PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Singularities of vector fields

Publications mathématiques de l'I.H.É.S., tome 43 (1974), p. 47-100
<http://www.numdam.org/item?id=PMIHES_1974__43__47_0>

SINGULARITIES OF VECTOR FIELDS

by Floris TAKENS [1]

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[1] Research supported by the Organization of American States (O.A.S.).

I. — INTRODUCTION AND STATEMENT OF THE RESULTS

1. Introduction.

The main aim of this paper is to classify codimension two singularities of vector fields (i.e. those singularities which generically occur in 2 parameter families of vector fields) and to show that, in a certain sense, such a classification is impossible for codimension three singularities (at least in the dimensions ≥ 5). A more precise statement of the results is given in § 3 of this chapter.

This study was mainly motivated by the desire to extend the Thom-Mather catastrophy theory [19; see also the appendix by J. Mather in the English translation] to systems which are not " regulated " by a potential function (or rather by its gradient flow) but which are regulated by an arbitrary (but generic) flow. For such an extension of the Thom-Mather theory, the next thing to be done is to study the " unfoldings " of the codimension 2 singularities which are classified in this paper. I hope to deal with these unfoldings in future publications.

I want to thank R. Thom for the many stimulating discussions, related with the topic of this paper, I had with him. I also want to thank M. Shub and C. C. Pugh for introducing me to their techniques on invariant manifolds which were essential in chapter IV. Finally, I want to thank the Instituto de Matemâtica Pura e Aplicada (I.M.P.A.) for the hospitality they offered me in the time I prepared this paper.

2. Definitions.

We shall study germs of singularities of C^∞, or C^ℓ, vector fields on $\mathbf{R}^n$ in 0.

Definition (**1.1**). — A *germ of a singularity* of a C^∞- (or C^ℓ-) vector field on $\mathbf{R}^n$ in 0 is an equivalence class in the set of all C^∞- (or C^ℓ-) vector fields X on $\mathbf{R}^n$ with $X(0)=0$; X_1 and X_2 are (germ-)equivalent if there is a neighbourhood U of $0\in\mathbf{R}^n$ such that $X_1|U=X_2|U$. $\mathscr{G}^n$ (or $\mathscr{G}^{n,\ell}$) denotes the set of all these C^∞- (or C^ℓ-) germs.

Definition (**1.2**). — Let $X_1, X_2\in\mathscr{G}^n$ (or $\mathscr{G}^{n,\ell}$, $\ell\geq k$). Then X_1 and X_2 are *k-jet equivalent* if all the partial derivatives up to, and including, order k of the component functions of X_1 in 0 coincide with those of X_2. The equivalence classes are called *k-jets*. There is a natural 1-1 correspondence between k-jets of singularities of vector fields on $\mathbf{R}^n$ and vector fields X on $\mathbf{R}^n$ with $X(0)=0$ and with component functions polynomials of degree $\leq k$. The set J^n_k of k-jets of singularities of vector fields on $\mathbf{R}^n$ is a vector space (and hence it has the structure of an algebraic manifold); $\pi_k : \mathscr{G}^n \to J^n_k$ are the induced projections; also the natural projections $J^n_\ell \to J^n_k$, $\ell \geq k$, are denoted by π_k. We take on $\mathscr{G}^n$ the *topology* induced by $\{\pi_k\}_{k=1}^\infty$; i.e. if $X \in \mathscr{G}^n$, then a basis of neighbourhoods of X is obtained by taking $\{\pi_k^{-1}(U_{k,i})\}_{k,i=1}^\infty$, where, for each k, $\{U_{k,i}\}_{i=1}^\infty$ is a basis of neighbourhoods of $\pi_k(X)$ in J^n_k. A (smooth or algebraic) *submanifold* $W \subset \mathscr{G}^n$ is a subspace which is, for some k, of the form $W = \pi_k^{-1}(W_k)$, where W_k is some (smooth or algebraic) submanifold of J^n_k; (semi-)algebraic subsets of $\mathscr{G}^n$ are similarly defined.

Definition **(1.3)**. — $X_1, X_2 \in \mathscr{G}^{n,\ell}$ are C^i-equivalent if for some (and hence all) representatives $\widetilde{X}_1$, $\widetilde{X}_2$ of X_1 and X_2 there are neighbourhoods U_1, U_2 of 0 in $\mathbf{R}^n$ and a C^i-homeomorphism $\varphi : U_1 \to U_2$ which maps integral curves of $\widetilde{X}_1$ to integral curves of $\widetilde{X}_2$ preserving the " sense " but not necessarily the precise parametrization; more precisely: if $p \in U_1$ and $\mathscr{D}_{\widetilde{X}_1}(p, [0, t_1]) \subset U_1$, $t_1 > 0$, then there is some $t_2 > 0$ such that $\mathscr{D}_{\widetilde{X}_2}(\varphi(p), [0, t_2]) = \varphi(\mathscr{D}_{\widetilde{X}_1}(p, [0, t_1]))$. (As usual, $\mathscr{D}_{\widetilde{X}_i} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ denotes the integral of X_i, i.e. $\mathscr{D}_{\widetilde{X}_i}(p, 0) = p$ and $\dfrac{\partial}{\partial t}(\mathscr{D}_{\widetilde{X}_i}(p, t)) = \widetilde{X}_i(\mathscr{D}_{\widetilde{X}_i}(p, t))$.)

Definition **(1.4)**. — Let X be a vector field on $\mathbf{R}^n$ with $X(0) = 0$ and let U be some bounded neighbourhood of 0 in $\mathbf{R}^n$. Then $L_{\omega, X, U}(p)$, the ω-limit of p with respect to U, for $p \in U$, is $\emptyset$ if the positive integral curve of X, starting at p, leaves U and otherwise is the set of those $q \in \overline{U}$ for which there is a sequence $t_1, t_2, \ldots \to +\infty$ with $\lim\limits_{i \to \infty} \mathscr{D}_X(p, t_i) = q$.

$L_{\alpha, X, U}(p)$ is defined analogously, i.e. by replacing " positive integral curve " and " $t_1, t_2, \ldots \to +\infty$ " by " negative integral curve " and " $t_1, t_2, \ldots \to -\infty$ ".

Definition **(1.5)**. — $X_1, X_2 \in \mathscr{G}^{n,\ell}$ are weakly-C^i-equivalent if for some (and hence all) representatives $\widetilde{X}_1$, $\widetilde{X}_2$ of X_1 and X_2, there are two bounded neighbourhoods U_1 and U_2 of 0 in $\mathbf{R}^n$ and a C^i-homeomorphism $\varphi : U_1 \to U_2$ such that for any $V \subset U$, with $0 \in V$, and any $p \in V$:

$$L_{\omega, \widetilde{X}_2, \varphi(V)}(\varphi(p)) = \varphi(L_{\omega, \widetilde{X}_1, V}(p)) \quad \text{and} \quad L_{\alpha, \widetilde{X}_2, \varphi(V)}(\varphi(p)) = \varphi(L_{\alpha, \widetilde{X}_1, V}(p)).$$

Definition (1.5) has the advantage that it is clear that " being weakly-C^i-equivalent " does not depend on the choice of the representatives X_1, X_2. In the following lemma we shall give a useful criterium for φ to realize a weak-C^i-equivalence.

Lemma **(1.6)**. — *Let X_1 and X_2 be two vector fields on $\mathbf{R}^n$ with $X_1(0) = X_2(0) = 0$. Let $\varphi : U_1 \to U_2$ be a C^i-homeomorphism; U_1, U_2 are bounded neighbourhoods of 0 in $\mathbf{R}^n$. Let K_i be the set of those points $p \in U_i$ for which $L_{\omega, X_i, U_i}(p)$ or $L_{\alpha, X_i, U_i}(p)$ is non-empty. Then φ realizes a weak-C^i-equivalence between the germs of X_1 and X_2 if and only if:*

1) $\varphi(K_1) = K_2$;
2) $\varphi \mid K_1$ *maps integral curves of X_1, " sense " preserving, to integral curves of X_2.*

Proof. — It is clear that if φ satisfies 1) and 2) then φ realizes a weak-C^i-equivalence. Also if φ does not satisfy 1) then φ does not realize a weak-C^i-equivalence.

We suppose now that φ does not satisfy 2) (but satisfies 1)). We then can find two points $p_1, p_2 \in K_1$ such that p_2 lies on the positive X_1 integral curve (in U_1) starting in p_1 but $\varphi(p_2)$ not on the positive X_2 integral curve (in U_2) starting in $\varphi(p_1)$. If $L_{\omega, X_1, U_1}(p_1) \neq \emptyset$, resp. $L_{\alpha, X_1, U_1}(p_2) \neq \emptyset$, we take $V_\omega = U_1 \setminus \{p_2\}$, resp. $V_\alpha = U_1 \setminus \{p_1\}$. We then see that $L_{\omega, X_1, V_\omega}(p_1) = \emptyset$, resp. $L_{\alpha, X_1, V_\alpha}(p_2) = \emptyset$, but, if φ realizes a weak-C^i-equivalence, $L_{\omega, X_2, \varphi(V_\omega)}(\varphi(p_1)) \neq \emptyset$, resp. $L_{\alpha, X_2, \varphi(V_\alpha)}(\varphi(p_2)) \neq \emptyset$. By the definition of K_1, we have $L_{\omega, X_1, U_1}(p_1) \neq \emptyset$ or $L_{\alpha, X_1, U_1}(p_2) \neq \emptyset$; this shows that φ does not realize a weak-C^i-equivalence.

Remark (**1.7**). — If $X_1, X_2 \in \mathcal{G}^n$ are C^i-equivalent, then it is clear that they are also weakly-C^i-equivalent. The converse is not true; for example, if X_1 and X_2 are germs of non-degenerate singularities on $\mathbf{R}^2$ of saddle type, then X_1 and X_2 are weakly-C^∞-equivalent (the equivalence is realized by a diffeomorphism φ with $\varphi(W^u_{X_1}) = W^u_{X_2}$ and $\varphi(W^s_{X_1}) = W^s_{X_2}$; $W^u_{X_i}$, resp. $W^u_{X_i}$, is the unstable-, resp. stable-, manifold of X_i in 0) but in general not C^2-equivalent (take for example $X_1 = 2x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}$ and $X_2 = 2x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_2^2\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2}$, see also [17]).

Definition (**1.8**). — Let $K \subset \mathcal{G}^n$ and $X \in K$. We say that X is K-(weakly)-C^i-stable if there is a neighbourhood U of X in $\mathcal{G}^n$ such that every $X' \in K \cap U$ is (weakly)-C^i-equivalent with X.

3. Statement of the results.

In this paper we shall prove the following two theorems:

Theorem 1. — There are, for each n, closed semi-algebraic subsets $V_1 \supset V_2 \supset V_3$ *in* $\mathcal{G}^n$ *of codimension* 1, 2 *and* 3 *respectively, such that, with* $V_0 = \mathcal{G}^n$, $i = 1, 2, 3$, *each* $X \in V_{i-1} \setminus V_i$ *is* V_{i-1}*-weakly-*C^0*-stable; moreover, each* $V_{i-1} \setminus V_i$ *is a non-singular open codimension* $(i-1)$*-manifold.*

Theorem 2. — If $n \geq 5$, *then there is no sequence* $V_1 \supset V_2 \supset V_3 \supset V_4$ *of closed semi-algebraic subsets of* $\mathcal{G}^n$ *as in theorem 1.*

Remark (**1.9**). — We shall take V_1 to be the set of germs of those vector fields $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ on $\mathbf{R}^n$ which have at least one eigenvalue on the imaginary axis (these eigenvalues are the eigenvalues of the matrix $\left(\left(\frac{\partial X_i}{\partial x_j}\right)_{i,j}\right)$. The fact that each $X \in \mathcal{G}^n \setminus V_1$ is $\mathcal{G}^n$-weakly-C^0-stable, even $\mathcal{G}^n$-C^0-stable, is a reformulation of the theorem of Hartman and Grobman ([5], [4]).

The singularities occurring for $X \in V_1 \setminus V_2$ are described by Sotomayor in [16] where they were proved to be V_1-C^0-stable and where their unfoldings were studied.

In the proof of theorem 1 we shall also give a complete classification of the singularities occurring for $X \in V_2 \setminus V_3$.

Remark **(1.10)**. — From [18] it follows that there is some positive k such that there is no sequence $V_1 \supset V_2 \supset \ldots \supset V_k$ of closed semi-algebraic subsets of $\mathcal{G}^4$ as in theorem 1; I do not know what the lowest such k is. On the other hand it seems reasonable to conjecture that in $\mathcal{G}^2$ there is an infinite sequence $V_1 \supset V_2 \supset V_3 \supset \ldots$ as in theorem 1. I have no idea about the situation in $\mathcal{G}^3$.

It seems likely that theorem 1 remains true if we replace weakly-C^0-stable by C^0-stable; however, I do not see how to prove that without having to go through very long computations.

4. Reduction to the completely non-hyperbolic case.

We first restate two theorems which we then shall use to prove that it is enough, in order to prove theorems 1 and 2, to prove certain theorems, namely (1.15), (1.16) and (1.19), concerning germs of vector fields $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ for which all the eigenvalues of $\left(\frac{\partial X_i}{\partial x_j}\right)_{i,j}$ are on the imaginary axis.

Theorem **(1.11)** ([6], [8]). — *Let* $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ *be a* C^∞*-vector field on* $\mathbf{R}^n$ *such that* c *eigenvalues of* $\left(\frac{\partial X_i}{\partial x_j}\right)_{i,j}$ *are on the imaginary axis and let* ℓ *be some positive integer. Then there is a* C^ℓ*-*c*-dimensional manifold* W^c_X*, containing the origin, and a neighbourhood* U *of* $0 \in \mathbf{R}^n$ *such that for any* $p \in W^c_X \cap U$*,* $X(p)$ *is tangent to* W^c_X*, and such that, if* $\tilde{X} = \sum_{i=1}^{c} \tilde{X}_i \frac{\partial}{\partial y_i}$ *is the restriction of* X *to* W^c_X *(*$y_1, \ldots, y_c$ *coordinates on* W^c_X*), then all the eigenvalues of* $\left(\frac{\partial \tilde{X}_i}{\partial y_j}\right)_{i,j}$ *are on the imaginary axis.*

Remark **(1.12)**. — It should be noted that W^c_X in theorem (1.11), the so-called center-manifold of X, is not unique. However, the ℓ-jet of the restriction of X to W^c_X is unique in the following sense:

Let $x_1, \ldots, x_n$ be coordinate functions on $\mathbf{R}^n$ such that $x_1, \ldots, x_c$, restricted to W^c_X, form, near the origin, a coordinate system on W^c_X; let W'^c_X be another C^ℓ-center-manifold of X and let $\tilde{X} = \sum_{i=1}^{c} \tilde{X}_i \frac{\partial}{\partial x_i}$ and $\tilde{X}' = \sum_{i=1}^{c} \tilde{X}'_i \frac{\partial}{\partial x_i}$ be the restrictions of X to W^c_X and W'^c_X respectively expressed in the coordinates $x_1, \ldots, x_c$ restricted to W^c_X, resp. W'^c_X. Then the ℓ-jets of $\tilde{X}_i$ *and* $\tilde{X}'_i$ *are equal for* $i = 1, \ldots, c$.

