde{B}\times_A B} & \xrightarrow{\widetilde{\tilde{\Delta}\times\Delta}} & \widetilde{\widetilde{(B\times_A B)}\times_A(B\times_A B)} \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{\mathscr{S}} & \widetilde{\tilde{B}\times_A B} \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle \widetilde{\tilde{I}\times\mathscr{S}}} \\ B=\tilde{\tilde{B}} & \xleftarrow{\tilde{\theta}} & \widetilde{\tilde{B}\times_A \mathrm{End}\,A} \end{array}$$

To make use of the Ess we must study algebras over A, (U, i) where i is injective and $\mathrm{Im}\, i=\int^x {}_x U_x$. In this case if we identified A with $\mathrm{Im}\, i$ we would have that A is a maximal commutative subalgebra of U.

Lemma **(9.3)**. — *Suppose* (U, i) *and* (V, j) *are algebras over* A.

a) *If j is injective, $\int^x {}_x V_x=\mathrm{Im}\, j$ and there is an* A*-bimodule isomorphism* $\sigma : U \to V$, *then*

1. *There is a unique invertible element* $b\in A$ *where* $\sigma i=jb^\ell$.

2. *i is injective and* $\int^x {}_x U_x=\mathrm{Im}\, i$.

3. U *and* V *have the same center in the sense that if* Z *is the center of* V, *then* $Z\subset \mathrm{Im}\, j$ *and* $ij^{-1}(Z)$ *is the center of* U. *Moreover* $j^{-1}(Z)=\{a\in A \mid av=va,\ v\in V\}$.

b) *If* $A \xrightarrow{h} U\times_A V$ *is injective then both i and j are injective.*

c) *If* $A \xrightarrow{h} U\times_A V$ *is injective and* $\mathrm{Im}(A \xrightarrow{h} U\times_A V)=\int^x {}_x U\times_A V_x$ *then there is a map*

$$\mathscr{D} : \int^r\int^y\int_\ell ({}_{y\ell}U_r\otimes_A U_y)\otimes({}_{y\ell}V_r\otimes_A V_y) \to \mathrm{End}\,A$$

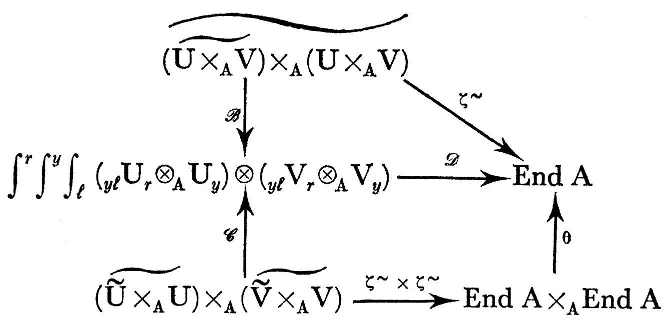
where for $\sum_q w_q\otimes x_q\otimes y_q\otimes z_q\in\int^r\int^y\int_\ell ({}_{y\ell}U_r\otimes_A U_y)\otimes({}_{y\ell}V_r\otimes_A V_y)$ *and* $a\in A$

$$h(\mathscr{D}(\sum_q w_q\otimes x_q\otimes y_q\otimes z_q)(a))=\sum_q (w_q a x_q)\otimes(y_q z_q)$$
$$=\sum_q (w_q x_q)\otimes(y_q a z_q)\in U\times_A V.$$

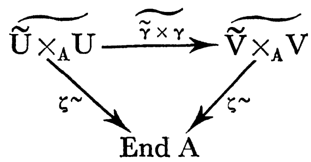
d) *Suppose* $A\xrightarrow{i} U$, $A\xrightarrow{j} V$ *and* $A\xrightarrow{h} U\times_A V$ *are injective,*

$$\mathrm{Im}\, i=\int^x {}_x U_x, \quad \mathrm{Im}\, j=\int^x {}_x V_x \quad \textit{and} \quad \mathrm{Im}(A\xrightarrow{h} U\times_A V)=\int^x {}_x U\times_A V_x.$$

Use i to identify A *with* L *in* (3.4) *so that* $\zeta : \tilde{U}\times_A U \to \mathrm{End}\,A$ *is an antihomomorphism of algebras over* A. *Thus* $\zeta^\sim : \widetilde{\tilde{U}\times_A U} \to \mathrm{End}\,A$ *is a map of algebras over* A. *Similarly* $\zeta^\sim : \widetilde{\tilde{V}\times_A V} \to \mathrm{End}\,A$ *and* $\zeta^\sim : \widetilde{\widetilde{(U\times_A V)}\times_A (U\times_A V)} \to \mathrm{End}\,A$ *are homomorphisms of algebras over* A. *The following diagram commutes*:



e) *Suppose* $A \xrightarrow{i} U$, $A \xrightarrow{j} V$ *are injective and* $\operatorname{Im} i = \int^x {}_x U_x$, $\operatorname{Im} j = \int^x {}_x V_x$ *and* $\gamma : U \to V$ *is a map of algebras over* A. *Then the following diagram commutes*:



Proof. — *a)* Let $v = \sigma(1) \in V$. Since σ is an A-bimodule map $v \in \int^x {}_x V_x = \operatorname{Im} j$ and $v = j(b)$ for some $b \in A$. Then $\sigma i = j b^\ell$ and b is the unique element of A with this property. Let $u = \sigma^{-1}(1) \in \int^x {}_x U_x$. Then $bu = b\sigma^{-1}(1) = \sigma^{-1}(b.1) = \sigma^{-1}(v) = 1$. Since $u \in \int^x {}_x U_x$ also $u^2 \in \int^x {}_x U_x$ and $\sigma(u^2) \in \int^x {}_x V_x = \operatorname{Im} j$. Thus

$$b\sigma(u^2) = \sigma(bu^2) = \sigma(u) = 1.$$

This implies that $c \in A$ is the inverse to b where c is determined by $j(c) = \sigma(u^2)$. This proves Part 1.

Injectivity of i follows from $\sigma i = j b^\ell$ with b invertible in A. This also implies that $\sigma(\operatorname{Im} i) = \operatorname{Im} j = \int^x {}_x V_x$. Since σ is an A-bimodule isomorphism it follows that $\int^x {}_x U_x = \operatorname{Im} i$. This proves Part 2.

Certainly the center of V centralizes $\operatorname{Im} j$. Thus $Z \subset \int^x {}_x V_x = \operatorname{Im} j$. For $a \in A$, $j(a) \in Z$ if and only if $av = va$ for all $v \in V$. By Part 2 the center of U is characterized similarly. Then the fact that $U \cong V$ as A-bimodules gives Part 3.

b) The composite $A \xrightarrow{h} U \times_A V \hookrightarrow \int_x {}_x U \otimes_x V$ is given by $a \mapsto i(a) \otimes 1 = 1 \otimes j(a)$. If this map is injective so are i and j.

c) The map

$$\int_\ell ({}_\ell U \otimes_A U) \otimes ({}_\ell V \otimes_A V) \xrightarrow{\text{mult} \otimes \text{mult}} \int_\ell {}_\ell U \otimes_\ell V$$

carries $\int^y \int_\ell ({}_{y\ell} U \otimes_A U_y) \otimes ({}_{y\ell} V \otimes_A V_y)$ to

$$\int^y \int_\ell ({}_{y\ell} U_y \otimes_{y\ell} V_y) \subset \int^z \int_\ell ({}_\ell U_z \otimes_\ell V_z) = U \times_A V.$$

As a submodule of $U\times_A V$, $\int^y\int_\ell ({}_{y\ell}U_y\otimes{}_{y\ell}V_y)$ is $\int^x {}_xU\times_A V_x$. By hypothesis $\int^x {}_xU\times_A V_x=\mathrm{Im}(A\xrightarrow{h} U\times_A V)$ and $A\xrightarrow{h} U\times_A V$ is injective. (The map $A\xrightarrow{h} U\times_A V$ is given by $A\cong A\times_A A\xrightarrow{i\times j} U\times_A V$.)

This implies the existence of a unique map d making the diagram commute:

$$\begin{array}{ccc}
\int_\ell ({}_\ell U\otimes_A U)\otimes({}_\ell V\otimes_A V) & \xrightarrow{\text{mult}\otimes\text{mult}} & \int_\ell {}_\ell U\otimes{}_\ell V \\
\uparrow & & \uparrow \\
 & & U\times_A V \\
\uparrow & & \uparrow h \\
\int^y\int_\ell ({}_{y\ell}U\otimes_A U_y)\otimes({}_{y\ell}V\otimes_A V_y) & \xrightarrow{\quad d \quad} & A
\end{array}$$

From $\sum_q w_q\otimes x_q\otimes y_q\otimes z_q\in\int^r\int^y\int_\ell ({}_{y\ell}U_r\otimes_A U_y)\otimes({}_{y\ell}V_r\otimes_A V_y)$ and $a\in A$ it follows that the element t defined as $t=\sum_q w_q a\otimes x_q\otimes y_q\otimes z_q=\sum_q w_q\otimes x_q\otimes y_q a\otimes z_q$ lies in

$$\int^y\int_\ell ({}_{y\ell}U\otimes_A U_y)\otimes({}_{y\ell}V\otimes_A V_y).$$

Thus $d(t)\in A$ and this element is $\mathscr{D}(\sum_q w_q\otimes x_q\otimes y_q\otimes z_q)\,(a)$.

d) Verification of the commutativity of the upper triangle and lower rectangle in the diagram is straightforward and left to the reader.

e) As between (2.2) and (2.3) we identify $\widetilde{U\times_A U}$ with $\int^y {}_yU\otimes_A U_y$ and $\widetilde{V\times_A V}$ with $\int^y {}_yV\otimes_A V_y$. With this identification $\widetilde{\gamma\times\gamma}$ corresponds to the map induced by $\gamma\otimes\gamma$. For $z=\sum_\alpha u_\alpha\otimes u'_\alpha\in\int^y {}_yU\otimes_A U_y$ and $a\in A$, the element $\zeta^\sim(z)\,(a)$ is the unique element b in A such that $i(b)=\sum_\alpha u_\alpha i(a)\,u'_\alpha\in\mathrm{Im}\,i=\int^x {}_xU_x\subset U$. $\zeta^\sim$ for $\widetilde{V\times_A V}$ works similarly. Thus $(\zeta^\sim(\widetilde{\gamma\times\gamma})\,(z))\,(a)$ is the unique element c in A with

$$j(c)=\sum_\alpha\gamma(u_\alpha)j(a)\,\gamma(u'_\alpha).$$

We have

$$\begin{aligned} j(b)=\gamma i(b)=\gamma(\sum_\alpha u_\alpha i(a)\,u'_\alpha)&=\sum_\alpha\gamma(u_\alpha)\,\gamma i(a)\,\gamma(u'_\alpha)\\ &=\sum_\alpha\gamma(u_\alpha)j(a)\,\gamma(u'_\alpha)\end{aligned}$$

which proves that $b=c$. Q.E.D.

The significance of the Ess is captured in part *c)* of the following proposition:

Proposition **(9.4)**. — *Suppose* $(B,\Delta,\mathscr{I},\mathscr{S})$ *is a* $\times_A$*-bialgebra with* Ess *where*

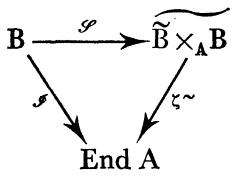
$$\Delta : B\to B\times_A B \quad \textit{and} \quad \mathscr{S} : B\to\widetilde{B\times_A B}$$

are isomorphisms. Furthermore suppose that $A \to B$ *is injective and* $\mathrm{Im}(A \to B) = \int^x {}_x B_x$. *Let* (U, i) *and* (V, j) *be algebras over* A *which are* A*-bimodule isomorphic to* B. *Then*

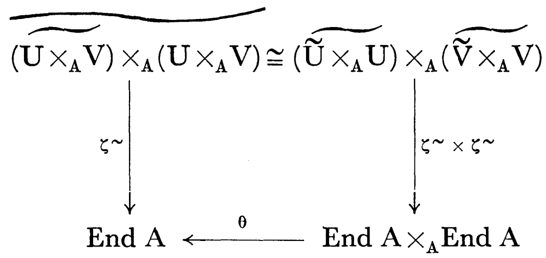
a) $i : A \to U$, $j : A \to V$, $A \xrightarrow{h} U \times_A V$ *are injective, and* $\mathrm{Im}\, i = \int^x {}_x U_x$, $\mathrm{Im}\, j = \int^x {}_x V_x$,

$$\mathrm{Im}(A \to U \times_A V) = \int^x {}_x U \times_A V_x.$$

b) *The diagram commutes*

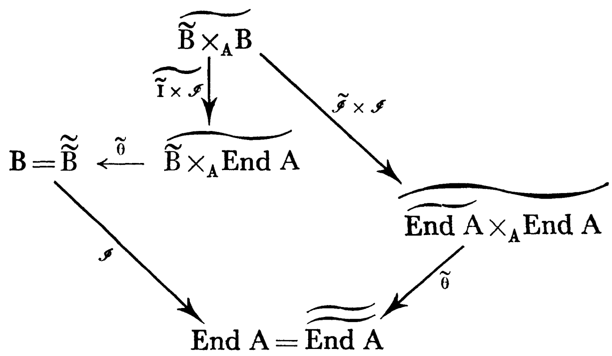


c) *Suppose* A *is a faithful* B*-module* (5.7), i.e. $\mathscr{I}$ *is injective, then there is an* A*-bimodule isomorphism* $\widetilde{\widetilde{(U \times_A V)} \times_A (U \times_A V)} \cong (\widetilde{\widetilde{U} \times_A U}) \times_A (\widetilde{\widetilde{V} \times_A V})$ *making the diagram commute*:

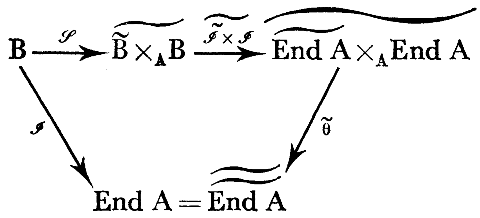


Proof. — *a)* By (9.3), *a)* $i : A \to U$ and $j : A \to V$ are injective and the images are $\int^x {}_x U_x$ and $\int^x {}_x V_x$ respectively. Since by hypothesis $B \cong B \times_A B$ as an algebra over A and $B \times_A B \cong U \times_A V$ as an A-bimodule it follows from (9.3), *a)* that $A \xrightarrow{h} U \times_A V$ has the desired properties.

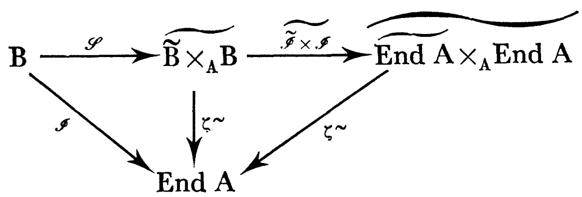
b) The second diagram in (9.2) may be added to the upper left of the commutative diagram:



to give the commutative diagram



By (3.8) the $\widetilde{\theta}:\widetilde{\widetilde{\text{End A}}\times_A\text{End A}}\to\widetilde{\widetilde{\text{End A}}}$ may be replaced by $\zeta^{\sim}$ and then by (9.3), *e)* the right side triangle commutes in the commutative diagram:



The remaining left side triangle is exactly what we wish to show.

c) Consider the first commutative diagram in (9.2). By the present hypotheses Δ and $\mathscr{S}$ are isomorphisms. Thus $\mathscr{S}\times\mathscr{S}$ and $\widetilde{\widetilde{\Delta}\times\Delta}$ are also isomorphisms. Hence, $\mathscr{C}$ and $\mathscr{B}$ have the same image and if we prove that $\mathscr{C}$ is injective then so is $\mathscr{B}$.

Consider the composite map from $\mathscr{B}$ to End A

$$\mathscr{D}\mathscr{C}(\mathscr{S}\times\mathscr{S})\,\Delta.$$

By (9.3), *d)* the composite is the same as the composite

$$\theta(\zeta^{\sim}\times\zeta^{\sim})\,(\mathscr{S}\times\mathscr{S})\,\Delta=\theta((\zeta^{\sim}\mathscr{S})\times(\zeta^{\sim}\mathscr{S}))\,\Delta$$

which by (9.4), *b)* equals

$$\theta(\mathscr{I}\times\mathscr{I})\,\Delta.$$

By the co-unit condition for $\times_A$-coalgebras $\theta(\mathscr{I}\times\mathscr{I})\,\Delta=\mathscr{I}$. Since A is assumed to be a faithful B-module $\mathscr{I}$ is injective and hence $\mathscr{C}$ is injective.

Since $\mathscr{B}$ and $\mathscr{C}$ are determined by A-bimodule structure alone the preceding paragraph shows that if L, M, N, P are A-bimodules which are A-bimodule isomorphic to B then the maps

$$\widetilde{(\widetilde{L\times_A M})\times_A(N\times_A P)}\xrightarrow{\mathscr{B}}\int^r\int^y\int_\ell({}_{y\ell}L_r\otimes_A N_y)\otimes({}_{y\ell}M_r\otimes_A P_y)$$

$$(\widetilde{L\times_A N})\times_A(\widetilde{M\times_A P})\xrightarrow{\mathscr{C}}\int^r\int^y\int_\ell({}_{y\ell}L_r\otimes_A N_y)\otimes({}_{y\ell}M_r\otimes_A P_y)$$

are injective and have the same image. They induce the isomorphism in the diagram of part *c)* where $U=L=N$ and $V=M=P$. The diagram commutes by (9.3), *d)*.

Q.E.D.

10. Simplicity of algebras in $\mathscr{G}\langle\mathfrak{B}\rangle$

The purpose of this section is to prove that if $(\mathfrak{B}, \Delta, \mathscr{I}, \mathscr{S})$ is a cocommutative $\times_A$-bialgebra with Ess where $\mathscr{I}$ is injective and Δ and $\mathscr{S}$ are isomorphisms, then for U an algebra over A with $\langle U\rangle \in \mathscr{G}\langle\mathfrak{B}\rangle$, the map $\widetilde{U}\times_A U \xrightarrow{\zeta} \operatorname{End} A$ is injective and has image $\mathscr{I}(B)$. This will induce an isomorphism $\widetilde{U}\times_A U \cong \widetilde{B}$ of algebras over A. As a consequence such algebras U must be simple algebras when B is simple and satisfies some module theoretic properties.

Lemma (**10.1**). — *Suppose* $(C, \Delta, \mathscr{I})$ *is a* $\times_A$*-coalgebra and* $E = \operatorname{Im} \mathscr{I} \subset \operatorname{End} A$.

a) *If* $f : C \to \operatorname{End} A$ *is an* A*-bimodule map then* $\operatorname{Im} f \subset E$.

In fact there is an A*-bimodule map* $f^0 : C \to C$ *with* $\mathscr{I} f^0 = f$. *If* $\mathscr{I}$ *is injective then of course* f^0 *is uniquely determined by* f.

b) *Suppose* C *is cocommutative,* Δ *is an isomorphism and* $g, h : C \to C$ *are* A*-bimodule maps. Then* $gh = hg : C \to C$. *Moreover* $gh = hg$ *is the same as the composite*

$$C \xrightarrow{\Delta} C\times_A C \xrightarrow{g\times h} C\times_A C \xrightarrow{\Delta^{-1}} C.$$

c) *Suppose* C *is cocommutative,* Δ *is an isomorphism and* M, N, R, S *are* A*-bimodules isomorphic to* C. *Let* $\sigma : M \to R$ *and* $\gamma : N \to S$ *be* A*-bimodule maps. Then*

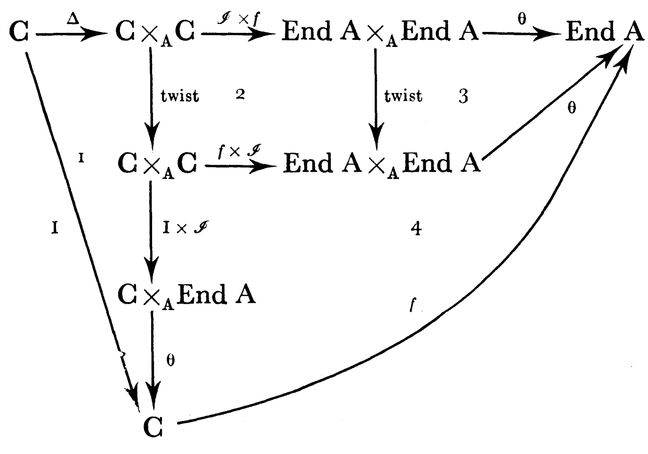
$$\sigma\times\gamma : M\times_A N \to R\times_A S$$

is an isomorphism if and only if both σ *and* γ *are isomorphisms.*

Proof. — *a)* Let f^0 be the composite

$$C \xrightarrow{\Delta} C\times_A C \xrightarrow{I\times f} C\times_A \operatorname{End}_A \xrightarrow{\theta} C.$$

Since all the maps in the composite are A-bimodule maps so is f^0. Then $\mathscr{I} f^0$ is the same as the top row in the diagram:



In the diagram triangle 1 commutes by the co-unit condition for $\times_A$-algebras. Rectangle 2 obviously commutes. Triangle 3 commutes by the remarks between (5.2) and (5.3). It is a straightforward computation to show that region 4 commutes. Therefore the outer diagram commutes and since the top row is the same as $\mathscr{I}f^0$ we have proved part *a)*.

b) In (2.5) the A-bimodule isomorphism $\text{twist}: C\times_A C \to C\times_A C$ is defined. Since Δ is cocommutative $(\text{twist})\Delta = \Delta$. Since Δ is an isomorphism it follows that twist is the identity map from $C\times_A C$ to $C\times_A C$. This gives the third equality in:

$$(*)\left\{\begin{array}{l}\Delta^{-1}(g\times h)\Delta = \Delta^{-1}(g\times I)(I\times h)\Delta = \\ \Delta^{-1}(g\times I)(\text{twist})(h\times I)(\text{twist})\Delta = \\ \Delta^{-1}(g\times I)(h\times I)\Delta = \Delta^{-1}(gh\times I)\Delta.\end{array}\right.$$

Similarly

$$(\substack{*\\*})\left\{\begin{array}{l}\Delta^{-1}(g\times h)\Delta = \Delta^{-1}(I\times h)(g\times I)\Delta = \\ \Delta^{-1}(\text{twist})(h\times I)(\text{twist})(g\times I)\Delta = \\ \Delta^{-1}(h\times I)(g\times I)\Delta = \Delta^{-1}(hg\times I)\Delta.\end{array}\right.$$

The map $C\times_A C \xrightarrow{I\times\mathscr{I}} C\times_A \text{End}\, A \xrightarrow{\theta} C$ is easily checked to be Δ^{-1} using the co-unit condition for C. If $\ell: C\to C$ is an A-bimodule map then

$$({}_*{}^*{}_*)\left\{\begin{array}{l}\Delta^{-1}(\ell\times I)\Delta = \theta(I\times\mathscr{I})(\ell\times I)\Delta \\ \quad = \theta(\ell\times\mathscr{I})\Delta = \ell\theta(I\times\mathscr{I})\Delta \\ \quad = \ell\Delta^{-1}\Delta = \ell.\end{array}\right.$$

Putting together $(*)$, $(\substack{*\\*})$ and $({}_*{}^*{}_*)$ gives part *b)*.

c) The "if" follows from (2.4), 1). To prove the "only if", M, N, R and S may all be replaced by C since they are assumed to be A-bimodule isomorphic to C. Thus we may assume that $\sigma, \gamma: C\to C$ are A-bimodule maps with $\sigma\times\gamma: C\times_A C \to C\times_A C$ an isomorphism. Then the composite of isomorphisms $\Delta^{-1}(\sigma\times\gamma)\Delta$ is an isomorphism and by part *b)* it follows that $\sigma\gamma = \gamma\sigma$ is an isomorphism. This implies that both σ and γ are isomorphisms. Q.E.D.

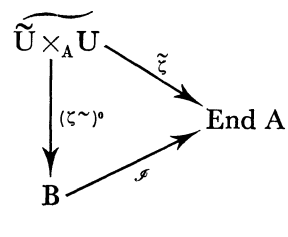
It follows from part *a)* that if M is an A-bimodule which is isomorphic to C and $f: M \to \text{End}\, A$ is an A-bimodule map then there is an A-bimodule map $f^0: M\to C$ with $\mathscr{I}f^0 = f$. And of course f^0 is uniquely determined by f if $\mathscr{I}$ is injective. We apply this to the following situation:

Theorem **(10.2)**. — *Let* $(\mathfrak{B}, \Delta, \mathscr{I}, \mathscr{S})$ *be a* $\times_A$*-bialgebra with* Ess *where* $\mathscr{I}$ *is injective and* $\mathscr{S}$ *is an isomorphism.*

a) *Suppose* U *is an algebra over* A *which is* A*-bimodule isomorphic to* B. *Then* $\widetilde{\tilde{U}\times_A U}$ *is* A*-bimodule isomorphic to* $\mathfrak{B}$. *If* $\zeta^\sim: \widetilde{\tilde{U}\times_A U} \to \text{End}\, A$ *is as defined in* (9.3), d) *then* $(\zeta^\sim)^0: \widetilde{\tilde{U}\times_A U} \to \mathfrak{B}$ *is a map of algebras over* A *where* $\mathscr{I}(\zeta^\sim)^0 = \zeta^\sim$.

b) *Suppose* Δ *is an isomorphism and* U *and* V *are algebras over* A *which are* A-*bimodule isomorphic to* B *and where* $U\times_A V\cong B$ *as an algebra over* A. *Then* $(\zeta^\sim)^0 : \widetilde{\tilde{U}\times_A U}\to B$ *is an isomorphism of algebras over* A.

Proof. — *a)* Since U is A-bimodule isomorphic to B, $\widetilde{\tilde{U}\times_A U}$ is A-bimodule isomorphic to $\widetilde{\tilde{B}\times_A B}$ which is A-bimodule isomorphic to B since $\mathscr{I}$ is an isomorphism. Thus by the lines just above (10.2) there is a unique A-bimodule map $(\zeta^\sim)^0$ making the diagram commute.



Since $\mathscr{I}$ and $\zeta^\sim$ are maps of algebras over A and $\mathscr{I}$ is injective it follows that $(\zeta^\sim)^0$ is a map of algebras over A.

b) Since Δ is an isomorphism and $\mathscr{I}$ is injective B is cocommutative by (6.9). As observed in part *a),* $\widetilde{\tilde{U}\times_A U}\cong B$ as an A-bimodule. Similarly $\widetilde{\tilde{V}\times_A V}\cong B$ as an A-bimodule. Thus by (10.1), *c)* it follows that $(\zeta^\sim)^0 : \widetilde{\tilde{U}\times_A U}\to B$ is an isomorphism if the map

$$(\widetilde{\tilde{U}\times_A U})\times_A(\widetilde{\tilde{V}\times_A V}) \xrightarrow{(\zeta^\sim)^0\times(\zeta^\sim)^0} B\times_A B$$

is an isomorphism. Since Δ is an isomorphism it suffices to prove that

$$(\widetilde{\tilde{U}\times_A U})\times_A(\widetilde{\tilde{V}\times_A V}) \xrightarrow{(\zeta^\sim)^0\times(\zeta^\sim)^0} B\times_A B \xrightarrow{\Delta^{-1}} B$$

is an isomorphism. Since $\mathscr{I}$ is injective it suffices to proves that the map

$$(*)\qquad (\widetilde{\tilde{U}\times_A U})\times_A(\widetilde{\tilde{V}\times_A V}) \xrightarrow{(\zeta^\sim)^0\times(\zeta^\sim)^0} B\times_A B \xrightarrow{\Delta^{-1}} B \xrightarrow{\mathscr{I}} \text{End } A$$

is injective with image Im $\mathscr{I}$. As mentioned between $(^*_*)$ and $({}_*{}^*{}_*)$ in the proof of (10.1) $\Delta^{-1}=\theta(I\times\mathscr{I})$. From this it is easily shown that $\mathscr{I}\Delta^{-1}=\theta(\mathscr{I}\times\mathscr{I})$. Thus the composite $(*)$ above becomes

$$(\widetilde{\tilde{U}\times_A U})\times_A(\widetilde{\tilde{V}\times_A V}) \xrightarrow{(\zeta^\sim)^0\times(\zeta^\sim)^0} \mathfrak{B}\times_A B \xrightarrow{\mathscr{I}\times\mathscr{I}} \text{End } A\times_A\text{End } A \xrightarrow{\theta} \text{End } A$$
$$= (\widetilde{\tilde{U}\times_A U})\times_A(\widetilde{\tilde{V}\times_A V}) \xrightarrow{(\mathscr{I}(\zeta^\sim)^0)\times(\mathscr{I}(\zeta^\sim)^0)} \text{End } A\times_A\text{End } A \xrightarrow{\theta} \text{End } A.$$

Since $\mathscr{I}(\zeta^\sim)^0=\zeta^\sim$ by part *a),* what we must prove is that

$$(\widetilde{\tilde{U}\times_A U})\times_A(\widetilde{\tilde{V}\times_A V}) \xrightarrow{\zeta^\sim\times\zeta^\sim} \text{End } A\times_A\text{End } A \xrightarrow{\theta} \text{End } A$$

is injective and has image Im $\mathscr{I}$.

By (9.4), *c)* it suffices to prove that

$$\widetilde{(\widetilde{\widetilde{U}\times_A V})\times_A(\widetilde{U\times_A V})} \xrightarrow{\zeta^\sim} \text{End A}$$

is injective and has image Im $\mathscr{I}$. By hypothesis $U\times_A V\cong B$ as an algebra over A so that it suffices to prove that

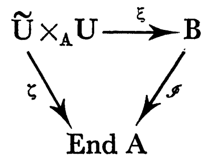
$$\widetilde{\widetilde{B}\times_A B} \xrightarrow{\zeta^\sim} \text{End A}$$

is injective and has image Im $\mathscr{I}$. This follows from (9.4), *b)*. Q.E.D.

We conclude the section by studying the consequences of (10.2), *b)*. Notice that the composite

$$\widetilde{U}\times_A U \xrightarrow{\sim} \widetilde{\widetilde{U}\times_A U} \xrightarrow{(\zeta^\sim)^0} B$$

is an anti-homomorphism of algebras over A. And if we denote this composite by ξ then the diagram:



commutes.

If ξ is an isomorphism then ζ is injective if $\mathscr{I}$ is injective. In other words U is Jake (3.5). Moreover A is a simple $\widetilde{U}\times_A U$-module if and only if A is a simple B-module.

Theorem **(10.3)**. — *Let* $(\mathfrak{B}, \Delta, \mathscr{I}, \mathscr{S})$ *be a* $\times_A$*-bialgebra with* Ess *where* $\mathscr{I}$ *is injective and* Δ *and* $\mathscr{S}$ *are isomorphisms. Furthermore assume that* B *is flat as a left (right)* A*-module and* $0\neq\widetilde{M}\times_A B$ $(0\neq\widetilde{B}\times_A M)$ *for any* A*-bimodule* $M\subset B$. *The following statements are equivalent*:

a) A *is a simple* $\mathfrak{B}$*-module*;
b) B *is a simple algebra*;
c) *if* U *is any algebra over* A *with* $\langle U\rangle\in\mathscr{G}\langle B\rangle$, *then* U *is a simple algebra.*

Proof. — By (10.2), *b)* and the lines just above this theorem A is a simple $\widetilde{U}\times_A U$-module if and only if A is a simple B-module. Since $U\cong B$ as an A-bimodule it follows from (3.7) that *a)* implies *c)*.