Theorem **(1.13)** ([6], [14]). — *Let* X, c, W_X^c *and* $\widetilde{X}=\sum_{i=1}^{c}\widetilde{X}_i(y_1, \ldots, y_c)\dfrac{\partial}{\partial y_i}$ *be as in theorem* (1.11). *Then there is an* m, *with* $0 \leq m \leq n-c$, *such that the germ of*:

$$Y=\sum_{i=1}^{c}\widetilde{X}_i(y_1, \ldots, y_c)\frac{\partial}{\partial y_i}+\sum_{i=c+1}^{c+m} y_i\frac{\partial}{\partial y_i}-\sum_{i=c+m+1}^{n} y_i\frac{\partial}{\partial y_i}$$

is C^0*-equivalent to the germ of* X.

Definition **(1.14)**. — $W_1^n, W_2^n, \ldots, W_5^n$ are the following semi-algebraic subsets of $\mathscr{G}^n$ (or $\mathscr{G}^{n,\ell}$): W_i^n is the set of those germs of vector fields $X=\sum_{j=1}^{n} X_j\dfrac{\partial}{\partial x_j}$ on $\mathbf{R}^n$ for which all the eigenvalues of $\left(\dfrac{\partial x_j}{\partial X_{j'}}\right)_{j,j'}$ have non-zero real parts except:

$i=1$: one eigenvalue is zero;
$i=2$: two non-zero (complex conjugate) eigenvalues are on the imaginary axis;
$i=3$: two eigenvalues are zero;
$i=4$: one eigenvalue is zero and two other non-zero (complex conjugate) eigenvalues are on the imaginary axis;
$i=5$: four non-zero eigenvalues are on the imaginary axis.

In chapter VI we shall prove the following two theorems, using the theory developed in the chapters II, III and IV.

Theorem **(1.15)**. — *There are closed semi-algebraic subsets* $W_1^1=V_{1,1}\supset V_{1,2}\supset V_{1,3}$ *and* $W_2^2=V_{2,1}\supset V_{2,2}\supset V_{2,3}$ *of* W_1^1 *and* W_2^2 *such that each* $X\in V_{i,j}\backslash V_{i,j+1}$ *is* $V_{i,j}$*-weakly-*C^0*-stable, for* $i=1, 2$, $j=1, 2$; $V_{i,j}$ *has codimension* j *in* $\mathscr{G}^i$.

This holds also if we consider W_1^1 and W_2^2 as subspaces of $\mathscr{G}^{n,\ell}$ for ℓ sufficiently large. The sets $V_{1,j}$, resp. $V_{2,j}$, have also the property that for any $X\in V_{1,j}$, resp. $V_{2,j}$, and any C^∞-diffeomorphism $\varphi : (\mathbf{R}, 0)\to(\mathbf{R}, 0)$, resp. $\varphi : (\mathbf{R}^2, 0)\to(\mathbf{R}^2, 0)$, $\varphi_*(X)\in V_{1,j}$, resp. $\in V_{2,j}$.

Theorem **(1.16)**. — *There are closed semi-algebraic sets* $W_3^2=V_{3,2}\supset V_{3,3}$, $W_4^3=V_{4,2}\supset V_{4,3}$ *and* $W_5^4=V_{5,2}\supset V_{5,3}$ *such that each* $X\in V_{i,2}\backslash V_{i,3}$ *is* $V_{i,2}$*-weakly-*C^0*-stable,* $i=3, 4, 5$; $V_{i,j}$ *has codimension* j *in* $\mathscr{G}^{i-1}$.

As in theorem (1.15) this also holds in $\mathscr{G}^{n,\ell}$ for ℓ sufficiently large and the sets $V_{i,3}$ are also invariant under the action of diffeomorphisms.

Remark **(1.17)**. — The number of different (non-weakly-C^0-equivalent) singularities in each of the above $V_{i,j}\backslash V_{i,j+1}$ is as follows:

$V_{1,1}\backslash V_{1,2} : 1$ $\quad$ $V_{2,1}\backslash V_{2,2} : 2$

$V_{1,2}\backslash V_{1,3} : 2$ $\quad$ $V_{2,2}\backslash V_{2,3} : 2$

$V_{3,2}\backslash V_{3,3} : 1$ $\quad$ $V_{4,2}\backslash V_{4,3} : 5$ $\quad$ $V_{5,2}\backslash V_{5,3} : 10$

We now give the

Proof of theorem 1 assuming theorems (1.15) *and* (1.16).

As in remark (1.9) we define V_1 to be the set of germs of those vector fields $X \in \mathcal{G}^u$ such that $\left(\dfrac{\partial X_i}{\partial x_j}\right)_{i,j}$ has at least one eigenvalue on the imaginary axis. $V'_i \subset V_{i-1}$, $i=2, 3$, is the closure of the subset of those germs X in V_{i-1} such that X, restricted to a center-manifold, is not in $V_{j,i-1} \backslash V_{j,i}$, $j=1, 2$ if $i=2$ and $j=1, 2, 3, 4, 5$ if $i=3$. V_i is the union of V'_i and the set of those points in V_{i-1} where V_{i-1} is not a non-singular codimension $(i-1)$-manifold. Note that $V_2 = V'_2$.

We first prove that V_1, V_2 and V_3 are closed and semi-algebraic sets. We recall that a set is semi-algebraic if it is the union of a finite number of sets which can be defined by polynomial equalities and polynomial inequalities. Our proof will be based on a corollary [9], due to Thom, of the Seidenberg-Tarski theorem [15] which, restated in a form which is convenient for our purposes, reads as follows:

Proposition **(1.18)**. — *Let* K *be a semi-algebraic subset of* J^n_k, *then the set*:

$K' = \{\alpha \in J^n_k \mid$ *there is a k-jet of a diffeomorphism* $\varphi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ *such that* $\varphi_*(\alpha) \in K\}$ *is also semi-algebraic.*

Using this proposition it is easy to show that V_1 is semi-algebraic: the set K of those $\alpha \in J^n_1$ which can be represented by $X = \sum_{i,j=1}^{n} A_{ij} x_i \dfrac{\partial}{\partial x_j}$ with A_{ij} in Jordan normal form and with A_{ij} having at least one eigenvalue on the imaginary axis, is clearly semi-algebraic; from (1.18) it now follows that the set K′ of those $\alpha \in J^n_1$ which can be represented by $X = \sum_{i,j=1}^{n} A_{ij} x_i \dfrac{\partial}{\partial x_j}$, with A_{ij} having at least one eigenvalue on the imaginary axis, is semi-algebraic; $V_1 = \pi_1^{-1}(K')$.

Now we prove that V_2 is closed and semi-algebraic; the proof for V_3 goes in the same way and is omitted. Let N be an integer, so large that each of the semi-algebraic sets $V_{i,j}$ is of the form $\pi_N^{-1}(V_{ij})$. Then we define $K \subset J^n_N$ to be the set of jets of vector fields $X = \sum_i X_i \dfrac{\partial}{\partial x_i}$ in V_1 with:

a) $\left(\dfrac{\partial X_i}{\partial x_j}(0)\right)_{i,j}$ is in Jordan normal form and has ν (>0) eigenvalues on the imaginary axis;

b) $X_i(x_1, \dots, x_\nu, 0, \dots, 0) \equiv 0$ for $i > \nu$, all the eigenvalues of $\left(\dfrac{\partial X_i}{\partial x_j}(0)\right)_{i,j \leq \nu}$ are on the imaginary axis;

c) $\tilde{X} = \sum_{j=1}^{\nu} X_j(x_1, \dots, x_\nu, 0, \dots, 0) \dfrac{\partial}{\partial x_j} \in V_{\mu,2}$ for some μ.

(Note that ν and μ are dependent: $\nu = \mu$ if $\mu \leq 2$ and $\nu + 1 = \mu$ if $\mu \geq 3$ (see definition (1.14)).) It is easy to see that K is semi-algebraic. Let K′ be again the orbit of K under the action of diffeomorphisms; then $\pi_N^{-1}(K')=V_2'$ is semi-algebraic. The set of points where V_1 is not a non-singular manifold of maximal dimension is closed and semi-algebraic. So $V_2=\overline{V}_2'\cup V_1$ is closed and semi-algebraic. (The closure of V_2' can be obtained by replacing $<$ in defining equations everywhere by $\leq$.)

The fact that the codimension of V_i is i follows from (1.15) and (1.16).

Finally we prove that each $X\in V_i\backslash V_{i+1}$ is V_i-C^0-weakly-stable for $i=0, 1, 2$. For $i=0$ see remark (1.9). We shall prove the stability of $X\in V_1\backslash V_2$; the case $X\in V_2\backslash V_3$ goes in the same way. Let N be again an integer, so big that the sets $V_{i,j}$ are all of the form $\pi_N^{-1}(V_{i,j})$ and let $X\in V_1\backslash V_2$. Let α be the N-jet of X, restricted to a center-manifold; α is unique "up to coordinate transformations".

According to the definition of V_1 and V_2, α is a point of $\pi_N(V_{i,1}\backslash V_{i,2})$, $i=1$ or 2, which has a neighbourhood U in $V_{i,1}\backslash V_{i,2}$ which is a non-singular manifold. Let $\hat{U}\subset J_N^n$ be the set of those jets of vector fields X, for which the N-jet of its restriction to a center-manifold is in U; $\hat{U}$ is a codimension 1 manifold. $\pi_N(V_1\backslash V_2)$ and $\hat{U}$ coincide locally, i.e. there is a neighbourhood U_1 of $\pi_N(X)$ such that $U_1\cap\hat{U}=U_1\cap\pi_N(V_1\backslash V_2)$. We now assume that we have taken U so small that if X_1 and X_2 are C^N-vector fields whose N-jets are in U, then X_1 and X_2 are weakly-C^0-equivalent; this assumption is justified, if N is large enough, by theorem (1.15). We also assume that U_1 is so small that if X_1 and X_2 are vector fields whose N-jets are in $U_1\cap\pi_N(V_1\backslash V_2)$, then the number of eigenvalues with negative, or positive, real part is equal for $\left(\frac{\partial X_{1i}}{\partial x_j}\right)_{i,j}$ and $\left(\frac{\partial X_{2i}}{\partial x_j}\right)_{i,j}$. Now it follows from theorem (1.13) that any $X_1, X_2\in\mathcal{G}^n$ with $\pi_N(X_i)\in U_1\cap\pi_N(V_1\backslash V_2)$ are weakly-C^0-equivalent. This proves theorem 1.

As to the proof of theorem 2, we shall show that that can be reduced to proving the following theorem (1.19), the proof of which will be given in chapter V using the theory developed in the chapters II and III.

Theorem **(1.19)**. — *There is a submanifold* $W\subset J_2^5$ *of codimension* 3 *which contains a residual subset* P *such that for any* $\beta\in P$ *and any* $\beta'\in J_k^5$, $k\geq 2$, *with* $\pi_2(\beta')=\beta$, *there are two representatives* X_1, X_2 *of* β', *such that for any two bounded neighbourhoods* U_1, U_2 *of* $0\in\mathbf{R}^5$ *the sets* L_1, L_2, *where* L_i *is the set of those points* $q\in U_i$ *for which both* $L_{\alpha,X_i,U_i}(q)$ *and* $L_{\omega,X_i,U_i}(q)$ *are the origin (in* $\mathbf{R}^5$*), are not homeomorphic.*

Proof of theorem 2 assuming theorem (1.19).

Suppose there is, for some $n\geq 5$, a sequence $V_1\supset V_2\supset V_3\supset V_4$ in $\mathcal{G}^n$ as in theorem 1. Because V_4 has codimension 4, there must be some $X\in\mathcal{G}^n$, $X\notin V_4$ such that the 2-jet of X, restricted to some center-manifold, is in P; let X_0 be such a germ and assume $X_0\in V_i\backslash V_{i-1}$, $i\leq 3$. According to the assumption there must be a neighbourhood U of X_0 in V_i such that each $X'\in U$ is weakly-C^0-equivalent with X_0. According to the definition of the topology in $\mathcal{G}^n$ and the definition of algebraic sets in $\mathcal{G}^n$ there is an integer k, such that every $X'\in\mathcal{G}^n$, which has the same k-jet as X_0, is in U (and hence weakly-C^0-equivalent to X_0). In order to construct a contradiction we take two representatives X_1, X_2 of the k-jet of X_0 such that the restrictions X_1, X_2 to their center-manifolds $W^c_{X_1}$, $W^c_{X_2}$ are as in the conclusion of theorem (1.19), i.e. for any two neighbourhoods U_1, U_2 of o in $W^c_{X_1}$, $W^c_{X_2}$ respectively the sets L_1, L_2 are topologically different (L_i is the set of those $q \in U_i$ for which both $L_{\alpha, X_i, U_i}(q)$ and $L_{\omega, X_i, U_i}(q)$ are the origin); such a choice of X_1 and X_2 is possible by theorem (1.19). According to theorem (1.13), if U is a neighbourhood of $o \in \mathbf{R}^n$ (small enough) and if $q \in U$ has the property that both $L_{\alpha, X_i, U}(q)$ and $L_{\omega, X_i, U}(q)$ are the origin, then q must be a point of the center-manifold $W^c_{X_i}$.

The above facts, together with theorem (1.13), show that X_1 and X_2 are not weakly-C^0-equivalent. This is the required contradiction proving theorem 2.