Since we may choose $U=B$ in *c)* it follows that *c)* implies *b)*. By (3.9) *b)* implies *a)*. Q.E.D.

Notice that (9.3), *a)* shows that the centers of the algebras in $\mathscr{G}\langle B\rangle$ are all the same.

When the map ξ just above (10.3) is an anti-isomorphism of algebras over A it follows that $\widetilde{U} \times_A U \cong B$ as an algebra over A. When $\mathscr{S}$ is an isomorphism it follows that $\widetilde{B} \times_A B = \widetilde{\widetilde{\widetilde{B} \times_A B}} \cong \widetilde{B}$ as algebras over A. And in fact from the second commutative diagram in (9.2) it follows that

$$\widetilde{B} \times_A B \xrightarrow{I \times \mathscr{S}} \widetilde{B} \times_A \text{End } A \xrightarrow{\theta} \widetilde{B}$$

is an isomorphism of algebras over A.

Proposition **(10.4)**. — *Suppose* B *is a subalgebra over* A *of* End A *and* B *is idempotent and associative as an algebra over* A. *Suppose* $(\widetilde{B}, B, B)$ *associates as* A-*bimodules and* $\widetilde{B} \times_A B \xrightarrow{\theta} \widetilde{B}$ *is an isomorphism of algebras over* A. *Let* U *be an algebra over* A *where* $\widetilde{U} \times_A U \cong \widetilde{B}$ *as an algebra over* A.

1. *If* $\langle U \rangle \in \mathscr{G}\langle B \rangle$, *then* $\widetilde{U} \cong \widetilde{B} \times_A U^{-1}$ *as algebras over* A.
2. *If* $B \cong \widetilde{B}$ *as an algebra over* A *and* $U \cong B$ *as* A-*bimodules, then both* $\langle U \rangle$ *and* $\langle \widetilde{U} \rangle$ *lie in* $\mathscr{G}\langle B \rangle$ *and* $\langle \widetilde{U} \rangle = \langle U \rangle^{-1}$.
3. *Suppose* $(B, \Delta, \iota, \mathscr{S})$ *is a* $\times_A$-*bialgebra with* Ess *where* $\mathscr{S}$ *and* Δ *are isomorphisms and* $B \cong \widetilde{B}$ *as algebras over* A. *Then* $\langle \widetilde{W} \rangle = \langle W \rangle^{-1}$ *for* $\langle W \rangle \in \mathscr{G}\langle B \rangle$.

Proof. — U^{-1} in part 1 denotes an algebra over A where $U^{-1} \times_A U \cong B$ as algebras over A and $U^{-1} \cong B$ as A-bimodule. By (4.9) $U^{-1} \times_A B \cong U^{-1}$ as algebras over A automatically holds. Applying " $\times_A U^{-1}$ " to both sides of $\widetilde{U} \times_A U \cong \widetilde{B}$ yields

$$(\widetilde{U} \times_A U^{-1}) \times_A U \cong \widetilde{B} \times_A U^{-1}$$

as algebras over A. By the associativity isomorphism (2.6) the left hand side is isomorphic to $\widetilde{U} \times_A (U^{-1} \times_A U) \cong \widetilde{U} \times_A B$ as algebras over A. The map $\widetilde{U} \times_A B \xrightarrow{\theta} \widetilde{U}$ is a map of algebras over A. Bijectivity of θ depends on the A-bimodule structure of $\widetilde{U}$ and not on the algebra structure. Thus the assumption that $\widetilde{B} \times_A B \xrightarrow{\theta} \widetilde{B}$ is bijective implies that $\theta : \widetilde{U} \times_A B \to \widetilde{U}$ is an equivalence of algebras over A. This proves part 1.

The assumption $B \cong \widetilde{B}$ as algebras over A implies that $\widetilde{U} \cong B$ as A-bimodules. Hence by (4.9) $\langle U \rangle \langle B \rangle = \langle U \rangle$ and $\langle \widetilde{U} \rangle \langle B \rangle = \langle \widetilde{U} \rangle$. By assumption

$$\langle \widetilde{U} \rangle \langle U \rangle = \langle \widetilde{B} \rangle = \langle B \rangle.$$

This proves part 2. Part 3 follows from part 2 and (10.2), *b)*. Q.E.D.

Notice that part 3 gives the usual Brauer Group relation between opposite algebras and inverse classes. See (12.4), *b)*.

11. Existence of the Ess

Some results are developed which can be used to ascertain when a $\times_A$-bialgebra has an Ess.

Lemma (**11.1**). — *Suppose* C *is a* $\times_A$*-coalgebra and* D *is an* A*-coalgebra. Then* $C\otimes_A D$ *has an* A*-coalgebra structure with diagonal*

$$C\otimes_A D \xrightarrow{\Delta\otimes\Delta} \int_x (C\times_A C)_x \otimes_x D\otimes_x D$$
$$\xrightarrow{\varphi} \int_x {}_xC\otimes_A D\otimes_x C\otimes_A D$$

where φ *is defined in* (2.9). *The co-unit of* $C\otimes_A D$ *is given by*

$$C\otimes_A D \xrightarrow{\mathscr{S}\otimes\varepsilon} \mathrm{End}\,A\otimes_A A = \mathrm{End}\,A \xrightarrow{\epsilon} A.$$

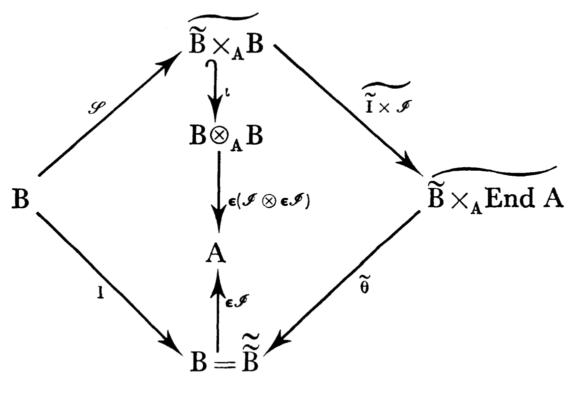
If C *and* D *are cocommutative then so is* $C\otimes_A D$. *If* D *is actually a* $\times_A$*-coalgebra then the diagonal map on* $C\otimes_A D$ *actually has image in* $(C\otimes_A D)\times_A(C\otimes_A D)$.

Proof. — Left to the reader.

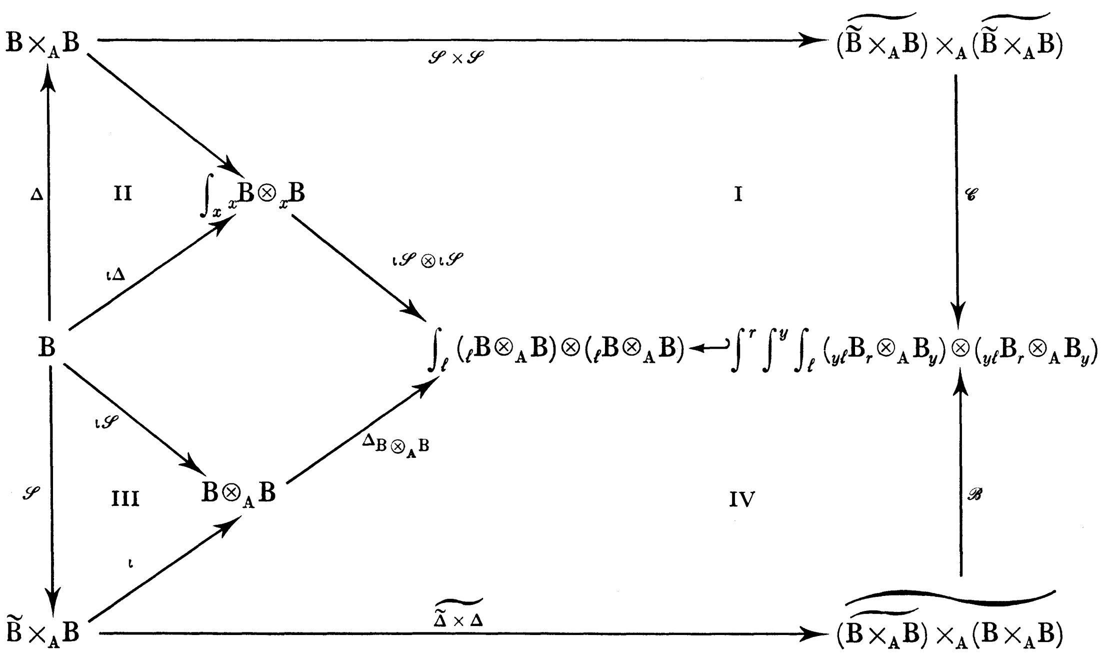
Note. — The coalgebra structure on $C\otimes_A D$ actually uses the fact that C is a $\times_A$-coalgebra. The coalgebra structure on $C\otimes_A D$ is *not* the usual tensor product coalgebra structure as defined in [17, page 49].

Proposition (**11.2**). — *Let* (B, Δ, ι) *be a* $\times_A$*-bialgebra. We consider* B *as an* A*-coalgebra as in (or just after)* (5.4), *and consider* $B\otimes_A B$ *as an* A*-coalgebra by* (11.1). *Let* $\mathscr{S}: B\to\widetilde{\tilde{B}\times_A B}$ *be a map of algebras over* A. *Then* $\mathscr{S}$ *is an* Ess *for* B *if and only if* $\iota\mathscr{S}: B\to B\otimes_A B$ *is a map of* A*-coalgebras, where* ι *is the inclusion* $\widetilde{\tilde{B}\times_A B}\xrightarrow{\iota} B\otimes_A B$ *defined above* (2.3).

Proof. — In the diagram below the outer diamond, $\diamondsuit$, is the second diagram in (9.2). The left triangle, $\lhd$, expresses the fact that $\iota\mathscr{S}$ preserves the co-units. It is left to the reader to show that the right triangle, $\rhd$, commutes. Thus preserving co-units is equivalent to the second diagram in (9.2) commuting.



In the diagram below the outer rectangle, $\square$, is the first diagram in (9.2). The inner diamond, $\diamondsuit$, expresses the fact that $\iota\mathscr{S}$ preserves diagonalizations. It is left to the reader to show that the side regions I, II, III, IV commute. Thus $\mathscr{S}$ preserving diagonalization is equivalent to the first diagram in (9.2) commuting. (The maps $\mathscr{B}$ and $\mathscr{C}$ are defined in (9.1).)



Q.E.D.

Lemma (**11.3**). — *Let* D *and* D′ *be sub-*A*-bimodules of* End A *where*

$$\Lambda : D\otimes_A D' \to \mathrm{Hom}(A, D)$$

is injective (1.4). *Consider* $\widetilde{\tilde{D}\times_A D'} \subset D\otimes_A D'$ *as above* (2.3).

a) *For* $x=\sum_i d_i\otimes d_i' \in D\otimes_A D'$, *x lies in* $\widetilde{\tilde{D}\times_A D'}$ *if and only if*

$$\sum_i bd_i(ad_i'(c)) = \sum_i d_i(ad_i'(bc))$$

for all $a, b, c \in A$.

b) $\widetilde{\tilde{D}\times_A D'} \xrightarrow{\tilde{\theta}} \tilde{\tilde{D}} = D$ *is injective.*

Proof. — Part *b)* follows from (1.6), 2). The proof of part *a)* is similar to the proof of (6.4), 3) and is left to the reader. Q.E.D.

As observed previously, when D is a subalgebra over A of End A then

$$\tilde{\theta} : \widetilde{\tilde{D}\times_A D} \to \tilde{\tilde{D}} = D$$

is a map of algebras over A. Hence if $\widetilde{\theta}$ is an isomorphism, then $\widetilde{\theta}^{-1}$ is an isomorphism of algebras over A. This raises the question of when $\widetilde{\theta}^{-1}$ is an Ess.

If $D \subset \mathrm{End}\, A$ is a subalgebra over A and (D, Δ, ι) is a $\times_A$-bialgebra with $\mathscr{S}$ as Ess then $\widetilde{\theta}\mathscr{S} = I$. This follows from the second diagram in the definition of Ess (9.2). If $\widetilde{\theta}$ is injective it follows from $\widetilde{\theta} = I$ that $\widetilde{\theta}$ is an isomorphism and $\mathscr{S}$ is uniquely determined as $\widetilde{\theta}^{-1}$.

Proposition **(11.5)**. — *Suppose* C *is a sub*-A-*bimodule of* End A *where* (C, Δ, ι) *is a* $\times_A$-*coalgebra and* $\widetilde{\theta} : \widetilde{\widetilde{C} \times_A C} \to \widetilde{\widetilde{C}} = C$ *is an* A-*bimodule isomorphism. Then the composite*

$$C \xrightarrow{\widetilde{\theta}^{-1}} \widetilde{\widetilde{C} \times_A C} \xrightarrow{\iota} C \otimes_A C$$

is a coalgebra map if and only if $\widetilde{\widetilde{C} \times_A C}$ *is a subcoalgebra of* $C \otimes_A C$. *If* C *is actually a* $\times_A$-*bialgebra, then* $\widetilde{\theta}^{-1}$ *is an* Ess *if and only if* $\widetilde{\widetilde{C} \times_A C}$ *is a subcoalgebra of* $C \otimes_A C$.

Remark. — Generally speaking there are difficulties in dealing with *subcoalgebras* when working over rings. However the map $C \otimes_A C \xrightarrow{I \otimes \epsilon} C$ satisfies $(I \otimes \epsilon) \mid \widetilde{\widetilde{C} \times_A C} = \widetilde{\theta}$. Since $\widetilde{\theta}$ is assumed to be an isomorphism it follows that $C \otimes_A C = \mathrm{Ker}(I \otimes \epsilon) \oplus \widetilde{\widetilde{C} \times_A C}$ as a direct sum of left A-modules. This gives injectivity of the map

$$\int_x {}_x(\widetilde{\widetilde{C} \times_A C}) \otimes_x (\widetilde{\widetilde{C} \times_A C}) \xrightarrow{\iota \otimes \iota} \int_x ({}_x C \otimes_A C) \otimes ({}_x C \otimes_A C).$$

If this map is taken for an identification then $\widetilde{\widetilde{C} \times_A C}$ is considered a subcoalgebra of $C \otimes_A C$ if under the diagonalization of $C \otimes_A C$ the submodule $\widetilde{\widetilde{C} \times_A C}$ is carried to $\int_x {}_x(\widetilde{\widetilde{C} \times_A C}) \otimes_x (\widetilde{\widetilde{C} \times_A C})$. This induces the subcoalgebra diagonalization on $\widetilde{\widetilde{C} \times_A C}$. The co-unit of $\widetilde{\widetilde{C} \times_A C}$ is the restriction of the co-unit of $C \otimes_A C$.

Proof. — Suppose C is a $\times_A$-bialgebra. As observed above $\widetilde{\theta}^{-1}$ is a map of algebras over A. By (11.2) $\widetilde{\theta}^{-1}$ is an Ess if and only if $\iota\widetilde{\theta}^{-1}$ a map of A-coalgebras.

Now assume that C is merely a $\times_A$-coalgebra. Clearly if $\iota\widetilde{\theta}^{-1}$ is a coalgebra map then $\mathrm{Im}\, \iota\widetilde{\theta}^{-1} = \widetilde{\widetilde{C} \times_A C}$ is a subcoalgebra of $C \otimes_A C$. Conversely suppose $\widetilde{\widetilde{C} \times_A C}$ is a subcoalgebra of $C \otimes_A C$. It is easily checked that $C \otimes_A C \xrightarrow{I \otimes \epsilon} C$ is a coalgebra map. This map restricted to $\widetilde{\widetilde{C} \times_A C}$ is $\widetilde{\theta}$. Then $\widetilde{\theta}$ is a coalgebra map and so is $\widetilde{\theta}^{-1} : C \to \widetilde{\widetilde{C} \times_A C}$. Hence $\iota\widetilde{\theta}^{-1}$ is a coalgebra map. Q.E.D.

Suppose C is merely a sub-A-bimodule of End A. Let

$$\Omega : \int_x {}_x C \otimes_A C \otimes_x C \otimes_A C \to \mathrm{Hom}(A \otimes A \otimes A \otimes A, A)$$

be determined by

$$\Omega(c_1\otimes c_2\otimes c_3\otimes c_4)\,(a_1\otimes a_2\otimes a_3\otimes a_4)$$
$$=(c_1(a_1c_2(a_2)))\,(c_3(a_3c_4(a_4))), \quad \text{for} \quad \{c_i\}_1^4\subset C, \quad \{a_i\}_1^4\subset A.$$

Lemma (**11.6**). — *Suppose* C *is a sub-*A*-bimodule of* End A.

1. *If* $s : C\to\widetilde{\widetilde{C}\times_A C}$ *is an* A*-bimodule map with* $\widetilde{\theta}s=I$, *then, for* $c\in C$ *with*

$$s(c)=\sum_i c_i\otimes c_i'\in\widetilde{\widetilde{C}\times_A C}\subset C\otimes_A C \quad \text{and} \quad a, b\in A,$$

(i) $\sum_i c_ic_i'(a)^\ell=ac$

(ii) $\sum_i c_i(bc_i'(a))=ac(b)$.

2. *Suppose* (C, Δ, ι) *is a* $\times_A$*-coalgebra and* $\widetilde{\theta} : \widetilde{\widetilde{C}\times_A C}\to\widetilde{\widetilde{C}}=C$ *is an isomorphism. If* Ω *is injective then* $\iota\widetilde{\theta}^{-1} : C\to C\otimes_A C$ *is a coalgebra map.*

3. *Suppose* C *is a subalgebra over* A *of* End A *and* (C, Δ, ι) *makes* C *into a* $\times_A$*-bialgebra. If* Ω *is injective and* $\widetilde{\theta} : \widetilde{\widetilde{C}\times_A C}\to\widetilde{\widetilde{C}}=C$ *is an isomorphism then* $\widetilde{\theta}^{-1}$ *is an* Ess *for* C.

Proof. — 1. With the notation of part 1

$$\sum_i c_ic_i'(a)^\ell=\sum_i ac_ic_i'(1)^\ell=a\widetilde{\theta}s(c)=ac.$$

The first equality follows from the fact that $\sum_i c_i\otimes c_i'\in\int^y {}_yC\otimes_A C_y$. The last equality follows from the assumption $\widetilde{\theta}s=I$. This proves 1), (i). 1), (ii) follows immediately from 1), (i).

2. Suppose $c\in C$ and $\widetilde{\theta}^{-1}(c)=\sum_i c_i\otimes c_i'\in\widetilde{\widetilde{C}\times_A C}\subset C\otimes_A C$. Then

$$c=\widetilde{\theta}(\sum_i c_i\otimes c_i')=\sum_i c_i\epsilon(c_i') \quad \text{and so} \quad \epsilon(c)=\sum_i \epsilon(c_i\epsilon(c_i')).$$

This shows that $\iota\widetilde{\theta}^{-1}$ preserves co-units. Thus it remains to verify commutativity of the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \iota\widetilde{\theta}^{-1}\ } & C\otimes_A C \\ \Big\downarrow{\scriptstyle \iota\Delta} & & \Big\downarrow{\scriptstyle \varphi(\Delta\otimes\Delta)} \\ \int_x {}_xC\otimes_x C & \xrightarrow{\ \iota\widetilde{\theta}^{-1}\otimes\iota\widetilde{\theta}^{-1}\ } & \int_x ({}_xC\otimes_A C)\otimes({}_xC\otimes_A C) \end{array}$$

where the right hand vertical map is the coalgebra structure on $C\otimes_A C$, see (11.1).

By injectivity of Ω it suffices to show that for $c\in C$

$$\Omega(\iota\widetilde{\theta}^{-1}\otimes\iota\widetilde{\theta}^{-1})\iota\Delta(c)=\Omega\varphi(\Delta\otimes\Delta)\iota\widetilde{\theta}^{-1}(c).$$

This is done by applying each side to $a_1 \otimes a_2 \otimes a_3 \otimes a_4 \in A \otimes A \otimes A \otimes A$. The result in each case is $a_2 a_4 c(a_1 a_3)$. The calculation uses part 1 and (5.8), *c)* and is left to the reader.

3. Since $\widetilde{\theta}$ is a map of algebras over A so is $\widetilde{\theta}^{-1}$. Then by part 2 and (11.2) it follows that $\widetilde{\theta}^{-1}$ is an Ess for C. Q.E.D.

The map Ω admits many different factorings. Here is one:
Let f_1 be the Λ-map

$$\int_x {}_x C \otimes_A C \otimes_x C \otimes_A C = \int_y \left(\int_x {}_x C \otimes_A C \otimes_x C_y \right) \otimes_y C$$
$$\xrightarrow{\Lambda} \operatorname{Hom}\left(A, \int_x {}_x C \otimes_A C \otimes_x C\right).$$

Let f_2 be the Λ'-map

$$\int_x {}_x C \otimes_A C \otimes_x C \to \operatorname{Hom}(A, C \otimes_A C).$$

Let f_3 be the Λ-map

$$C \otimes_A C \xrightarrow{\Lambda} \operatorname{Hom}(A, C).$$

Let f_4 be the inclusion

$$C \xrightarrow{\iota} \operatorname{Hom}(A, A).$$

Note Λ is defined in (1.4) and Λ' is defined in (6.2).
Then Ω admits the factorization

$$\begin{aligned} \int_x {}_x C \otimes_A C \otimes_x C \otimes_A C &\xrightarrow{f_1} \operatorname{Hom}\left(A, \int_x {}_x C \otimes_A C \otimes_x C\right) \\ &\xrightarrow{\operatorname{Hom}(I, f_2)} \operatorname{Hom}(A, \operatorname{Hom}(A, C \otimes_A C)) \\ &\xrightarrow{\operatorname{Hom}(I, \operatorname{Hom}(I, f_3))} \operatorname{Hom}(A, \operatorname{Hom}(A, \operatorname{Hom}(A, C))) \\ &\xrightarrow{\operatorname{Hom}(I, \operatorname{Hom}(I, \operatorname{Hom}(I, f_4)))} \operatorname{Hom}(A, \operatorname{Hom}(A, \operatorname{Hom}(A, \operatorname{Hom}(A, A)))) \\ &\cong \operatorname{Hom}(A \otimes A \otimes A \otimes A, A) \end{aligned}$$

where the last isomorphism is the appropriate adjointness, resulting from the relation $\operatorname{Hom}(X \otimes Y, Z) = \operatorname{Hom}(X, \operatorname{Hom}(Y, Z))$.

By left exactness of Hom it follows that Ω is injective if f_1, f_2, f_3 and f_4 are injective. Of course f_4 is injective. Conditions for f_1, f_2 and f_3 to be injective are given by (1.5).

Lemma (**11.7**). — *Suppose* C *is a left sub-*A*-module of* End A. *If* C *is a sub-*A*-bimodule of* End A *and* $\Lambda : M \otimes_A C \to \operatorname{Hom}(A, M)$ *is injective for all right* A*-modules* M *then*

$$\Omega : \int_x {}_x C \otimes_A C \otimes_x C \otimes_A C \to \operatorname{Hom}(A \otimes A \otimes A \otimes A, A)$$

is injective.

Proof. — If all Λ-maps for C are injective then so are all Λ'-maps. Thus by the factoring of Ω above this lemma it follows that Ω is injective. Q.E.D.

12. Examples of $\times_A$-bialgebras with Ess

Consider End A as an $A\otimes A$-module and let $\{L_\alpha\}$, $\{C_\alpha\}$, C be as above (6.6). In (6.6) it is shown that C is a $\times_A$-coalgebra and C is a $\times_A$-bialgebra if it happens to be a subalgebra over A of End A.

Let twist : $A\otimes A \to A\otimes A$, $a_1\otimes a_2 \mapsto a_2\otimes a_1$.

Theorem **(12.1)**. — a) *Let* N *be an* A-*bimodule and hence an* $A\otimes A$-*module. Suppose there is* L_α *with* $(\text{twist}(L_\alpha)).N = 0$. *Then the maps* $\widetilde{N}\times_A C_\alpha \xrightarrow{\theta} \widetilde{N}$ *and* $\widetilde{N}\times_A C \xrightarrow{\theta} \widetilde{N}$ *are isomorphisms.*

Assume that for each L_α *there is an* L_β *where* $\text{twist}(L_\beta)\subset L_\alpha$.

b) *Then* $\widetilde{\widetilde{C}\times_A C} \xrightarrow{\tilde\theta} \widetilde{\widetilde{C}} = C$ *is an isomorphism and* $\iota\tilde\theta^{-1}$ *is a coalgebra map. If* C *is a subalgebra over* A *of* End A *and hence a* $\times_A$-*bialgebra, then* $\tilde\theta^{-1}$ *is an* Ess *for* C.

c) *If* M *is any sub*-A-*bimodule of* C, *then* $\widetilde{M}\times_A C \xrightarrow{\theta} \widetilde{M}$ *is an isomorphism. Hence* $\widetilde{M}\times_A C \neq 0$ *if* $M\neq 0$.

d) *If* C *is a* $\times_A$-*bialgebra, then the following statements are equivalent*:

(i) A *is a simple* C-*module.*
(ii) C *is a simple algebra.*
(iii) *If* U *is any algebra over* A *with* $\langle U\rangle \in \mathscr{G}\langle C\rangle$, *then* U *is a simple algebra.*

e) *If* C *is a* $\times_A$-*bialgebra and for each ideal* $0\neq I \subsetneqq A$ *there is an* L_α *with*

$$A\otimes I \not\subset I\otimes A + L_\alpha,$$

then A *is a simple* C-*module.*

f) *The center of* C *lies in* A^ℓ *and is*

$$(\{a\in A \mid 1\otimes a - a\otimes 1 \in \bigcap_\alpha L_\alpha\})^\ell.$$

Proof. — *a)* Since $L_\alpha.\widetilde{N} = 0$ is equivalent to $(\text{twist}(L_\alpha)).N = 0$, part a follows from (6.6), *b)*.

c) By (2.12), *e)* the map $\widetilde{M}\times_A C \xrightarrow{\theta} \widetilde{M}$ is injective and it suffices to prove that θ is surjective. Consider the diagram

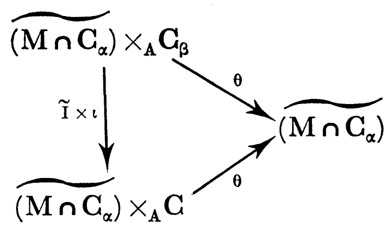
$$\begin{array}{ccc}
\widetilde{(M\cap C_\alpha)}\times_A C & \xrightarrow{\theta} & \widetilde{M\cap C_\alpha} \\
\Big\downarrow {\scriptstyle \tilde\iota\times 1} & & \Big\downarrow {\scriptstyle \tilde\iota} \\
\widetilde{M}\times_A C & \xrightarrow{\theta} & \widetilde{M}
\end{array}$$

Since C is the union of the (C_α)'s it follows that M is the union of the $\widetilde{(M \cap C_\alpha)}$'s. Thus the above diagram shows that it suffices to prove that the map $\widetilde{(M\cap C_\alpha)} \times_A C \xrightarrow{\theta} \widetilde{M \cap C_\alpha}$ is surjective.

Choose L_β where $\text{twist}(L_\beta) \subset L_\alpha$. Then $(\text{twist}(L_\beta)).C_\alpha = 0$ and so

$$(\text{twist}(L_\beta)).(M \cap C_\alpha) = 0.$$

By part *a)* this implies that the top θ in the diagram below is an isomorphism



Hence the bottom θ is surjective and part *c)* is proved.

b) By part *c)* the map $\tilde{C}\times_A C \xrightarrow{\theta} \tilde{C}$ is an isomorphism. Hence $\tilde{\theta}$ is an isomorphism. Since all Λ-maps for C are injective, (11.7) and (11.6), 2) imply that $\iota\tilde{\theta}^{-1}$ is a coalgebra map. Moreover if C is a subalgebra over A of End A then by (11.6), 3) $\tilde{\theta}^{-1}$ is an Ess for C.

d) Follows from (10.3).

e) Let I be an ideal in A with $A\otimes I \not\subset I\otimes A + L_\alpha$. Let ρ be the composite $A\otimes I \to A\otimes A \to (A\otimes A)/L_\alpha$. Then $\text{Im}\,\rho \not\subset I.((A\otimes A)/L_\alpha)$. Since $(A\otimes A)/L_\alpha$ is a projective left A-module there is a left A-module map $F : (A\otimes A)/L_\alpha \to A$ with $F(\text{Im}\,\rho) \not\subset I$. By (2.12), *a)*, F arises from an $f \in C_\alpha$ and it is left to the reader to show that $f(I) \not\subset I$. Thus no proper ideal of A is C-stable. Since $A^\ell \subset C$ any C-submodule of A must be an ideal. This proves part *e)*.

f) Since A^ℓ is a maximal commutative subring of C or End A, the center of C lies in A^ℓ. If $1\otimes a - a\otimes 1 \in L_\alpha$ for each L_α, then $(1\otimes a - a\otimes 1).c = 0$ for all $c \in C$. This means that $a^\ell c = c a^\ell$ and a^ℓ lies in the center of C. Conversely suppose that for some L_α, $1\otimes a - a\otimes 1 \notin L_\alpha$. Then the image z of $1\otimes a - a\otimes 1$ under $A\otimes A \to (A\otimes A)/L_\alpha$ is not zero. Since $(A\otimes A)/L_\alpha$ is a projective left A-module there is a left A-module map $G : (A\otimes A)/L_\alpha \to A$ with $G(z) \neq 0$. By (2.12), *a)* G arises from $g \in C_\alpha$ and it is left to the reader to show that $(1\otimes a - a\otimes 1).g \neq 0$. Thus $a^\ell g \neq g a^\ell$ and a^ℓ does not lie in the center of C. Q.E.D.

Remark **(12.2)**. — Suppose A has almost finite projective differentials, and let $\{L_\alpha\}$ be as in (8.5). As explained above (8.7) the C which arises from $\{L_\alpha\}$ is equal to D_A and is a $\times_A$-bialgebra. Since $\text{twist}(1\otimes a - a\otimes 1) = -(1\otimes a - a\otimes 1)$ it follows that $\text{twist}(\mathfrak{M}^n) \subset \mathfrak{M}^n$. Since $\{L_\alpha\}$ is cofinal with $\{\mathfrak{M}^n\}$ it follows that given L_α there is L_β with $\text{twist}(L_\beta) \subset L_\alpha$. Thus (12.1) applies with $D_A = C$.

Remark **(12.3)**. — End A arises as C in (12.1) as shown above (8.8). The one element set of ideals $\{0\}$ satisfies $\text{twist}(0)\subset 0$. Thus (12.1) applies to End A when A is a finite projective R-module. Suppose for convenience that A is a faithful R-module. Then A is a simple End A-module if and only if R is a field. The center of End A is R.