II. — NORMAL FORMS

The main theorem of this chapter is very close to the formal part in [17] and was probably known to Sternberg and others; the applications are, however, new (as far as I know) and rather surprising. We develop the theory of normal forms in § 1 in the generality we need for the final results of this paper; in appendices 1 and 2 we give some extensions of the theory. Applications are given in § 2, § 3 and § 4; § 5 deals with the geometric interpretation of the results of § 4.

1. The normal form theorem.

Let X be a C^k-vector field on $\mathbf{R}^n$ with $X(0)=0$. We want to put X, or rather its ℓ-jet, $\ell \leq k$, in a simple form by " changing the coordinates " in $\mathbf{R}^n$. For this purpose we define X_1 to be the vector field on $\mathbf{R}^n$ which has the same 1-jet in o as X and whose coefficient functions are linear. H^h denotes the vector space of those vector fields on $\mathbf{R}^n$ whose coefficient functions are homogeneous polynomials of degree h.

$[X_1, -]_h : H^h \to H^h$ is the linear map which assigns to each $Y \in H^h$ the Lie-product $[X_1, Y]$ which is again in H^h. For X_1 fixed, we define a splitting $H^h = B^h + G^h$ such that $B^h = \operatorname{Im}([X_1, -]_h)$ and such that G^h is some supplementary space.

Theorem (**2.1**). — *Let X, X_1, B^h, G^h be as above. Then, for $\ell \leq k$, there is a C^∞-diffeomorphism $\varphi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $\varphi_*(X) = X'$ is of the form:*

$$X' = X_1 + g_2 + \ldots + g_\ell + R_\ell,$$

where $g_i \in G^i$, $i = 2, \ldots, \ell$ and R_ℓ is a vector field, the component functions of which have all zero ℓ-jet; $\ell = k = \infty$ is not excluded. (φ can be used to define new coordinates $x'_1, \ldots, x'_n$ with respect to which X is of the form of X'.)

Proof. — We prove the theorem by induction on ℓ. For $\ell=1$, the proposition is trivially true. Suppose now we have $X=X_1+g_2+\ldots+g_{\ell-1}+R_{\ell-1}$ as in the conclusion of the theorem. We can then write $R_{\ell-1}=g_\ell+b_\ell+R_\ell$, with $g_\ell\in G^\ell$, $b_\ell\in B^\ell$ and R_ℓ a vector field, the component functions of which have all zero ℓ-jet. We take $Y\in H^\ell$ with $[X_1, Y]=b_\ell$ and consider $(\mathscr{D}_{Y,t})_*X=X_t$, $((\mathscr{D}_{Y,t})$ is the diffeomorphism $(\mathbf{R}^n, 0)\to(\mathbf{R}^n, 0)$ obtained by integrating Y over time t (see definition (1.3))). The $(\ell-1)$-jets of X and X_t are equal, $\ell\geq 2$, so $[Y, X_t]=-b_\ell+R_{\ell,t}$, where $R_{\ell,t}$ is C^{k-1}, but for some positive A, and all $|t|\leq 1$ and all $x\in\mathbf{R}^n$ with $||x||\leq 1$:

$$||R_{\ell,t}(x)||\leq A.||x||^{\ell+1}.$$

From differential geometry we know that $-\dfrac{d}{dt}(X_t)=[Y, X_t]$, so X_t has the form $X_t=X_1+g_2+\ldots+g_\ell+b_\ell+tb_\ell+R'_{\ell,t}$ with $R'_{\ell,t}$ C^k because $X_t=(\mathscr{D}_{Y,t})_*X$ and $\mathscr{D}_{Y,t}$ is C^∞. It is now clear that if we take $\varphi_\ell=\mathscr{D}_{Y,-1}$ we have $\varphi_\ell(X)=X_1+g_2+\ldots+g_\ell+R_\ell$ as in the conclusion of the theorem. This induction proves the theorem for the case $\ell<\infty$.

Suppose now $\ell=\infty$; the above construction gives a sequence of diffeomorphisms $\Psi_{\ell'}:(\mathbf{R}^n, 0)\to(\mathbf{R}^n, 0)$, $\ell'=1, 2, \ldots$, such that $\Psi_{\ell'*}(X)=X_1+g_2+\ldots+g_{\ell'}+R_{\ell'}$, as in the conclusion of the theorem with ℓ' instead of ℓ, and such that the $(\ell'-1)$-jets of $\Psi_{\ell'-1}$ and $\Psi_{\ell'}$ are equal (the $\varphi_{\ell'}$ constructed above is so that we can take $\Psi_{\ell'}=\varphi_{\ell'}\Psi_{\ell'-1}$ and the $(\ell'-1)$-jet of $\varphi_{\ell'}$ was the $(\ell'-1)$-jet of the identity). By the theorem of E. Borel, see [11], there is a diffeomorphism Ψ such that for each ℓ', the ℓ'-jets of $\Psi_{\ell'}$ and Ψ are equal. It then follows that $\Psi_*(X)=X_1+g_2+g_3+\ldots+R_\infty$ and the theorem is proved.

2. The singularity " $x_1\dfrac{\partial}{\partial x_2}$ ".

We apply in this paragraph theorem (2.1) to the case where X is a C^k-vector field on $\mathbf{R}^2$ which has the same 1-jet as $X_1=x_1\dfrac{\partial}{\partial x_2}$. The image of $[X_1, -]_\ell$ in H^ℓ, or B^ℓ, is determined by the following formulas:

$$\left[x_1\frac{\partial}{\partial x_2}, x_1^{n_1}x_2^{n_2}\frac{\partial}{\partial x_1}\right]=n_2x_1^{n_1+1}x_2^{n_2-1}\frac{\partial}{\partial x_1}-x_1^{n_1}x_2^{n_2}\frac{\partial}{\partial x_2},\qquad\text{for}\quad n_2\geq 1$$

$$\left[x_1\frac{\partial}{\partial x_2}, x_1^{n_1}x_2^{n_2}\frac{\partial}{\partial x_2}\right]=n_2x_1^{n_1+1}x_2^{n_2-1}\frac{\partial}{\partial x_2},\qquad\text{for}\quad n_2\geq 1$$

and:

$$\left[x_1\frac{\partial}{\partial x_2}, x_1^{n_1}\frac{\partial}{\partial x_1}\right]=-x_1^{n_1}\frac{\partial}{\partial x_2}$$

$$\left[x_1\frac{\partial}{\partial x_2}, x_1^{n_1}\frac{\partial}{\partial x_2}\right]=0.$$

From this it is clear that H^ℓ is spanned by B^ℓ, $x_2^\ell \frac{\partial}{\partial x_1}$ and $x_2^\ell \frac{\partial}{\partial x_2}$. B^ℓ has codimension 2 in H^ℓ because the kernel of $[X_1, -]_\ell$ has dimension 2 $\left(\text{it is spanned by } x_1^\ell \frac{\partial}{\partial x_2} \text{ and } x_1^\ell \frac{\partial}{\partial x_1} + x_1^{\ell-1} x_2 \frac{\partial}{\partial x_2}\right)$. From this it follows that we may take G^ℓ to be the subspace of H^ℓ, spanned by $x_2^\ell \frac{\partial}{\partial x_1}$ and $x_2^\ell \frac{\partial}{\partial x_2}$, and we have the following:

Proposition **(2.2)**. — *Let* X *be a* C^k*-vector field on* $\mathbf{R}^2$ *whose* 1*-jet in* 0 *equals the* 1*-jet of* $x_1 \frac{\partial}{\partial x_2}$. *Then there is a* C^∞*-map* $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *such that*:

$$\varphi_*(X) = x_1 \frac{\partial}{\partial x_2} + \sum_{\ell=2}^{k} \left(a_\ell x_2^\ell \frac{\partial}{\partial x_1} + b_\ell x_2^\ell \frac{\partial}{\partial x_2} \right) + R_k,$$

where the k*-jet of* R_k *is zero.*

3. A single rotation.

We take X again as vector field on $\mathbf{R}^2$, but now so that its 1-jet in 0 equals the 1-jet of $X_1 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$. In order to determine B^ℓ and to choose G^ℓ it is convenient to have a basis of eigenvectors in H^ℓ; in order to be able to make such a basis we complexify H^ℓ to $H^\ell \otimes \mathbf{C}$. The elements of $H^\ell \otimes \mathbf{C}$ can be written in the form $Y_1 + iY_2$ with $Y_1, Y_2 \in H^\ell$; the action of $[X_1, -]_\ell$ on $H^\ell \otimes \mathbf{C}$ is given by:

$$[X_1, Y_1 + iY_2]_\ell = [X_1, Y_1]_\ell + i[X_1, Y_2]_\ell.$$

In order to construct a basis of eigenvectors in $H^\ell \otimes \mathbf{C}$ we define the following vector fields:

$$Z_1 = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \qquad Z_{-1} = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2},$$

and the following functions:

$$V^{r,k}(x_1, x_2) = (x_1^2 + x_2^2)^r (x_1 + ix_2)^k \quad \text{for} \quad k \geq 0, \quad r \geq 0,$$
$$V^{r,k}(x_1, x_2) = (x_1^2 + x_2^2)^r (x_1 - ix_2)^{-k} \quad \text{for} \quad k \leq 0, \quad r \geq 0.$$

It is easy to see that $[X_1, Z_{\pm 1}] = \pm i . Z_{\pm 1}$ and $\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) V^{r,k} = k . i . V^{r,k}$. From this it follows that $\{V^{r,k} . Z_j\}_{\substack{r \geq 0 \\ j = \pm 1 \\ 2r + |k| = \ell}}$ forms a basis of $H^\ell \otimes \mathbf{C}$, consisting of eigenvectors of $[X_1, -]_\ell$. The eigenvalue of $V^{r,k} . Z_j$ is $(k+j) . i$; so the kernel of:

$$[X_1, -] : H^\ell \otimes \mathbf{C} \to H^\ell \otimes \mathbf{C}$$

is 0 if ℓ is even and 2-dimensional if ℓ is odd; in that case the kernel is generated by $V^{\frac{1}{2}(\ell-1),1}.Z_{-1}$ and $V^{\frac{1}{2}(\ell-1),-1}.Z_{+1}$. Next we observe that:

$$\frac{1}{2}(V^{\frac{1}{2}(\ell-1),1}.Z_{-1}+V^{\frac{1}{2}(\ell-1),-1}.Z_{+1})=W_+^\ell=(x_1^2+x_2^2)^{\frac{1}{2}(\ell-1)}\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right)$$

and $$\frac{1}{2}(i.V^{\frac{1}{2}(\ell-1),1}.Z_{-1}-i.V^{\frac{1}{2}(\ell-1),-1}.Z_{+1})=W_-^\ell=(x_1^2+x_2^2)^{\frac{1}{2}(\ell-1)}\left(x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right).$$

So the kernel of $[X_1, -]_\ell$, in the complexified case, for ℓ odd, is generated by the "real vectors" W_+^ℓ and W_-^ℓ. It is now an easy exercise in linear algebra to show that we may take $G^\ell=0$ for ℓ even and $G^\ell=\{$the 2-dimensional vector space spanned by W_+^ℓ and $W_-^\ell\}$ for ℓ odd; so we proved:

Proposition **(2.3)**. — *Let X be a C^k-vector field on $\mathbf{R}^2$ whose 1-jet in 0 equals the 1-jet of $X_1=x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}$. Then there is a C^∞-diffeomorphism $\varphi:(\mathbf{R}^2,0)\to(\mathbf{R}^2,0)$ such that*

$$\varphi_*(X)=X_1+\sum_{\ell\geq 1}^{2\ell<k}a_\ell(x_1^2+x_2^2)^\ell\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right)+\sum_{\ell\geq 1}^{2\ell<k}b_\ell(x_1^2+x_2^2)^\ell\left(x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right)+R_k,\quad \textit{where}$$

the k-jet of R_k is zero.

An equivalent, but more geometric form of the above proposition can be given as follows:

Proposition **(2.4)**. — *Let X be a C^k-vector field on $\mathbf{R}^2$ whose 1-jet equals the 1-jet of $X_1=x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}$. Let $R_\theta:\mathbf{R}^2\to\mathbf{R}^2$ be the linear rotation, given by:*

$$R_\theta(x_1,x_2)=(x_1.\cos\theta-x_2.\sin\theta,\ x_1.\sin\theta+x_2.\cos\theta).$$

Then there is a C^∞-diffeomorphism $\varphi:(\mathbf{R}^2,0)\to(\mathbf{R}^2,0)$ such that $X'=\varphi_*(X)$ has the property that for each θ, the k-jets of $(R_\theta)_*(X')$ and X' are equal (or "the k-jet of X' is invariant under the rotations R_θ").

4. Several rotations.

In this paragraph we generalize § 3 to the case where X is a C^k-vector field on $\mathbf{R}^n$ such that the 1-jet of X in 0 equals the 1-jet of $X_1=\sum_{i=1}^m\lambda_i\left(x_{2i-1}\frac{\partial}{\partial x_{2i}}-x_{2i}\frac{\partial}{\partial x_{2i-1}}\right)$ where $2m\leq n$. We assume that $\sum_{i=1}^m\alpha_i\lambda_i\neq 0$ whenever $\alpha_i\in\mathbf{Z}$ and $1\leq\sum_{i=1}^m|\alpha_i|\leq k+1$; this corresponds to excluding "strong resonances".