In the beginning of Section 9 it is shown how an Ess may arise from a $\times_A$-antipode. Such an $\mathcal{S}$ is given for $A\#H$ at the end of this section. What we show now is that " often " End A has no $\times_A$-antipode since " often " $\text{End A} \ncong \widetilde{\text{End A}}$ as *algebras over* A.

Theorem **(12.4)**. — *Suppose* A *is a commutative* R-*algebra which is a finite projective* R-*module.*

a) *If* R *is a field and* $\text{End A}\cong\widetilde{\text{End A}}$ *as algebras over* A, *then* A *is a Frobenius* R-*algebra.*

b) *If* A *is a Frobenius* R-*algebra, then* $\text{End A}\cong\widetilde{\text{End A}}$ *as algebras over* A. *Moreover for* $\langle W\rangle\in\mathcal{G}\langle \text{End A}\rangle$, $\langle\widetilde{W}\rangle$ *also lies in* $\mathcal{G}\langle\text{End A}\rangle$ *and* $\langle\widetilde{W}\rangle=\langle W\rangle^{-1}$.

Proof. — *a)* If R is a field then minimal left ideals in End A are isomorphic to A as End A-modules; hence as A^ℓ-modules. Minimal right ideals in End A are isomorphic to A^* as End A-modules; hence as right A^ℓ-modules. An anti-automorphism $\sigma:\text{End A}\to\text{End A}$ carries minimal left ideals to minimal right ideals. If σ fixes A^ℓ then it induces an isomorphism between a minimal left ideal with the left A^ℓ action and σ of that ideal with the right A^ℓ action; hence A^* is a free rank one A-module and A is Frobenius. Certainly $\text{End A}\cong\widetilde{\text{End A}}$ as algebras over A is equivalent to there being an anti-isomorphism σ fixing A^ℓ.

b) If A is a Frobenius R-algebra there is an element $f\in A^*$ which induces a bijective map $\sigma: A\xrightarrow{\sigma} A^*$, determined by $\sigma(a)(b)=f(ab)$. Thus $A\otimes A\xrightarrow{I\otimes\sigma}A\otimes A^*=\text{End A}$ is an A-bimodule isomorphism denoted F. It is left to the reader to verify that the composition

$$\tau:\ \text{End A}\xrightarrow{F^{-1}}A\otimes A\xrightarrow{\text{twist}}A\otimes A\xrightarrow{F}\text{End A}$$

is an anti A-bimodule isomorphism which is anti-multiplicative; i.e.

$$\tau(gh)=\tau(h)\tau(g),\qquad \tau(agb)=b\tau(g)a,$$

$g, h\in\text{End A}$, $a, b\in A$. Since τ is bijective it must preserve the unit. Thus $\text{End A}\to\widetilde{\text{End A}}$, $g\to\widetilde{\tau(g)}$, is an algebra isomorphism and an A-bimodule isomorphism. Hence, the map must be an isomorphism of algebras over A.

Since End A is a $\times_A$-bialgebra it is associative as an A-bimodule. Since

$$\text{End A}\cong\widetilde{\text{End A}}$$

as algebras over A it follows that $(\widetilde{\text{End A}}, \text{End A}, \text{End A})$ associates as A-bimodules. By (12.3) $\widetilde{\text{End A}}\times_A\text{End A}\xrightarrow{\theta}\text{End A}$ is an isomorphism and $\widetilde{\theta}^{-1}=\mathcal{S}$ is an isomorphism.

By (8.8) the $\theta^{-1}=\Delta$ is an isomorphism. Hence by (10.4), 3) if $\langle W\rangle=\mathcal{G}\langle \text{End } A\rangle$ then $\langle \widetilde{W}\rangle\in\mathcal{G}\langle \text{End } A\rangle$ and $\langle \widetilde{W}\rangle=\langle W\rangle^{-1}$. Q.E.D.

In Section 7 the situation where A is a field and R is a subfield is studied. It is shown in (7.1) that there is a unique maximal $\times_A$-coalgebra or $\times_A$-bialgebra $B\subset \text{End } A$. In (7.3) it is shown that there is a unique maximal $\times_A$-bialgebra $E\subset \text{End } A$ such that $\widetilde{E}\times_A E \xrightarrow{\theta} \widetilde{E}$ is an isomorphism.

Theorem **(12.5)**. — a) *In the above setting* $\widetilde{\theta}:\widetilde{\widetilde{E}\times_A E}\to\widetilde{\widetilde{E}}=E$ *is an isomorphism and* $\widetilde{\theta}^{-1}$ *is an* Ess *for* E.

b) *If* $F\subset\text{End } E$ *is a* $\times_A$*-bialgebra with* Ess *then* $\widetilde{\widetilde{F}\times_A F}\xrightarrow{\widetilde{\theta}}\widetilde{\widetilde{F}}$ *is an isomorphism and the* Ess *is* $\widetilde{\theta}^{-1}$.

c) E *is the unique maximal* $\times_A$*-bialgebra with* Ess *in* End A.

Proof. — Suppose C is any left A-submodule of End A. By (1.5), 4) all Λ-maps for C are injective.

a) By (11.6), 3) and (11.7) $\widetilde{\theta}^{-1}$ is an Ess for E.

b) By (1.6) the map $\widetilde{\theta}:\widetilde{\widetilde{F}\times_A F}\to\widetilde{\widetilde{F}}=F$ is injective. Thus by the remark above (11.5) $\widetilde{\theta}$ is an isomorphism and $\widetilde{\theta}^{-1}$ is the Ess of F.

c) If F is a $\times_A$-bialgebra with Ess, then, by part *b)*, $\widetilde{\theta}$ is an isomorphism and hence $\widetilde{F}\times_A F\xrightarrow{\theta}\widetilde{F}$ is an isomorphism. By the defining property of E it follows that $F\subset E$. Q.E.D.

At the end of section 5 the example $A\# H$ is given. Suppose H happens to be a Hopf algebra with antipode s. Then $A\# H$ has what would be considered a $\times_A$-antipode. (See the beginning of Section 9.) This is the map $S: A\# H\to A\# H$ given by

$$a\# h\to(1\# s(h))(a\# 1)=\sum_{(h)}(s(h_{(1)}).a)\# s(h_{(2)}).$$

(Recall H is assumed cocommutative.) As in the beginning of Section 9 we use S to define $\mathcal{S}$ as the composite

$$A\# H\xrightarrow{\sim S}\widetilde{A\# H}\xrightarrow{\widetilde{\Delta}}\widetilde{(A\# H)\times_A(A\# H)}\xrightarrow{(\sim S)\times I}\widetilde{\widetilde{(A\# H)}\times_A(A\# H)}.$$

This composite $\mathcal{S}$ maps $a\# h$ to $\sum_{(h)}(a\# h_{(1)})\otimes(1\# S(h_{(2)}))\in(A\# H)\otimes_A(A\# H)$. Here we are considering $\widetilde{\widetilde{(A\# H)}\times_A(A\# H)}\subset(A\# H)\otimes_A(A\# H)$ as indicated between (2.2) and (2.3). It is left to the reader to show that $\mathcal{S}$ is an Ess for $A\# H$.

13. The module of differentials

In view of (12.2) one wishes to know when A has almost finite projective differentials. The purpose of this section is to study the module of differentials and arrive at some classes of algebras which have almost finite projective differentials.

The construction of $J_n(A)$, the module of n-th order differentials, and j_n, the universal n-th order differential operator, appear after (8.3).

We begin the section by proving a number of " extension " results about modules of differential operators.

Uniqueness lemma **(13.1)**. — *Suppose* B *is a commutative algebra,* M *a left* B-*module and* $f: A \to B$ *an algebra homomorphism.* M *is considered an* A-*module by means of* f. *Suppose that*:

if $d: B \to M$ *is a* B-*derivation with* $df = 0$ *then* $d = 0$.

Then if $d_1, d_2: B \to M$ *are* B-*differential operators of any order with* $d_1 f = d_2 f$ *then* $d_1 = d_2$.

Proof. — Let $\mathfrak{N} = \mathrm{Ker}(B \otimes_A B \xrightarrow{\text{mult}} B)$. Then $\mathfrak{N}/\mathfrak{N}^2$ is the Kaehler module of B over A. If $\partial: B \to \mathfrak{N}/\mathfrak{N}^2$ is the universal derivation then $\mathfrak{N}/\mathfrak{N}^2$ is generated as a left B-module by Im ∂. Moreover, $\partial f = 0$. Hence by hypothesis $\partial = 0$ and so $\mathfrak{N}/\mathfrak{N}^2 = 0$. This implies that $\mathfrak{N} = \mathfrak{N}^t$ for $0 < t \in \mathbf{Z}$.

By considering $e = d_1 - d_2$ it suffices to prove that if $e: B \to M$ is a B-differential operator of any order with $ef = 0$ then $e = 0$. If e has order smaller than or equal to one (and $e(1) = 0$ since $ef(1) = 0$) then e is a derivation and by hypothesis $e = 0$. Suppose e has order n (with $n > 1$) and the result has been proved for lower order than n.

For $a, a' \in A$

$$[f(a), e](f(a')) = f(a)ef(a') - ef(aa') = 0$$

where $[f(a), e]$ is defined in the beginning of section eight. The differential operator $[f(a), e]$ has order less than n, and we have just shown $[f(a), e]f = 0$. Hence by induction $[f(a), e] = 0$. This shows that $e: B \to M$ is an A-module map. Thus $e \in \mathrm{Hom}_A(B, M)$.

$\mathrm{Hom}_A(B, M)$ is a $B \otimes_A B$-module in the following manner: For $b_1 \otimes b_2 \in B \otimes_A B$, $g \in \mathrm{Hom}_A(B, M)$, $b_3 \in B$

$$((b_1 \otimes b_2).g)(b_3) = b_1 \otimes g(b_2 b_3)).$$

Since e is an n-th order differential operator $\mathfrak{N}^{n+1}.e = 0$. Since $\mathfrak{N} = \mathfrak{N}^{n+1}$, it follows $\mathfrak{N}.e = 0$. Thus e is actually a zero order differential operator. Q.E.D.

Local extension lemma **(13.2)**. — *Let* S *be a multiplicative system in* A *and* $\varphi: A \to A_S$ *the natural algebra map. If* M *is a left* A_S-*module and* $f \in D_A^n(M)$ *then there is unique* $f_S \in D_{A_S}^n(M)$ *where* $f_S \varphi = f$.

Proof. — Uniqueness: Suppose $d: A_S \to M$ is a derivation with $d\varphi = 0$. The usual " quotient rule "

$$d\left(\frac{a}{s}\right) = \frac{1}{s^2}(\varphi(s)d\varphi(a) - \varphi(a)d\varphi(s))$$

shows that $d = 0$. By (13.1) this gives uniqueness of extensions.

Existence: The proof goes by induction on n. For $n = 0$,

$$D_A^0(M) = \mathrm{Hom}_A(A, M) = M = \mathrm{Hom}_{A_S}(A_S, M) = D_{A_S}^0(M).$$

This gives existence (and uniqueness) when $n = 0$.

Suppose by induction that the lemma is true for $n-1$, and $f \in D_A^n(M)$. For

$$x, y \in A, \quad (x \otimes 1)(y \otimes 1 - 1 \otimes y) + (1 \otimes y)(x \otimes 1 - 1 \otimes x) = xy \otimes 1 - 1 \otimes xy.$$

Thus for $g \in \mathrm{Hom}(A, N)$ the identity holds

$$(*) \qquad (x \otimes 1) . [y, g] + (1 \otimes y) . [x, g] = [xy, g],$$

where N is any left A-module.

The idea is to define f_S inductively by

$$(13.3) \qquad f_S(a/b) = (1/b)([b, f]_S(a/b) + f(a))$$

$a \in A$, $b \in S$. For specific $a \in A$, $b \in S$ the right hand side of (13.3) makes sense since $[b, f] \in D_A^{n-1}(M)$ and by induction $[b, f]_S$ is uniquely defined. Let $f'(a/b)$ denote the right hand side of (13.3).

Suppose $z \in A_S$ has representations $a/b = z = c/d$, $a, c \in A$, $b, d \in S$. There is $s \in S$ with $sad = sbc$.

$$(**) \qquad \begin{cases} [sbd, f]_S(z) + f(sad) = [sdb, f]_S(z) - [sd, f](a) + sdf(a) \\ \qquad = [sbd, f]_S(z) - [sd, f]_S(\varphi(a)) + sdf(a) \\ \qquad = [sbd, f]_S(z) - [sd, f]_S(\varphi(b)(a/b)) + sdf(a). \end{cases}$$

Using sd for x, b for y and f for g in $(*)$ yields $(sd \otimes 1)[b, f] = [sbd, f] - (1 \otimes b)[sd, f]$. By induction this implies that $(sd \otimes 1)[b, f]_S = [sdb, f]_S - (1 \otimes \varphi(b))[sd, f]_S$. Applying both sides to z and substituting in the right hand side of $(**)$ yields

$$(***) \qquad sd[b, f]_S(z) + sdf(a) = sbdf'(a/b).$$

The left hand side of $(**)$ equals $[sbd, f]_S(z) + f(sbc)$. In the same manner as above this equals

$$(****) \qquad sbdf'(c/d).$$

Since $sbd \in S$ it follows from the right hand side of $(***)$ and from $(****)$ that $f'(a/b) = f'(c/d)$. Thus f' is a well-defined map from A_S to M. Notice that if f is actually an $(n-1)$-th order differential operator, i.e. $f \in D_A^{n-1}(M)$, then by the induction f_S is defined and (13.3) is an identity. In this case $f' = f_S$.

That $f(a) = f'\varphi(a)$, $a \in A$ is left to the reader. That f' is additive is immediate once the two elements being added are expressed with common denominator. That f' commutes with scalars (from R) is immediate. Thus $f' \in \mathrm{Hom}(A_S, M)$. Next to show that $f' \in D^n_{A_S}(M)$.

By the definition (13.3) of f' it is easily verified that for $a \in A$, $f, g \in D^n_A(M)$

$$(\varphi(a)\otimes 1)f' = ((a\otimes 1)f)',$$
$$(1\otimes\varphi(a))f' = ((1\otimes a)f)',$$
$$(f+g)' = f'+g'.$$

For $a \in A$, $b \in S$ apply both sides of

$$((a/b)\otimes 1 - 1\otimes(a/b)) = (1\otimes(1/b))(\varphi(a)\otimes 1 - 1\otimes\varphi(a)) - ((a/b)\otimes(1/b))(\varphi(b)\otimes 1 - 1\otimes\varphi(b))$$

to f' to obtain

$$(*****)\qquad \begin{cases} [(a/b), f'] = (1\otimes(1/b))[\varphi(a), f'] - ((a/b)\otimes(1/b))[\varphi(b), f'] \\ \qquad = (1\otimes(1/b))[a,f]' - ((a/b)\otimes(1/b))[b,f]'. \end{cases}$$

Since $[a,f]$ and $[b,f]$ are $(n-1)$-th order differential operators the right hand side of (*****) equals

$$(1\otimes(1/b))[a,f]_S - ((a/b)\otimes(1/b))[b,f]_S$$

which lies in $D^{n-1}_{A_S}(M)$. Thus $f' \in D^n_{A_S}(M)$ and the induction is completed by setting $f_S = f'$. Q.E.D.

Localization theorem **(13.4)**. — *The pair* $(A_S \otimes_A J_n(A), (\iota\otimes j_n)_S)$ *has the same universal property for* A_S *as* $(J_n(A_S), j_n)$. *Hence* $J_n(A_S) \cong A_S \otimes_A J_n(A)$ *as left* A_S*-modules.*

Note. — $(\iota\otimes j_n) : A \to A_S \otimes_A J_n(A)$, $(a \mapsto 1\otimes j_n(a))$ is an element of $D^n_A(A_S\otimes_A J_n(A))$ and $(\iota\otimes j_n)_S$ is as defined in (13.2).

Proof. — By (13.2) $(A_S\otimes_A J_n(A), (\iota\otimes j_n)_S)$ has the desired universal property. As the universal property characterizes $(J_n(A_S), j_n)$ so the second assertion holds. Q.E.D.

Base extension lemma **(13.5)**. — *Suppose* S *is a commutative* R*-algebra, and* M *is a left* $S\otimes A$*-module. Let* $\lambda : A \to S\otimes A$, $(a \mapsto 1\otimes a)$. *Then for* $f \in D^n_A(M)$ *the canonical extension* $S\otimes f : S\otimes A \to M$, $(s\otimes a \mapsto sf(a))$ *gives the unique element* $g \in \mathrm{Hom}_S(S\otimes A, M)$ *where*

$$g \in D^n_{S\otimes A,\, S}(M) \quad \textit{and} \quad g\lambda = f.$$

Note. — $D^n_{S\otimes A,\, S}(M)$ denotes n-th order differential operators with respect to the base ring S. There are inclusions

$$D^n_{S\otimes A}(M) \subset \mathrm{Hom}(S\otimes A, M) \supset \mathrm{Hom}_S(S\otimes A, M) \supset D^n_{S\otimes A,\, S}(M).$$

It is directly verified that

$$D^n_{S\otimes A}(M) \cap \mathrm{Hom}_S(S\otimes A, M) = D^n_{S\otimes A,\, S}(M).$$

Proof. — Left to the reader.

Base change theorem **(13.6)**. — *The pair* $(S\otimes J_n(A), S\otimes j_n')$ *has the same universal property for* $S\otimes A$ *as* $(J_{n,S}(S\otimes A), j_n)$. *Hence* $S\otimes J_n(A)\cong J_{n,S}(S\otimes A)$ *as left* $S\otimes A$-*modules.*

Note. — $j_n' : A\to S\otimes J_n(A)$, $(a\mapsto 1\otimes j_n(A))$ is an element of $D_A^n(S\otimes J_n(A))$ and $S\otimes j_n$ is as defined in (13.5). Also, $(J_{n,S}(S\otimes A), j_n)$ is the J_n module of $S\otimes A$ as an algebra over the base ring S.

Proof. — Left to the reader.

Lemma **(13.7)**. — *Let* A *and* B *be commutative algebras,* $\sigma : A\to B$ *an algebra map,* I *an ideal of* A *and* M *a left* A-*module.*

1. *If* $f\in D_A^n(M)$ *then* $F(I^{n+m})\subset I^m M$ *for* $0\leq m\in\mathbf{Z}$.
2. $J_n(A)$ *is generated by* $j_n(A)$ *as a left* A-*module.*
3. $J_n^+(A)$ *is generated by* $j_n^+(A)$ *as a left* A-*module.*
4. *If* $0\leq n\leq m\in\mathbf{Z}$ *there is a unique left* A-*module map* $J_m(A)\xrightarrow{J(\sigma)} J_n(B)$ *making the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ \big\downarrow{\scriptstyle j_m} & & \big\downarrow{\scriptstyle j_n} \\ J_m(A) & \xrightarrow{J(\sigma)} & J_n(B) \end{array}$$

commute; $J(\sigma)$ *is an algebra homomorphism.*

5. *The map* $J(\sigma)$ *is surjective if* σ *is surjective.*
6. *If* σ *is surjective and* $J_m(A)$ *is a finitely generated* A-*module, then* $J_n(B)$ *is a finitely generated* B-*module.*

Proof. — Suppose $n=0$. Then $f\in \mathrm{Hom}_A(A, M)$ and $f(a)=af(1)$. Thus

$$f(I^m)=I^m f(1)\subset I^m M.$$

Suppose by induction on n that for $g\in D_A^{n-1}(M)$, $g(I^{n-1+m})\subset I^m M$. Let $f\in D_A^n(M)$.

Induct on m. For $m=0$, $f(I^{n+0})\subset M=AM=I^0 M$ since $I^0=A$. Suppose that $f(I^{n+m-1})\subset I^{m-1}M$. For $x\in I$, $y\in I^{n+m-1}$, $f(xy)=xf(y)-[x,f](y)$. Since $f(y)\subset I^{m-1}M$, $xf(y)\in I^m M$. Since $[x,f]\in D_A^{n-1}(M)$, by the induction on n, $[x,f](y)\in I^m M$. Thus $f(xy)\in I^m M$ and part 1 is proved.

Part 2 follows from the construction of $(J_n(A), j_n)$ or the observation that $(J_n(A), j_n)$ could be replaced by $(Aj_n(A), j_n)$. Part 3 follows from part 2 and (8.4).

The composite $A\xrightarrow{\sigma} B\xrightarrow{j_n} J_n(B)$ is an element of $D_A^m(J_n(B))$. Thus there is a unique element

$$J(\sigma)\in \mathrm{Hom}_A(J_m(A), J_n(B))$$

making the diagram commute. The map $A\otimes A\xrightarrow{\sigma\otimes\sigma} B\otimes B$ is an algebra map and induces a unique algebra map $\gamma : J_m(A)\to J_n(B)$ making the diagram:

$$\begin{array}{ccc} J_m(A)=(A\otimes A)/\mathfrak{M}_A^{m+1} & \xrightarrow{\ \gamma\ } & (B\otimes B)/\mathfrak{M}_B^{n+1}=J_n(B) \\ \uparrow & & \uparrow \\ A\otimes A & \xrightarrow{\ \sigma\otimes\sigma\ } & B\otimes B \end{array}$$

commute. The map γ satisfies $\gamma j_m = j_n \sigma$ and $\gamma \in \mathrm{Hom}_A(J_m(A), J_n(B))$. Thus γ must equal $J(\sigma)$ and $J(\sigma)$ is an algebra map. This proves part 4.

Part 5 follows from the explicit description of $J(\sigma)$ as γ. Part 6 follows from part 5 since the quotient of a finitely generated A-module is again such, and if $J_n(B)$ is finitely generated as an A-module it is finitely generated as a B-module. Q.E.D.

Complete extension lemma **(13.8)**. — *Let* I *be an ideal of* A *and* $\hat{A}$ *the completion of* A *in the* I*-adic topology. Let* $\lambda : A \to \hat{A}$ *be the natural algebra map. Let* $\hat{I}$ *denote the closure of* $\lambda(I)$ *in* $\hat{A}$. *Suppose* M *is a left* $\hat{A}$*-module and* M *is complete in the* $\hat{I}$*-adic topology. Then for* $f \in D_A^n(M)$ *there is a unique* $\hat{f} \in D_{\hat{A}}^n(M)$ *where* $\hat{f} : \hat{A} \to M$ *is continuous and* $\hat{f}\lambda = f$.

Proof. — By (13.7), part 1, $f : A \to M$ is continuous where A has the I-adic topology and M has the $\hat{I}$-adic topology. Thus there is a unique continuous map $\hat{f} : \hat{A} \to M$ where $\hat{f}\lambda = f$. That $\hat{f} \in D_{\hat{A}}^n(M)$ is left to the reader. Q.E.D.

(13.8) does not imply that $J_n(\hat{A}) = \widehat{J_n(A)}$ since not all $\hat{A}$-modules are complete in the $\hat{I}$-adic topology.

Let $\mathscr{A}$ be a commutative algebra with ideal $\mathfrak{J}$ and suppose $\mathscr{A}$ is complete in the $\mathfrak{J}$-adic topology. There is a left $\mathscr{A}$-module $\mathscr{J}_n(\mathscr{A})$ which is complete in the $\mathfrak{J}$-adic topology and a " universal " element $j_n \in D_{\mathscr{A}}^n(\mathscr{J}_n(\mathscr{A}))$ such that for any left $\mathscr{A}$-module M which is complete in the $\mathfrak{J}$-adic topology and for any $f \in D_{\mathscr{A}}^n(M)$ there is a unique (continuous) $J(f) \in \mathrm{Hom}_{\mathscr{A}}(\mathscr{J}_n(\mathscr{A}), M)$ where $f = J(f) j_n$. In fact the construction of $(\mathscr{J}_n(\mathscr{A}), j_n)$ is easy since by (13.7), part 1, differential operators are continuous in the $\mathfrak{J}$-adic topology and $\mathscr{A}$-module maps are automatically continuous in the $\mathfrak{J}$-adic topology. $\mathscr{J}_n(A)$ is just the completion of $J_n(A)$ with respect to the $\mathfrak{J}$-adic topology and j_n is the composite

$$\mathscr{A} \xrightarrow{\ j_n\ } J_n(\mathscr{A}) \longrightarrow \widehat{J_n(\mathscr{A})} = \mathscr{J}_n(\mathscr{A}).$$

Completion theorem **(13.9)**. — *Let* I *be an ideal of* A *and* $\mathscr{A}$ *the completion of* A *in the* I*-adic topology. Let* $\lambda : A \to \mathscr{A}$ *be the natural algebra map and let* $\mathfrak{J}$ *be the closure of* $\lambda(I)$. *Let* $\widehat{J_n(A)}$ *be the completion of* $J_n(A)$ *in the* I*-adic topology with the natural* $\mathscr{A}$*-module structure. The pair* $(\widehat{J_n(A)}, \widehat{j_n''})$ *has the same universal property for* $\mathscr{A}$ *as* $(\mathscr{J}_n(\mathscr{A}), j_n)$. *Hence*

$$\widehat{J_n(A)} \cong \mathscr{J}_n(\mathscr{A})$$

as left $\mathscr{A}$*-modules.*

Note. — $j_n'' : A \to \widehat{J_n(A)}$ is the composite $A \xrightarrow{j_n} J_n(A) \to \widehat{J_n(A)}$ and is an element of $D_A^n(\widehat{J_n(A)})$. Since $\widehat{J_n(A)}$ is a complete $\mathscr{A}$-module in the $\mathfrak{J}$-adic topology, $\widehat{j_n''}$ is defined in (13.8).

Proof. — Left to the reader.

Definition **(13.10)**. — Suppose A and B are commutative algebras and $\varphi : A \to B$ is an algebra map, making B into an A-algebra. B is called a *finite separable extension of* A if B is finitely generated and projective as an A-module and B is a projective $B \otimes_A B$-module.

This usage of finite separable is a special case of [2, p. 369], and when A and B are fields, finite separable in the sense of (13.10) is equivalent to the classical notion of a finite degree separable extension.

Lemma **(13.11)**. — *Suppose* B *is a finite separable extension of* A, D *is a commutative* A-*algebra and* $\pi : D \to B$ *a surjective* A-*algebra map where* $(\operatorname{Ker} \pi)^n = 0$ *for some* $n \in \mathbf{Z}$. *Then there is a unique* A-*algebra map* $B \to D$ *splitting* π.

Proof. — The proof is a standard application of (relative) homological algebra and is included for the reader's convenience.

Uniqueness. — Suppose σ is a splitting of π. Then $\sigma : B \to D$ is an A-algebra map and gives D and $\operatorname{Ker} \pi$ a B-bimodule structure. Let γ be another splitting. γ determines a projection $P : D \to \operatorname{Ker} \pi$, $(d \mapsto d - \gamma\pi(d))$ and the composite $B \xrightarrow{\sigma} D \xrightarrow{P} \operatorname{Ker} \pi$ is a Hochschild 1-cocycle. Since B is a projective $B \otimes_A B$-module $\operatorname{Ext}^1_{B \otimes_A B}(B, \operatorname{Ker} \pi) = 0$ and by [5, Proposition (4.1), p. 170] $P\sigma$ is inner. Thus there is $x \in \operatorname{Ker} \pi$ where $P\sigma(b) = bx - xb$ for $b \in B$. Since D is commutative this implies that $P\sigma = 0$ and hence $\operatorname{Im} \sigma \subset \operatorname{Ker} P$. This implies that $\sigma = \gamma$.

Existence. — The proof goes by induction on n where $(\operatorname{Ker} \pi)^n = 0$. Say $n = 2$. Then $\operatorname{Ker} \pi$ has a natural B-bimodule (or $B \otimes_A B$-module) structure. Since B is projective as an A-module there is an A-module splitting $s : B \to D$ of π. The map $\lambda : B \times B \to \operatorname{Ker} \pi$, $((b_1, b_2) \mapsto (s(b_1 b_2) - s(b_1)s(b_2))$ is a Hochschild 2-cocycle. Since B is a projective $B \otimes_A B$-module, $\operatorname{Ext}^2_{B \otimes_A B}(B, \operatorname{Ker} \pi) = 0$ and by [5, p. 175, last sentence], λ is a coboundary of say $t : B \to \operatorname{Ker} \pi$. Then $s + t : B \to D$ is the desired A-algebra splitting. Next suppose $n > 2$ and the result is true for values less than n. The map $D \xrightarrow{\pi} B$ factors

$$D \xrightarrow{\pi_1} D/(\operatorname{Ker} \pi)^{n-1} \xrightarrow{\pi_2} B.$$

By the induction π_2 admits an A-algebra splitting σ_2 so that $\sigma_2(B)$ is an A-subalgebra of $D/(\operatorname{Ker} \pi)^{n-1}$ which is isomorphic to B as an A-algebra. Let E be $\pi_1^{-1}(\sigma_2(B))$. Then $E \xrightarrow{\pi_1 | E} \sigma_2(B) \cong B$ is a surjective A-algebra map and $(\operatorname{Ker}(\pi_1 | E))^{n-1} = 0$. Again by the induction this map has an A-algebra splitting σ_1 and the composite $\sigma_1 \sigma_2$ is the desired A-algebra splitting of π. Q.E.D.

Separable extension theorem **(13.12)**. — *Suppose* B *is a finite separable extension of* A. *The natural surjective algebra map* $J_n(A) \to A$ (*mentioned above* (8.4)) *is an* A-*bimodule map and induces a surjective algebra map* $B \otimes_A J_n(A) \xrightarrow{\pi} B \otimes_A A = B$.

1. $(\operatorname{Ker} \pi)^{n+1} = 0$.

2. $A \to B \otimes_A J_n(A)$, $(a \mapsto 1 \otimes j_n(a))$ *is an algebra map making* $B \otimes_A J_n(A)$ *into an* A-*algebra and* π *into an* A-*algebra map.*

3. *If* $\sigma : B \to B \otimes_A J_n(A)$ *is the unique splitting of* π *guaranteed by* (13.11) *then* $(B \otimes_A J_n(A), \sigma)$ *has the same universal property for* B *as* $(J_n(B), j_n)$.

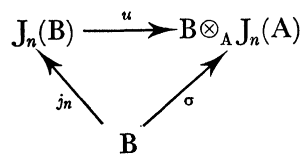
4. $B \otimes_A J_n(A) \cong J_n(B)$ *as left* B-*modules.*

5. *If* M *is a left* B-*module and* $d \in D^n_A(M)$ *then there is a unique* $d' \in D^n_B(M)$ *where* $d'\varphi = d$, ($\varphi : A \to B$ *being the map making* B *an extension of* A).

Proof. — Parts 2 and 5 are left to the reader. Since the kernel of $J_n(A) \to A$ is $J^+_n(A)$ and $J^+_n(A)^{n+1} = 0$, part 1 is proved.

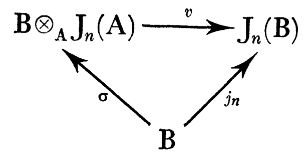
For $b \in B$, $b \otimes 1 - \sigma(b) \in B \otimes_A J_n(A)$ is in $\operatorname{Ker} \pi$. This implies that $\sigma : B \to B \otimes_A J_n(A)$ is in $D^n_B(B \otimes_A J_n(A))$. Thus there is unique $u \in \operatorname{Hom}_B(J_n(B), B \otimes_A J_n(A))$ such that the diagram

(*)



commutes. The algebra map $A \xrightarrow{\varphi} B$ induces an algebra map $J_n(A) \xrightarrow{J(\varphi)} J_n(B)$, (13.7), part 4. Thus $v : B \otimes_A J_n(A) \to J_n(B)$, $(b \otimes z \mapsto b(J(\varphi)(z)))$ is an algebra map. The diagram

(**)



commutes. This is because the maps j_n, $v\sigma : B \to J_n(B)$ are A-algebra maps where φ makes B an A-algebra and j_n makes $J_n(B)$ an A-algebra. Moreover, both split the natural A-algebra map $J_n(B) \to B$. The kernel of this map is $J^+_n(B)$ and $J^+_n(B)^{n+1} = 0$. Thus by uniqueness in (13.11) $j_n = v\sigma$.

From the two commutative diagrams (*), (**), $vu \in \operatorname{Hom}_B(J_n(B), J_n(B))$ and $j_n = vuj_n$. By the universal property of $(J_n(B), j_n)$ it follows that $vu = I$.

The map $B \otimes B \to B \otimes_A J_n(B)$, $(b \otimes \beta \mapsto b\sigma(\beta))$ is an algebra map which carries $\mathfrak{M}$ to $\operatorname{Ker} \pi$. Thus $\mathfrak{M}^{n+1}$ maps to zero and $B \otimes B \to B \otimes_A J_n(A)$ factors to an algebra map $(B \otimes B)/\mathfrak{M}^{n+1} = J_n(B) \xrightarrow{u'} B \otimes_A J_n(A)$, where $u'j_n = \sigma$ and $u' \in \operatorname{Hom}_B(J_n(B), B \otimes_A J_n(A))$. By the universal property of $(J_n(B), j_n)$ it follows that $u = u'$. As a left A-module $J_n(A)$ is generated by $j_n(A)$, (13.7), part 2. Since $\sigma\varphi = j_n$ it follows that u' is surjective.

With $vu=I$ it follows that v and u are inverse isomorphisms. From the commutative diagram $(*)$, σ corresponds to j_n under the isomorphism and part 3 is proved. Q.E.D.

Corollary **(13.13).** — *The map* $B\otimes B\to B\otimes_A J_n(B)$, $(b\otimes\beta\mapsto b\sigma(\beta))$ *factors to an isomorphism* $J_n(B)\to B\otimes_A J_n(A)$.

Proof. — This is the map u' in the proof of (13.12). Q.E.D.

Definition **(13.14).** — A is a *purely inseparable* R-algebra if the kernel of $A\otimes A\xrightarrow{\text{mult}}A$ consists of nilpotent elements.

In case A and R are fields the notion of purely inseparable in (13.14) coincides with the usual definition. If $\mathfrak{M}=\operatorname{Ker}(A\otimes A\xrightarrow{\text{mult}}A)$ is finitely generated as an ideal and consists of nilpotent elements then there is an $n\in\mathbf{Z}$ with $\mathfrak{M}^n=0$.

Example **(13.15).** — Suppose A contains an ideal I where $I^n=0$ and $A=R+I$. Then $\mathfrak{M}$ is generated by elements $1\otimes x-x\otimes 1$ with $x\in I$. Hence $\mathfrak{M}\subset A\otimes I+I\otimes A$ and $\mathfrak{M}^{2n-1}=0$. This implies that A is a purely inseparable R-algebra. Thus for example $R[x_1,\ldots,x_n]/\langle\{X_i^{e_i}\}\rangle$ with $0<e_i\in\mathbf{Z}$ is a purely inseparable R-algebra. Even if R is a field of characteristic zero.

Purely inseparable theorem **(13.16).** — *Suppose* A *is a purely inseparable* R-*algebra and is a finite projective* R-*module. Then* A *has finite projective differentials. In fact there is an n where* $J_m(A)=A\otimes A$, $D_A^m=D_A=\operatorname{End}A$ *for all* $m>n$.

Proof. — Since A is a finitely generated R-module, $\mathfrak{M}$ is a finitely generated ideal. Hence there is an n with $\mathfrak{M}^n=0$. Clearly for this n, $D_A^n=D_A=\operatorname{End}A$. For $m>n$ $J_m(A)=A\otimes A$ which is a finite projective left A-module since A is a finite projective R-module. Q.E.D.

Tensor product theorem **(13.17).** — *Let* A *and* B *be commutative* R-*algebras. Suppose both* A *and* B *have almost finite projective differentials* (8.5). *Then* $A\otimes B$ *has almost finite projective differentials. Moreover the natural map*

$$D_A\otimes D_B\to\operatorname{End}(A\otimes B)$$
$$d\otimes e\mapsto(a\otimes b\to d(a)\otimes e(b))$$

induces an isomorphism between $D_A\otimes D_B$ *and* $D_{A\otimes B}$.

Proof. — Let $t: A\otimes A\otimes B\otimes B\xrightarrow{\cong}A\otimes B\otimes A\otimes B$, $a_1\otimes a_2\otimes b_1\otimes b_2\mapsto a_1\otimes b_1\otimes a_2\otimes b_2$. Then

$$\mathfrak{M}_{A\otimes B}=\operatorname{Ker}((A\otimes B)\otimes(A\otimes B)\xrightarrow{\text{mult}_{A\otimes B}}A\otimes B)$$
$$=t(\mathfrak{M}_A\otimes B\otimes B+A\otimes A\otimes\mathfrak{M}_B).$$

$$(*)\begin{cases}\text{Thus } \mathfrak{M}^n_{A\otimes B}\supset t(\mathfrak{M}_A^n\otimes B\otimes B+A\otimes A\otimes\mathfrak{M}_B^n)\\ \text{and } \mathfrak{M}^n_{A\otimes B}\subset t(\mathfrak{M}_A^s\otimes B\otimes B+A\otimes A\otimes\mathfrak{M}_A^r)\quad\text{for}\quad r+s=n+1.\end{cases}$$

Let $\{L_\alpha\}$ be as in (8.5) for A and $\{M_\gamma\}$ as in (8.5) for B. Since $\{L_\alpha\}$ is cofinal with $\{\mathfrak{M}_A^i\}$ and $\{M_\gamma\}$ is cofinal with $\{\mathfrak{M}_B^j\}$ it follows from (*) that

$$\{t(L_\alpha \otimes B \otimes B + A \otimes A \otimes M_\gamma)\}_{\alpha,\gamma}$$

is cofinal with $\{\mathfrak{M}_{A\otimes B}^n\}$.

Since t is an isomorphism it induces an isomorphism

$$\frac{A\otimes A\otimes B\otimes B}{L_\alpha\otimes B\otimes B + A\otimes A\otimes M_\gamma} \cong \frac{A\otimes B\otimes A\otimes B}{t(L_\alpha\otimes B\otimes B + A\otimes A\otimes M_\gamma)}$$

and the left hand side is isomorphic to

$$\frac{A\otimes A}{L_\alpha}\otimes\frac{B\otimes B}{M_\gamma}.$$

If

$$\left({}_x\frac{A\otimes A}{L_\alpha}\right)\otimes\left({}_x\frac{B\otimes B}{M_\gamma}\right)$$

has the x A⊗B-module structure, then the isomorphism

$$(**)\qquad \left(\frac{A\otimes A}{L_\alpha}\right)\otimes\left(\frac{B\otimes B}{M_\gamma}\right) \cong \frac{A\otimes B\otimes A\otimes B}{t(L_\alpha\otimes B\otimes B + A\otimes A\otimes M_\gamma)}$$

is as A⊗B-modules. Since $(A\otimes A)/L_\alpha$ is a finite projective A-module and $(B\otimes B)/M_\gamma$ is a finite projective B-module it follows that

$$\frac{A\otimes B\otimes A\otimes B}{t(L_\alpha\otimes B\otimes B + A\otimes A\otimes M_\gamma)}$$

is a finite projective A⊗B-module. Thus A⊗B has almost finite projective differentials.

By (**) we may identify $\mathrm{Hom}_{A\otimes B}\left(\frac{A\otimes B\otimes A\otimes B}{t(L_\alpha\otimes B\otimes B + A\otimes A\otimes M_\gamma)}, A\otimes B\right)$ with $\mathrm{Hom}_{A\otimes B}\left(\frac{A\otimes A}{L_\alpha}\otimes\frac{B\otimes B}{M_\gamma}, A\otimes B\right)$. There is a natural map

$$(*{*}*)\qquad \mathrm{Hom}_A\left(\frac{A\otimes A}{L_\alpha}, A\right)\otimes \mathrm{Hom}_B\left(\frac{B\otimes B}{M_\gamma}, B\right) \to \mathrm{Hom}_{A\otimes B}\left(\frac{A\otimes A}{L_\alpha}\otimes\frac{B\otimes B}{M_\gamma}, A\otimes B\right)$$

$$c\otimes d \mapsto (x\otimes y \to c(x)\otimes d(y))$$

for

$$c\in \mathrm{Hom}_A\left(\frac{A\otimes A}{L_\alpha}, A\right),\quad d\in \mathrm{Hom}_B\left(\frac{B\otimes B}{M_\gamma}, B\right),\quad x\in\frac{A\otimes A}{L_\alpha},\quad y\in\frac{B\otimes B}{M_\gamma}.$$

This map is an isomorphism since $(A\otimes A)/L_\alpha$ is a finite projective A-module and $(B\otimes B)/M_\gamma$ is a finite projective B-module. In view of (2.12), *a)* (***) induces the desired isomorphism $D_A\otimes D_B \cong D_{A\otimes B}$. Q.E.D.

For a commutative algebra A the algebra $J_n(A)$ is filtered by powers of the ideal $J_n^+(A)$. As pointed out earlier $A \cong J_n(A)/J_n^+(A)$. It is easily verified that the Kaehler module of A, $J_1^+(A)$ is isomorphic to $J_n^+(A)/J_n^+(A)^2$.

Definition **(13.18)**. — $J_n(A)$ is of *graded type* if $J_n(A) \cong \operatorname{gr} J_n(A)$ as an A-algebra. $J_n(A)$ is of (finite) projective graded type if it is of graded type and (finite) projective as a left A-module.

If $J_n(A)$ is of graded type there is a left A-module $V \subset J_n^+(A)$ where

$$J_n^+(A)^i = V^i \oplus J_n^+(A)^{i+1}, \quad i = 1, \ldots, n.$$

In this case $J_n(A)$ is projective as a left A-module if and only if each V^i is projective as a left A-module.

Graded type theorem **(13.19)**. — *Suppose* B *is a commutative algebra and* A *is a commutative algebra where* $J_n(A)$ *is of graded type.*

1. $J_n(B\otimes A) \cong \bigoplus_{i=0}^{n} (J_i(B) \otimes \operatorname{gr} J_n(A)_{n-i})$ *as a left* $B\otimes A$*-module, where* $\operatorname{gr} J_n(A)_i$ *is the* i-th *graded part of* $\operatorname{gr} J_n(A)$.

2. *If* $J_0(B), \ldots, J_n(B)$ *are finite projective* B*-modules and* $J_n(A)$ *is a finite projective* A*-module, then* $J_n(B\otimes A)$ *is a finite projective* $B\otimes A$*-module.*

3. *Suppose* R *is a ring of characteristic* p, i.e. *either* p *is a prime and* $p.1 = 0$ *in* R *or* $p = 0$. *In what follows* t *is assumed to be zero if* $p = 0$. *Let* C *be the algebra*

$$R[X_1, \ldots, X_s, Y_1, \ldots, Y_t]/\langle\{Y_i^{p^{e_i}}\}\rangle \qquad 0 < e_1, \ldots, e_t \in \mathbf{Z}.$$

Recall that for $c \in C$, $j_n^+(c) = j_n(c) - cj_n(1)$. *Let* V *be the left* C*-submodule of* $J_n^+(C)$ *spanned by* $B = \{j_n^+(X_i)\} \cup \{j_n^+(Y_i)\}$. *Then* V *is a free* C*-module with basis* B. *Moreover* V^i *is a free* C*-module with basis consisting of monomials*

$$j_n^+(X_1)^{f_1} \ldots j_n^+(X_s)^{f_s} j_n^+(Y_1)^{g_1} \ldots j_n^+(Y_t)^{g_t}$$

where $f_1 + \ldots + f_s + g_1 + \ldots + g_t = i$ *and* $g_1 < p^{e_1}, \ldots, g_t < p^{e_t}$. *Finally*

$$J_n^+(C)^i = V^i \oplus J_n^+(C)^{i+1},$$

so that $J_n(C)$ *is of finite projective (actually free) graded type.*

4. *Suppose* R *is a field and* L *is a field extension of* R *which is finitely and separably generated as a field over* R; i.e. L *is a finite separable extension of* $R(X_1, \ldots, X_s)$. *Then* $J_n(L)$ *is of finite projective graded type.*

Proof. — Part 2 follows immediately from part 1.

For part 3 observe that the algebra map $C[U_1, \ldots, U_s, V_1, \ldots, V_t] \to C\otimes C$ determined by $c \mapsto c\otimes 1$, $U_i \mapsto 1\otimes X_i - X_i \otimes 1$, $V_i \mapsto 1 \otimes Y_i - Y_i \otimes 1$ induces an algebra isomorphism

$$D = \frac{C[U_1, \ldots, U_s, V_1, \ldots, V_t]}{\langle\{V_i^{p^{e_i}}\}\rangle} \to C\otimes C$$

and this isomorphism is a left C-module map. Under the isomorphism, $\mathfrak{M} \subset C \otimes C$ corresponds to the ideal in D generated by $\{U_i\} \cup \{V_i\}$. Thus $J_n(C)$ corresponds to a truncation of D, factoring out degree $n+1$ and higher ones. The specific assertions of part 3 now follow easily and are left to the reader.

For part 1 identify $(B \otimes A) \otimes (B \otimes A)$ with $(B \otimes B) \otimes (A \otimes A)$ by

$$(b \otimes a \otimes \beta \otimes \alpha) \leftrightarrow (b \otimes \beta \otimes a \otimes \alpha), \qquad b, \beta \in B, \qquad a, \alpha \in A.$$

Let

$$\mathfrak{M}_B = \operatorname{Ker}(B \otimes B \xrightarrow{\text{mult}} B), \qquad \mathfrak{M}_A = \operatorname{Ker}(A \otimes A \xrightarrow{\text{mult}} A),$$
$$\mathfrak{M}_{B \otimes A} = \operatorname{Ker}((B \otimes A) \otimes (B \otimes A) \xrightarrow{\text{mult}} B \otimes A).$$

Then $\mathfrak{M}_{B \otimes A}$ corresponds to $B \otimes B \otimes \mathfrak{M}_A + \mathfrak{M}_B \otimes A \otimes A$ and $\mathfrak{M}_{B \otimes A}^{n+1}$ corresponds to

$$K = B \otimes B \otimes \mathfrak{M}_A^{n+1} + \sum_{i=1}^{n} \mathfrak{M}_B^i \otimes \mathfrak{M}_A^{n+1-i} + \mathfrak{M}_B^{n+1} \otimes A \otimes A.$$

This gives an isomorphism between $J_n(B \otimes A)$ and $(B \otimes B \otimes A \otimes A)/K$ as algebras and left $B \otimes A$-modules. The surjection $B \otimes B \otimes A \otimes A \to (B \otimes B \otimes A \otimes A)/K$ factors

$$\begin{array}{ccc} B \otimes B \otimes A \otimes A & \longrightarrow & (B \otimes B \otimes A \otimes A)/K \\ \downarrow & \nearrow & \\ (B \otimes B \otimes A \otimes A)/(B \otimes B \otimes \mathfrak{M}_A^{n+1}) = B \otimes B \otimes J_n(A) & & \end{array}$$

This induces a surjection $B \otimes B \otimes J_n(A) \to (B \otimes B \otimes A \otimes A)/K$ which is a left $B \otimes A$-module map. The kernel is T^{n+1} if T is the ideal $\mathfrak{M}_B \otimes J_n(A) + B \otimes B \otimes J_n^+(A) \subset B \otimes B \otimes J_n(A)$. Since $J_n(A)$ is of graded type there is a left A-module $V \subset J_n^+(A)$ where

$$J_n^+(A)^i = V^i \oplus J_n^+(A)^{i+1}.$$

This gives $J_n(A)$ the grading $A \oplus V \oplus V^2 \oplus \ldots \oplus V^n$. V certainly generates $J_n^+(A)$ as an ideal. Thus T is generated by $\mathfrak{M}_B \otimes A + B \otimes B \otimes V$ as an ideal. The grading on $J_n(A)$ induces a grading on $B \otimes B \otimes J_n(A)$ and T is a homogeneous ideal, since it is generated by homogeneous components. Thus T^{n+1} is homogeneous and in fact is the direct sum of graded components

$$T^{n+1} = (\mathfrak{M}_B^{n+1} \otimes A) \oplus (\mathfrak{M}_B^n \otimes V) \oplus \ldots \oplus (\mathfrak{M}_B \otimes V^n).$$

Thus $(B \otimes B \otimes J_n(A))/T^{n+1}$ is isomorphic to

$$\bigoplus_{i=0}^{n} ((B \otimes B)/\mathfrak{M}_B^{n+1-i}) \otimes V^i$$

as a left $B\otimes A$-module, where V^0 denotes A. By choice of V, $V^i \cong \operatorname{gr} J_n(A)_i$ and by definition $(B\otimes B)/\mathfrak{M}_B^{n+1-i} \cong J_{n-i}(B)$. This proves part 1.

Now part 4. Choose indeterminates $X_1, \ldots, X_s \in L$ so that L is a finite separable extension of $R(X_1, \ldots, X_s) \subset L$. By (13.13) it suffices to prove the result for $R(X_1, \ldots, X_s)$. The map $R[X_1, \ldots, X_s] \xrightarrow{\iota} R(X_1, \ldots, X_s)$ induces an algebra map

$$J(\iota) : J_n(R[X_1, \ldots, X_s]) \to J_n(R(X_1, \ldots, X_s)),$$

(13.2). By (13.4)

$$R(X_1, \ldots, X_s) \otimes_{R[X_1, \ldots, X_s]} J_n(R[X_1, \ldots, X_s]) \to J_n(R(X_1, \ldots, X_s))$$
$$u \otimes v \mapsto u(J(\iota)(v))$$

$u \in R(X_1, \ldots, X_s)$, $v \in J_n(R[X_1, \ldots, X_s])$, is an algebra isomorphism. Thus is suffices to prove the result for $R[X_1, \ldots, X_s]$ and this is done in part 3. Q.E.D.

Corollary **(13.20)**. — 1. *If* A *is a localization of a finitely generated* R-*algebra then* $J_n(A)$ *is finitely generated as a left* A-*module for all* n.

2. *Suppose* R *is a field and* K *is a field extension of* R *which is finitely generated* as a field over R. *Then* $J_n(K)$ *is a finite projective left* K-*module for all* n.

Proof. — 1. Suppose A is finitely generated as an R-algebra. Then for some $0 < m \in \mathbf{Z}$ there is an algebra surjection

$$R[X_1, \ldots, X_m] \to A.$$

By (13.19), 3), $J_n(R[X_1, \ldots, X_m])$ is finitely generated as a left $R[X_1, \ldots, X_m]$-module. By (13.7), 6) $J_n(A)$ is finitely generated as a left A-module. By (13.4), if S is any multiplicative system in A, then $J_n(A_S)$ is finitely generated as a left A_S-module. This proves part 1.

2. The projectivity is clear and the finiteness follows from part 1. Q.E.D.

Proposition **(13.21)**. — *Suppose* R *is a field of characteristic* p *and* K *is a field extension of* R *which is finitely generated as a field. Let*

$$A = \frac{K[X_1, \ldots, X_s, Y_1, \ldots, Y_t]}{\langle\{Y_i^{p^{e_i}}\}\rangle}$$

$0 < e_i \in \mathbf{Z}$. *Note,* t *is assumed to be zero if* $p = 0$. *Then* $J_n(A)$ *is a finite projective* A-*module for all* n.

Proof. — Let $A_1 = R[X_1, \ldots, X_s, Y_1, \ldots, Y_t]/\langle\{Y_i^{p^{e_i}}\}\rangle$ and $B_1 = K$. By (13.20), 2), $J_n(B_1)$ is a finite projective B_1-module for all n. By (13.19), 3), $J_n(A_1)$

is of finite projective graded type for all n. By (13.19), 2), $J_n(B_1 \otimes A_1)$ is a finite projective $B_1 \otimes A_1$-module for all n. Since $A \cong B_1 \otimes A_1$ we are done. Q.E.D.

Proposition **(13.22)**. — *Suppose* R *is a field of characteristic* p *and* K *is a field extension of* R *which is finitely generated as a field. Let*

$$A = \frac{K[X_1, \ldots, X_s, Y_1, \ldots, Y_t]}{\langle \{Y_i^{p^{e_i}}\} \rangle}$$

$0 < e_i \in \mathbf{Z}$, *where* t *is assumed to be zero if* $p = 0$. *Let* I *be the maximal ideal in* A *generated by the cosets of* $\{X_i\} \cup \{Y_i\}$. $\mathscr{A}$ *denotes the completion of* A *in the* I*-adic topology,* λ *the natural map* $A \to \mathscr{A}$ *and* $\mathfrak{I}$ *the closure of* $\lambda(I)$ *in* $\mathscr{A}$.

1. $\mathscr{A} \cong K[[X_1, \ldots, X_s, Y_1, \ldots, Y_t]]/\langle \{Y_i^{p^{e_i}}\} \rangle$ *as an algebra,* λ *corresponds to the natural map*

$$\frac{K[X_1, \ldots, X_s, Y_1, \ldots, Y_t]}{\langle \{Y_i^{p^{e_i}}\} \rangle} \to \frac{K[[X_1, \ldots, X_s, Y_1, \ldots, Y_t]]}{\langle \{Y_i^{p^{e_i}}\} \rangle}$$

and $\mathfrak{I}$ *corresponds to the ideal generated by the cosets of* $\{X_i\} \cup \{Y_i\}$ *in*

$$K[[X_1, \ldots, X_s, Y_1, \ldots, Y_t]]/\langle \{Y_i^{p^{e_i}}\} \rangle.$$

2. $\mathscr{J}_n(\mathscr{A})$ *is finitely generated and projective as a left* $\mathscr{A}$*-module for all* n, *where* $\mathscr{J}_n(\mathscr{A})$ *is defined just above* (13.9).

3. $D^n_{\mathscr{A}} = \mathrm{Hom}_{\mathscr{A}}(\mathscr{J}_n(\mathscr{A}), \mathscr{A})$ *for all* n.

Proof. — Part 1 is left to the reader. By (13.21), $J_n(A)$ is a finite projective A-module for all n. By (13.9), $\mathscr{J}_n(\mathscr{A})$ is isomorphic to the I-adic completion of $J_n(A)$ as left $\mathscr{A}$-modules. Since $J_n(A)$ is finitely generated as a left A-module it follows [4, Th. 3, p. 68] that $\mathscr{J}_n(\mathscr{A}) \cong \mathscr{A} \otimes_A J_n(A)$ as left $\mathscr{A}$-modules. This proves part 2.

Part 3 is implied by the definition of $\mathscr{J}_n(\mathscr{A})$ since $\mathscr{A}$ is complete in the $\mathfrak{I}$-adic topology. Q.E.D.

Definition **(13.23)**. — Suppose $\mathscr{A}$ is a commutative local algebra with maximal ideal $\mathfrak{I}$. Assume that for each $0 \leq n \in \mathbf{Z}$, $\mathscr{J}_n(\mathscr{A})$ is a finite projective left $\mathscr{A}$-module, where $\mathscr{J}_n(\mathscr{A})$ is defined above (13.9). Then $\mathscr{A}$ is called a *formal algebra*.

The $\mathscr{A}$ in (13.22), 1) is a formal algebra by (13.22), 2).

Proposition **(13.24)**. — *Let* R *be a field and* A *an algebra which is a localization of a finitely generated* R*-algebra. Assume that for each maximal ideal* $\mathfrak{N}$ *of* A *the completion of* A *in the* $\mathfrak{N}$*-adic topology is a formal algebra. Then for each* n $J_n(A)$ *is a finite projective left* A*-module.*

Proof. — $J_n(A)$ is a finitely generated left A-module by (13.20).

To prove that $J_n(A)$ is projective as a left A-module it suffices to do so locally. I.e. to show that for each maximal ideal $\mathfrak{N} \subset A$, $A_{\mathfrak{N}} \otimes J_n(A)$ is a projective (free) left $A_{\mathfrak{N}}$-module. By (13.4) it suffices to show that $J_n(A_{\mathfrak{N}}) \cong A_{\mathfrak{N}} \otimes_A J_n(A)$ is a projective left $A_{\mathfrak{N}}$-module. Note that $J_n(A_{\mathfrak{N}})$ is a finitely generated left $A_{\mathfrak{N}}$-module. Let $\widehat{A_{\mathfrak{N}}}$ and $\widehat{J_n(A_{\mathfrak{N}})}$ denote the completions of $A_{\mathfrak{N}}$ and $J_n(A_{\mathfrak{N}})$ in the $\mathfrak{N}$-adic topology. Since $J_n(A_{\mathfrak{N}})$ is a finitely generated left $A_{\mathfrak{N}}$-module, $\widehat{J_n(A_{\mathfrak{N}})} \cong \widehat{A_{\mathfrak{N}}} \otimes_{A_{\mathfrak{N}}} J_n(A_{\mathfrak{N}})$ as left $\widehat{A_{\mathfrak{N}}}$-modules. Since $A_{\mathfrak{N}}$ is noetherian and $J_n(A_{\mathfrak{N}})$ is finitely generated as an $A_{\mathfrak{N}}$-module it suffices to prove that $J_n(A_{\mathfrak{N}})$ is flat as an $A_{\mathfrak{N}}$-module [3, Cor. 2, p. 140]. By [4, Prop. 9, p. 72] and [3, Prop. 6, p. 48], it suffices to prove that $\widehat{A_{\mathfrak{N}}} \otimes_{A_{\mathfrak{N}}} J_n(A_{\mathfrak{N}}) \cong \widehat{J_n(A_{\mathfrak{N}})}$ is flat as a left $\widehat{A_{\mathfrak{N}}}$-module. By (13.9), $\widehat{J_n(A_{\mathfrak{N}})} \cong \mathscr{J}_n(\widehat{A_{\mathfrak{N}}})$ as left $\widehat{A_{\mathfrak{N}}}$-modules. By the assumption on $\widehat{A_{\mathfrak{N}}}$ being a formal algebra it follows that $\mathscr{J}_n(\widehat{A_{\mathfrak{N}}})$ is a projective hence flat left $\widehat{A_{\mathfrak{N}}}$-module. Q.E.D.

Proposition **(13.25)**. — *Suppose* R *is a field,* A *a localization of a finitely generated* R-*algebra and for each maximal ideal* $\mathfrak{N}$ *of* A, $A_{\mathfrak{N}}$ *is a regular local ring. Then for each n,* $J_n(A)$ *is a finite projective left* A-*module.*

Proof. — In view of (13.22), 2) this is a special case of (13.24). Q.E.D.

Proposition **(13.26)**. — *Suppose* R *is noetherian,* S *is a commutative* R-*algebra and* A *is a localization of a finitely generated* R-*algebra.*

1. $S \otimes A$ *is a localization of a finitely generated* S-*algebra.*
2. *Let* $J_{n,S}(S \otimes A)$ *denote* J_n *of* $S \otimes A$ *as an* S-*algebra. If* $J_{n,S}(S \otimes A)$ *is flat as a left* $S \otimes A$-*module and* S *is faithfully flat as an* R-*module then* $J_n(A)$ *is a finite projective left* A-*module.*
3. *If* R *and* S *are fields and for each maximal ideal* $\mathfrak{N}$ *in* $S \otimes A$, *the completion of* $S \otimes A$ *in the* $\mathfrak{N}$-*adic topology is a formal* S-*algebra then* $J_n(A)$ *is a finite projective left* A-*module.*

Proof. — Part 1 is left to the reader.

The finiteness in part 2 follows from (13.20). Since A is noetherian, by [3, Cor. 2, p. 140], it suffices to prove that $J_n(A)$ is a flat left A-module. This is implied by $S \otimes J_n(A)$ being a flat left $S \otimes A$-module. By (13.6), $S \otimes J_n(A) \cong J_{n,S}(S \otimes A)$ as $S \otimes A$-modules. This proves part 2.

Part 3 is implied by parts 1 and 2 and (13.24). Q.E.D.

(13.26) permits passage to the perfect closure or algebraic closure of R.

14. Simplicity and the center of D_A

In view of (12.1), *d)* and (12.2) one wishes to know when A is a simple D_A-module. In view of (9.3), *a)* one wishes to know the center of D_A since then one knows the centers of the algebras U over A with $\langle U \rangle \in \mathcal{G}\langle D_A \rangle$. These questions are partially answered in this section. Throughout the section A is a commutative R-algebra.

For a commutative ring S and $s \in S$ we let J_s denote the ideal $\{x \in S \mid xs = 0\}$. If T is a subset of S we let J(T) denote $\bigcup_{0 \neq t \in T} J_t$.

Lemma (**14.1**). — *Let* S *be a commutative ring with subset* T.

a) *For* $x \in J_s$, *if* $1 + x$ *is invertible in* S, *then the inverse is of the form* $1 + y$ *with* $y \in J_s$.

b) *For* $x \in J(T)$, *let* I *be the ideal* $S(1+x)$. *Then there is* $0 \neq t \in T \cap (\bigcap_{n=1}^{\infty} I^n)$.

c) *If* S *is Noetherian and* I *is an ideal of* S *where* $0 \neq t \in T \cap (\bigcap_{n=1}^{\infty} I^n)$, *then* I *contains an element of the form* $1 + x$ *with* $x \in J(T)$.

d) *If, for each proper ideal* $K \subset S$, *the intersection* $\bigcap_{n=1}^{\infty} K^n$ *is zero, then* $1 + d$ *is invertible for each zero divisor* $d \in S$. *If* S *is Noetherian and* $1 + d$ *is invertible for each zero divisor* $d \in S$, *then for each ideal* $K \subsetneq S$ *the intersection* $\bigcap_{n=1}^{\infty} K^n$ *is zero.*

e) *Suppose* B *is a commutative* R-*algebra which is flat as an* R-*module. Then in* $B \otimes A$, $J_{1 \otimes a} = B \otimes J_a$.