As in § 3 we consider the action of $[X_1,-]_\ell$ on $H^\ell\otimes\mathbf{C}$. In order to describe a basis of eigenvectors in $H^\ell\otimes\mathbf{C}$ we use the notation of § 3, i.e. $V^{r,k}(x_{2j-1},x_{2j})$ will have the same meaning as in § 3 but with x_1, x_2 replaced by x_{2j-1}, x_{2j}; $Z_{\pm 1}(x_{2j-1}, x_{2j})$ denotes the vector field $\dfrac{\partial}{\partial x_{2j-1}} \pm i \dfrac{\partial}{\partial x_{2j}}$. As in § 3 we have that:

$$[X_1, Z_{\pm 1}(x_{2j-1}, x_{2j})] = \pm i.\lambda_j.Z_{\pm 1}(x_{2j-1}, x_{2j}) \quad \text{for} \quad j \leq m$$

and:

$$\left(\sum_{\nu=1}^{m} \lambda_\nu \left(x_{2\nu-1}\frac{\partial}{\partial x_{2\nu}} - x_{2\nu}\frac{\partial}{\partial x_{2\nu-1}}\right)\right) \mathrm{V}^{r,k}(x_{2j-1}, x_{2j}) = k.\lambda_j.i.\mathrm{V}^{r,k}(x_{2j-1}, x_{2j}).$$

With this we can give a basis of eigenvectors in $\mathrm{H}^\ell \otimes \mathbf{C}$ as the union of the following two sets:

$$\{\mathrm{V}^{r_1,k_1}(x_1, x_2), \ldots, \mathrm{V}^{r_m,k_m}(x_{2m-1}, x_{2m}).\mathrm{W}(x_{2m+1}, \ldots, x_n).Z_{\mathrm{I}}(x_{2j_0-1}, x_{2j_0})\},$$

where $\mathrm{I} = \pm 1$, $j_0 = 1, 2, \ldots, m$ and where W is a homogeneous polynomial of degree $\ell - 2\sum_{j=1}^{m} r_j - \sum_{j=1}^{m} |k_j|$, and:

$$\left\{\mathrm{V}^{r_1,k_1}(x_1, x_2), \ldots, \mathrm{V}^{r_m,k_m}(x_{2m-1}, x_{2m}).\mathrm{W}(x_{2m+1}, \ldots, x_n).\frac{\partial}{\partial x_\nu}\right\},$$

where $\nu \geq 2m+1$ and where W is a homogeneous polynomial of degree $\ell - 2\sum_{j=1}^{m} r_j - \sum_{j=1}^{m} |k_j|$.

The corresponding eigenvalue is $(\sum_{j=1}^{m} \lambda_j.k_j.i) + \mathrm{I}\lambda_{j_0}.i$ for an eigenvector of the first set and $\sum_{j=1}^{m} \lambda_j.k_j.i$ for an eigenvector of the second set.

We now assume $\ell \leq k$ and absence of " strong resonances " (see the beginning of the paragraph); then the eigenvectors with eigenvalue zero are all of the form:

$$\mathrm{W}((x_1^2 + x_2^2), \ldots, (x_{2m-1}^2 + x_{2m}^2), x_{2m+1}, \ldots, x_n).\mathrm{V}^{0,\pm 1}(x_{2j_0-1}, x_{2j_0}).Z_{\mp 1}(x_{2j_0-1}, x_{2j_0})$$

and: $$\mathrm{W}((x_1^2 + x_2^2), \ldots, (x_{2m-1}^2 + x_{2m}^2), x_{2m+1}, \ldots, x_n)\frac{\partial}{\partial x_\nu}, \quad \nu \geq 2m+1,$$

with W an appropriate polynomial.

As in § 3 we conclude that we can choose G^ℓ to be the subspace of H^ℓ consisting of those elements which can be written as:

$$\begin{aligned}\sum_{i=1}^{m} f_i((x_1^2 + x_2^2), \ldots, (x_{2m-1}^2 + x_{2m}^2), x_{2m+1}, \ldots, x_n)\left(x_{2i-1}\frac{\partial}{\partial x_{2i}} - x_{2i}\frac{\partial}{\partial x_{2i-1}}\right) \\ + \sum_{i=1}^{m} g_i((x_1^2 + x_2^2), \ldots, (x_{2m-1}^2 + x_{2m}^2), x_{2m+1}, \ldots, x_n)\left(x_{2i-1}\frac{\partial}{\partial x_{2i-1}} + x_{2i}\frac{\partial}{\partial x_{2i}}\right) \\ + \sum_{i=2m+1}^{m} h_i((x_1^2 + x_2^2), \ldots, (x_{2m-1}^2 + x_{2m}^2), x_{2m+1}, \ldots, x_n)\frac{\partial}{\partial x_i}.\end{aligned}$$

So we proved:

Proposition **(2.5)**. — *Let X be a C^k-vector field on $\mathbf{R}^n$ which has in* 0 *the same* 1*-jet as*:

$$X_1=\sum_{i=1}^{m}\lambda_i\left(x_{2i-1}\frac{\partial}{\partial x_{2i}}-x_{2i}\frac{\partial}{\partial x_{2i-1}}\right),\quad 2m\leq n,\quad \text{with}\quad \sum_{i=1}^{m}\alpha_i\lambda_i\neq 0$$

whenever $\alpha_i\in\mathbf{Z}$ *and* $1\leq\sum_{i=1}^{m}|\alpha_i|\leq k+1$.

Then there is a C^∞*-diffeomorphism* $\varphi:(\mathbf{R}^n,0)\to(\mathbf{R}^n,0)$ *such that* $\varphi_*(X)=X'$ *is of the form*:

$$\begin{aligned}X'=&\sum_{i=1}^{m}f_i((x_1^2+x_2^2),\ldots,(x_{2m-1}^2+x_{2m}^2),x_{2m+1},\ldots,x_n)\left(x_{2i-1}\frac{\partial}{\partial x_{2i}}-x_{2i}\frac{\partial}{\partial x_{2i-1}}\right)\\&+\sum_{i=1}^{m}g_i((x_1^2+x_2^2),\ldots,(x_{2m-1}^2+x_{2m}^2),x_{2m+1},\ldots,x_n)\left(x_{2i-1}\frac{\partial}{\partial x_{2i-1}}+x_{2i}\frac{\partial}{\partial x_{2i}}\right)\\&+\sum_{i=2m+1}^{n}h_i((x_1^2+x_2^2),\ldots,(x_{2m-1}^2+x_{2m}^2),x_{2m+1},\ldots,x_n)\frac{\partial}{\partial x_i}+R_k,\end{aligned}$$

with $f_i(0,\ldots,0)=\lambda_i$, $g_i(0,\ldots,0)=0$, $h_i(0,\ldots,0)=0$, $\dfrac{\partial h_i}{\partial x_j}(0,\ldots,0)=0$ *for all i and* $j\geq 2m+1$, *and R_k a vector field with zero k-jet.*

Analogous to proposition (2.4) in § 3 we have here:

Proposition **(2.6)**. — *Let X, X_1 be as in the assumptions of proposition* (2.5); *let*:

$$R_\theta^i:(\mathbf{R}^n,0)\to(\mathbf{R}^n,0),\qquad i\leq m,$$

be the rotation: $R_\theta^i(x_1,\ldots,x_n)=(x_1',\ldots,x_n')$ *with* $x_j'=x_j$ *if* $j\neq 2i-1,2i$, *and*:

$$x_{2i-1}'=x_{2i-1}\cos\theta-x_{2i}\sin\theta,\qquad x_{2i}'=x_{2i-1}\sin\theta+x_{2i}\cos\theta.$$

Then there is a C^∞*-diffeomorphism* $\varphi:(\mathbf{R}^n,0)\to(\mathbf{R}^n,0)$ *such that* $X'=\varphi_*(X)$ *has the property*:

For any $i\leq m$ *and* θ, $(R_\theta^i)_*(X')$ *and X' have the same k-jet.*

5. Jet reductions.

Definition **(2.7)**. — Let X be a vector field on $\mathbf{R}^n$ having a k-jet as X' in the conclusion of proposition (2.5). Then the *reduced k-jet of X*, is the k-jet of the vector field X'' on $\mathbf{R}^{n-m}$:

$$X''=\sum_{i=1}^{m}g_i(y_1^2,\ldots,y_m^2,x_{2m+1},\ldots,x_n)\cdot y_i\frac{\partial}{\partial y_i}+\sum_{i=2m+1}^{n}h_i(y_1^2,\ldots,y_m^2,x_{2m+1},\ldots,x_n)\frac{\partial}{\partial x_i}.$$

If the germ of X is as X' in the conclusion of proposition (2.5) with $R_k\equiv 0$ and f_i, g_i, h_i C^k-functions, then the *reduced germ of X* is defined by the same formula which defined the reduced jet.

Remark **(2.8).** — In order to see clearly the relation between a jet (or a germ) of a vector field and its reduction, one has to consider the multivalued " map " $\Phi : \mathbf{R}^n \to \mathbf{R}^{n-m}$ defined by:

$$(x_1, \ldots, x_n) \xrightarrow{\Phi} (\pm\sqrt{x_1^2+x_2^2}, \ldots, \pm\sqrt{x_{2m-1}^2+x_{2m}^2}, x_{2m+1}, \ldots, x_n).$$

If X is a germ on $\mathbf{R}^n$ and X'' its reduction, then Φ maps integral curves of X to integral curves of X''; X'' then contains all the information about X except for the " speed of rotation ". For jets, of course, the same type of observation holds.

Proposition **(2.9).** — *We consider* $\mathbf{R}^n$, *with coordinates* $x_1, \ldots, x_n$, *and* $\mathbf{R}^{n-m}$, *with coordinates* $y_1, \ldots, y_m, x_{2m+1}, \ldots, x_n$. *The rotations* R_θ^i *on* $\mathbf{R}^n$ *are defined as in proposition* (2.6). $\mathrm{T}_i : \mathbf{R}^{n-m} \to \mathbf{R}^{n-m}$, $i=1, \ldots, m$, *is defined by*:

$$\mathrm{T}_i(y_1, \ldots, y_m, x_{2m+1}, \ldots, x_n) = (y_1, \ldots, y_{i-1}, -y_i, y_{i+1}, \ldots, y_m, x_{2m+1}, \ldots, x_n).$$

1) *If X'' is a germ (or a k-jet) of a vector field on $\mathbf{R}^{n-m}$ which is obtained by reducing a germ (or a k-jet) of a vector field X on $\mathbf{R}^n$, then X'' is invariant under the maps* T_i.

2) *If X'' is a germ (or a k-jet) of a C^∞-vector field on $\mathbf{R}^{n-m}$ which is invariant under the maps T_i, then there is a germ (or a k-jet) of a vector field X on $\mathbf{R}^n$, such that X'' is the reduction of X.*

3) *Let X be a germ (or a k-jet) of a vector field on $\mathbf{R}^n$ and let X'' be its reduction. If $\varphi'' : (\mathbf{R}^{n-m}, 0) \to (\mathbf{R}^{n-m}, 0)$ is a germ (or a k-jet) of a C^∞-diffeomorphism, commuting with $\mathrm{T}_1, \ldots, \mathrm{T}_m$, then there is a germ (or a k-jet) of a diffeomorphism $\varphi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $\varphi''_*(X'')$ is the reduction of $\varphi_*(X)$; φ commutes with all R_θ^i.*

Proof. — 1) follows directly from the definitions.

We prove 2) only for the case that X'' is a germ of a vector field (the case where X'' is a k-jet then follows). As X'' is invariant under $\mathrm{T}_1, \ldots, \mathrm{T}_m$, we can write X'' in the form:

$$X'' = \sum_{i=1}^{m} g_i(y_1, \ldots, y_m, x_{2m+1}, \ldots, x_n) y_i \frac{\partial}{\partial y_i} + \sum_{i=2m+1}^{n} h_i(y_1, \ldots, y_m, x_{2m+1}, \ldots, x_n) \frac{\partial}{\partial x_i},$$

where h_i and g_i are C^∞-functions on $\mathbf{R}^n$, invariant under $\mathrm{T}_1, \ldots, \mathrm{T}_m$. So 2) is proved once we know that if f is some C^∞-function on $\mathbf{R}^{n-m}$, invariant under $\mathrm{T}_1, \ldots, \mathrm{T}_m$ then there is a C^∞-function $\widetilde{f} : \mathbf{R}^{n-m} \to \mathbf{R}$ such that:

$$f(y_1, \ldots, y_m, x_{2m+1}, \ldots, x_n) = \widetilde{f}(y_1^2, \ldots, y_m^2, x_{2m+1}, \ldots, x_n).$$

The existence of such an $\widetilde{f}$, up to ∞-jet equivalence is easily verified (none of the y_i can occur in an odd power in the Taylor expansion of f). The existence of the function $\widetilde{f}$ is then proved using Malgrange's preparation theorem as in [10, see I, § 4].

To prove 3), it is enough to show that if $\varphi'' : (\mathbf{R}^{n-m}, 0) \to (\mathbf{R}^{n-m}, 0)$ is a germ of a diffeomorphism, commuting with $\mathrm{T}_1, \ldots, \mathrm{T}_m$, then there is a germ of a diffeomorphism $\varphi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, commuting with all R_θ^i and such that $\Phi\varphi = \varphi''\Phi$

where Φ is as in remark (2.8) (the choices of $\pm$ in the definition of Φ must be the same at both sides of the $=$ sign).

To obtain such φ, we first write φ'' in the form:

$$\varphi''(y_1, \ldots, y_m, x_{2m+1}, \ldots, x_n) = (y_1, \ldots, y_m, x_{2m+1}, \ldots, x_n)$$
$$+ (y_1 \cdot \mu_1(y_1, \ldots, x_n), \ldots, y_m \cdot \mu_m(y_1, \ldots, x_n), \mu_{m+1}(y_1, \ldots, x_n), \ldots, \mu_{n-m}(y_1, \ldots, x_n)).$$

Because φ'' commutes with all the T_i, $\mu_1, \ldots, \mu_{n-m}$ must be invariant under $T_1, \ldots, T_m$; this means that there are functions $\bar{\mu}_i$ such that:

$$\bar{\mu}_i(y_1^2, \ldots, y_m^2, x_{2m+1}, \ldots, x_n) = \mu_i(y_1, \ldots, x_n).$$

We can now define φ as the map which sends $(x_1, \ldots, x_n)$ to:

$$(x_1, \ldots, x_n) + (x_1 \cdot \bar{\mu}_1((x_1^2 + x_2^2), \ldots, (x_{2m-1}^2 + x_{2m}^2), x_{2m+1}, \ldots, x_n),$$
$$x_2 \cdot \bar{\mu}_1((x_1^2 + x_2^2), \ldots, (x_{2m-1}^2 + x_{2m}^2), x_{2m+1}, \ldots, x_n), x_3 \cdot \bar{\mu}_2((x_1^2 + x_2^2), \ldots, x_n),$$
$$\ldots, x_{2m} \cdot \bar{\mu}_m((x_1^2 + x_2^2), \ldots, x_n), \bar{\mu}_{m+1}((x_1^2 + x_2^2), \ldots, x_n), \ldots, \bar{\mu}_{n-m}((x_1^2 + x_2^2), \ldots, x_n));$$

this φ has the required properties.