Proof. — *a)* Suppose $1 + x$ has inverse z. Multiply $(1+x)s = s$ on both sides by z to obtain $s = zs$. Then $z - 1 \in J_s$.

b) Given $x \in J(T)$ let $0 \neq t \in T$ with $xt = 0$. Then $t = (1+x)^n t \in I^n$ for all n. Thus $0 \neq t \in T \cap (\bigcap_n I^n)$.

c) By the Krull intersection theorem there is a $y \in I$ with $yt = t$. Then $y - 1 \in J(T)$.

d) The first statement follows from part *b)* with $T = S$. The second statement follows from part *c)* with $T = S$.

e) Let $I = A.a$. The sequence $0 \to J_a \to A \xrightarrow{a^\ell} I \to 0$ is exact. Tensoring by the flat R-module B gives the exact sequence

$$0 \longrightarrow B \otimes J_a \longrightarrow B \otimes A \xrightarrow{1 \otimes (a^\ell)} B \otimes I \longrightarrow 0$$

Since $1 \otimes (a^\ell) = (1 \otimes a)^\ell$ it follows that $J_{1 \otimes a} = B \otimes J_a$. Q.E.D.

Definition (**14.2**). — An element $0 \neq a \in A$ has the *strong intersection property* if for each commutative R-algebra B the elements $x \in B \otimes J_a$ are such that $1 + x$ is invertible in

$B \otimes A$. The algebra A has the strong intersection property if each $0 \neq a \in A$ has the strong intersection property.

Suppose $0 \neq a \in A$ has the strong intersection property, B is a commutative flat R-algebra where $B \otimes A$ is Noetherian and I is an ideal in $B \otimes A$. Furthermore suppose that $0 \neq 1 \otimes a \in \bigcap_{n=1}^{\infty} I^n$. Applying (14.1), *c)* with $T = \{1 \otimes a\}$ it follows that there is $x \in J_{1 \otimes a}$ with $1 + x \in I$. By (14.1), *e)* $x \in B \otimes J_a$ and hence, by definition of the strong intersection property, $1 + x$ is invertible. This implies that $I = B \otimes A$ and gives part *a)* in:

Lemma **(14.3)**. — *Let* B *be a commutative flat* R*-algebra where* $B \otimes A$ *is Noetherian and* I *is an ideal in* $B \otimes A$.

a) *If* $a \in A$ *has the strong intersection property and* $1 \otimes a \in \bigcap_{n=1}^{\infty} I^n$, *then either* $1 \otimes a = 0$ *or* $I = B \otimes A$.

b) *If* A *has the strong intersection property and we let* $\widetilde{A}$ *denote* $\operatorname{Im}(A \xrightarrow{a \mapsto 1 \otimes a} B \otimes A)$, *then either* $\widetilde{A} \cap (\bigcap_{n=1}^{\infty} I^n) = 0$ *or* $I = B \otimes A$.

Proof. — Part *a)* is proved just above (14.3). Part *b)* follows from part *a)*. Q.E.D.

Proposition **(14.4)**. — a) *An element* $0 \neq a \in A$ *has the strong intersection property if* J_a *consists of nilpotent elements.*

b) *Suppose* R *is an algebraically closed field and* $0 \neq a \in A$. *Assume that for each* $x \in J_a$ *the element* $1 + x$ *is invertible in* A *and that* J_a *is contained in a finitely generated subalgebra of* A. *Then a has the strong intersection property.*

Proof. — *a)* Suppose B is a commutative R-algebra. Since J_a consists of nilpotent elements the image of $B \otimes J_a \to B \otimes A$ consists of nilpotent elements. Thus $x \in B \otimes J_a$ is nilpotent and $1 + x$ is invertible. This proves part *a)*.

b) Let C be any subalgebra of A containing J_a. For $x \in J_a$, $1 + x$ is invertible in A by hypothesis. By (14.1), *a)* the inverse actually lies in C.

Let B be a commutative R-algebra and $z \in B \otimes J_a$. If $z = \sum_i b_i \otimes a_i$, then there is a finitely generated subalgebra D of B with $\{b_i\} \subset D$. Since by hypothesis J_a lies in a finitely generated subalgebra of A, there is a finitely generated subalgebra C of A with $a \in C$, $J_a \subset C$ and $\{a_i\} \subset C$. We shall show that $1 + z$ is an invertible element of $D \otimes C$.

If $1 + z$ is not invertible in $D \otimes C$ it lies in some maximal ideal. Since $D \otimes C$ is a finitely generated R-algebra and R is algebraically closed, we can apply the Hilbert Nullstellensatz to conclude that there is an R-algebra homomorphism $f : D \otimes C \to R$ with $f(1 + z) = 0$. Clearly f is of the form $D \otimes C \xrightarrow{\sigma \otimes \rho} R \otimes R = R$ where $\sigma : D \to R$ and $\rho : C \to R$ are R-algebra homomorphisms.

Let $\tilde{\sigma} : D\otimes C \to C$, $d\otimes c \mapsto \sigma(d)c$. This is an R-algebra homomorphism and $\rho\tilde{\sigma} = f$. Since $z(1\otimes a) = 0$, $\tilde{\sigma}(1\otimes a) = a \in C$ we have $\tilde{\sigma}(z)a = \tilde{\sigma}(z)\tilde{\sigma}(1\otimes a) = \tilde{\sigma}(z(1\otimes a)) = 0$. Hence $\tilde{\sigma}(z) \in J_a$ and by hypothesis $1 + \tilde{\sigma}(z)$ is invertible. This implies $\rho(1+\tilde{\sigma}(z)) \neq 0$. But $\rho(1+\tilde{\sigma}(z)) = \rho\tilde{\sigma}(1+z) = f(1+z) = 0$. The contradiction shows that $1+z$ must be invertible and $a \in A$ has the strong intersection property. Q.E.D.

Corollary **(14.5)**. — a) A *has the strong intersection property if all zero divisors in* A *are nilpotent. In particular integral domains have the strong intersection property.*

b) *If* R *is an algebraically closed field,* A *is a finitely generated* R*-algebra and for each proper ideal* $I \subset A$ *the intersection* $\bigcap_n I^n$ *is zero, then* A *has the strong intersection property.*

Proof. — *a)* Since J_a consists of zero divisors for $0 \neq a \in A$ part *a)* follows from (14.4), *a)*.

b) By (14.1), *d)*, for each zero divisor $d \in A$, the element $1 + d$ is invertible. Hence by (14.4), *b)* A has the strong intersection property. Q.E.D.

Proposition **(14.6)**. — *Suppose* A *has the strong intersection property and let*

$$\mathfrak{M} = \operatorname{Ker}(A\otimes A \xrightarrow{\text{mult}} A).$$

If $0 \neq I$ *is a proper ideal of* A *where* I *and* A/I *are flat* R*-modules and* $(A/I)\otimes A$ *is a Noetherian ring, then there is* $0 < n \in \mathbf{Z}$ *such that*

$$(*) \qquad A\otimes I \not\subset I\otimes A + \mathfrak{M}^n.$$

Proof. — Let $\pi : A \to A/I$ be the natural map. In $(A/I)\otimes A$ let $\mathfrak{N} = (\pi\otimes I)(\mathfrak{M})$. Since $\mathfrak{N}$ lies in the kernel of the composite

$$(A/I)\otimes A \xrightarrow{I\otimes\pi} (A/I)\otimes(A/I) \xrightarrow{\text{mult}} A/I, \qquad \mathfrak{N} \neq (A/I)\otimes A.$$

To verify $(*)$ it suffices to prove that for some $0 < n \in \mathbf{Z}$

$$(\pi\otimes I)(A\otimes I) \not\subset (\pi\otimes I)(I\otimes A + \mathfrak{M}^n)$$

which reduces to showing

$$(A/I)\otimes I \not\subset \mathfrak{N}^n.$$

for some $0 < n \in \mathbf{Z}$.

By flatness of I and A/I the following composite is injective

$$I = R\otimes I \to (A/I)\otimes I \to (A/I)\otimes A.$$

Thus for $0 \neq x \in I$, it follows that $0 \neq 1\otimes x \in (A/I)\otimes A$. By (14.3), *b)* there is $0 < n \in \mathbf{Z}$ with $1\otimes x \notin \mathfrak{N}^n$. Q.E.D.

Theorem **(14.7).** — *Suppose* A *has the strong intersection property and almost finite projective differentials* (8.5). *Furthermore suppose that for each ideal* $0 \neq I \subsetneq A$ *both* I *and* A/I *are flat* R-*modules and* $(A/I) \otimes A$ *is a Noetherian ring. Then* A *is a simple* D_A-*module.*

Note. — If $A \otimes A$ is Noetherian then $(A/I) \otimes A$ is Noetherian for any ideal $I \subset A$.

Proof. — By (12.2) we may apply (12.1). Let $\{L_\alpha\}$ be as in (8.5). Suppose $0 \neq I$ is a proper ideal of A. By (14.6) there is $0 < n \in \mathbf{Z}$ where $A \otimes I \not\subset I \otimes A + \mathfrak{M}^n$. Since $\{L_\alpha\}$ is cofinal with $\{\mathfrak{M}^i\}$ there is L_α with $A \otimes I \not\subset I \otimes A + L_\alpha$. By (12.1), *e)* A is a simple D_A-module. Q.E.D.

Now we wish to study the center of D_A. In $A \otimes A$ let $\mathfrak{M}^\infty = \bigcap_n \mathfrak{M}^n$. Let f and g denote the composites:

$$f : A \xrightarrow{a \mapsto 1 \otimes a} A \otimes A \longrightarrow (A \otimes A)/\mathfrak{M}^\infty$$

$$g : A \xrightarrow{a \mapsto a \otimes 1} A \otimes A \longrightarrow (A \otimes A)/\mathfrak{M}^\infty.$$

Definition **(14.8).** — Let $Z_R(A) = \{a \in A \mid f(a) = g(a)\}$.

Clearly $Z_R(A)$ is a subalgebra of A. Often we write Z(A) for $Z_R(A)$.

Theorem **(14.9).** — *Suppose* A *has almost finite projective differentials. Then* Z(A) *is the center of* D_A. (*Here we are identifying* Z(A) *with* $Z(A)^\ell$ *and considering* $Z(A)^\ell \subset A^\ell \subset D_A$.)

Proof. — Let $\{L_\alpha\}$ be as in (8.5). By (12.2) we may characterize the center of D_A by (12.1), *f)*. Since $\{L_\alpha\}$ is cofinal with $\{\mathfrak{M}^i\}$, (12.1), *f)* implies that the center of D_A is $(\{a \in A \mid 1 \otimes a - a \otimes 1 \in \mathfrak{M}^i \text{ for all } i\})^\ell$. Thus the center of D_A is

$$(\{a \in A \mid 1 \otimes a - a \otimes 1 \in \mathfrak{M}^\infty\})^\ell = Z(A)^\ell. \qquad \text{Q.E.D.}$$

Lemma **(14.10).** — a) *Suppose* A *and* B *are commutative* R-*algebras and* $\varphi : A \to B$. *Then* $\varphi(Z(A)) \subset Z(B)$.

b) *Suppose* C *is an* R-*algebra which is a finite separable extension of* R (13.10). *Then* $C = Z(C)$.

c) *Suppose* A *is a commutative* R-*algebra with subalgebra* C *where* C *is a finite separable extension of* R. *Then* $C \subset Z(A)$.

d) *Suppose* R *is a field and* S *is a commutative* R-*algebra. We can consider* $S \otimes A$ *as an* S-*algebra and so* $Z_S(S \otimes A)$ *is defined. The natural map* $S \otimes Z_R(A) \to S \otimes A$ *carries* $S \otimes_R Z(A)$ *isomorphically to* $Z_S(S \otimes A)$.

e) *Suppose* R *is an algebraically closed field,* $A \otimes A$ *is Noetherian and* $a \in A$ *where* a *is transcendental over* R. *Then* $a \notin Z(A)$.

Proof. — Let $\mathfrak{M}_A = \mathrm{Ker}(A \otimes A \xrightarrow{\text{mult}} A)$ and $\mathfrak{M}_B = \mathrm{Ker}(B \otimes B \xrightarrow{\text{mult}} B)$. Then

$$(\varphi \otimes \varphi)(\mathfrak{M}_A) \subset \mathfrak{M}_B$$

so that $(\varphi\otimes\varphi)(\mathfrak{M}_A^\infty)\subset\mathfrak{M}_B^\infty$. Clearly the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ A\otimes A & \xrightarrow{\varphi\otimes\varphi} & B\otimes B \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ A\otimes A & \xrightarrow{\varphi\otimes\varphi} & B\otimes B \end{array}$$

commute.

Thus for $a\in A$, if $(f-g)(a)\in\mathfrak{M}_A^\infty$, then $(f-g)\,\varphi(a)\in\mathfrak{M}_B^\infty$ and it follows that

$$\varphi(Z(A))\subset Z(B).$$

b) Since C is a projective $C\otimes C$-module the map $C\otimes C \xrightarrow{\text{mult}} C$ has a $C\otimes C$-module splitting. Then $C\otimes C$ is the direct sum of $\mathfrak{M}_C$ and the ideal which is the image of the splitting. The component of 1 in $\mathfrak{M}_C$ is an idempotent which generates $\mathfrak{M}_C$. Thus $\mathfrak{M}_C^2=\mathfrak{M}_C$ and $\mathfrak{M}_C^\infty=\mathfrak{M}_C$. For all $c\in C$, $(f-g)(c)=1\otimes c-c\otimes 1\in\mathfrak{M}_C$ and so $C=Z(C)$.

c) follows from *a)* and *b)* when we consider the inclusion map $C\hookrightarrow A$.

d) There is a natural isomorphism $(S\otimes A)\otimes_S(S\otimes A)\cong S\otimes A\otimes A$. Under this isomorphism $\mathfrak{M}_{S\otimes A}$ corresponds to $S\otimes\mathfrak{M}_A$. Then $\mathfrak{M}_{S\otimes A}^n$ corresponds to $S\otimes\mathfrak{M}_A^n$. Since R is a field $\bigcap_{n=1}^{\infty}(S\otimes\mathfrak{M}_A^n)=S\otimes(\bigcap_{n=1}^{\infty}\mathfrak{M}_A^n)$. Thus $\mathfrak{M}_{S\otimes A}^\infty$ corresponds to $S\otimes\mathfrak{M}_A^\infty$. Finally under the isomorphism the f and g maps from $S\otimes A$ to $(S\otimes A)\otimes_S(S\otimes A)$ correspond to $S\otimes A\xrightarrow{1\otimes f}S\otimes A\otimes A$ and $S\otimes A\xrightarrow{1\otimes g}S\otimes A\otimes A$ respectively. Thus

$$\{x\in S\otimes A\,|\,(f-g)(x)\in\mathfrak{M}_{S\otimes A}^\infty\}$$

equals the kernel of the composite

$$S\otimes A\xrightarrow{1\otimes(f-g)}S\otimes A\otimes A\longrightarrow S\otimes((A\otimes A)/\mathfrak{M}_A^\infty).$$

The kernel of this map is $S\otimes Z(A)$.

e) Let T be the multiplicative system in A generated by $\{a-r\}_{r\in R}$. If a is transcendental over R then no product $(a-r_1)\,\dots\,(a-r_n)$ is equal to zero. Otherwise a would be a zero of the polynomial $(X-r_1)\,\dots\,(X-r_n)$. Hence $0\notin T$ and there is an ideal $\mathfrak{P}\subset A$ which is maximal with respect to $\mathfrak{P}\cap T=\emptyset$. As is well known $\mathfrak{P}$ is prime. Let $\pi: A\to A/\mathfrak{P}$. By part *a)* it suffices to prove that $\pi(a)\notin Z(A/\mathfrak{P})$. By choice of the multiplicative system T we know that $\pi(a)\notin R$. Hence

$$0\neq 1\otimes\pi(A)-\pi(a)\otimes 1\in(A(\mathfrak{P})\otimes(A/\mathfrak{P})).$$

Since R is algebraically closed and $A/\mathfrak{P}$ is an integral domain we have that $(A/\mathfrak{P})\otimes(A/\mathfrak{P})$ is an integral domain. Since $A\otimes A$ is Noetherian so is $(A/\mathfrak{P})\otimes(A/\mathfrak{P})=(\pi\otimes\pi)(A\otimes A)$. Hence by (14.1), *d)*, $\mathfrak{M}_{A/\mathfrak{P}}^\infty=0$ and $1\otimes\pi(a)-\pi(a)\otimes 1\notin\mathfrak{M}_{A/\mathfrak{P}}^\infty$. Thus $\pi(a)\notin Z(A/\mathfrak{P})$.

Q.E.D.

To further characterize $Z(A)$ we must assume that R is a field.

Definition (**14.11**). — For a commutative algebra A over a field R, $\mathrm{Sep}_R A$ denotes $\{a \in A \mid$ there is a separable polynomial $0 \neq f \in R[X]$ with $f(a) = 0\}$.

We often write Sep A for $\mathrm{Sep}_R A$.

Sep A has the following properties:

(**14.12**)

1. Sep A is a subalgebra of A.
2. If B is another commutative algebra, then the inclusion
$$\mathrm{Sep}\,A \otimes \mathrm{Sep}\,B \to A \otimes B$$
has image $\mathrm{Sep}(A \otimes B)$. Thus $\mathrm{Sep}\,A \otimes \mathrm{Sep}\,B$ is naturally isomorphic to $\mathrm{Sep}(A \otimes B)$.
3. If B is another commutative algebra, then the inclusion
$$\mathrm{Sep}\,A \oplus \mathrm{Sep}\,B \to A \oplus B$$
has image $\mathrm{Sep}(A \oplus B)$. Thus $\mathrm{Sep}\,A \oplus \mathrm{Sep}\,B$ is naturally isomorphic to $\mathrm{Sep}(A \oplus B)$.
4. If S is a field extension of R and $\mathrm{Sep}_S(S \otimes A)$ is the Sep of $S \otimes A$ as an S-algebra, then the map $S \otimes \mathrm{Sep}\,A \to S \otimes A$ maps $S \otimes \mathrm{Sep}\,A$ isomorphically to $\mathrm{Sep}_S(S \otimes A)$.
5. If R is algebraically closed, then Sep A is spanned by idempotents.
6. If Sep A is spanned by idempotents and A is Noetherian, then Sep A is finite dimensional.
7. If Sep A is spanned by idempotents and is finite dimensional, then there is a unique set $\{e_1, \ldots, e_n\}$ of minimal orthogonal idempotents in Sep A which is a basis for Sep A and any idempotent in Sep A is of the form $e_{i_1} + \ldots + e_{i_m}$ for a set $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$.
8. Suppose $\sigma : A \to B$ is an algebra map where B is a commutative algebra. Then $\sigma(\mathrm{Sep}\,A) \subset \mathrm{Sep}\,B$. If σ is surjective and $\mathrm{Ker}\,\sigma$ consists of nilpotent elements, then σ maps Sep A isomorphically to Sep B.

Theorem (**14.13**). — *Suppose* A *is a commutative algebra over a field* R *and* S *is the algebraic closure of* R. *If* $S \otimes A \otimes A$ *is Noetherian then* $Z(A) = \mathrm{Sep}\,A$ *and* Sep A *is finite dimensional.*

Proof. — By (14.10), *d)* and (14.12), 4) it suffices to prove that
$$Z_S(S \otimes A) = \mathrm{Sep}_S(S \otimes A).$$

Since $S \otimes A \otimes A$ is Noetherian we have that $(S \otimes A) \otimes_S (S \otimes A)$ is Noetherian. Thus we may assume that R is algebraically closed and $A \otimes A$ is Noetherian.

Let $n \in Z(A)$. If n is invertible it is not nilpotent. Suppose n is not invertible and $\mathfrak{N} = A.n$. Let $\pi : A \to A/\mathfrak{N}$.

Since $A \otimes A$ is Noetherian and $1 \otimes n - n \otimes 1 \in \mathfrak{M}_A^\infty$ it follows from the Krull intersection theorem that there is $u \in \mathfrak{M}_A$ with $u(1 \otimes n - n \otimes 1) = 1 \otimes n - n \otimes 1$. Hence

$$(u-1)(1\otimes n-n\otimes 1)=0$$

and $$0=((\pi\otimes I)(u-1))(\pi\otimes I(1\otimes n-n\otimes 1))=((\pi\otimes I)(u-1))(1\otimes n).$$

Thus by (14.1), *e)* $(\pi\otimes I)(u-1)\in(A/\mathfrak{N})\otimes J_n$ and so $u-1\in\mathfrak{N}\otimes A+A\otimes J_n$.

Since $u\in\mathfrak{M}_A$, $\text{mult}(u)=0$ and $\text{mult}(u-1)=-1$. Thus $-1\in\mathfrak{N}_A+AJ_n=\mathfrak{N}+J_n$ and $A=\mathfrak{N}+J_n$. Say $1=an+b$ with $an\in\mathfrak{N}$, $b\in J_n$. Since $nb=0$ this proves that an is idempotent. Since an is idempotent n cannot be nilpotent.

Thus far we have shown that Z(A) contains no nilpotent elements. By (14.10), *e)* Z(A) contains no transcendental elements. Let $z\in Z(A)$. Since z is not transcendental it generates a finite dimensional subalgebra $C\subset Z(A)$. Since C contains no nilpotent elements it is semi-simple over R. Since R is algebraically closed C is the direct sum of copies of R. Hence C is spanned by idempotents which are separable and $C\subset \text{Sep}\,A$. Thus $Z(A)\subset\text{Sep}\,A$.

Since $A\otimes A$ is Noetherian so is the homomorphic image $A=\text{mult}(A\otimes A)$. Thus by (14.12), 5) and 6), Sep A is finite dimensional.

It can be shown that Sep A is a finite separable extension of R in the sense of (13.10). In which case $\text{Sep}\,A\subset Z(A)$ by (14.10), *c)*. Alternatively, by (14.12, 5), Sep A is spanned by idempotents. If e is an idempotent in A, then direct calculation proves that $(1\otimes e-e\otimes 1)^3=1\otimes e-e\otimes 1$. Thus $1\otimes e-e\otimes 1=(1\otimes e-e\otimes 1)^{3^n}\in\mathfrak{M}_A^{3^n}$ for all n and $1\otimes e-e\otimes 1\in\mathfrak{M}_A^{\infty}$. This shows that $e\in Z(A)$. Since Sep A is spanned by idempotents $\text{Sep}\,A\subset Z(A)$. Q.E.D.

15. Cohomology of a $\times_A$-bialgebra

Throughout this section A is a commutative algebra and $(B, \Delta, \mathscr{I})$ is a cocommutative $\times_A$-bialgebra. If C is a $\times_A$-coalgebra and D is an A-coalgebra then $C\otimes_A D$ has an A-coalgebra structure described in (11.1). This coalgebra structure shall be used frequently. If D is an A-coalgebra and K an A-algebra then $\text{Hom}_A(D, K)$ has an A-algebra structure [17, p. 69-70]. The unit is the composite $D\xrightarrow{\varepsilon} A\xrightarrow{(a\mapsto a.1)} K$ for $f, g\in\text{Hom}_A(D, K)$ the product $f*g$ is given by

$$D\xrightarrow{\Delta}\int_x {}_xD\otimes_x D\xrightarrow{f\otimes g}\int_x {}_xK\otimes_x K\xrightarrow{\text{mult}} K.$$

When D is cocommutative and K is commutative then $\text{Hom}_A(D, K)$ is a commutative A-algebra.

Definition **(15.1).** — $\text{Reg}_A(D, K)$ denotes the group of invertible elements in $\text{Hom}_A(D, K)$.

When D is cocommutative and K is commutative then $\text{Reg}_A(D, K)$ is an abelian group.

When E is another A-coalgebra and $D \to E$ is a coalgebra map, the induced map $\mathrm{Hom}_A(E, K) \to \mathrm{Hom}_A(D, K)$ is an algebra map. Hence it induces a group map $\mathrm{Reg}_A(E, K) \to \mathrm{Reg}_A(D, K)$.

Lemma **(15.2)**. — a) *Suppose* C *is a* $\times_A$*-coalgebra,* D *and* E *are* A*-coalgebras and* $f: D \to E$ *is a coalgebra map. Then* $I \otimes f : C \otimes_A D \to C \otimes_A E$ *is a coalgebra map.*

b) *Suppose* D *is an* A*-coalgebra and* $C_1, \ldots, C_r$ *are* $\times_A$*-coalgebras. Then*

$$C_1 \otimes_A C_2 \otimes_A \ldots \otimes_A C_r \otimes_A D$$

is an A*-coalgebra by* (11.1) *iterated.*

c) *Suppose* C_1 *and* C_2 *are* $\times_A$*-coalgebras,* $f: C_1 \otimes_A C_2 \to C_1$ *is a coalgebra map and an* A*-bimodule map and* D *is an* A*-coalgebra. Then* $f \otimes I : C_1 \otimes_A C_2 \otimes_A D \to C_1 \otimes_A D$ *is a coalgebra map.*

d) *For the* $\times_A$*-bialgebra* B *the map* $B \otimes_A B \xrightarrow{\text{mult}} B$ *is an* A*-bimodule map and a coalgebra map. The map* $B \xrightarrow{\varepsilon} A$ *is a coalgebra map.*

e) *The following two maps* e_0 *and* e_1 *from* A *to* $\mathrm{Hom}_A(B, A)$ *are algebra maps. For* $a \in A$, $b \in B$

$$e_0(a)\,(b) = b\,.\,a$$

$$e_1(a)\,(b) = a\varepsilon(b).$$

Note. — *In e (and f),* $b\,.\,a$ *or* $b_0.f(b_1 \otimes \ldots \otimes b_n)$ *is the natural action of* B *on* A (5.7). *Also* $\mathrm{Hom}_A(B \otimes_A \ldots \otimes_A B, A)$ *denotes* $\int^y \mathrm{Hom}({}_yB \otimes_A \ldots \otimes_A B, {}_yA)$.

f) *For* $0 < n \in \mathbf{Z}$ *the following* $n+2$ *maps from*

$$\mathrm{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n}, A) \quad \textit{to} \quad \mathrm{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n+1}, A)$$

are algebra maps: for $f \in \mathrm{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n}, A)$, $\{b_i\}_0^n \subset B$

$$e_0(f)\,(b_0 \otimes \ldots \otimes b_n) = b_0 f(b_1 \otimes \ldots \otimes b_n)$$

$$e_1(f)\,(b_0 \otimes \ldots \otimes b_n) = f(b_0 b_1 \otimes b_2 \otimes \ldots \otimes b_n)$$

$$e_2(f)\,(b_0 \otimes \ldots \otimes b_n) = f(b_0 \otimes b_1 b_2 \otimes b_3 \otimes \ldots \otimes b_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$e_n(f)\,(b_0 \otimes \ldots \otimes b_n) = f(b_0 \otimes \ldots \otimes b_{n-2} \otimes b_{n-1} b_n)$$

$$e_{n+1}(f)\,(b_0 \otimes \ldots \otimes b_n) = f(b_0 \otimes \ldots \otimes b_{n-2} \otimes b_{n-1} \varepsilon(b_n)).$$

Proof. — *a)* and *c)* are left to the reader.

b) This is obvious once $C_1 \otimes_A \ldots \otimes_A C_r \otimes_A D$ is viewed as

$$C_1 \otimes_A (C_2 \otimes_A (C_3 \ldots \otimes_A (C_{r-1} \otimes_A (C_r \otimes_A D)) \ldots)).$$

d) For any A-coalgebra D, $\varepsilon : D \to A$ is a coalgebra map. Hence $\varepsilon : B \to A$ is a coalgebra map. For $b \otimes c \in B \otimes_A B$, $\varepsilon(b \otimes c) = \varepsilon(b\varepsilon(c))$ by definition of the coalgebra structure on $B \otimes_A B$ (11.1). By the remark following (5.7), $\varepsilon(b\varepsilon(c)) = \varepsilon(bc)$. Hence $\text{mult} : B \otimes_A B \to B$ preserves co-unit. Since B is a $\times_A$-bialgebra, $\Delta : B \to B \times_A B$ is multiplicative. It is left to the reader to show that the multiplicativity of Δ implies that $\text{mult} : B \otimes_A B \to B$ preserves diagonalization.

e) The e_1 map is simply $A \to \text{Hom}_A(B, A)$, $(a \mapsto a.1)$ and hence is an A-algebra map. For $a, a' \in A$, $b \in B$ let $\Delta(b) = \sum_i b_i \otimes b'_i \in B \times_A B \subset \int_x {}_x B \otimes_x B$. Then

$$(e_0(a)\, e_0(a'))\,(b) = \sum_i (b_i . a)\,(b'_i . a')$$

which equals $b(aa')$ by (5.8), *c)* and $b(aa') = e_0(aa')\,(b)$. Thus e_0 is multiplicative. For $b \in B$, $e_0(1)(b) = b.1 = \mathscr{I}(b)(1) = \varepsilon\mathscr{I}(b) = \varepsilon(b)$. Hence $e_0(1)$ is the unit in $\text{Hom}_A(B, A)$.

f) If D is an A-coalgebra and K and L are A-algebras and $K \to L$ is an algebra map, then the induced map $\text{Hom}_A(D, K) \to \text{Hom}_A(D, L)$ is an algebra map. We apply this where $K = A$, $L = \text{Hom}_A(B, A)$ and $A \xrightarrow{e_0} \text{Hom}_A(B, A)$ with e_0 as in part *e)*. This gives the algebra map

$$(*) \qquad \text{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n}, A) \to \text{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n}, \text{Hom}_A(B, A)).$$

Identify $\text{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n}, \text{Hom}_A(B, A))$ with $\text{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n+1}, A)$ by the usual adjointness relation; the map $(*)$ becomes e_0 in part *f)*. Thus e_0 is an algebra map.

By parts *b)* and *d)* the map

$$\overbrace{B \otimes_A \ldots \otimes_A B}^{i+1} \xrightarrow{\text{mult} \otimes 1 \otimes \ldots \otimes 1} \overbrace{B \otimes_A \ldots \otimes_A B}^{i}$$

is a coalgebra map. Thus by part *a)* the map

$$\overbrace{B \otimes_A \ldots \otimes_A B}^{n+1} \xrightarrow{1 \otimes \ldots \otimes 1 \otimes \text{mult} \otimes 1 \otimes \ldots \otimes 1} \overbrace{B \otimes_A \ldots \otimes_A B}^{n}$$

is a coalgebra map. This shows that $e_1, \ldots, e_n$ are algebra maps.

By parts *d)* and *a)* the map

$$\overbrace{B \otimes_A \ldots \otimes_A B}^{n+1} \xrightarrow{1 \otimes \ldots \otimes 1 \otimes \varepsilon} \overbrace{B \otimes_A \ldots \otimes_A B}^{n} \otimes_A A = \overbrace{B \otimes_A \ldots \otimes_A B}^{n}$$

is a coalgebra map. Thus e_{n+1} is an algebra map. Q.E.D.

It is left to the reader to verify that

$$\textbf{(15.3)} \qquad \{\text{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n}, A), e_0, \ldots, e_{n+1}\}_{n=0}^{\infty}$$

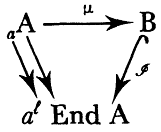
forms a semi-co-simplicial complex.

Note. — $\overbrace{B\otimes_A\ldots\otimes_A B}^{0}$ denotes A.

Definition **(15.4).** — The homology of the above semi-co-simplicial complex with respect to the " units " functor is denoted $H^*(B)$.