Remark **(2.10)**. — If X'' is a k-jet of a vector field on $\mathbf{R}^{n-m}$, obtained by reducing a k-jet X on $\mathbf{R}^n$, then we have seen that X'' is invariant under the involutions $T_1, \ldots, T_m$. A consequence of this is that all the hyperplanes $y_i = 0$, $i = 1, \ldots, m$, are kept invariant by the flow of X'' (as far as k-jets go); hence the codimension two subspaces $x_{2i-1} = x_{2i} = 0$, $i = 1, \ldots, m$, in $\mathbf{R}^n$ are kept fixed by the flow of X, as far as k-jets go. It will be one of the basic questions in the following chapters to decide whether there are really invariant submanifolds for X, tangent in 0 to $x_{2i-1} = x_{2i} = 0$, for each $i = 1, \ldots, m$.

Appendix 1: On unicity.

Let X, X_1 and $H^h = B^h + G^h$ be again as in § 1. We assume that the dimensions of $G^2, \ldots, G^{h_0-1}$ are all zero. According to theorem (2.1) one can find a diffeomorphism φ, such that the h_0-jet of $\varphi_*(X) = X_1 + g_{h_0}$ for some $g_{h_0} \in G^{h_0}$. In this appendix we want to investigate the "extent to which g_{h_0} is unique".

Let $\mathscr{A}$ be the group of linear isomorphisms $A : \mathbf{R}^n \to \mathbf{R}^n$ such that $A_*(X_1) = X_1$. There is a natural $\mathscr{A}$ action on H^h for each h, defined by $A(\bar{h}) = A_*(\bar{h})$ for $A \in \mathscr{A}$ and $\bar{h} \in H^h$.

Proposition **(2.11)**. — *Let X be as above. Suppose there are two diffeomorphisms φ and φ' such that $\varphi_*(X) = X_1 + g_{h_0}$ and $\varphi'_*(X) = X_1 + g'_{h_0}$ (up to h_0-jets). Then there is an* A *in* $\mathscr{A}$ *(see above) so that* $A(g_{h_0}) = g'_{h_0} \bmod (B^{h_0})$.

Proof. — Without loss of generality we may assume that φ is the identity (i.e. that X was already in normal form). The 1-jet of φ' must obviously be an element of $\mathscr{A}$, say A. From the proof of theorem (2.1) it follows that the (h_0-1)-jet of φ' must be the (h_0-1)-jet of the linear map A in order to have the (h_0-1)-jet of $\varphi'_*(X)$ equal to the (h_0-1)-jet of X_1. We now observe that $A_*(X) = X_1 + A(g_{h_0})$; as G^{h_0} is in general not invariant under the $\mathscr{A}$ action, $A(g_{h_0})$ does not necessarily belong to G^{h_0}, so we write $A(g_{h_0})=b+g$ with $b \in B^{h_0}$ and $g \in G^{h_0}$. According to the proof of theorem (2.1), for any φ', with the same (h_0-1)-jet as A, we have $\varphi'_*(X)=X_1+b'+g$ (up to h_0-jets) for some $b' \in B^{h_0}$; from this the proposition follows.

Appendix 2: Normal forms in case $X_1 \equiv 0$.

In the case where $X_1 \equiv 0$, see § 1, the conclusion of theorem (2.1) is trivial. In that case however, one can proceed as follows. Let s be the smallest integer such that the s-jet of X is non-zero; X_s denotes the vector field, the component functions of which are homogeneous polynomials of degree s and which has the same s-jet as X. As in § 1 we define a map:

$$[X_s, -]_h : H^h \to H^{h+s-1}.$$

For $h > s$ we get $H^h = B^h + G^h$ with $B^h = \mathrm{Im}[X_k, -]_{h-s+1}$ and G^h a supplementary space. With these modified definitions theorem (2.1) remains true if we replace $X' = X_1 + g_2 + \ldots + g_\ell + R_\ell$ by $X' = X_s + g_{s+1} + \ldots + g_\ell + R_\ell$.

The proof is completely analogous to the proof of theorem (2.1) and is hence omitted.

III. — THE " BLOWING UP " CONSTRUCTION FOR VECTOR FIELDS

The construction to be described in § 1 of this chapter can also be found in [18] for the C^∞-case. The C^k-case is practically the same, but, for the sake of completeness, we repeat the construction here anyhow. Our blowing up construction can be seen as a refinement of the method used by Gomory in [3]; also the treatment of 2-dimensional singularities in [12] suggests our blowing up construction. The examples we give in §§ 3, 4 and 5, will all be used in the proofs of theorem (1.15) and (1.16); the technique of " blowing up " will also be used in the proof of theorem (1.19).

1. The construction and its properties.

Proposition **(3.1)**. — *Let X be a C^k-vector field on $\mathbf{R}^n$ with $X(0)=0$. Let:*

$$\Phi : S^{n-1} \times \mathbf{R} \to \mathbf{R}^n$$

be the map defining polar coordinates (i.e. *if* $\bar{x}_1, \ldots, \bar{x}_n$, *with* $\sum_{i=1}^{n} \bar{x}_i^2 = 1$, *are coordinates on* S^{n-1} *and r is the coordinate function on* $\mathbf{R}$, *then* $\Phi(\bar{x}_1, \ldots, \bar{x}_n, r) = (r\bar{x}_1, \ldots, r\bar{x}_n)$). *Then there is a* C^{k-1}-*vector field* $\widetilde{X}$ *on* $S^{n-1} \times \mathbf{R}$ *such that in each* $q \in S^{n-1} \times \mathbf{R}$, $\Phi_*(\widetilde{X}(q)) = X(\Phi(q))$ (or $\Phi_*(\widetilde{X}) = X$).

Proof. — On $\mathbf{R}^n$ we define the following vector fields:

$$R=\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i} \quad \text{and} \quad V_{ij}=\frac{1}{2}\left(x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}\right);$$

note that $V_{ii}=0$ and $V_{ij}=-V_{ji}$. From a direct calculation it follows that:

$$\left(\sum_{i=1}^{n} x_i^2\right).X=\langle R, X\rangle.R+\sum_{i,j=1}^{n}\langle V_{ij}, X\rangle.V_{ij}.$$

Next we define on $S^{n-1}\times\mathbf{R}$ the vector fields $\widetilde{R}$ and $\widetilde{V}_{ij}$ by $\Phi_*(\widetilde{R})=R$ and $\Phi_*(\widetilde{V}_{ij})=V_{ij}$, i.e. $\widetilde{R}=r\dfrac{\partial}{\partial r}$ and $\widetilde{V}_{ij}$ is the vector field, the integral of which is a rotation in S^{n-1} in "the $\bar{x}_i$, $\bar{x}_j$ plane". We introduce the functions $\alpha_r=\langle R, X\rangle:\mathbf{R}^n\to\mathbf{R}$ and $\alpha_{ij}=\langle V_{ij}, X\rangle:\mathbf{R}^n\to\mathbf{R}$. It is now clear that $\Phi_*((\alpha_r\Phi).\widetilde{R}+\sum_{i,j}(\alpha_{ij}\Phi).\widetilde{V}_{ij})=(\sum_{i=1}^{n}x_i^2).X$, or $\Phi_*\left(\dfrac{1}{r^2}((\alpha_r\Phi).\widetilde{R}+\sum_{i,j}(\alpha_{ij}\Phi).\widetilde{V}_{ij})\right)=X$; this last equation, however, does not necessarily make sense for $r=0$.

Because in $0\in\mathbf{R}^n$, X, R and V_{ij} are zero, $\langle X, R\rangle$ and $\langle X, V_{ij}\rangle$ have their 1-jet zero. This implies that the 1-jet of $\alpha_r\Phi$ and $\alpha_{ij}\Phi$ is zero in each point of:

$$S^{n-1}\times\{0\}=\{r=0\}.$$

By the division theorem it then follows that $\dfrac{1}{r^2}(\alpha_r\Phi)$ and $\dfrac{1}{r^2}(\alpha_{ij}\Phi)$ are C^{k-2} if we take for $r=0$ the limit.

As J. J. Duistermaat pointed out to me, this argument can be refined to obtain $\dfrac{1}{r^2}(\alpha_r\Phi)$ and $\dfrac{1}{r^2}(\alpha_{ij}\Phi)$ to be C^{k-1} as follows:

We write $X=\sum_{i=1}^{n}X_i\dfrac{\partial}{\partial x_i}$, the X_i's are C^k-functions. α_r is then $\sum_{i=1}^{n}x_iX_i(x_1, \ldots, x_n)$ and $\alpha_r\Phi=\sum_{i=1}^{n}\bar{x}_i.r.(X_i\Phi)$ and $\dfrac{1}{r^2}.(\alpha_r\Phi)=\sum_{i=1}^{n}\bar{x}_i.\dfrac{1}{r}.(X_i\Phi)$; so we have to divide only once by r and hence the resulting function $\dfrac{1}{r^2}(\alpha_r\Phi)$ is C^{k-1}. A similar argument works for $\dfrac{1}{r^2}(\alpha_{ij}\Phi)$.

We now have that $\widetilde{X}=\dfrac{1}{r^2}((\alpha_r\Phi).\widetilde{R}+\sum_{i,j}(\alpha_{ij}\Phi).\widetilde{V}_{ij})$ is a C^{k-1}-vector field; we had that for $r\neq 0$ $\Phi_*(\widetilde{X})=X$; hence, by continuity, we have it everywhere. This proves the proposition.

Remark **(3.2)**. — If the vector field X in proposition (3.1) has a sufficiently degenerate singularity in $0\in\mathbf{R}^n$, i.e. if its 1-jet is zero, then $\widetilde{X}$ will be identically zero on $S^{n-1}\times\{0\}$. In such cases the geometrical structure of X, or $\widetilde{X}$, can be made more "visible" by dividing $\widetilde{X}$ by a sufficiently high power of r (i.e. so often that $\dfrac{\widetilde{X}}{r^k}$ is not identically zero in $S^{n-1}\times\{0\}$). This division is possible in $S^{n-1}\times\mathbf{R}$, loosing each time only one degree of differentiability, because the vector field is zero on a manifold of codimension one.

Proposition (3.3). — *Let $\widetilde{X}$ be a vector field on $S^{n-1}\times\mathbf{R}$ which is of the form:*

$$\widetilde{X}=\sum_{\ell\leq N} r^\ell . g_\ell(\bar{x}_1, \ldots, \bar{x}_n) . \widetilde{R}+\sum_{\substack{i,j=1\\ k\leq N}}^{n} r^k . f_{ijk}(\bar{x}_1, \ldots, \bar{x}_n) . \widetilde{V}_{ij},$$

where f_{ijk} and g_ℓ are polynomials in $\bar{x}_1, \ldots, \bar{x}_n$, $\widetilde{R}$ and $\widetilde{V}_{ij}$ are the vector fields introduced in the proof of proposition (3.1) *and $\bar{x}_1, \ldots, \bar{x}_n$, r are the coordinate functions introduced in the statement of proposition* (3.1). $T: S^{n-1}\times\mathbf{R}\to S^{n-1}\times\mathbf{R}$ *is the involution defined by*:

$$T(\bar{x}_1, \ldots, \bar{x}_n, r)=(-\bar{x}_1, \ldots, -\bar{x}_n, -r).$$

If $T_*(\widetilde{X})=\widetilde{X}$ *or* $T_*(\widetilde{X})=-\widetilde{X}$, *then there is a vector field Y on $\mathbf{R}^n$ and an integer $m\geq 0$ such that $\widetilde{Y}=r^m\widetilde{X}$, where $\widetilde{Y}$ is such that $\Phi_*(\widetilde{Y})=Y$ as in proposition* (3.1).

Proof. — We assume $T_*(\widetilde{X})=\widetilde{X}$ (the case $T_*(\widetilde{X})=-\widetilde{X}$ goes in the same way). Because $T_*(\widetilde{R})=\widetilde{R}$ and $T_*(\widetilde{V}_{ij})=\widetilde{V}_{ij}$, the functions f_{ijk} and g_ℓ satisfy:

$$f_{ijk}(\bar{x}_1, \ldots, \bar{x}_n)=(-1)^k f_{ijk}(-\bar{x}_1, \ldots, -\bar{x}_n)$$

and:

$$g_\ell(\bar{x}_1, \ldots, \bar{x}_n)=(-1)^\ell g_\ell(-\bar{x}_1, \ldots, -\bar{x}_n).$$

This means that g_ℓ say, for ℓ even, can be written as $g_\ell(\bar{x}_1, \ldots, \bar{x}_n)=\sum_{s\leq s_0} g_{\ell s}(\bar{x}_1, \ldots, \bar{x}_n)$, for some s_0, with $g_{\ell s}$ homogeneous of degree $2s$. Because $\sum_{i=1}^{n}\bar{x}_i^2=1$, we may replace g_ℓ by the homogeneous polynomial $\bar{g}_\ell$ of degree $2s_0$:

$$\bar{g}_\ell(\bar{x}_1, \ldots, \bar{x}_n)=\sum_{s\leq s_0}\Big(\Big(\sum_{i=1}^{n}\bar{x}_i^2\Big)^{s_0-s} . g_{\ell s}(\bar{x}_1, \ldots, \bar{x}_n)\Big)$$

without changing the vector field $\widetilde{X}$. The same holds for f_{ijk} with k even; for k and ℓ odd, f_{ijk} and g_ℓ can be replaced by homogeneous polynomials $\bar{f}_{ijk}$ and $\bar{g}_\ell$ of odd degree.

Now we choose m so that $(m+k-\deg(\bar{f}_{ijk}))$ and $(m+1-\deg(\bar{g}_\ell))$ are all positive and even for all i, j, k and ℓ.

The vector field:

$$Y=\sum_{\ell\leq N}(x_1^2+\ldots+x_n^2)^{\frac{1}{2}(m+\ell-\deg(\bar{g}_\ell))} . \bar{g}_\ell(x_1, \ldots, x_n) . R + \sum_{\substack{i,j\\ k\leq N}}(x_1^2+\ldots+x_n^2)^{\frac{1}{2}(m+k-\deg(\bar{f}_{ijk}))} . \bar{f}_{ijk}(x_1, \ldots, x_n) . V_{ij}$$

will then have the required properties.

Remark (3.4). — If X is a C^k-vector field on $\mathbf{R}^n$ with $X(0)=0$ and if $\widetilde{X}$ is the corresponding vector field on $S^{n-1}\times\mathbf{R}$, then $\widetilde{X}$ can be written in the form:

$$\widetilde{X}=\Big(\sum_{\ell\leq k-1} r^{\ell}.g_{\ell}(\bar{x}_1, \ldots, \bar{x}_n)+\mathrm{G}\Big).\widetilde{R}+\Big(\sum_{\substack{i,j\\ \ell'\leq k-1}} r^{\ell'}.f_{ij\ell'}(\bar{x}_1, \ldots, \bar{x}_n)+\mathrm{F}_{ij}\Big).\widetilde{V}_{ij},$$

where g_ℓ and $f_{ij\ell'}$ are homogeneous polynomials of degree $(\ell+2)$, resp. $(\ell'+2)$, and where the $(k-1)$-jet of G and F_{ij} is zero in each point of $\mathrm{S}^{n-1}\times\{0\}$; the polynomials g_ℓ, $f_{ij\ell'}$, $(\ell, \ell'\leq k-1)$ determine the k-jet of X and vice versa.

2. The homogeneous case (in $\mathbf{R}^2$).

Let X be a vector field on $\mathbf{R}^2$ which is at least C^{k+1} and such that the $(k-1)$-jet of X in $0\in\mathbf{R}^2$ is zero. Let X_k denote the vector field whose component functions are homogeneous polynomials of degree k and which has in 0 the same k-jet as X. We consider the following two functions:

$$f=\left\langle X_k, x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right\rangle \qquad \text{and} \qquad g=\left\langle X_k, x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right\rangle.$$

In the following we assume that 0 is not the only 0-point of g.

Proposition **(3.5)**. — *For X, X_k, f and g as above, and f, g in " general position "* (i.e. *for each $p\neq 0$, $p\in\mathbf{R}^2$ with $g(p)=0$, we have $dg(p)\neq 0$ and $f(p)\neq 0$) each C^{k+1}-vector field Y which has the same k-jet as X, is C^0-equivalent with X* (i.e. *the germs of X and Y are C^0-equivalent*).

Proof. — We blow up our vector field X as in § 1 to obtain a vector field on $\mathrm{S}^1\times\mathbf{R}$. On $\mathrm{S}^1\times\mathbf{R}$ we take coordinates φ (mod 2π), such that $\bar{x}_1=\cos\varphi$ and $\bar{x}_2=\sin\varphi$, and r; we have $\Phi(\varphi, r)=(r.\cos\varphi, r.\sin\varphi)$: usual polar coordinates. $\widetilde{X}$ then gets the form:

$$\widetilde{X}=\frac{1}{r^2}\left((g(r.\cos\varphi, r.\sin\varphi)+r^{k+1}.\widetilde{g}(r,\varphi))\frac{\partial}{\partial\varphi}+(f(r.\cos\varphi, r.\sin\varphi)+r^{k+1}.\widetilde{f}(r,\varphi))r\frac{\partial}{\partial r}\right),$$

where f, g are homogeneous of degree $k+1$ in r and where $\widetilde{f}(0,\varphi)=\widetilde{g}(0,\varphi)=0$; f and g are at least C^1. Following remark (3.2) we now define $\overline{X}$ as:

$$\overline{X}=\frac{1}{r^{k+1}}\widetilde{X}=(g(\cos\varphi, \sin\varphi)+\widetilde{g}(r,\varphi))\frac{\partial}{\partial\varphi}+(f(\cos\varphi, \sin\varphi)+\widetilde{f}(r,\varphi))r\frac{\partial}{\partial r}.$$

Because f, g are in general position, $\overline{X}\,|\,\mathrm{S}^1\times\{0\}$ is a Morse-Smale system and also, in each point $(\varphi_0, 0)\in\mathrm{S}^1\times\{0\}$ where $\overline{X}$ is zero, $\overline{X}$ has a hyperbolic singularity.

If Y has the same k-jet as X and is at least C^{k+1}, and if Y and $\overline{Y}$ are defined analogous to X and $\overline{X}$, then all the above remarks concerning $\overline{X}$ also hold for $\overline{Y}$. This means that, using the techniques of [1], we can make a homeomorphism h of a neighbourhood U_1 of $\mathrm{S}^1\times\{0\}$ in $\mathrm{S}^1\times\{r\in\mathbf{R}\,|\,r\geq 0\}$ onto another such neighbourhood U_2 such that h maps integral curves of $\overline{X}$ to integral curves of $\overline{Y}$, i.e. if $p\in\mathrm{U}_1$ and $\mathscr{D}_{\overline{X}}(p, [0, t_1])$, $t_1>0$, is contained in U_1, then there is a $t_2>0$ such that $h(\mathscr{D}_{\overline{X}}(p, [0, t_1]))=\mathscr{D}_{\overline{Y}}(h(p), [0, t_2])$.

Using h, we construct a C^0-equivalence between the germs by taking:

$$\varphi: \Phi(\mathrm{U}_1)\to\Phi(\mathrm{U}_2)$$

defined by $\varphi(0)=0$ and $\varphi(p)=\Phi h \Phi^{-1}(p)$ for $p \neq 0$, where $\Phi^{-1}(p)$ has to be chosen so that its r-coordinate is positive. The fact that φ is a homeomorphism which sends X integral curves to Y integral curves follows immediately.

In order to give an idea of what kind of singularities can occur under the assumption of proposition (3.5), we consider a neighbourhood in $S^1 \times \{r \in \mathbf{R} \mid r \geq 0\}$ of an arc ℓ in $S^1 \times \{0\}$ joining two succeeding points p_1, p_2 on $S^1 \times \{0\}$ where $\bar{X}$ is zero. At each such point the direction of $\bar{X}$ changes; without loss of generality we may assume that between p_1 and p_2, $\bar{X}$ " flows " from p_2 to p_1. Four different situations can occur in a neighbourhood of ℓ, according to whether $\bar{X}$ is normally expanding or contracting in p_1 and p_2:

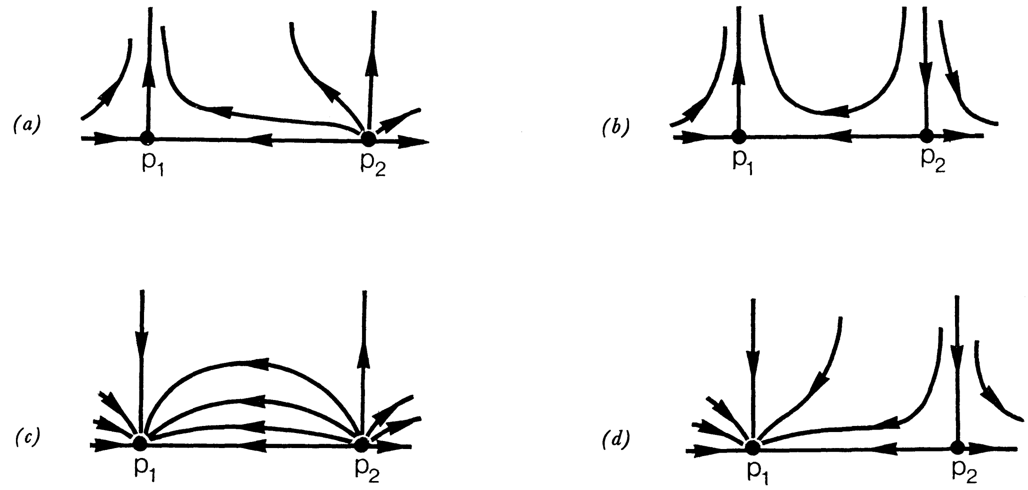

FIG. 1

Below we show how the integral curves of X look in the corresponding " sector " of $\mathbf{R}^2$:

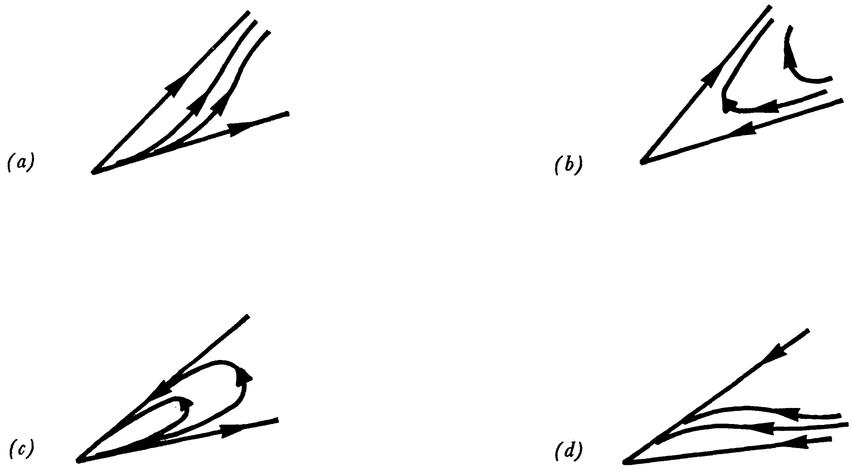

FIG. 2

Remark **(3.6)**. — Proposition (3.5) can be somewhat sharpened in the sense that it is sufficient to require that the $(k-1)$-jet of Y is zero and that the k-jet of Y is close to the k-jet of X. The same proof applies in this case.

Remark **(3.7)**. — For C^{k+1}-vector fields X on $\mathbf{R}^n$ with zero $(k-1)$-jet we can do also our blowing up construction and obtain a vector field on S^{n-1} $(=S^{n-1}\times\{0\})$. This vector field may be very complicated (say non structurally stable); on the other hand it is not a C^0-invariant in the following sense: Let X_1 and X_2 be two vector fields whose $(k-1)$-jets are zero. Then it is possible that the germs of X_1 and X_2 are C^0-(but not C^k-)equivalent, without their corresponding vector fields on S^{n-1} being C^0-equivalent.

3. The homogeneous case with one symmetry in $\mathbf{R}^2$.

We consider vector fields X on $\mathbf{R}^2$ which are at least C^3 and such that $X(0)=0$, the 1-jet of X is zero in 0 and $T_{1*}(X)=X$, where $T_1:(\mathbf{R}^2, 0)\to(\mathbf{R}^2, 0)$ is defined by $T_1(x_1, x_2)=(-x_1, x_2)$.

Germs and jets of vector fields as above occur as germ-, or jet-reductions of vector fields on $\mathbf{R}^3$, with one eigenvalue zero and two non-zero eigenvalues on the imaginary axis, which are in normal form (see chapter II).

In this paragraph we shall apply the method of § 2 to the above type of vector fields under the assumption that the 2-jet is in " general form " (to be specified below). However, as it may not always be possible to bring germs in normal form, we cannot " easily " carry over our results on vector fields on $\mathbf{R}^2$ invariant under T_1 as above to vector fields on $\mathbf{R}^3$ which have one eigenvalue zero and two non-zero eigenvalues on the imaginary axis.

Because of the above requirements the 2-jet X_2 of X must be of the form:

$$X_2=(ax_1^2+bx_2^2)\frac{\partial}{\partial x_2}+cx_1x_2\frac{\partial}{\partial x_1}.$$

We may, and do, assume that $a\geq 0$; because if not we replace the x_2 coordinate by $-x_2$.

Proposition **(3.8)**. — *Let X, X_2, a, b and c be as above* (i.e. *also* $a\geq 0$). *Suppose the following conditions are satisfied*:

1. $a\neq 0$;
2. $(b-c)\neq 0$;
3. $b\neq 0$;
4. *if* $(b-c)<0$ *then* $c\neq 0$.

(*Observe that these conditions are satisfied on an open dense subset of* $\{(a, b, c)\in\mathbf{R}^3 \mid a\geq 0\}$.)

Then every C^3*-vector field on* $\mathbf{R}^2$ *with zero* 1*-jet and* 2*-jet close enough to* X_2 *is* C^0*-equivalent with* X.

Moreover, there are 5 *topological types* (i.e. C^0-*equivalence classes*) *possible for* X, *according to which of the following conditions is satisfied*:

I. $a>0 \quad (b-c)>0 \quad b>0$;
II. $a>0 \quad (b-c)>0 \quad b<0$;
III. $a>0 \quad (b-c)<0 \quad b>0 \quad c>0$;
IV. $a>0 \quad (b-c)<0 \quad b<0 \quad c>0$;
V. $a>0 \quad (b-c)<0 \quad b<0 \quad c<0$.