H^0 *theorem* **(15.5).** — *Suppose* $\mathscr{I}: B\rightarrow \text{End}\, A$ *is injective and* $\mu: A\rightarrow B$, $a\mapsto a.1$. *Then* μ *maps* $H^0(B)$ *isomorphically to the group of units in the center of* B.

Proof. — Since $\mathscr{I}$ is a map of algebras over A the diagram



commutes.

This shows that μ is injective, and since $\mathscr{I}$ is injective, $\mu(A)$ is a maximal commutative subring of B and thus contains the center of B.

For $b\in B$ $a, a'\in A$

$$(ba').a=(\mathscr{I}(b)a^\ell)(a')$$
$$a\varepsilon(ba')=a^\ell\mathscr{I}(b)(a').$$

Hence $e_0(a)=e_1(a)$ is equivalent to a^ℓ lying in the center of $\mathscr{I}(B)$. Since $\mathscr{I}$ is injective

$$(*)\qquad e_0(a)=e_1(a)$$

is equivalent to $\mu(a)$ lying in the center of B.

Applying the units functor to $(*)$ concludes the proof. Q.E.D.

Lemma **(15.6).** — *Suppose* C *is a* $\times_A$-*bialgebra,* $E=E_C$ *as in* (6.1) *and* ${}_xC\otimes_A\ldots\otimes_A C_y$ *is considered as an* A-*bimodule with the x left* A-*module structure and the y right* A-*module structure.*

a) *If* $f: C\otimes_A\ldots\otimes_A C\rightarrow A$ *is a left* A-*module map and* $f^t: C\otimes_A\ldots\otimes_A C\rightarrow \text{End}\, A$ *as in* (5.2), *then* $\text{Im} f^t\subset E$. *In fact for* $c, \ldots, d\in C$ *with*

$$\Delta c=\sum_i c_i\otimes c_i', \ \ldots, \ \Delta d=\sum_j d_j\otimes d_j'\in C\times_A C\subset\int_x {}_xC\otimes_x C$$

define $f^{to}: C\otimes_A\ldots\otimes_A C\rightarrow C$ *by*

$$f^{to}(c\otimes\ldots\otimes d)=\sum_i\ldots\sum_j f(c_i'\otimes\ldots\otimes d_j')c_i\ldots d_j.$$

Then $f^t=\mathscr{I}f^{to}$, f^{to} *is an* A-*bimodule map.*

b) *If* $g: C\otimes_A\ldots\otimes_A C\rightarrow \text{End}\, A$ *is any* A-*bimodule map, then* $\text{Im}\, g\subset E$.

Proof. — *b)* follows from *a)* by the remark below (5.2). *a)* is a jazzed-up version of (10.1), *a)* and is proved as follows:

For $c, \ldots, d$ as in *a)* and $a \in \mathrm{A}$

$$\begin{aligned} f^t(c\otimes\ldots\otimes d)(a) &= f(c\otimes\ldots\otimes da) \\ &= \sum_i \ldots \sum_j f(\mathscr{I}(c_i)(\ldots\mathscr{I}(d_j)(a)\ldots)c'_i\otimes\ldots\otimes d'_j) \\ &= \sum_i \ldots \sum_j \mathscr{I}(c_i)(\ldots\mathscr{I}(d_j)(a)\ldots)f(c'_i\otimes\ldots\otimes d'_j) \\ &= \sum_i \ldots \sum_j f(c'_i\otimes\ldots\otimes d'_j)\mathscr{I}(c_i\ldots d_j)(a) \\ &= \mathscr{I}f^{to}(c\otimes\ldots\otimes d)(a). \end{aligned}$$

The first equality follows from the definition of f^t. The second equality follows from (5.8), *c)* iterated. The third equality follows from f being a left A-module map. The fourth equality follows from $\mathscr{I}: \mathrm{C} \to \mathrm{End\,A}$ being an algebra homomorphism. The last equality follows from the definition of f^{to}. Q.E.D.

Lemma **(15.7)**. — a) *Suppose* $\mathrm{Hom_A(A, A)}$ *is identified with* A *in the usual way and* $f \in \mathrm{Hom_A(A, A)}$ *corresponds to* $a \in \mathrm{A}$ *where* $f(1) = a$. *Then* $a^\ell = f^t \in \mathrm{End\,A}$.

Suppose B *is a* $\times_\mathrm{A}$*-bialgebra and* $\mathrm{E} = \mathrm{E_B}$.

b) *For* $a \in \mathrm{A}$, $b \in \mathrm{B}$

$$e_0(a)^t(b) = \mathscr{I}(b)a^\ell \in \mathrm{E}$$
$$e_1(a)^t(b) = a^\ell\mathscr{I}(b) \in \mathrm{E}.$$

c) *For* $f \in \mathrm{Hom_A}(\overbrace{\mathrm{B}\otimes_\mathrm{A}\ldots\otimes_\mathrm{A}\mathrm{B}}^{n}, \mathrm{A})$, $\{b_i\}_0^n \subset \mathrm{B}$,

$$\begin{aligned} e_0(f)^t(b_0\otimes\ldots\otimes b_n) &= \mathscr{I}(b_0)f^t(b_1\otimes\ldots\otimes b_n) \\ e_1(f)^t(b_0\otimes\ldots\otimes b_n) &= f^t(b_0b_1\otimes b_2\otimes\ldots\otimes b_n) \\ e_2(f)^t(b_0\otimes\ldots\otimes b_n) &= f^t(b_0\otimes b_1b_2\otimes b_3\otimes\ldots\otimes b_n) \\ &\ldots\ldots\ldots\ldots \\ e_n(f)^t(b_0\otimes\ldots\otimes b_n) &= f^t(b_0\otimes\ldots\otimes b_{n-2}\otimes b_{n-1}b_n) \\ e_{n+1}(f)^t(b_0\otimes\ldots\otimes b_n) &= f^t(b_0\otimes\ldots\otimes b_{n-1})\mathscr{I}(b_n). \end{aligned}$$

d) *For* $f, g \in \mathrm{Hom_A(B, A)}$, $b, c \in \mathrm{B}$

$$(e_0(f) * e_2(g))^t(b\otimes c) = g^t(b)f^t(c) \in \mathrm{E}.$$

e) *For* $f, g \in \mathrm{Hom_A(B, A)}$

$$(f * g)^{to} = f^{to}g^{to} \in \mathrm{Hom_{A\otimes A}(B, B)}.$$

Note. — In parts *d)* and *e)* the $*$-product refers to the product discussed at the beginning of this section. The $(\ \ )^{to}$ in part *e)* is defined in (15.6), *a)*.

Proof. — Parts *a)*, *b)* and *c)* are left to the reader.

d) By the remark below (5.2) it suffices to prove that

$$\boldsymbol{\epsilon}(e_0(f) * e_2(g))^t(b\otimes c) = \boldsymbol{\epsilon}(g^t(b)f^t(c)) \in \mathrm{A}.$$

Hence by the same remark it suffices to prove that

$$(e_0(f)*e_2(g))(b\otimes c)=\epsilon(g^t(b)f^t(c)).$$

By definition of ϵ

$$\begin{aligned}\epsilon(g^t(b)f^t(c)) &= g^t(b)f^t(c)(1)\\ &= g^t(b)(\epsilon f^t(c))\end{aligned}$$

which equals

$$(*)\qquad g^t(b)(f(c))$$

by the remark below (5.2).

Suppose $\Delta b=\sum_i b_i\otimes b'_i$, $\Delta c=\sum_j c_j\otimes c'_j\in \mathrm{B}\times_{\mathrm{A}}\mathrm{B}\subset\int_x {}_x\mathrm{B}\otimes_x\mathrm{B}$. Then

$$(e_0(f)*e_2(g))(b\otimes c)=\sum_{i,j} b_i.f(c_j)g(b'_i\,\varepsilon(c'_j))$$

which equals $\sum_i b_i.f(c)g(b'_i)$ since ε is the co-unit of B. Since g is a left A-module map we have the first equality in

$$(**)\qquad \begin{aligned}\sum_i b_i.f(c)g(b'_i) &= \sum_i g((b_i.f(c))b'_i)\\ &= g(bf(c)),\end{aligned}$$

the second equality following from (5.8), *c)*. By (5.2) $(*)=(**)$ and part *d)* is proved.

For part *e)*, direct computation shows that for $b\in\mathrm{B}$ with $\Delta b=\sum_i b_i\otimes b'_i\in\int_x {}_x\mathrm{B}\otimes_x\mathrm{B}$ and $(\mathrm{I}\otimes\Delta)\,\Delta b=\sum_j \beta_j\otimes\beta'_j\otimes\beta''_j\in\int_x {}_x\mathrm{B}\otimes_x\mathrm{B}\otimes_x\mathrm{B}$ then

$$(+)\qquad (f*g)^{to}(b)=\sum_j f(\beta'_j)g(\beta''_j)\beta_j$$

and

$$g^{to}(b)=\sum_i g(b'_i)\,b_i.$$

Then $f^{to}g^{to}(b)=\sum_i g(b'_i)f^{to}(b_i)$. By coassociativity $(\Delta\otimes\mathrm{I})\,\Delta(h)=\sum_j \beta_j\otimes\beta'_j\otimes\beta''_j$ so that

$$(++)\qquad \sum_i g(b'_i)f^{to}(b_i)=\sum_j g(\beta''_j)f(\beta'_j)\,\beta_j.$$

Comparing (+) and (++) gives *e)*. Q.E.D.

H^1 *theorem* **(15.8).** — a) *If* $f\in\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A})$ *is a* 1*-cocycle, then* $f^t\in\mathrm{Hom}_{\mathrm{A}\otimes\mathrm{A}}(\mathrm{B},\mathrm{E})$ *is a homomorphism of algebras over* A.

Suppose $\mathscr{I}:\mathrm{B}\to\mathrm{End}\,\mathrm{A}$ *is injective. Then:*

b) $f\in\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A})$ *is a* 1*-cocycle if and only if* $f^{to}\in\mathrm{Hom}_{\mathrm{A}\otimes\mathrm{A}}(\mathrm{B},\mathrm{B})$ *is an isomorphism of algebras over* A.

c) $f\in\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{A})$ *is a* 1*-coboundary if and only if* $f^{to}\in\mathrm{Hom}_{\mathrm{A}\otimes\mathrm{A}}(\mathrm{B},\mathrm{B})$ *is an inner automorphism of* B *induced by an invertible element of* A.

d) *The correspondence* $f \mapsto f^{to}$ *induces an isomorphism between* $H^1(B)$ *and the group of automorphisms of* B *as an algebra over* A *modulo the subgroup of inner automorphisms of* B *induced by invertible elements of* A.

Note by (10.1), *b)* the group of automorphisms of B as an algebra over A is a commutative group.

Proof. — *a)* If f is a 1-cocycle then $e_0(f) * e_2(f) = e_1(f)$. Thus

$$(e_0(f) * e_2(f))^t = e_1(f)^t.$$

By (15.7), *c)* and *d)*, this proves that $f^t : B \to E$ is multiplicative. Since f is an A-bimodule map it remains to prove that $f(1) = 1$. Applying the identity $f * f^{-1} = \varepsilon \otimes \varepsilon$ to $1 \otimes 1 \in \int_x {}_xB \otimes_x B$ shows that $f(1 \otimes 1)$ and $f^{-1}(1 \otimes 1)$ are inverse elements of each other in A. Then applying the 1-cocycle identity $e_0(f) * e_1(f^{-1}) * e_2(f) = \varepsilon \otimes \varepsilon \otimes \varepsilon$ to $1 \otimes 1 \otimes 1 \in \int_x {}_xB \otimes_x B \otimes_x B$ shows that $f(1 \otimes 1) = 1$.

b) When $\mathscr{I} : B \to \operatorname{End} A$ is injective, then $\mathscr{I}$ gives an isomorphism between B and E. For $f \in \operatorname{Hom}_A(B, A)$ $\mathscr{I} f^{to} = f^t$. Thus, by part *a)*, if f is a 1-cocycle, then $f^{to} B \to B$ is a homomorphism of algebras over A. When f is a 1-cocycle so is f^{-1}. Thus by (15.7), *e)*, f^{to} is an automorphism with inverse $(f^{-1})^{to}$.

Conversely suppose $f \in \operatorname{Hom}(B, A)$ and f^{to} is an automorphism of B as an algebra over A. Let G be the inverse automorphism to f^{to} and let $g = \varepsilon \mathscr{I} G$. Then $G = g^{to}$ and by (15.7), *e)*, g is inverse to f in $\operatorname{Hom}_A(B, A)$. Thus f is invertible in the A-algebra $\operatorname{Hom}_A(B, A)$.

Since f^{to} is multiplicative so is $\mathscr{I} f^{to} = f^t$. Thus for $b_0, b_1 \in B$

$$f^t(b_0) f^t(b_1) = f^t(b_0 b_1).$$

By (15.7), *c)* and *d)* this proves that $(e_0(f) * e_2(f))^t = e_1(f)^t$. Hence $e_0(f) * e_2(f) = e_1(f)$. Together with invertibility of f this proves that f is a 1-cocycle.

c) Follows from (15.7), *b)*.

d) Follows from *b)* and *c)*. Q.E.D.

16. $H^2(B)$

In this section we show when $H^2(B) \cong \mathscr{G}\langle B \rangle$ as abelian groups. Throughout the section A is a commutative algebra and $(B, \Delta, \mathscr{I})$ is a cocommutative $\times_A$-bialgebra.

Definition (**16.1**). — Suppose M, N are A-bimodules $f \in \operatorname{Hom}_{A \otimes A}(M \otimes_A M, M)$, $g \in \operatorname{Hom}_{A \otimes A}(N \otimes_A N, N)$. We call (M, f) and (N, g) *equivalent* and write $(M, f) \sim (N, g)$ if there is an A-bimodule isomorphism $\sigma : M \to N$ such that $f = \sigma^{-1} g (\sigma \otimes \sigma)$.

Definition (**16.2**). — Suppose M is an A-bimodule and $f \in \operatorname{Hom}_{A \otimes A}(M \otimes_A M, M)$.

We call (M, f) a *non-associative algebra over* A if there is an A-bimodule map $\gamma : \mathrm{A} \rightarrow \mathrm{M}$ making the diagrams below commute:

$$\begin{array}{ccc} \mathrm{A} \otimes \mathrm{M} & \xrightarrow{(a \otimes m \mapsto a.m)} & \mathrm{M} \\ \Big\downarrow{\scriptstyle \gamma \otimes \mathrm{I}} & & \Big\uparrow{\scriptstyle f} \\ \mathrm{M} \otimes \mathrm{M} & \longrightarrow & \mathrm{M} \otimes_{\mathrm{A}} \mathrm{M} \end{array}$$

$$\begin{array}{ccc} \mathrm{M} \otimes \mathrm{A} & \xrightarrow{(m \otimes a \rightarrow m.a)} & \mathrm{M} \\ \Big\downarrow{\scriptstyle \mathrm{I} \otimes \gamma} & & \Big\uparrow{\scriptstyle f} \\ \mathrm{M} \otimes \mathrm{M} & \longrightarrow & \mathrm{M} \otimes_{\mathrm{A}} \mathrm{M} \end{array}$$

The proof of the next lemma is left to the reader:

Lemma **(16.3)**. — *Suppose* M, N *are* A-*bimodules*,

$$f \in \mathrm{Hom}_{\mathrm{A} \otimes \mathrm{A}}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{M}, \mathrm{M}), \quad g \in \mathrm{Hom}_{\mathrm{A} \otimes \mathrm{A}}(\mathrm{N} \otimes_{\mathrm{A}} \mathrm{N}, \mathrm{N}).$$

Let $h \in \mathrm{Hom}_{\mathrm{A} \otimes \mathrm{A}}((\mathrm{M} \times_{\mathrm{A}} \mathrm{N}) \otimes_{\mathrm{A}} (\mathrm{M} \times_{\mathrm{A}} \mathrm{N}), \mathrm{M} \times_{\mathrm{A}} \mathrm{N})$ *be the composite*:

$$(\mathrm{M} \times_{\mathrm{A}} \mathrm{N}) \otimes_{\mathrm{A}} (\mathrm{M} \times_{\mathrm{A}} \mathrm{N}) \xrightarrow{\xi} (\mathrm{M} \otimes_{\mathrm{A}} \mathrm{M}) \times_{\mathrm{A}} (\mathrm{N} \otimes_{\mathrm{A}} \mathrm{N}) \xrightarrow{f \times g} \mathrm{M} \times_{\mathrm{A}} \mathrm{N}$$

where ξ *is defined in* (2.10). *Let* $h' \in \mathrm{Hom}_{\mathrm{A} \otimes \mathrm{A}}((\mathrm{N} \times_{\mathrm{A}} \mathrm{M}) \otimes_{\mathrm{A}} (\mathrm{N} \times_{\mathrm{A}} \mathrm{M}), \mathrm{N} \times_{\mathrm{A}} \mathrm{M})$ *be the composite*:

$$(\mathrm{N} \times_{\mathrm{A}} \mathrm{M}) \otimes_{\mathrm{A}} (\mathrm{N} \times_{\mathrm{A}} \mathrm{M}) \xrightarrow{\xi} (\mathrm{N} \otimes_{\mathrm{A}} \mathrm{N}) \times_{\mathrm{A}} (\mathrm{M} \otimes_{\mathrm{A}} \mathrm{M}) \xrightarrow{g \times f} \mathrm{N} \times_{\mathrm{A}} \mathrm{M}.$$

1. "$\sim$" *in* (16.1) *is an equivalence relation.*

2. *If* $\mathrm{M} \cong \mathrm{B}$ *as an* A-*bimodule, then there is* $r \in \mathrm{Hom}_{\mathrm{A} \otimes \mathrm{A}}(\mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}, \mathrm{B})$ *with* $(\mathrm{M}, f) \sim (\mathrm{B}, r)$.

3. *For* $r, s \in \mathrm{Hom}_{\mathrm{A} \otimes \mathrm{A}}(\mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}, \mathrm{B})$, $(\mathrm{B}, r) \sim (\mathrm{B}, s)$ *if and only if there is an* A-*bimodule isomorphism* $\sigma : \mathrm{B} \rightarrow \mathrm{B}$ *where* $r(\sigma \otimes \sigma) = \sigma s$.

4. (M, f) *is a non-associative algebra over* A *if and only if there is* $e \in \int^x {}_x\mathrm{M}_x$ *which is a* 2-*sided unit for* (M, f); i.e. *for all* $m \in \mathrm{M}$, $f(e \otimes m) = m = f(m \otimes e)$. *In this case* γ (*as in* (16.2)) *is uniquely determined as the map*

$$\mathrm{A} \rightarrow \mathrm{M} \quad (a \mapsto a.e = e.a).$$

5. *Suppose* $(\mathrm{M}, f) \sim (\mathrm{N}, g)$ *and* (M, f) *is a non-associative algebra over* A. *Then* (N, g) *is a non-associative algebra over* A.

6. *If* (M, f) *and* (N, g) *are non-associative algebras over* A, *then so is* $(\mathrm{M} \times_{\mathrm{A}} \mathrm{N}, h)$. *If* e *is the unit of* M *and* e' *the unit of* N, *then* $e \otimes e' \in \int^y \int_x {}_x\mathrm{M}_y \otimes {}_x\mathrm{N}_y = \mathrm{M} \times_{\mathrm{A}} \mathrm{N}$ *is the unit of* $\mathrm{M} \times_{\mathrm{A}} \mathrm{N}$.

7. *Suppose* M *and* N *are algebras over* A *(associative) and*

$$f : M \otimes_A M \to M, \quad g : N \otimes_A N \to N$$

are the multiplication maps. Then $h : (M \times_A N) \otimes_A (M \times_A N) \to M \times_A N$ *is the multiplication map.*

8. *The* " $\sim$ " *equivalence class of* $(M \times_A N, h)$ *depends only upon the* " $\sim$ " *equivalence classes of* (M, f) *and* (N, g).

9. *If* M, N *are algebras over* A *(associative) and* f *and* g *are the multiplication maps, then* $(M, f) \sim (N, g)$ *if and only if* $M \cong N$ *as algebras over* A.

10. *Suppose* $(M, f) \sim (B, r)$, $(N, g) \sim (B, s)$ *and* $\Delta : B \to B \times_A B$ *is an isomorphism; then* $(M \times_A N, h) \sim (B, t)$ *where* $t \in \mathrm{Hom}_{A \otimes A}(B \otimes_A B, B)$ *is the composite*

$$B \otimes_A B \xrightarrow{\Delta \otimes \Delta} (B \times_A B) \otimes_A (B \times_A B) \xrightarrow{\xi} (B \otimes_A B) \times_A (B \otimes_A B) \xrightarrow{r \times s} B \times_A B \xrightarrow{\Delta^{-1}} B.$$

11. *Suppose the products on* M *and* N *induced by* f *and* g *respectively are associative. Then the product on* $M \times_A N$ *induced by* h *is associative.*

12. *Suppose* $N = B$ *and* $g = \mathrm{mult} : B \otimes_A B \to B$. *The composite map*

$$M \times_A N = M \times_A B \xrightarrow{I \times \mathscr{I}} M \times_A \mathrm{End}\, A \xrightarrow{\theta} M$$

is multiplicative if $M \times_A N$ *has the* h *product and* M *has the* f *product.* (θ *is defined in* (2.8), *also see* (4.2).)

13. *Suppose* $(M, f) \sim (N, g)$ *and* f *induces an associative product on* M. *Then* g *induces an associative product on* N.

14. $(M \times_A N, h) \sim (N \times_A M, h')$.

Lemma **(16.4)**. — *Suppose* M *is an* A*-bimodule,* $f \in \mathrm{Hom}_{A \otimes A}(M \otimes_A M, M)$ *and the product induced on* M *by* f *is associative. For* $m, n \in M$ *denote the* f*-product* $f(m \otimes n)$ *by* $m \circ n$.

1. *Suppose there is* $m \in \int^y {}_y M_y$ *such that the map* $M \to M$, $(x \mapsto m \circ x)$ *is an isomorphism. Then there is* $z \in \int^y {}_y M_y$ *which is a left* " $\circ$ " *identity.* I.e. *for all* $n \in M$, $z \circ n = n$.

2. *Suppose there is* $m \in \int^y {}_y M_y$ *such that the map* $M \to M$, $(x \mapsto x \circ m)$ *is an isomorphism. Then there is* $z \in \int^y {}_y M_y$ *which is a right* " $\circ$ " *identity.*

3. *Suppose* $\Delta : B \to B \times_A B$ *is an isomorphism,* $\mathrm{Im}(A \to B) = \int^y {}_y B_y$, N *is an* A*-bimodule,* $g \in \mathrm{Hom}_{A \otimes A}(N \otimes_A N, N)$ *and* $M \cong B \cong N$ *as* A*-bimodules. For* $u, v \in N$ *denote the* g*-product* $g(u \otimes v)$ *by* $u \square v$. *Form* $(M \times_A N, h)$ *as in* (16.3).

If $(M \times_A N, h)$ *has an* h *unit in* $\int^y {}_y (M \times_A N)_y$ *then* M *has a* " $\circ$ " *unit in* $\int^y {}_y M_y$. *Hence by* (16.3), 4) (M, f) *is a non-associative algebra over* A *and since* f *is assumed associative there is a unique algebra map* $\gamma : A \to M$ *making* $((M, f), \gamma)$ *into an algebra over* A.

Proof. — 1. Choose $z \in M$ with $m \circ z = m$. For $a \in A$

$$m \circ (az) = (ma) \circ z = (am) \circ z = a(m \circ z) = am; \quad m \circ (za) = (m \circ z)a = ma.$$

These calculations use that $m\in\int^{y}{}_{y}M_{y}$ and $f\in \mathrm{Hom}_{A\otimes A}(M\otimes_{A} M, M)$. Hence

$$m\circ(az)=am=ma=m\circ(za).$$

Since " $m\circ$ " is an isomorphism it follows that $az=za$ and $z\in\int^{y}{}_{y}M_{y}$.

For $n\in M$, $m\circ(z\circ n)=(m\circ z)\circ n=m\circ n$, here we have used associativity of " $\circ$ ". Again since " $m\circ$ " is an isomorphism we conclude that $z\circ n=n$ and z is a left " $\circ$ " unit.

2) is proved similarly to 1).

3. Let $\sigma: B\to M$ and $\tau: B\to N$ be A-bimodule isomorphisms. Then

$$B \xrightarrow{\Delta} B\times_{A}B \xrightarrow{\sigma\times\tau} M\times_{A}N$$

is an A-bimodule isomorphism. Let $m=\sigma(1)\in\int^{y}{}_{y}M_{y}$ and $n=\tau(1)\in\int^{y}{}_{y}N_{y}$. Since $\Delta(1)=1\otimes 1$ it follows that $(\sigma\times\tau)\Delta(1)=m\otimes n$. Since $\mathrm{Im}(A\to B)=\int^{y}{}_{y}B_{y}$ it follows that $A.1=\int^{y}{}_{y}B_{y}$. Hence $A.(m\otimes n)=\int^{y}{}_{y}(M\times_{A}N)_{y}$. Suppose w is the h unit of $M\times_{A}N$ in $\int^{y}{}_{y}(M\times_{A}N)_{y}$. Then w can be written $a.(m\otimes n)=m\otimes an$ for some $a\in A$.

Let $r: M\to M$, $(x\mapsto m\circ x)$ and $s: N\to N$, $(x\mapsto(an)\circ x)$. Both r and s are A-bimodule maps. The map $r\times s: M\times_{A}N\to M\times_{A}N$ is left multiplication by w and hence is the identity. Thus $r\times s$ is an isomorphism and by (10.1), *c)* r is an isomorphism. Hence by part 1), M has a left unit in $\int^{y}{}_{y}M_{y}$.

Let $r': M\to M$, $(x\mapsto x\circ m)$ and $s': N\to N$, $(x\mapsto x\circ(an))$. Both r' and s' are A-bimodule maps and $r'\times s': M\times_{A}N\to M\times_{A}N$ is right multiplication by w, the identity. Thus by (10.1), *c)* r' is an isomorphism and part 2) M has a right unit in $\int^{y}{}_{y}M_{y}$. As usual left and right unit must be the same. Q.E.D.

Definition (**16.5**). — For (M,f) — as in (16.1) — let $\langle(M,f)\rangle$ denote the " $\sim$ " equivalence class. Let $\mathscr{P}$ denote the *set* of equivalence classes $\langle(M,f)\rangle$ where $M\cong B$ as an A-bimodule. Let

$$\mathscr{A}=\{\langle(M,f)\rangle\in\mathscr{P}\mid f \text{ induces an associative product on } M\}$$
$$\mathscr{B}=\{\langle(M,f)\rangle\in\mathscr{P}\mid (M,f) \text{ is a non-associative algebra over } A\}$$

Note. — That $\mathscr{P}$ is a set follows from (16.3), 2); that $\mathscr{A}$ and $\mathscr{B}$ are well defined follows from (16.3), 13) and (16.3), 5).

Lemma (**16.6**). — *Suppose* $\Delta: B\to B\times_{A}B$ *is an isomorphism and* $\mathrm{Im}(A\to B)=\int^{y}{}_{y}B_{y}$.

a) *For* $\langle M,f\rangle, \langle N,g\rangle\in\mathscr{P}$ *one may define the product* $\langle M,f\rangle\langle N,g\rangle$ *as*

$$\langle M\times_{A}N, h\rangle\in\mathscr{P}$$

with h as in (16.3). *This defines an associative commutative product on* $\mathscr{P}$ *which has* $\langle(B,\text{mult})\rangle$ *as unit.*

b) $\mathscr{A}$ *and* $\mathscr{B}$ *are submonoids of* $\mathscr{P}$ *and* $\mathscr{A}\cap\mathscr{B}=\mathscr{E}\langle B\rangle$ *which is defined in* (4.8).

c) *The subgroup of invertible elements of* $\mathscr{A}$ *coincides with the subgroup of invertible elements of* $\mathscr{E}\langle B\rangle$. *They equal* $\mathscr{G}\langle B\rangle$ *defined in* (4.8).

Proof. — *a)* (16.3), 8) implies that the product on $\mathscr{P}$ is well defined. (16.3), 14) gives that the product on $\mathscr{P}$ is commutative. We only outline the proof that the product on $\mathscr{P}$ is associative. Suppose $\langle(L, d)\rangle$, $\langle(M,f)\rangle$, $\langle(N, g)\rangle\in\mathscr{P}$. Since B is a $\times_A$-bialgebra, B is associative as an A-bimodule (2.7). Since $L\cong M\cong N\cong B$ as A-bimodules it follows that (L, M, N) associates (2.6). $\int^y\int_x {}_xL_y\otimes_x M_y\otimes_x N_y$ has a product where for $x=\sum_i \ell_i\otimes m_i\otimes n_i$, $y=\sum_j \ell'_j\otimes m'_j\otimes n'_j\in\int^y\int_x {}_xL_y\otimes_x M_y\otimes_x N_y$ the product xy is defined as

$$\sum_{i,j} d(\ell_i\otimes\ell'_j)\otimes f(m_i\otimes m'_j)\otimes g(n_i\otimes n'_j).$$

With this product the maps α and α' ((2.5), 2) and just above (2.6)) are multiplicative. Thus the association isomorphism is multiplicative as well as an A-bimodule isomorphism. Hence $(\langle(L, d)\rangle\langle(M,f)\rangle)\langle(N, g)\rangle\sim\langle(L, d)\rangle(\langle(M,f)\rangle\langle(N, g)\rangle)$ and the product on $\mathscr{P}$ is associative.

Since $B\to B\times_A B$ is an isomorphism the composite

$$B\times_A B \xrightarrow{I\times\mathscr{J}} B\times_A \operatorname{End} A \xrightarrow{\theta} B$$

must be the inverse isomorphism (see (5.1)). If $\langle(M,f)\rangle\in\mathscr{P}$, then $M\cong B$ as an A-bimodule and so the composite

$$M\times_A B \xrightarrow{I\times\mathscr{J}} M\times_A \operatorname{End} A \xrightarrow{\theta} M$$

is an isomorphism. By (16.3), 12) it follows that $\langle(M,f)\rangle\langle(B, \text{mult})\rangle\sim\langle(M,f)\rangle$. Hence $\langle(B, \text{mult})\rangle$ is a right unit for the product on $\mathscr{P}$. Since the product is commutative $\langle(B, \text{mult})\rangle$ is the unit.

b) (16.3), 11) implies that $\mathscr{A}$ is a submonoid and (16.3), 6) implies that $\mathscr{B}$ is a submonoid. That $\mathscr{A}\cap\mathscr{B}=\mathscr{E}\langle B\rangle$ is a matter of definition.

c) (16.4), 3) implies that the subgroup of invertible elements of $\mathscr{A}$ all lie in $\mathscr{B}$. Hence the subgroup of invertible elements of $\mathscr{A}$ coincides with the subgroup of invertible elements of $\mathscr{E}\langle B\rangle$. This group equals $\mathscr{G}\langle B\rangle$ by definition. Q.E.D.

Lemma **(16.7)**. — *Suppose* $f, g\in\operatorname{Hom}_A(B\otimes_A B, A)$, $\sigma, \gamma\in\operatorname{Hom}_A(B, A)$ *and* $b, c, d\in B$. *In what follows the* $*$*-product refers to the product defined at the beginning of Section 15. The* $(\ )^{to}$ *is defined in* (15.6, *a*). $E=E_B$.

a) $(e_0(f)*e_2(g))^t(b\otimes c\otimes d)=g^t(b\otimes f^{to}(c\otimes d))$.

b) $(e_1(f)*e_3(g))^t(b\otimes c\otimes d)=f^t(g^{to}(b\otimes c)\otimes d)$.

c) *If* Δ *is an isomorphism, then* $(f*g)^t$ *is equal to the composite*

$$(*)\quad B\otimes_A B \xrightarrow{\Delta\otimes\Delta} (B\times_A B)\otimes_A(B\times_A B) \xrightarrow{\xi} (B\otimes_A B)\times_A(B\otimes_A B) \xrightarrow{f^{to}\times g^{to}} B\times_A B \xrightarrow{\Delta^{-1}} B.$$

d) $(e_2(\sigma)*e_0(\gamma)*f)^t(b\otimes c)=f^t(\sigma^{to}(b)\otimes\gamma^{to}(c))$.

e) $(f*e_1(\sigma))^t(b\otimes c)=\sigma^t(f^{to}(b\otimes c))$.

Proof. — The proofs of *a)*, *b)*, *d)* and *e)* are similar to the proofs of (15.7), *d)* et *e)* and are left to the reader.

c) By (5.1) $\Delta^{-1}=\theta(I\times\mathscr{I})$. Hence $\epsilon\Delta^{-1}$ equals the composite

$$(**)\qquad B\times_A B \xrightarrow{\mathscr{I}\times\mathscr{I}} \mathrm{End}\times_A \mathrm{End}\ A \hookrightarrow \int_x {}_x\mathrm{End}\ A\otimes_x\mathrm{End}\ A \xrightarrow{\epsilon\otimes\epsilon} \int_x {}_xA\otimes_x A = A.$$

By the remark below (5.2) it suffices to prove that $\epsilon(f*g)^t=\epsilon C$ where C is the composite $(*)$ in part *c)*. By the same remark $\epsilon(f*g)^t=f*g$. By $(**)$ ϵC is equal to

$$B\otimes_A B \xrightarrow{\Delta\otimes\Delta} (B\times_A B)\otimes_A(B\times_A B) \xrightarrow{\xi} (B\otimes_A B)\times_A(B\otimes_A B) \xrightarrow{f^{to}\times g^{to}} B\times_A B$$
$$\xrightarrow{\mathscr{I}\times\mathscr{I}} \mathrm{End}\ A\times_A\mathrm{End}\ A \hookrightarrow \int_x {}_x\mathrm{End}\ A\otimes_x\mathrm{End}\ A \xrightarrow{\epsilon\otimes\epsilon} \int_x {}_xA\otimes_xA=A.$$

Using $\mathscr{I}f^{to}=f^t$ and $\mathscr{I}g^{to}=g^t$ this equals:

$$B\otimes_A B \xrightarrow{\Delta\times\Delta} (B\times_A B)\otimes_A(B\times_A B) \xrightarrow{\xi} (B\otimes_A B)\times_A(B\otimes_A B)$$
$$\xrightarrow{f^t\times g^t} \mathrm{End}\ A\times_A\mathrm{End}\ A \xrightarrow{\epsilon\otimes\epsilon} \int_x {}_xA\otimes_xA=A.$$

Using $\epsilon f^t=f$ and $\epsilon g^t=g$ this equals:

$$B\otimes_A B \xrightarrow{\Delta\otimes\Delta} (B\times_A B)\otimes(B\times_A B) \xrightarrow{\xi} (B\otimes_A B)\times_A(B\otimes_A B)$$
$$\xrightarrow{\iota} \int_x {}_x(B\otimes_A B)\otimes_x(B\otimes_A B) \xrightarrow{f\otimes g} \int_x {}_xA\otimes_xA=A.$$

Since the composite of the first three maps $\iota\xi(\Delta\otimes\Delta)$ equals the diagonalization in $B\otimes_A B$, the entire composite equals $f*g$. Q.E.D.

H^2 *theorem* **(16.8)**. — *Suppose* $\mathscr{I}:B\to\mathrm{End}\ A$ *is injective and* $\Delta:B\to B\times_A B$ *is an isomorphism. Note that injectivity of* $\mathscr{I}$ *implies that* $\mathrm{Im}(A\to B)=\int^y {}_yB_y$.

a) *For* $f, g\in\mathrm{Hom}_A(B\otimes_A B, A)$ *write* $f\sim\sim g$ *if there is a* $*$*-invertible element*

$$\sigma\in\mathrm{Hom}_A(B, A)$$

with $e_0(\sigma)*e_2(\sigma)*f=e_1(\sigma)*g$. *Then* $\sim\sim$ *is an equivalence relation. Let* $[f]$ *denote the "* $\sim\sim$ *" equivalence class of f and let* Q *denote the set of equivalence classes* $\{[f]\}$. *The* $*$*-product on* $\mathrm{Hom}_A(B\otimes_A B, A)$ *induces a commutative associative product on* Q *with unit* $[\varepsilon_{B\otimes_A B}]$. *There is a bijective product preserving correspondence*

$$\mathscr{P}\leftrightarrow Q$$
$$\langle(B,f^{to})\rangle\leftrightarrow[f].$$

b) *For* $f\sim\sim g\in\mathrm{Hom}_A(B\otimes_A B, A)$ *if*

$$e_0(f)e_2(f)=e_1(f)e_3(f)$$

then
$$e_0(g)e_2(g)=e_1(g)e_3(g).$$

Thus
$$X=\{[f]\in Q\mid e_0(f)e_2(f)=e_1(f)e_3(f)\}$$

is a well defined subset of Q. X *is actually a submonoid of* Q *and the correspondence in part* a) *induces an isomorphism between* X *and* $\mathscr{A}$.

c) *Let* $f \sim\sim g \in \mathrm{Hom}_A(B \otimes_A B, A)$ *and suppose there is* $a \in A$ *satisfying*

$$(**) \qquad f(a.1 \otimes b) = \epsilon\mathscr{I}(b) = f(b \otimes a.1), \qquad b \in B.$$

Then a *is an invertible element of* A *with inverse* $f(1 \otimes 1)$. *There is an element* $a' \in A$ *satisfying*

$$g(a'.1 \otimes b) = \epsilon\mathscr{I}(b) = g(b \otimes a'.1)$$

for all $b \in B$. *Thus*

$$Y = \{[f]\} \in Q \mid \text{there is } a \in A \text{ satisfying } (**)\}$$

is a well defined subset of Q. Y *is actually a submonoid of* Q *and the correspondence in part* a) *induces an isomorphism between* Y *and* $\mathscr{B}$.

d) *For* $f \sim\sim g \in \mathrm{Hom}_A(B \otimes_A B, A)$, *if* f *is* $*$*-invertible, then so is* g. *A class* $[f] \in Q$ *is invertible in* Q *if and only if each member of the class is* $*$*-invertible.*

e) *The correspondence in part* a) *induces an isomorphism between* $X \cap Y$ *and* $\mathscr{E}\langle B \rangle$. *The invertible elements in* $X \cap Y$ *coincide with the invertible elements of* X. *The correspondence in part* a) *induces a (group) isomorphism between the subgroup of invertible elements of* X *and* $\mathscr{G}(B)$. *The subgroup of invertible elements of* X *is naturally isomorphic to* $H^2(B)$.

Proof. — By (15.6) and the hypothesis that $\mathscr{I}: B \to \mathrm{End}\, A$ is injective, it follows that the correspondence

$$\mathrm{Hom}_{A \otimes A}(\overbrace{B \otimes_A \ldots \otimes_A B}^{n}, B) \leftrightarrow \mathrm{Hom}_A(\overbrace{B \otimes_A \ldots \otimes_A B}^{n}, A)$$
$$f^{to} \leftarrow f$$
$$g \to \epsilon\mathscr{I} g$$

is bijective. This is used throughout the proof.

a) For f, g, σ as in part *a)* it follows from (16.7), *d)* and *e)*, (16.3), 3), (15.7), *e)* that $f \sim\sim g$ if and only if $(B, \epsilon\mathscr{I} f) \sim (B, \epsilon\mathscr{I} g)$ with "$\sim$" as in (16.1). Since "$\sim$" is an equivalence relation "$\sim\sim$" must also be. Moreover with (16.2), 2) we have established the bijection $\mathscr{P} \leftrightarrow Q$ of part *a)*. By (16.7), *c)* and (16.3), 10) the product induced on Q by the bijection is the same as the product arising from "$*$".

b) This follows from (16.7), *a)* and *b)*, and (16.3), 13).

c) The condition on $a.1$ and f is equivalent to $a.1$ being the unit for f^{to}. Hence by (16.3), 4) and 5), it follows that a suitable a' exists. Using that $a.1$ is the unit for f^{to},

$$a.f^{to}(1 \otimes 1) = f^{to}(a.1 \otimes 1) = 1.$$

Clearly $f^{to}(1 \otimes 1) \in \int^y {}_yB_y = \mathrm{Im}(A \to B)$. Hence a is invertible with inverse

$$\epsilon\mathscr{I} f^{to}(1 \otimes 1) = f(1 \otimes 1).$$

That Y is a submonoid of Q follows from (16.3), 6) or the fact that $\mathscr{B}$ is a submonoid of $\mathscr{P}$.

d) Follows from (16.7), *c)* and the fact that e_0, e_1, e_2 are multiplicative.

e) Follows from the preceding four parts. Q.E.D.

Invertibility corollary **(16.9).** — *Suppose* $\mathscr{I} : B \to \operatorname{End} A$ *is injective and* $\Delta : B \to B \times_A B$ *is an isomorphism. In addition suppose that for* $f \in \operatorname{Hom}_A(B \otimes_A B, A)$, *if* $f(1 \otimes 1)$ *is an invertible element of* A, *then* f *is an* $*$*-invertible element of* $\operatorname{Hom}_A(B \otimes_A B, A)$. *Then* $\mathscr{E}\langle B \rangle = \mathscr{G}\langle B \rangle$.

Proof. — The condition on elements of $\operatorname{Hom}_A(B \otimes_A B, A)$ implies that Y consists of invertible elements. Hence $X \cap Y$ consists of invertible elements. By (16.8), *e)* this gives $\mathscr{E}\langle B \rangle = \mathscr{G}\langle B \rangle$. Q.E.D.

17. Examples of $\times_A$-bialgebras and their cohomology

The example $A \# H$ of a $\times_A$-bialgebra is presented toward the end of section 7. As a left A-module $A \# H \cong A \otimes H$. Hence, as a left A-module,

$$(A \# H) \otimes_A \ldots \otimes_A (A \# H) \cong A \otimes H \otimes \ldots \otimes H$$

and

$$\operatorname{Hom}_A(A \# H \otimes_A \ldots \otimes_A A \# H, A) \cong \operatorname{Hom}(H \otimes \ldots \otimes H, A).$$

This isomorphism induces an isomorphism of complexes between the complex (15.4) and the complex used to compute the Hopf algebra cohomology [16, § 2, p. 209]. The details are left to the reader. Hence we have:

Theorem **(17.1).** — *Suppose* $A \# H$ *is a* $\times_A$*-bialgebra; then the cohomology* $H^*(A \# H)$ *as in* (15.4) *is naturally isomorphic to the Hopf algebra cohomology of* H *in* A [16, § 2, p. 208].

Corollary **(17.2).** — *Suppose* $H = RG$, *the group algebra of a group* G *which acts as automorphisms of* A. *Then* $H^*(A \# H)$ *is naturally isomorphic to the group cohomology of* " G *acting on the group of invertible elements of* A ".

Proof. — Follows from (17.1) and the result [16, Theorem (3.1), p. 211] on Hopf algebra cohomology. Q.E.D.

Corollary **(17.3).** — *Suppose* $H = UL$ *the enveloping algebra of a Lie algebra which acts as derivations of* A. *Then, for* $i \geq 2$, $H^i(A \# H)$ *is naturally isomorphic to the Lie cohomology of* " L *acting on* A ". $H^0(A \# H)$ *is the group of invertible elements in* A^L, *the subalgebra of* A *consisting of* L*-constants.*

Proof. — Follows from (17.1) and the result [16, Theorem (4.3), p. 214] on Hopf algebra cohomology. Q.E.D.

***

Consider End A as an $A\otimes A$-module and let $\{L_\alpha\}$, $\{C_\alpha\}$, C be as above (6.6). In (6.6) it is shown that C is a $\times_A$-bialgebra if it happens to be a subalgebra over A of End A. In (6.6), *a)* it is shown that C is a subalgebra over A of End A when:

$$(17.4)\begin{cases}\text{(i) there is an } L_\tau \text{ contained in } \operatorname{Ker}(A\otimes A \xrightarrow{\text{mult}} A);\\ \text{(ii) for each } L_\alpha \text{ and } L_\beta \text{ there is } L_\gamma \text{ with } e(L_\gamma)\subset L_\alpha\otimes A + A\otimes L_\beta,\end{cases}$$

$$e: A\otimes A \to A\otimes A\otimes A, \qquad a\otimes b \mapsto a\otimes 1\otimes b.$$

Theorem **(17.5)**. — *Assume that* $\{L_\alpha\}$ *satisfy* (17.4) *in addition to the conditions above* (6.6).

a) *For* $1\leq n\in\mathbf{Z}$ *and* $L_{\alpha_1}, \ldots, L_{\alpha_n}\in\{L_\alpha\}$, *let* $L_{\alpha_1,\ldots,\alpha_n}$ *be the ideal in* $\overbrace{A\otimes\ldots\otimes A}^{n+1}$ *which is the kernel of the composite*

$$\left.\begin{array}{l}\overbrace{A\otimes\ldots\otimes A}^{n+1}=\overbrace{(A\otimes A)\otimes_A(A\otimes A)\otimes_A\ldots\otimes_A(A\otimes A)}^{(A\otimes A)\quad n\text{ times}}\\ a_0\otimes\ldots\otimes a_n=(a_0\otimes a_1)\otimes(1\otimes a_2)\otimes\ldots\otimes(1\otimes a_n)\end{array}\right\}\longrightarrow \frac{A\otimes A}{L_{\alpha_1}}\otimes_A\frac{A\otimes A}{L_{\alpha_2}}\otimes_A\ldots\otimes_A\frac{A\otimes A}{L_{\alpha_n}}.$$

Then

$$L_{\alpha_1,\ldots,\alpha_n}=L_{\alpha_1}\otimes\overbrace{A\otimes\ldots\otimes A}^{n-1}+A\otimes L_{\alpha_2}\otimes\overbrace{A\otimes\ldots\otimes A}^{n-2}+\ldots+\overbrace{A\otimes\ldots\otimes A}^{n-1}\otimes L_{\alpha_n}.$$

b) *For* $1\leq n\in\mathbf{Z}$ *and* $L_{\alpha_1}, \ldots, L_{\alpha_n}\in\{L_\alpha\}$, *the map*

$$\overbrace{A\otimes\ldots\otimes A}^{n+1}\xrightarrow{\varphi}\operatorname{Hom}_A(C_{\alpha_1}\otimes_A\ldots\otimes_A C_{\alpha_n}, A),$$
$$a_0\otimes\ldots\otimes a_n\mapsto(c_1\otimes\ldots\otimes c_n\mapsto a_0c_1(\ldots a_{n-1}c_n(a_n)\ldots))$$

is surjective with kernel $L_{\alpha_1,\ldots,\alpha_n}$. *This induces an isomorphism between*

$$\operatorname{Hom}_A(\overbrace{C\otimes_A\ldots\otimes_A C}^{n}, A) \quad \textit{and} \quad \widehat{A\otimes\ldots\otimes A},$$

the completion of $\overbrace{A\otimes\ldots\otimes A}^{n+1}$ *with respect to* $\{L_{\alpha_1,\ldots,\alpha_n}\}_{\alpha_1,\ldots,\alpha_n}$.

c) *The maps*

$$\bar{e}_i: \overbrace{A\otimes\ldots\otimes A}^{n}\to\overbrace{A\otimes\ldots\otimes A}^{n+1},$$
$$a_1\otimes\ldots\otimes a_n\mapsto a_1\otimes\ldots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\ldots\otimes a_n, \qquad i=0,\ldots,n$$

are continuous when $\overbrace{A\otimes\ldots\otimes A}^{m}$ *has the* $\{L_{\alpha_1,\ldots,\alpha_{m-1}}\}$ *topology. Let* $\{\hat{e}_i\}$ *be the induced maps on the completions.* $\{\widehat{A\otimes\ldots\otimes A}, \hat{e}_0, \ldots, \hat{e}_{n+1}\}_{n=0}^{\infty}$ *is a semi-cosimplicial complex. The isomorphism in part* b) *induces an isomorphism of complexes between this complex and the complex* (15.3).

d) $H^*(C)$ (15.4) *is naturally isomorphic to the homology of the complex in part* c) *with respect to the units functor.*

e) *The map* $\overbrace{A\otimes\ldots\otimes A}^{n}\xrightarrow{\text{mult}} A$, $a_1\otimes\ldots\otimes a_n\mapsto a_1\cdot\ldots\cdot a_n$ *is continuous when* $\overbrace{A\otimes\ldots\otimes A}^{n}$ *has the* $\{L_{\alpha_1,\ldots,\alpha_{n-1}}\}$ *topology and* A *has the discrete topology. Hence it induces a map*

$$\widehat{A\otimes\ldots\otimes A}\xrightarrow{\widehat{\text{mult}}} A \quad (=\hat{A}).$$

For each i the diagram below commutes:

$$\begin{array}{ccc} \widehat{A\otimes\ldots\otimes A} & \xrightarrow{\hat{e}_i} & \widehat{A\otimes\ldots\otimes A} \\ & \searrow^{\widehat{\text{mult}}} \quad \swarrow_{\widehat{\text{mult}}} & \\ & A & \end{array}$$

f) *Suppose that the ground ring contains a copy of the rational numbers and for*

$$x\in\mathfrak{M}=\operatorname{Ker}(A\otimes A\xrightarrow{\text{mult}} A)$$

and any L_β *there is* $0<n\in\mathbf{Z}$ (*depending on x and* L_β) *such that* $x^n\in L_\beta$ *and hence higher powers of x lie in* L_β. *Then from degree two onward the cohomology of the complex*

$$\{\widehat{A\otimes\ldots\otimes A}, \hat{e}_0, \ldots, \widehat{e_{n+1}}\}_{n=0}^{\infty}$$

with respect to the functor " *units* " *is naturally isomorphic to the cohomology of the same complex with respect to the functor* " *underlying additive group* ".

g) *Suppose* C *is projective as a left* A-*module* ($C=\bigcup_\alpha C_\alpha$ *as above* (6.6)). *Let* A *have the natural* B-*module structure* (5.7). *The homology of the complex*

$$\{\widehat{A\otimes\ldots\otimes A}, \hat{e}_0, \ldots, \widehat{e_{n+1}}\}_{n=0}^{\infty}$$

with respect to the functor " *underlying additive group* " *is naturally isomorphic to* $\operatorname{Ext}_C^*(A, A)$.

h) *Suppose there is a countable set* $\{M_i\}_{i=1}^{\infty}$ *of sets* $M_i\subset A\otimes A$ *where* $\{L_\alpha\}$ *and* $\{M_i\}_{i=1}^{\infty}$ *are cofinal. Then* C *is projective as a left* A-*module.*

Proof. — *a)* By the standard result for "$\otimes$" the kernel of

$$\overbrace{(A\otimes A)\otimes_A\cdot\ \cdot\otimes_A(A\otimes A)}^{(A\otimes A)\quad n\text{ times}}\to\frac{A\otimes A}{L_{\alpha_1}}\otimes_A\ldots\otimes_A\frac{A\otimes A}{L_{\alpha_n}}$$

is $$\sum_i \overbrace{(A\otimes A)\otimes_A \ldots \otimes_A (A\otimes A)}^{(A\otimes A)\quad i-1 \text{ times}} \otimes_A L_{\alpha_i} \otimes_A \overbrace{(A\otimes A)\otimes_A \ldots \otimes_A (A\otimes A)}^{(A\otimes A)\quad n-i \text{ times}}.$$

It is left to the reader to show that this corresponds to

$$\sum_i \overbrace{A\otimes \ldots \otimes A}^{i-1} \otimes L_{\alpha_i} \otimes \overbrace{A\otimes \ldots \otimes A}^{n-i}$$

under the first map $(=)$ in the composite.

b) For $n=1$ the result about φ follows from (2.12), *a)* and *b)*. Suppose by induction that the result about φ has been proved for $n-1$ and $n\geq 2$. To proceed we use the adjointness relation:

$$\mathrm{Hom}_S({}_SN, \mathrm{Hom}_R({}_RM_S, {}_RP)) = \mathrm{Hom}_R({}_RM_S \otimes_S {}_SN, {}_RP),$$

with $C_{\alpha_1}=M$, $C_{\alpha_2}\otimes_A \ldots \otimes_A C_{\alpha_n}=N$, $A=P$ and both R and S are A. Then

$$\begin{aligned}
\mathrm{Hom}_A(C_{\alpha_1}\otimes_A C_{\alpha_2}\otimes_A \ldots \otimes_A C_{\alpha_n}) &= \mathrm{Hom}_A(C_{\alpha_2}\otimes_A \ldots \otimes_A C_{\alpha_n}, \mathrm{Hom}_A(C_{\alpha_1}, A)) \\
&= \mathrm{Hom}_A(C_{\alpha_2}\otimes_A \ldots \otimes_A C_{\alpha_n}, (A\otimes A)/L_{\alpha_1}) \\
&= \left(\frac{A\otimes A}{L_{\alpha_1}}\right)\otimes_A \mathrm{Hom}_A(C_{\alpha_2}\otimes_A \ldots \otimes_A C_{\alpha_n}, A) \\
&= \left(\frac{A\otimes A}{L_{\alpha_1}}\right)\otimes_A \frac{\overbrace{A\otimes \ldots \otimes A}^{n}}{L_{\alpha_2,\ldots,\alpha_n}} \\
&= \frac{\overbrace{A\otimes \ldots \otimes A}^{n+1}}{L_{\alpha_1,\ldots,\alpha_n}}.
\end{aligned}$$

The first equality is the adjointness. The second and fourth equality rely on the induction. By (2.12), *b)* each C_{α_i} is a finite projective left A-module. It is easily shown that this implies that $C_{\alpha_2}\otimes_A \ldots \otimes_A C_{\alpha_n}$ is a finite projective left A-module. This gives the third equality. For the fifth equality we use part *a)* to identify $(A\otimes \ldots \otimes A)/L_{\alpha_2,\ldots,\alpha_n}$ with

$$\frac{A\otimes A}{L_{\alpha_2}}\otimes_A \ldots \otimes_A \frac{A\otimes A}{L_{\alpha_n}}.$$

Then the term to the left of the fifth equality becomes

$$\left(\frac{A\otimes A}{L_{\alpha_1}}\right)\otimes_A \ldots \otimes_A \left(\frac{A\otimes A}{L_{\alpha_n}}\right)$$

and by part *a)* this equals $(A\otimes \ldots \otimes A)/L_{\alpha_1,\ldots,\alpha_n}$. This concludes the induction.

The second part of *b)* follows from

$$\begin{aligned}\mathrm{Hom}_A(\overbrace{C\otimes_A\ldots\otimes_A C}^{n}, A) &= \mathrm{Hom}_A(\varinjlim C_{\alpha_1}\otimes_A\ldots\otimes_A C_{\alpha_n}, A)\\ &= \varprojlim \mathrm{Hom}_A(C_{\alpha_1}\otimes_A\ldots\otimes_A C_{\alpha_n}, A)\\ &= \varprojlim\left(\frac{\overbrace{A\otimes A\otimes\ldots\otimes A}^{n+1}}{L_{\alpha_1\ldots\alpha_n}}\right) = A\otimes\ldots\otimes A.\end{aligned}$$

c), d), e) The continuity of the maps $\bar{e}_i$ is assured by (17.4), (ii). The continuity of the map " mult " is assured by (17.4), (i). The diagram in part *e) without the hats* is easily checked to commute. Hence by continuity the diagram in part *e)* commutes. The rest of part *c)* as well as part *d)* is left to the reader.

f) Let $C_i = \mathrm{Ker}(\overbrace{\widehat{A\otimes\ldots\otimes A}}^{i+1} \xrightarrow{\widehat{\mathrm{mult}}} A)$ and $D_i = \{x\in \overbrace{\widehat{A\otimes\ldots\otimes A}}^{i+1} \mid \widehat{\mathrm{mult}}(x) = 1\}$ for $i\geq 1$. In degree zero let $C_0 = A$ and $D_0 =$ units of A. For $i\geq 1$, the elements of D_i are invertible being of the form $1-z$ where z lies in C_i, so the inverse is $1+z+z^2+\ldots$ By part *e)* it is a routine calculation to verify that the complex $\{C_i, \{\hat{e}_j \mid C_i\}\}$ is a " normal " subcomplex of $\{\widehat{A\otimes\ldots\otimes A}, \{\hat{e}_j\}\}$ and has the same homology with respect to the functor " underlying additive group ". Similarly the complex $\{D_i, \{\hat{e}_j \mid D_i\}\}$ is a " normal " subcomplex of $\{\widehat{A\otimes\ldots\otimes A}, \{\hat{e}_j\}\}$ and has the same homology with respect to the functor " units ".

For $i\geq 1$ there is the map $\exp : C_i \to D_i$, $z \mapsto \sum_{i=0}^{\infty}\frac{z^i}{i!}$. This map is bijective with inverse $\log : D_i \to C_i$, $1-z \mapsto -\sum_{i=1}^{\infty}\frac{z^i}{i}$. It is left to the reader to verify that exp, log induce an isomorphism of complexes. Since the complexes are isomorphic from degree one onward, the cohomology is isomorphic from degree two onward.

g) For each i let $\otimes_A^i C$ denote ${}_xC\otimes_A\ldots\otimes_A C$, i-times, considered as a (left) x C-module. By assumption that C is a projective left A-module it follows that $\otimes_A^i C$ is projective as a left C-module.

Since C is a $\times_A$-bialgebra the map $\varepsilon = \mathrm{End} : C\to A$ is a left C-module map. Denote this map by d_1. For $i\geq 1$ the maps $d_{i+1} : \otimes_A^{i+1}C \to \otimes_A^i C$ determined by

$$d_{i+1}(c_0\otimes\ldots\otimes c_i) = \sum_{n=0}^{i-1}(-)^n c_0\otimes\ldots\otimes c_n c_{n+1}\otimes\ldots\otimes c_i + (-)^i c_0\otimes\ldots\otimes c_{i-1}\varepsilon(c_i)$$

make $\{\otimes_A^i C, d_i\}$ into a projective resolution of A as a C-module. (The details are left to the reader.) Apply $\mathrm{Hom}_C(-, A)$ to this projective resolution to obtain a complex E^*. The homology of E is $\mathrm{Ext}_C^*(A, A)$ by definition of Ext.

The complex E^* is the cosimplicial complex resulting from the complex (15.3) with respect to the functor " underlying additive group ". Thus, by part *c)*, E^* is isomorphic to the cosimplicial complex resulting from the complex

$$\{\widehat{A\otimes\ldots\otimes A}, \hat{e}_0, \ldots, \widehat{e_{n+1}}\}_{n=0}^{\infty}$$

with respect to the functor " underlying additive group ".

h) Choose $N_1 \in \{L_\alpha\}$ where $N_1 \subset M_1$. Inductively choose N_i according to the rule: $N_i \in \{L_\alpha\}$ and $N_i \subset N_{i-1} \cap M_i$. Then actually N_i lies in $N_1, \ldots, N_{i-1}$ and in $M_1, \ldots, M_i$. Thus $\{N_i\}_{i=1}^{\infty}$ is a countable subcollection of $\{L_\alpha\}$ which is cofinal and which is nested decreasing. Replacing $\{L_\alpha\}$ with $\{N_i\}_{i=1}^{\infty}$ does not alter C.

The sequence

$$0 \to \frac{N_n}{N_{n+1}} \to \frac{A\otimes A}{N_{n+1}} \to \frac{A\otimes A}{N_n} \to 0$$

splits since $(A\otimes A)/N_n$ is a projective left A-module. By (2.12), *a)* this proves that C_{n-1} is a direct summand of C_n as a left A-module. Say $C_n = C_{n-1} \oplus D_n$. Since C_n is a projective left A-module it follows that D_n is projective. It is easily verified that $C = C_1 \oplus D_2 \oplus D_3 \oplus \ldots$ Hence C is projective as a left A-module. Q.E.D.

The exp-log technique used in the proof of part *f)* may be milked somewhat more. As in the following proposition:

Proposition **(17.6)**. — *Suppose the ground ring contains a copy of the rational numbers,* A *contains an ideal* I *such that* $A = I + \operatorname{Ker}(\hat{e}_0 - \hat{e}_1 : A \to \widehat{A\otimes A})$ *and* I *consists of nilpotent elements or* A *is complete in the* I*-adic topology. Furthermore assume that for* $x \in \mathfrak{M}$ *and any* L_β *there is* $0 < n \in \mathbf{Z}$ *such that* $x^n \in L_\beta$. *Then the cohomology of the complex* $\{\widehat{A\otimes\ldots\otimes A}, \{\hat{e}_j\}\}$ *with respect to the functor " units " is naturally isomorphic to the cohomology of the same complex with respect to the functor " underlying additive group " in degree one.*

Proof. — The proof of part *f)* established an isomorphism of normal subcomplexes from degree one onward. Since $A = I + \operatorname{Ker}(\hat{e}_0 - \hat{e}_1)$ it follows that if x is a degree one *additive* coboundary, then it is the coboundary of an element $a \in I$. By the nilpotence or completion assumption we may form $\exp a \in A$. The multiplicative coboundary of $\exp a$ is the same as $\exp x$. Using the log map shows that we have established a bijective correspondence between the additive one coboundaries and the multiplicative one coboundaries. Q.E.D.

In degree zero, there is a simple relation between the additive and multiplicative cohomologies.

Proposition **(17.7)**. — *The degree zero cohomology of the complex* $\{\widehat{A\otimes\ldots\otimes A}, \{\hat{e}_j\}\}$ *with respect to the functor " underlying additive group " is the subalgebra of* A

$$(*) \qquad \operatorname{Ker}(\hat{e}_0 - \hat{e}_1 : A \to \widehat{A\otimes A}).$$

The degree zero cohomology of the same complex with respect to the functor " units " is the group of units in the subalgebra $(*)$.

Proof. — Left to the reader.

18. Applications to Differential Operators

Theorem **(18.1)**. — *Suppose* A *has almost finite projective differentials* (8.5).

a) D_A *is a projective left* A*-module and hence the cohomology of the complex* $\{\widehat{A\otimes\ldots\otimes A}, \{\hat{e}_j\}\}$ *with respect to the functor " underlying additive group " is naturally isomorphic to* $\mathrm{Ext}_{D_A}(A, A)$.

b) $\mathscr{E}\langle D_A\rangle = \mathscr{G}\langle D_A\rangle$, i.e. *if* U *is an algebra over* A *such that* $U \cong D_A$ *as an* A*-bimodule, then* $\langle U\rangle$ *is automatically invertible in* $\mathscr{E}\langle D_A\rangle$.