(*Observe that* $a>0$, $(b-c)<0$, $b>0$, $c<0$ *cannot occur.*)

The diagrams below show the topological types:

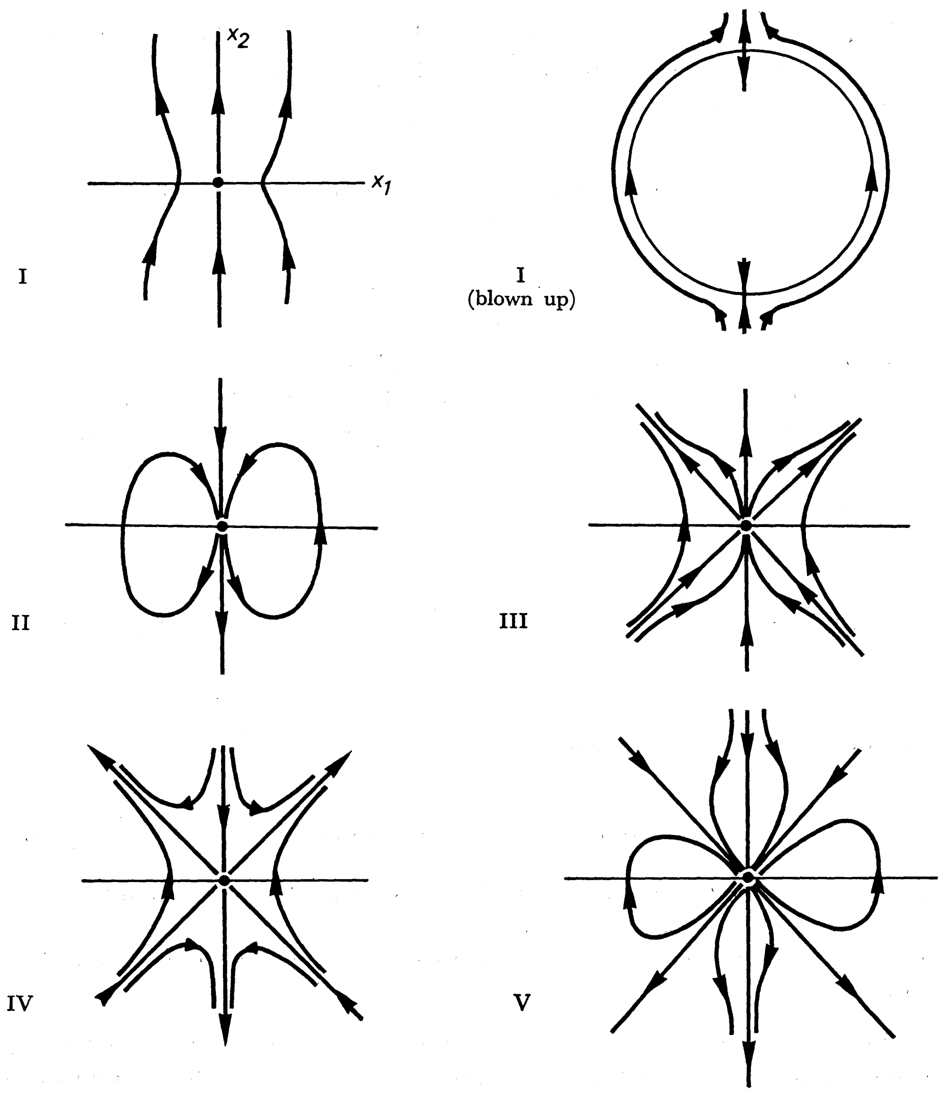

Fig. 3

Proof. — According to proposition (3.5) we first have to determine the zeros of $\left\langle X_2, x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right\rangle = ax_1^3 + bx_1x_2^2 - cx_1x_2^2 = x_1(ax_1^2 + (b-c)x_2^2)$. $\{x_1 = 0\}$ is clearly a line of zeros, and hence a line, invariant under the flow of X_2 (from now on we call these lines of zeros "*invariant lines*"). As in the statement of proposition (3.8) we assume that $a \geq 0$, and hence because of condition 1, $a > 0$; if $(b-c) > 0$ then $x_1 = 0$ is the only invariant line (cases I and II); if $(b-c) < 0$ then we have also a pair of invariant lines $x_1 = \pm\sqrt{\frac{c-b}{a}}x_2$ (cases III, IV and V).

Because we have $a > 0$, we always have $\left\langle X_2, x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right\rangle$ positive on the positive x_1-axis; it changes sign at every invariant line because the invariant lines have all multiplicity 1 (because of the conditions 1 and 2 in the proposition).

Finally, we have to determine the sign of:

$$f(x_1, x_2) = \left\langle X_2(x_1, x_2), x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right\rangle = ax_1^2x_2 + bx_2^3 + cx_1^2x_2$$

along the invariant lines in order to determine whether the flow goes in or out along these lines.

Along $\{x_1 = 0\}$, f equals $f(0, x_2) = bx_2^3$, so along $\{x_1 = 0\}$ the flow goes up, resp. down, whenever $b > 0$, resp. $b < 0$ (see cases I, III, resp. II, IV, V); $b = 0$ is excluded by condition 3. Now we assume $(b-c) < 0$ and determine the value of f in points of the form $x_1 = +\sqrt{\frac{c-b}{a}}.x_2$:

$$f\left(\sqrt{\frac{c-b}{a}}.x_2, x_2\right) = \left(a.\frac{c-b}{a} + b + c.\frac{c-b}{a}\right)x_2^3 = \frac{1}{a}.x_2^3(ac - ab + ab + c^2 - bc) = \frac{c}{a}.x_2^3(a + c - b).$$

Because according to our assumptions, $a > 0$, $(c-b) > 0$, $f\left(\sqrt{\frac{c-b}{a}}x_2, x_2\right)$ has the same sign as $c.x_2$. So the flow is going up, resp. down, along $x_1 = +\sqrt{\frac{c-b}{a}}.x_2$, whenever c is > 0, resp. < 0, as in the cases III, IV, resp. V. The corresponding statements about the invariant line $x_1 = -\sqrt{\frac{c-b}{a}}.x_2$ follow from the fact that X, and hence X_2, is invariant under T_*.

From the above considerations it follows that, because of the conditions 1, 2, 3 and 4 in proposition (3.8) the assumptions in proposition (3.5) are satisfied. Our proposition follows now directly from proposition (3.5) (and remark (3.6)).

4. The homogeneous case with 2 symmetries in $\mathbf{R}^2$.

We consider now vector fields X on $\mathbf{R}^2$ which are invariant under:

$$T_1 \ (T_1(x_1, x_2)=(-x_1, x_2)) \quad \text{and} \quad T_2 \ (T_2(x_1, x_2)=(x_1, -x_2))$$

and which have 1-jet zero in the origin; throughout this paragraph X will be assumed to be at least C^4. We shall carry out, for these vector fields, a program analogous to that of § 3. From the above assumptions it follows that the 2-jet of X is zero; its 3-jet X_3 can be written in the form:

$$X_3 = x_1(a_{11}x_1^2 + a_{12}x_2^2)\frac{\partial}{\partial x_1} + x_2(a_{21}x_1^2 + a_{22}x_2^2)\frac{\partial}{\partial x_2}.$$

It will turn out that we have to distinguish between the following cases:

I. $(a_{21}-a_{11})>(a_{22}-a_{12})>0$;
II. $(a_{21}-a_{11})>0>(a_{22}-a_{12})$;
III. $0>(a_{21}-a_{11})>(a_{22}-a_{12})$;
IV. $(a_{21}-a_{11})<(a_{22}-a_{12})<0$;
V. $(a_{21}-a_{11})<0<(a_{22}-a_{12})$;
VI. $0<(a_{21}-a_{11})<(a_{22}-a_{12})$.

In order to reduce the number of cases we actually have to investigate, we first notice that if we replace X_3 by $-X_3$, then cases I, II, III are changed to the cases IV, V and VI respectively and vice versa. If we interchange the x_1 and x_2 coordinates, then case I is carried over to case III and case II is carried over to itself. So by changing, if necessary, the sign of X, and X_3, and permuting, if necessary, x_1 and x_2, we can always come down to case I or case II; if we come in case II we can arrange, by permuting x_1 and x_2 if necessary, that $a_{11} \geq a_{22}$.

From the above it is clear that we only have to consider the cases:

I. $(a_{21}-a_{11})>(a_{22}-a_{12})>0$ and
II′. $(a_{21}-a_{11})>0>(a_{22}-a_{12})$, $\quad a_{11} \geq a_{22}$.

Remark **(3.9)**. — Replacing X by $-X$ may change the C^0-equivalence class of the germ of X; this must, and will, be taken into account when we list all the C^0-equivalence classes of singularities of vector fields as considered in this paragraph (with generic 3-jet). It is clear that permuting x_1 and x_2 does not change the C^0-equivalence class of the singularity.

Proposition **(3.10)**. — *Let X, X_3 and a_{ij} be as above (such that X_3 belongs to case* I *or* II′*). Then generically one of the following nine conditions is satisfied (in which* $A=a_{11}a_{22}-a_{12}a_{21}$*):*

I *a*	$(a_{21}-a_{11})>(a_{22}-a_{12})>0$	*and*	$a_{11}, a_{22}>0$;
I *b*	—	*and*	$a_{11}>0>a_{22}$;
I *c*	—	*and*	$a_{22}>0>a_{11}$;
I *d*	—	*and*	$0>a_{11}, a_{22}$;

II′ *a*	$(a_{21}-a_{11})>0>(a_{22}-a_{12}),\ a_{11}\geq a_{22}$	*and*	$a_{11}, a_{22}>0$;
II′ *b*	—	*and*	$a_{11}>0>a_{22}$, $A>0$;
II′ *c*	—	*and*	$a_{11}>0>a_{22}$, $A<0$;
II′ *d*	—	*and*	$0>a_{11}, a_{22}$, $A>0$;
II′ *e*	—	*and*	$0>a_{11}, a_{22}$, $A<0$.

For any of these nine conditions there is an X_3 satisfying that condition.

For any X, such that the corresponding X_3 satisfies one of the above conditions, any C^4-vector field Y, with 2-jet zero and 3-jet close enough to X_3, has a germ which is C^0-equivalent to the germ of X.

Below it is indicated how the vector field X_3 (or rather " its integral ") looks if one of the above nine conditions is satisfied (only the part in $x_1, x_2\geq 0$ is indicated, the rest follows by symmetry).

Proof. — First we determine the zeros of:

$$g=\left\langle X_3,\ x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right\rangle = x_1x_2((a_{21}-a_{11})x_1^2+(a_{22}-a_{12})x_2^2).$$

In case I $((a_{21}-a_{11})>(a_{22}-a_{12})>0)$ there are only two lines of zeros, or invariant lines, namely $\{x_1=0\}$ and $\{x_2=0\}$. In case II there are two more, namely:

$$\{(a_{21}-a_{11})x_1^2=(a_{12}-a_{22})x_2^2\}.$$

First we consider case I. — For $x_1, x_2>0$, g is positive, so the flow is " turning to the left " (figure 4). Now we determine the sign of:

$$f=\left\langle X_3,\ x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right\rangle = a_{11}x_1^4+(a_{12}+a_{21})x_1^2x_2^2+a_{22}x_2^4$$

on the invariant lines: on $\{x_1=0\}$ this sign is positive, resp. negative, if a_{22} is positive, resp. negative (cases I *a*, I *c*, resp. I *b*, I *d*); on $\{x_2=0\}$ this sign is positive, resp. negative, if a_{11} is positive, resp. negative (cases I *a*, I *b*, resp. I *c*, I *d*). This proves, using proposition (3.5), that if X is so that X_3 satisfies one of the conditions I *a*, ..., *d*, then every C^4-vector field Y with 2-jet zero and 3-jet close to X_3 has a germ which is C^0-equivalent to the germ of X.

The case II′. — All our considerations will be restricted to the region $x_1, x_2\geq 0$. The invariant lines we have are the x_1-axis, the x_2-axis and one in between, which we shall denote by ℓ. From the formula above for g and the fact that $(a_{21}-a_{11})>0>(a_{22}-a_{12})$ it follows that g is positive between the x_1-axis and ℓ and negative between ℓ and the x_2-axis (figure 4).

Now we determine the sign of f on the three invariant lines. Using the above formula for f (see case I), we see that f has on the x_1-axis the same sign as a_{11} and on the x_2-axis the same sign as a_{22}; this agrees with figure 4. To determine the value of f

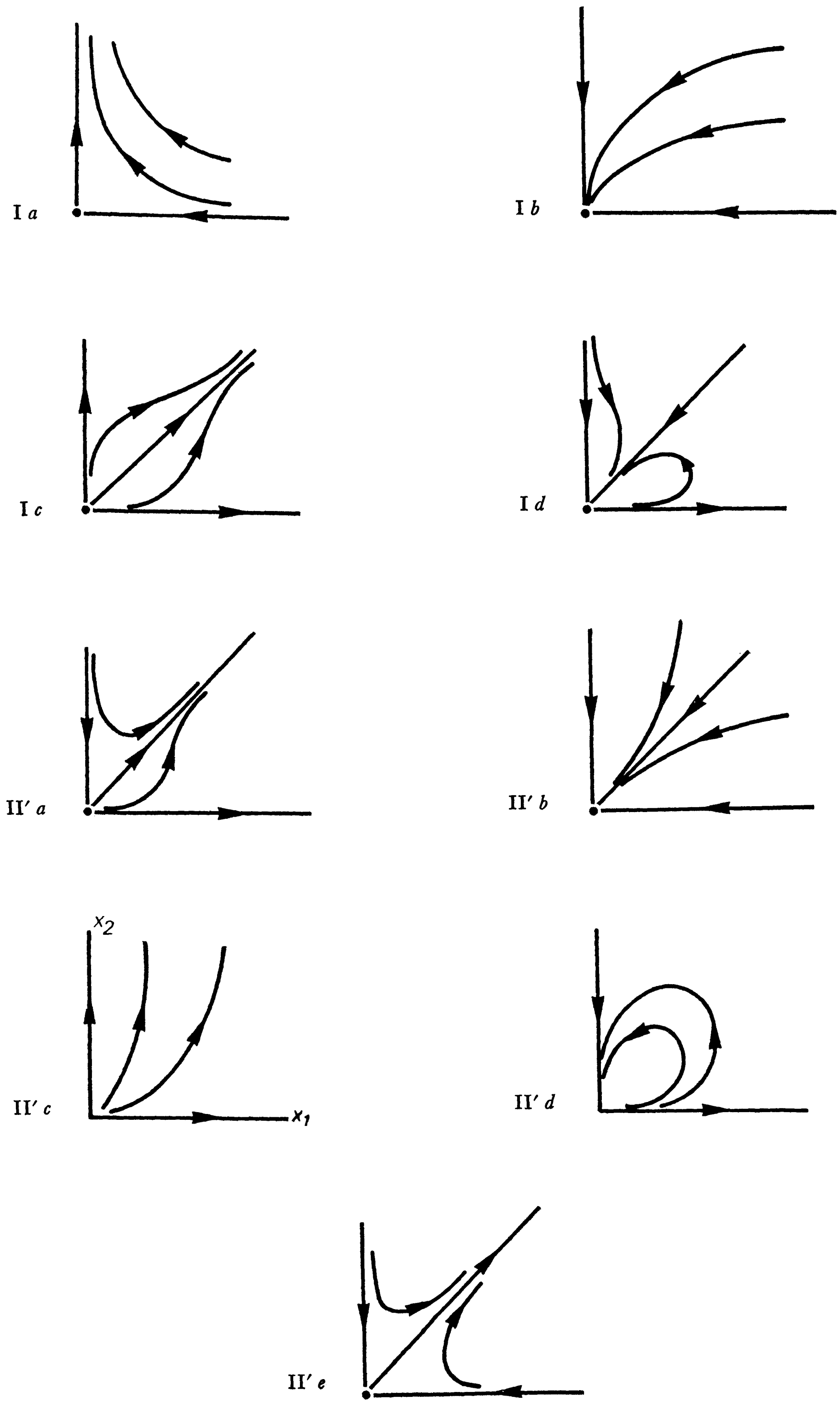

FIG. 4

on ℓ we substitute $x_1^2=\lambda.(a_{12}-a_{22})$ and $x_2^2=\lambda.(a_{21}-a_{11})$ in the formula for f and obtain:

$$f(\lambda)=\lambda^2(a_{11}(a_{12}-a_{22})^2+(a_{12}+a_{21})(a_{12}-a_{22})(a_{21}-a_{11})+a_{22}(a_{21}-a_{11})^2)$$
$$=\lambda^2.\mathrm{A}.((a_{22}-a_{12})-(a_{21}-a_{11})),$$

where A (as in the proposition) is $(a_{11}a_{22}-a_{12}a_{21})$. Because $\lambda^2>0$, $(a_{22}-a_{12})<0$ and $(a_{21}-a_{11})>0$, the sign of f on ℓ equals the sign of $-\mathrm{A}$.