Proof. — *a)* Suppose A has almost finite projective differentials and $\{L_\alpha\}$ is as in (8.5). Since $\{L_\alpha\}$ is cofinal with $\{\mathfrak{M}^i\}$ which is countable, by (17.5), *h)* D_A is projective as a left A-module. By (17.5), *g)* the result follows.

b) Since $\{L_\alpha\}$ is cofinal in $\{\mathfrak{M}^i\}$ the completion of $A\otimes A$ with respect to $\{L_\alpha\}$ is the same as $\varprojlim(A\otimes A)/\mathfrak{M}^n$. Hence by (17.5), *b)*

$$\mathrm{Hom}_A(D_A\otimes_A D_A, A) \cong \varprojlim(A\otimes A)/\mathfrak{M}^n = \widehat{A\otimes A}.$$

It is easily shown that under this isomorphism the elements $f\in\mathrm{Hom}_A(D_A\otimes_A D_A, A)$ for which $f(1\otimes 1)=1$ correspond to the elements $x\in\widehat{A\otimes A}$ with $\widehat{\mathrm{mult}}(x)=1$. The element x can be written $x=1-z$ with $z\in\widehat{\mathfrak{M}}$ and has inverse $1+z+z^2+\ldots$ Hence by (16.9) the result follows. Q.E.D.

The filtration $D_A^0\subset D_A^1\subset D_A^2\subset\ldots$ has such properties as $D_A^i D_A^j\subset D_A^{i+j}$ and for $f\in D_A^i$, $g\in D_A^i$, $fg-gf\in D_A^{i+j-1}$ [9, (2.1.1), *b)*, p. 210]. Thus the associated graded algebra $\mathrm{gr}\, D_A$ is a commutative algebra. The zeroth graded component $\mathrm{gr}\,{}^0 D_A$ is $D_A^0 = A^\ell$ which is identified with A. Thus $\mathrm{gr}\, D_A$ is an A-algebra. Let Der A denote the left A-module consisting of R-algebra derivations of A. It is easily verified that $\mathrm{Der}\, A\oplus A^\ell = D_A^1$. Thus $\mathrm{gr}^1 D_A$ is naturally isomorphic to Der A as a left A-module if $\mathrm{gr}^1 D_A$ has the module structure induced by $\mathrm{gr}\, D_A$ being an A-algebra.

Since $D_A^1 = \mathrm{Der}\, A\oplus A^\ell$ is a direct sum of left A-modules it follows that Der A is a projective left A-module if and only if D_A^1 is. Let M be Der A considered only as a left A-module and let $\mathbf{S}_A M$ denote the symmetric A-algebra on M. Since $\mathrm{gr}^1 D_A = M$ there is a natural graded A-algebra homomorphism

$$\mathbf{S}_A M \to \mathrm{gr}\, D_A$$

induced by the (identity) map of M to Der A.

In the next theorem it is not assumed that A has projective differentials.

Theorem **(18.2)**. — *Suppose that* R *is a ring containing a copy of* $\mathbf{Q}$, *the rational numbers, and* A *is a commutative* R-*algebra such that* $J_n(A)$ *is a finitely presented left* A-*module for all n and* $J_1(A)$ *is a projective left* A-*module. Then*:

1. $D_A^n = D_A^1 \ldots D_A^1$ *(n-times) for all n.*
2. $J_n(A)$ *and* D_A^n *are finite projective left* A-*modules for all n. Hence* A *does have finite projective differentials.*
3. *The natural map* $S_A M \to \operatorname{gr} D_A$ *is an isomorphism.*

Proof. — Suppose $\mathfrak{N}$ is a maximal ideal of A. By (13.4), $J_1(A_\mathfrak{N}) \cong A_\mathfrak{N} \otimes_A J_1(A)$ as a left $A_\mathfrak{N}$-module. Thus $J_1(A_\mathfrak{N})$ is a projective left $A_\mathfrak{N}$-module. Since

$$J_1(A_\mathfrak{N}) = A_\mathfrak{N} \oplus J_1^+(A_\mathfrak{N})$$

as left $A_\mathfrak{N}$-modules it follows that $J_1^+(A_\mathfrak{N})$ is a projective left $A_\mathfrak{N}$-module. Since $A_\mathfrak{N}$ is a local ring $J_1^+(A_\mathfrak{N})$ is a free $A_\mathfrak{N}$-module. Since R has characteristic zero [8, (16.12.2), p. 55] applies (since Grothendieck's $\Omega^1_{X/S}$ is the same as our $J_1^+(A_\mathfrak{N})$ in this case). Thus Spec $A_\mathfrak{N}$ is differentially smooth over Spec R (the morphism induced by the canonical map $R \to A_\mathfrak{N}$) in the sense of [8, (16.10.1), p. 51] and for all n (by the sentence after [8, (16.10.1), p. 51]) $J_n(A_\mathfrak{N})$ is a free $A_\mathfrak{N}$-module (since Grothendieck's $P^n_{X/S}$ here coincides with our $J_n(A_\mathfrak{N})$).

Since $J_n(A_\mathfrak{N}) \cong A_\mathfrak{N} \otimes_A J_n(A)$ as a left A-module by (13.4), it follows that $J_n(A)$ is a finite projective left A-module for all n. Since $D_A^n = \operatorname{Hom}_A(J_n(A), A)$ as left A-modules part 2 is proved.

Using the isomorphism $J_n(A_\mathfrak{N}) \cong A_\mathfrak{N} \otimes_A J_n(A)$, (13.4), it follows that

$$\begin{aligned} D^n_{A_\mathfrak{N}} &= \operatorname{Hom}_{A_\mathfrak{N}}(A_\mathfrak{N} \otimes_A J_n(A), A_\mathfrak{N}) = \operatorname{Hom}_A(J_n(A), A_\mathfrak{N}) \\ &= A_\mathfrak{N} \otimes_A \operatorname{Hom}_A(J_n(A), A) = A_\mathfrak{N} \otimes_A D_A^n, \end{aligned}$$

where the next to last equality uses the fact that $J_n(A)$ is a finite projective left A-module (part 2). This map is given more explicitly as follows: for $d \in D_A^n$ the map $A \xrightarrow{d} A \to A_\mathfrak{N}$ is in $D_A^n(A_\mathfrak{N})$ and by (13.2) has a unique "lifting" to $\widetilde{d} \in D^n_{A_\mathfrak{N}}(A_\mathfrak{N}) = D^n_{A_\mathfrak{N}}$. Then the above isomorphism between $D^n_{A_\mathfrak{N}}$ and $A_\mathfrak{N} \otimes_A D_A^n$ is given by

$$\begin{gathered} A_\mathfrak{N} \otimes_A D_A^n \to D^n_{A_\mathfrak{N}} \\ z \otimes d \mapsto z\widetilde{d} \end{gathered}$$

$z \in A_\mathfrak{N}$, $d \in D_A^n$. Let $\widetilde{D}_A^n$ denote $\{\widetilde{d} \in D^n_{A_\mathfrak{N}} \mid d \in D_A^n\}$. Then $D^n_{A_\mathfrak{N}} = A_\mathfrak{N} \widetilde{D}_A^n$. Moreover if $d \in D_A^i$, $e \in D_A^j$ with $i + j \leq n$, then $de \in D_A^n$, and by uniqueness of the lifting "$\sim$" it follows that $\widetilde{de} = \widetilde{d}\widetilde{e}$.

To prove that $D_A^n = D_A^1 \ldots D_A^1$ (n times) it suffices to prove that for all maximal ideals $\mathfrak{N} \subset A$, $A_\mathfrak{N} \otimes_A D_A^n = A_\mathfrak{N} \otimes_A D_A^1 \ldots D_A^1$. Under the isomorphism $A_\mathfrak{N} \otimes_A D_A^n \cong D^n_{A_\mathfrak{N}}$ the right hand side maps to

$$A_\mathfrak{N}(\widetilde{D_A^1 \ldots D_A^1}) \quad (n \text{ times})$$

which equals $A_{\mathfrak{N}}\widetilde{D}^1_A\ldots\widetilde{D}^1_A$ because of the multiplicative property of " $\sim$ " mentioned above.

When $n=1$ the isomorphism $A_{\mathfrak{N}}\otimes_A D^1_A \cong D^1_A$ shows that $A_{\mathfrak{N}}\widetilde{D}^1_A = D^1_{A_{\mathfrak{N}}}$. Clearly $D^1_{A_{\mathfrak{N}}} = D^1_{A_{\mathfrak{N}}}A_{\mathfrak{N}}$ and thus

$$A_{\mathfrak{N}}\overbrace{\widetilde{D}^1_A\ldots\widetilde{D}^1_A}^{n} = D^1_{A_{\mathfrak{N}}}\overbrace{\widetilde{D}^1_A\ldots\widetilde{D}^1_A}^{n-1} = D^1_{A_{\mathfrak{N}}}A_{\mathfrak{N}}\overbrace{\widetilde{D}^1_A\ldots\widetilde{D}^1_A}^{n-1} = D^1_{A_{\mathfrak{N}}}D^1_{A_{\mathfrak{N}}}\overbrace{\widetilde{D}^1_A\ldots\widetilde{D}^1_A}^{n-2}$$

$$D^1_{A_{\mathfrak{N}}}D^1_{A_{\mathfrak{N}}}A_{\mathfrak{N}}\overbrace{\widetilde{D}^1_A\ldots\widetilde{D}^1_A}^{n-2} = D^1_{A_{\mathfrak{N}}}D^1_{A_{\mathfrak{N}}}D^1_{A_{\mathfrak{N}}}\overbrace{\widetilde{D}^1_A\ldots\widetilde{D}^1_A}^{n-3} = \ldots = D^1_{A_{\mathfrak{N}}}\ldots D^1_{A_{\mathfrak{N}}} \quad (n \text{ times}).$$

As observed in the first paragraph of the proof Spec $A_{\mathfrak{N}}$ is differentially smooth over Spec R and $J_1^+(A_{\mathfrak{N}})$ is a free left $A_{\mathfrak{N}}$-module. Thus [8, (16.11.2), p. 54] applies. Using the notation in [8, (16.11.2), p. 54], since $D_q \in D^1_A$ when $|q| \leq 1$, and by (16.11.2.2) and the fact that the characteristic is zero, $D_q \in D^1_{A_{\mathfrak{N}}}\ldots D^1_{A_{\mathfrak{N}}}$ (n times) if $|q|=n$. Thus by the lines following (16.11.2.2), $D^n_{A_{\mathfrak{N}}} = D^1_{A_{\mathfrak{N}}}\ldots D^1_{A_{\mathfrak{N}}}$ (n times). Thus part 1 is proved.

By part 1 it follows that the natural map $\mathbf{S}_A M \to \operatorname{gr} D_A$ is surjective and injectivity must be established. Again it suffices to prove that for each maximal ideal $\mathfrak{N} \subset A$ the map

$$(*) \qquad A_{\mathfrak{N}}\otimes_A \mathbf{S}_A M \to A_{\mathfrak{N}}\otimes_A \operatorname{gr} D_A$$

is injective.

From the isomorphisms $A_{\mathfrak{N}}\otimes_A D^n_A \to D^n_{A_{\mathfrak{N}}}$ for all n it follows that

$$A_{\mathfrak{N}}\otimes_A D_A \to D_{A_{\mathfrak{N}}}, \qquad z\otimes d \mapsto z\widetilde{d}$$

is an isomorphism. The left hand side at $(*)$ is naturally isomorphic to $\mathbf{S}_{A_{\mathfrak{N}}}(A_{\mathfrak{N}}\otimes_A M)$. For $y, z \in A_{\mathfrak{N}}$, $d \in D^n_A$, $e \in D^m_A$, $(y\widetilde{d})(z\widetilde{e}) = yz\widetilde{d}\widetilde{e} - y[z, \widetilde{d}]\widetilde{e}$. The element $[\widetilde{d}, z] \in D^{n-1}_{A_{\mathfrak{N}}}$ and so $y[\widetilde{d}, z]\widetilde{e} \in D^{n+m-1}_{A_{\mathfrak{N}}}$. This shows that $A_{\mathfrak{N}}\otimes_A \operatorname{gr} D_A$ is naturally isomorphic to $\operatorname{gr} D_{A_{\mathfrak{N}}}$. $(*)$ corresponds to the natural isomorphism

$$\mathbf{S}_{A_{\mathfrak{N}}}(A_{\mathfrak{N}}\otimes_A M) \to \operatorname{gr} D_{A_{\mathfrak{N}}}.$$

Note $A_{\mathfrak{N}}\otimes_A M$ is naturally isomorphic to Der $A_{\mathfrak{N}}$, the isomorphism being induced by the isomorphism $A_{\mathfrak{N}}\otimes_A D^1_A \xrightarrow{\cong} D^1_{A_{\mathfrak{N}}}$.

Again the differential smoothness of $A_{\mathfrak{N}}$ and [8, (16.11.2), p. 54] will give the desired result. In the notation of [8, (16.11.2), p. 54] the operators $\{D_q\}$ form an $A_{\mathfrak{N}}$-basis for the free $A_{\mathfrak{N}}$-module $D_{A_{\mathfrak{N}}}$. Thus $\{q!\,D_q\}$ is also an $A_{\mathfrak{N}}$-basis since the characteristic is zero. Since $D_q \in D^{|q|}_A$ the images $\{q!\,\overline{D}_q\}$ form an $A_{\mathfrak{N}}$-basis for $\operatorname{gr} D_{A_{\mathfrak{N}}}$. By (16.11.2.2) the images $\{q!\,\overline{D}_q\}$ are the usual polynomial monomials. This proves injectivity of $\mathbf{S}_{A_{\mathfrak{N}}}(A_{\mathfrak{N}}\otimes_A M) \to \operatorname{gr} D_{A_{\mathfrak{N}}}$ and part 3. Q.E.D.

Proposition **(18.3)**. — *Suppose* A *is an algebra such that* $J_n(A)$ *is a finite projective left* A-*module for all* n *and the natural map* $\mathbf{S}_A M \to \operatorname{gr} D_A$ *is an isomorphism, where* $M = \operatorname{Der} A$ *as a left* A-*module. Let* I *be an ideal of* A, *let* $\mathscr{A}$ *be the completion of* A *in the* I-*adic topology and let* $N = \operatorname{Der} \mathscr{A}$ *as a left* $\mathscr{A}$-*module. Then*

1. $D^n_{\mathscr{A}} = D^1_{\mathscr{A}} \dots D^1_{\mathscr{A}}$ (n times) *for all* n.
2. $D^n_{\mathscr{A}}$ *is a finite projective left* $\mathscr{A}$-*module for all* n.
3. *The natural map* $\mathbf{S}_A N \to \operatorname{gr} D_{\mathscr{A}}$ *is an isomorphism.*
4. $D_{\mathscr{A}}$ *is a projective left* $\mathscr{A}$-*module.*

Proof. — By (13.9) $\mathscr{J}_n(\mathscr{A})$ is isomorphic to the completion of $J_n(A)$ in the I-adic topology. Since $J_n(A)$ is a finitely generated module this is isomorphic to $\mathscr{A} \otimes_A J_n(A)$. Thus

$$\begin{aligned} D^n_{\mathscr{A}} &= \operatorname{Hom}_{\mathscr{A}}(\mathscr{J}_n(\mathscr{A}), \mathscr{A}) \cong \operatorname{Hom}_{\mathscr{A}}(\mathscr{A} \otimes_A J_n(A), \mathscr{A}) \\ &= \operatorname{Hom}_A(J_n(A), \mathscr{A}) = \mathscr{A} \otimes_A \operatorname{Hom}_A(J_n(A), A) \\ &= \mathscr{A} \otimes_A D^n_A. \end{aligned}$$

The first equality holds by definition of $\mathscr{J}_n(\mathscr{A})$ and the fact that $\mathscr{A}$ is complete in the I-topology. The next to last equality holds because $J_n(A)$ is a finite projective A-module. The above isomorphism is now displayed explicitly: for $d \in D^n_A$ the map $A \xrightarrow{d} A \to \mathscr{A}$ is in $D^n_A(\mathscr{A})$ and by (13.8) has a unique "lifting" to an element $\tilde{d} \in D^n_{\mathscr{A}}(\mathscr{A}) = D^n_{\mathscr{A}}$. The above isomorphism is then given by

$$(*) \qquad \begin{array}{ccc} \mathscr{A} \otimes_A D^n_A & \xrightarrow{\cong} & D^n_{\mathscr{A}} \\ x \otimes d & \longmapsto & x\tilde{d} \end{array}$$

for $x \in \mathscr{A}$, $d \in D^n_A$. The map at $(*)$ yields the isomorphism $\mathscr{A} \otimes_A D_A \to D_{\mathscr{A}}$, $(x \otimes d \mapsto x\tilde{d})$, $x \in \mathscr{A}$, $d \in D_A$.

For $n = 1$ the map at $(*)$ induces an isomorphism between $\mathscr{A} \otimes_A M$ and N.

For $x, y \in \mathscr{A}$, $d \in D^n_A$, $e \in D^m_A$, $(x\tilde{d})(y\tilde{e}) = xy\tilde{d}\tilde{e} - x[y, \tilde{d}]\tilde{e}$. Since $[y, \tilde{d}] \in D^{n-1}_{\mathscr{A}}$ the term $x[y, \tilde{d}]\tilde{e} \in D^{n+m-1}_{\mathscr{A}}$. Thus the isomorphism $\mathscr{A} \otimes_A D_A \xrightarrow{\cong} D_{\mathscr{A}}$ induces an algebra isomorphism $\mathscr{A} \otimes_A \operatorname{gr} D_A \xrightarrow{\cong} \operatorname{gr} D_{\mathscr{A}}$. This proves part 3.

Since the map in part 3 is a graded algebra map it follows that $\mathbf{S}^n_{\mathscr{A}} N$ maps onto $\operatorname{gr}^n D_{\mathscr{A}}$ for all n. By an easy induction left to the reader this proves part 1.

Since $J_1(A)$ is a finite projective A-module so is $D^1_A = \operatorname{Hom}_A(J_1(A), A)$. Thus $\operatorname{Der} A$ is a finite projective left A-module since $D^1_A = A^\ell \oplus \operatorname{Der} A$ as left A-modules. Thus $N = \mathscr{A} \otimes_A \operatorname{Der} A$ is a finite projective left $\mathscr{A}$-module. Then for all n

$$D^n_{\mathscr{A}} / D^{n-1}_{\mathscr{A}} \cong \mathbf{S}^i_{\mathscr{A}} N$$

is a finite projective left $\mathscr{A}$-module. Hence there exist finite projective left $\mathscr{A}$-modules $P_i \subset D^i_{\mathscr{A}}$ where $D^i_{\mathscr{A}} = P_i \oplus D^{i-1}_{\mathscr{A}}$ for $i \geq 1$. Let $P_0 = \mathscr{A}^\ell = D^0_{\mathscr{A}}$. Then

$$D^n_{\mathscr{A}} = \bigoplus_{i=0}^{n} P_i$$

which proves part 2. And part 4 follows from

$$D_{\mathscr{A}} = \bigoplus_{i=0}^{\infty} P_i$$

Q.E.D.

Corollary **(18.4)**. — *Let* B *be a commutative algebra such that* $L = \text{Der } B$, L *is a (finite) projective left* B*-module and the natural map* $S_B L \to \text{gr } D_B$ *is an isomorphism. Then*

1. $D_B^n = D_B^1 \dots D_B^1$ *(n times) for all n.*
2. D_B^n *is a (finite) left* B*-module for all n.*
3. D_B *is a projective left* B*-module.*

Proof. — The steps needed to prove the corollary are contained in the proof of (18.3). Q.E.D.

When B and D_B satisfy the hypotheses of (18.4) it is possible to characterize D_B as a certain universal enveloping algebra. In [14, § 2, p. 197] (K, R)-Lie algebras are introduced. If B is a commutative R-algebra then $L = \text{Der } B$ is an (R, B)-Lie algebra. L has an enveloping algebra V(B, L) [14, § 2, p. 197]. There is a canonical algebra map $B \to V(B, L)$ which is injective. There is a map $L \to V(B, L)$ which is a Lie algebra map to $V(B, L)^-$. Let $\bar{L}$ denote the image of $L \to V(B, L)$ and identify B with its image in V(B, L). There is a filtration on V(B, L) such that

$$\begin{aligned} &V_0(B, L) = B \\ &V_1(B, L) = B + \bar{L} \\ &\dots\dots\dots\dots\dots \\ &V_n(B, L) = B + \bar{L} + \dots + \overbrace{\bar{L} \dots \bar{L}}^{n}. \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

Note that V(B, L) is not the usual universal enveloping algebra of L as a Lie algebra but is the enveloping algebra of L as an (R, B)-Lie algebra. See [14, § 2-3, p. 197-200].

By the universal property of V(B, L) there is a natural algebra map $V(B, L) \to D_B$ which is a map of algebras over B. It is induced by the natural inclusion $L = \text{Der } B \hookrightarrow D_B$.

Proposition **(18.5)**. — *Suppose* B *is a commutative* R*-algebra such that* $L = \text{Der } B$ *is projective as a left* B*-module and the natural algebra map* $S_B L \to \text{gr } D_B$ *is an isomorphism. Then the natural map of algebras over* B, $V(B, L) \to D_B$ *is an isomorphism.*

Proof. — By (18.4), part 1, $V_n(B, L)$ maps onto D_B^n so that $V(B, L) \to D_B$ is surjective and preserves filtration. Thus it suffices to prove that the natural map $\text{gr } V(B, L) \to \text{gr } D_B$ is injective. This is true by [14, (3.1), p. 198] and the assumption that $S_B L \to \text{gr } D_B$ is an isomorphism. Q.E.D.

The algebraic DeRham complex of a commutative algebra is the exterior algebra of the Kaehler module together with the unique exterior derivation. More precisely, for a commutative algebra B let K denote the Kaehler module $J_1^+(B)$. Then $j_1^+ : B \to K$ is the universal derivation of B. Let $E_B(K)$ be the B-exterior algebra on K. It is shown in [13, Lemma (9.2), p. 155] that $E_B(K)$ has a unique degree 1 derivation δ satisfying $\delta\delta = 0$ and $\delta | B = j_1^+$. Explicitly δ is determined by

$$\textbf{(18.6)} \qquad \begin{cases} \delta(b_0) = j_1^+(b_0) \\ \delta(b_0\, \delta b_1 \wedge \ldots \wedge \delta b_n) = \delta b_0 \wedge \delta b_1 \wedge \ldots \wedge \delta b_n \end{cases}$$

for $\{b_i\} \subset B$.

If M is a finite projective left B-module and $N = \operatorname{Hom}_B(M, B)$, then in each degree $E_B^i(M)$ is a finite projective left B-module. Moreover $\operatorname{Hom}_B(E_B(M), B)$ is naturally isomorphic to $E_B(N)$. The isomorphism is given as follows: say $n_1, \ldots, n_t \in N$ and $m_1, \ldots, m_t \in M$. The element $n_1 \wedge \ldots \wedge n_t \in E_B^t N$ corresponds to a function f on $E_B M$ which vanishes on $E_B^i M$ for $i \neq t$. For $i = t$, f is determined by

$$\textbf{(18.7)} \qquad f(m_1 \wedge \ldots \wedge m_t) = \det \begin{pmatrix} n_1(m_1) \ldots n_t(m_1) \\ \vdots \\ n_1(m_t) \ldots n_t(m_t) \end{pmatrix}.$$

Proposition **(18.8)**. — *Let* B *be a commutative algebra such that* $K = J_1^+(B)$ *is a finite projective left* B-*module. Let* $L = \operatorname{Der} B = \operatorname{Hom}_B(K, B)$ *and assume the natural algebra map* $S_B L \to \operatorname{gr} D_B$ *is an isomorphism. Then the cohomology* $\operatorname{Ext}^*_{D_B}(B, B)$ *is naturally isomorphic to the algebraic DeRham cohomology* $H^*_{DR}(B)$ *of* B.

Proof. — By (18.5) $\operatorname{Ext}^*_{D_B}(B, B) = \operatorname{Ext}^*_{V(B,L)}(B, B)$. In [14, (4.2) and (4.3), p. 202] it is shown that $\operatorname{Ext}^*_{V(B,L)}(B, B)$ is the cohomology of the R-module of strongly alternating B multilinear maps from L to B under the formal differentiation

$$\textbf{(18.9)} \qquad \begin{cases} (Df)(\ell_1, \ldots, \ell_n) = \sum_{i=1}^{n} (-)^{i-1} \ell_i(f(\ell_1, \ldots, \hat{\ell}_i, \ldots, \ell_n)) \\ \qquad + \sum_{j<k} (-)^{j+k} f([\ell_j, \ell_k], \ell_1, \ldots, \hat{\ell}_j, \ldots, \hat{\ell}_k, \ldots, \ell_n) \end{cases}$$

where f is a strongly alternating B $(n-1)$-linear map from L to B.

The strongly alternating B i-linear maps from L to B are the same as the maps from $E_B^i L$ to B. Thus the cohomology of $\operatorname{Ext}^*_{V(B,L)}(B, B)$ is computed from $\operatorname{Hom}_B(E_B L, B)$. Since $L = \operatorname{Hom}_B(K, B)$ and K is a finite projective B-module, L is also, and $\operatorname{Hom}_B(E_B L, B)$ is naturally isomorphic to $E_B K$.

Suppose f in (18.9) corresponds to the element $b_1\, \delta b_2 \wedge \ldots \wedge \delta b_n \in E_B^{n-1} K$. Using the duality (18.7) and the formula (18.9) it is a straightforward but lengthy calculation to verify that

$$(Df)(\ell_1 \wedge \ldots \wedge \ell_n) = \det \begin{pmatrix} \ell_1(b_1) \ldots \ell_n(b_1) \\ \vdots \qquad \vdots \\ \ell_1(b_n) \ldots \ell_n(b_n) \end{pmatrix}$$

which implies that $Df = \delta b_1 \wedge \delta b_2 \wedge \ldots \wedge \delta b_n$. Thus the cohomology of $\operatorname{Ext}^*_{V(B,L)}(B, B)$ is computed from the algebraic DeRham complex of B. Q.E.D.

Theorem **(18.10)**. — *Suppose that* R *contains a copy of* Q *and* A *is a commutative* R-*algebra such that* $J_n(A)$ *is a finitely presented left* A-*module for all n and* $J_1(A)$ *is a projective left* A-*module. Then the cohomology* $H^*(D_A)$ (15.4) *is naturally isomorphic to the algebraic DeRham cohomology of* A *from degree two onward.*

Proof. — By (18.2), 4) and (18.1) D_A is a projective left A-module. Thus by (17.5), *g), f)* and *d)* it follows that $H^*(D_A)$ is actually isomorphic to $\mathrm{Ext}^*_{D_A}(A, A)$ from degree two onward. By (18.2), 3) and (18.5) it follows that $H^*(D_A)$ is naturally isomorphic to $\mathrm{Ext}^*_{V(A,M)}(A, A)$ from degree two onward, where $M = \mathrm{Der}\, A$. By (18.8) the theorem is proved. Q.E.D.

The results (13.20), 2), (13.21) and (13.25) provide a supply of algebras A where $J_n(A)$ is a finitely presented (actually finite projective) left A-module for all n and where $J_1(A)$ is a projective left A-module.

Corollary **(18.11)**. — *Suppose* R *is a field of characteristic zero and* $A = R[X_1, \ldots, X_n]$. *If* U *is an algebra over* A *and* $U \cong D_A$ *as an* A-*bimodule, then* $U \cong D_A$ *as an algebra over* A; i.e. $\mathscr{E}_{D_A} = \{\langle D_A \rangle\}$.

Proof. — By (13.19), part 3, $J_n(A)$ is a finitely generated free A-module for all *n*. Hence (18.10) applies and $H^2(D_A) \cong H^2_{DR}(A)$. Since the DeRham cohomology of the polynomial ring is zero in positive degree, it follows that $H^2(D_A) = \{0\}$. By (18.10) $\mathscr{G}\langle D_A \rangle$ thus consists of only the identity and by (18.1), *b)* $\mathscr{E}\langle D_A \rangle$ consists of only $\langle D_A \rangle$. By (4.9), $\mathscr{E}\langle D_A \rangle = \mathscr{E}_{D_A}$. Q.E.D.

BIBLIOGRAPHY

[1] S. Amitsur, Simple algebras and cohomology groups of arbitrary fields, *Trans. Amer. Math. Soc.*, **90** (1959), 73-112.

[2] M. Auslander and O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.*, **97** (1960), 367-409.

[3] N. Bourbaki, *Eléments de Mathématique. Algèbre commutative*, Hermann, chap. 1.

[4] N. Bourbaki, *Eléments de Mathématique. Algèbre commutative*, Hermann, chap. 3-4.

[5] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press (1956).

[6] S. U. Chase, D. K. Harrison, Alex Rosenberg, Galois theory and cohomology of commutative rings, *Memoirs of the Amer. Math. Soc.*, **52** (1965).

[7] Jean Dieudonné, La théorie de Galois des anneaux simples et semi-simples, *Commentarii Mathematici Helvetici*, **21** (1948), 154-184.

[8] A. Grothendieck, Eléments de géométrie algébrique, IV, part 4, *Publ. math. I.H.E.S.*, **32** (1967).

[9] R. G. Heyneman and M. E. Sweedler, Affine Hopf Algebras, *J. of Algebra*, **13** (1969), 192-241.

[10] G. Hochschild, Simple algebras with purely inseparable splitting fields of exponent 1, *Trans. Amer. Math. Soc.*, **79** (1955), 477-489.

[11] N. Jacobson, *Lectures in abstract algebra*, vol. III, Van Nostrand, Princeton (1964).

[12] John C. McConnell and Moss E. Sweedler, Simplicity of smash products, *Proc. London Math. Soc.*, **23** (1971), 251-266.

[13] H. Osborn, Modules of differentials, II. *Math. Annalen*, **175** (1968), 146-158.

[14] G. S. Rinehart, Differential forms of general commutative algebras, *Trans. Amer. Math. Soc.*, **108** (1963), 195-222.
[15] A. Rosenberg and D. Zelinsky, On Amitsur's complex, *Trans. Amer. Math. Soc.*, **97** (1960), 327-356.
[16] M. E. Sweedler, Cohomology of algebras over Hopf algebras, *Trans. Amer. Math. Soc.*, **133** (1968), 205-239.
[17] M. E. Sweedler, *Hopf algebras*, New York, W. A. Benjamin, 1969.
[18] M. E. Sweedler, Multiplication alteration by two-cocycles, *Ill. J. Math.*, **15** (1971), 302-323.

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It has come to my attention that the work of Lieberman in [Generalizations of the DeRham complex with applications to duality theory and the cohomology of singular varieties, Lieberman, *Rice University Studies*, 59 (1973), 57-70], gives an alternative proof to parts of our theory relating X_A-bialgebra cohomology and algebraic DeRham cohomology.