In case II′ a, we have $a_{11}>0$ and $(a_{21}-a_{11})>0$, hence $a_{21}>a_{11}>0$, in the same way we have $a_{12}>a_{22}>0$; from this it follows that in case II′ a, $\mathrm{A}=a_{11}a_{22}-a_{12}a_{21}<0$.

This proves, according to proposition (3.5), that if X is such that X_3 satisfies one of the conditions II′ a, ..., e then every C^4-vector field Y, with zero 2-jet and 3-jet close to X_3, is C^0-equivalent with X.

It is immediate that if X_3 belongs to one of the cases I, II′, then, generically, one of the nine conditions I a, ..., II′ e is satisfied.

The fact that for each of the conditions I a, ..., d and II′ a there is an X_3 satisfying it, is also very easy to check because the conditions are all linear; for the condition II′ b, ..., e that fact is shown by the examples below:

	a_{11}	a_{12}	a_{21}	a_{22}
II′ b	1	$-\frac{1}{2}$	4	-1
II′ c	1	0	2	-1
II′ d	-1	0	0	-1
II′ e	-1	$\frac{1}{2}$	4	-1

This proves our proposition.

Remark **(3.11)**. — If X, X_3 and X', X'_3 are as in proposition (3.10) and if X_3, resp. X'_3, satisfies condition I d, resp. II′ d, then both X and X' are local contractions, i.e. for any point $p\in\mathbf{R}^2$, close to the origin, $\lim_{t\to\infty}\varphi_X(p,t)=0$, resp. $\lim_{t\to\infty}\varphi_{X'}(p,t)=0$. From this it is clear that the germs of X and X' are C^0-equivalent. A same type of remark holds for the types I a and II′ a. Using this we can make a list of all different topological types (C^0-equivalence classes) of germs of vector fields X on $\mathbf{R}^2$ which have zero 2-jet and the 3-jet X_3 of which is invariant under T_1 and T_2 and satisfies the generic non-degeneracy condition.

This generic non-degeneracy condition is satisfied whenever X_3 or $-X_3$, if necessary after interchanging x_1 and x_2, satisfies one of the nine conditions I a, ..., II′ e in proposition (3.10).

In the following list of C^0-types $X\in$ I b means that X_3, if necessary after interchanging x_1 and x_2, satisfies condition I b.

Type	*Occurs for*			
I	$X \in \mathrm{I}\ a$,	$X \in \mathrm{II}'\ a$,	$-X \in \mathrm{I}\ d$,	$-X \in \mathrm{II}'\ d$
II	$-X \in \mathrm{I}\ a$,	$-X \in \mathrm{II}'\ a$,	$X \in \mathrm{I}\ d$,	$X \in \mathrm{II}'\ d$
III	$X \in \mathrm{I}\ b$,	$-X \in \mathrm{I}\ b$		
IV	$X \in \mathrm{I}\ c$,	$-X \in \mathrm{I}\ c$		
V	$X \in \mathrm{II}'\ b$			
VI	$-X \in \mathrm{II}'\ b$			
VII	$X \in \mathrm{II}'\ c$			
VIII	$-X \in \mathrm{II}'\ c$			
IX	$X \in \mathrm{II}'\ e$			
X	$-X \in \mathrm{II}'\ e$			

As a matter of fact the last two types are C^0-equivalent; we want to consider them, however, as distinct types because there is no conjugacy φ, commuting with T_1 and T_2, which carries a germ of type IX over in a germ of type X.

The fact that all the other types are different is easily checked by comparing the sets of points in $\mathbf{R}^2$, which have as α- or ω-limit the origin, for all the different types.

5. The singularity " $x_1 \dfrac{\partial}{\partial x_2}$ ".

In this paragraph we analyse vector fields X on $\mathbf{R}^2$ which are at least C^5 and whose 1-jet equals the 1-jet of $x_1 \dfrac{\partial}{\partial x_2}$. According to proposition (2.2) we can choose our coordinates so that the 4-jet X_4 of X takes the form:

$$X_4 = x_1 \frac{\partial}{\partial x_2} + (a_2 x_2^2 + a_3 x_2^3 + a_4 x_2^4) \frac{\partial}{\partial x_1} + (b_2 x_2^2 + b_3 x_2^3 + b_4 x_2^4) \frac{\partial}{\partial x_2}. \tag{1}$$

From now on we shall *assume that* $a_2 \neq 0$. We then may also *assume that* $a_2 > 0$, because the case $a_2 < 0$ can be reduced to the former by the coordinate transformation $(x_1, x_2) \mapsto (-x_1, -x_2)$. We shall prove the following:

Proposition (**3.12**). — *Let X be a C^5-vector field on* $\mathbf{R}^2$ *as above* (i.e. *also with* $a_2 > 0$). *Then the germ of X is C^0-equivalent to the germ of* $X' = x_1 \dfrac{\partial}{\partial x_2} + x_2^2 \dfrac{\partial}{\partial x_1}$.

Proof. — Let X be as in proposition (3.12); we assume that X is already in the form (1). We shall analyse X by a sequence of three successive blow ups, which are illustrated in the following figure.

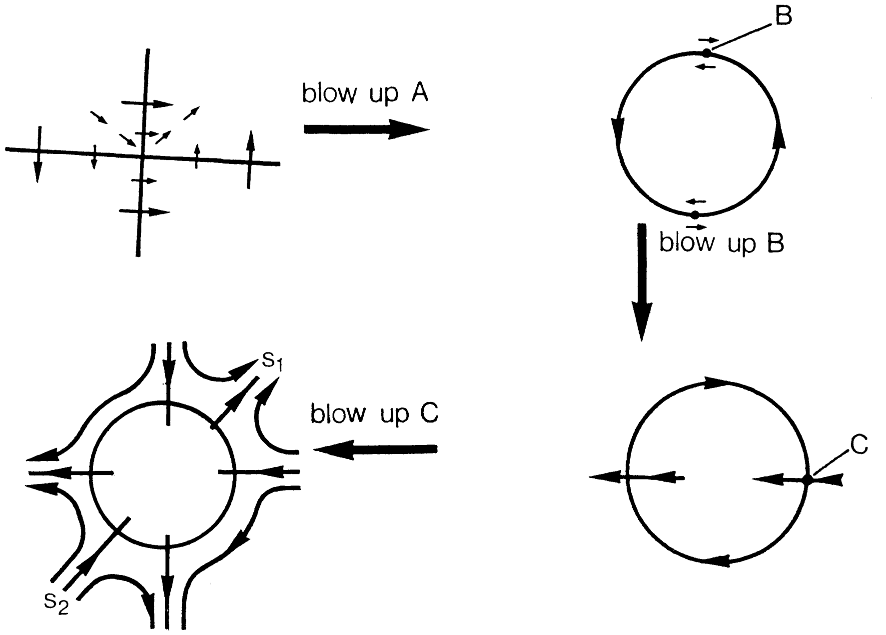

FIG. 5

We will calculate these blowing ups below, but want first to indicate how this sequence of blowing ups is used to determine the topological type of X.

After the last blowing up, and " dividing by r ", we have a situation as in proposition (3.5); hence, the topological type of C is fixed. The topological type of the other singular point in 5 *c)* is also determined because the vector field in figure 5 *c)* is invariant under $(\bar{x}_1, \bar{x}_2, r) \to (-\bar{x}_1, -\bar{x}_2, -r)$ (coordinates as in I, § 1) because it is obtained by the blowing up construction. So the topological type of the vector field near $S^1 \times \{0\}$ in figure 5 *c)* is as in figure 6.

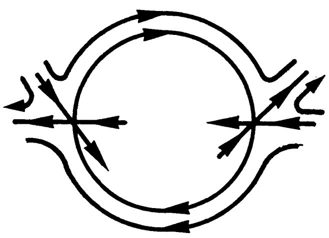

FIG. 6

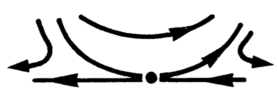

FIG. 7

From this it then follows that the topological type of the vector field near B in figure 5 *b)* must be as in figure 7. Using that the vector field in figure 5 *b)* is also invariant under the involution $(\bar{x}_1, \bar{x}_2, r) \to (-\bar{x}_1, -\bar{x}_2, -r)$ it is easy to see that the topological type of the vector field along $S^1 \times \{0\}$ in figure 5 *b)* must be as follows:

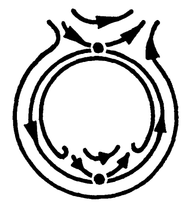

FIG. 8

So in 5 *a)* we have:

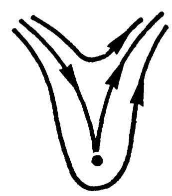

FIG. 9

The lines of points in figure 9, with α-, resp. ω-limit the origin, are the images, under the three successive blowing down maps, of the unstable, resp. stable manifolds s_1, resp. s_2, in figure 5 *d)*. The above considerations hold for any vector field X satisfying the assumptions of proposition (3.12); from this it easily follows that any such vector field is C^0-equivalent with any other such vector field, and hence with $X'=x_1\frac{\partial}{\partial x_2}+x_2^2\frac{\partial}{\partial x_1}$. We now come to the calculation of the three blow ups.

We consider the functions:

$$g(x_1, x_2)=\left\langle X(x_1, x_2), x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right\rangle=x_1^2-x_2(a_2x_2^2+a_3x_2^3+\dots)+x_1(b_2x_2^2+b_3x_2^3+\dots)$$

and:

$$f(x_1, x_2)=\left\langle X, x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right\rangle=x_1x_2+x_1(a_2x_2^2+a_3x_2^3+\dots)+x_2(b_2x_2^2+b_3x_2^3+\dots).$$

On $S^1\times\mathbf{R}$ we choose coordinates r, φ such that $\Phi(r, \varphi)=(-r\sin\varphi, r\cos\varphi)$. From § 1 it then follows that the vector field X_1, on $S^1\times\mathbf{R}$, obtained by blowing up X is $X_1=\frac{g(-r\sin\varphi, r\cos\varphi)}{r^2}\frac{\partial}{\partial\varphi}+\frac{f(-r\sin\varphi, r\cos\varphi)}{r}\frac{\partial}{\partial r}$.

It is easy to see that $X_1|S^1\times\{0\}$ has two singularities, namely $B=(r=0, \varphi=0)$ and $(r=0, \varphi=\pi)$. We now determine the 3-jet $[X_1]_3$ of X_1 in B using $\sin\varphi=\varphi-\frac{\varphi^3}{3!}+\dots$ and $\cos\varphi=1-\frac{\varphi^2}{2!}+\dots$ and obtain:

$$[X_1]_3=(\varphi^2-a_2r-a_3r^2-a_4r^3+1\frac{1}{2}a_2r\varphi^2-b_2r\varphi-b_3r^2\varphi)\frac{\partial}{\partial\varphi}+(-r\varphi-a_2r^2\varphi+b_2r^2+b_3r^3)\frac{\partial}{\partial r}.$$

This terminates the first blow up; we now change the notation in the sense that *we substitute* $-x_1$ *for* φ *and* x_2 *for* r, so that we get:

$$[X_1]_3 = (-x_1^2 + a_2x_2 + a_3x_2^2 + a_4x_2^3 - 1\tfrac{1}{2}a_2x_1^2x_2 - b_2x_1x_2 - b_3x_1x_2^2)\frac{\partial}{\partial x_1} + (x_1x_2 + x_2x_1x_2^2 + b_2x_2^2 + b_3x_2^3)\frac{\partial}{\partial x_2}.$$

In order to blow up this singularity we determine the 4-jet of:

$$g_1(x_1, x_2) = \left\langle X_1(x_1, x_2), x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right\rangle$$

and the 3-jet of:

$$f_1(x_1, x_2) = \left\langle X_1(x_1, x_2), x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right\rangle;$$

they are:

$$g_1(x_1, x_2) = -a_2x_2^2 + 2x_1^2x_2 + 2b_2x_1x_2^2 - a_3x_2^3 + 2b_3x_1x_2^3 + 2\tfrac{1}{2}a_2x_1^2x_2^2 - a_4x_2^4$$

$$f_1(x_1, x_2) = a_2x_1x_2 - x_1^3 - b_2x_1^2x_2 + (a_3 + 1)x_1x_2^2 + b_2x_2^3.$$

We take now coordinates r, φ in $S^1 \times \mathbf{R}$ such that Φ maps (r, φ) to $(r\cos\varphi, r\sin\varphi)$; after blowing up we then obtain (neglecting terms of order >2 in r):

$$X_2 = \frac{g_1(r\cos\varphi, r\sin\varphi)}{r^2}\frac{\partial}{\partial\varphi} + \frac{f_1(r\cos\varphi, r\sin\varphi)}{r}\frac{\partial}{\partial r}.$$

$X_2 \mid S^1 \times \{0\}$ has two singularities, namely in $C = (r = 0, \varphi = 0)$ and in $(r = 0, \varphi = \pi)$. We now calculate the 2-jet $[X_2]_2$ of X_2 in C and obtain:

$$[X_2]_2 = (-a_2\varphi^2 + 2r\varphi)\frac{\partial}{\partial\varphi} + (a_2r\varphi - r^2)\frac{\partial}{\partial r}.$$

This concludes the second blow up; we now change our notation again in the sense that *we substitute* x_1 *for* r *and* x_2 *for* φ, so we get:

$$[X_2]_2 = (2x_1x_2 - a_2x_2^2)\frac{\partial}{\partial x_2} + (-x_1^2 + a_2x_1x_2)\frac{\partial}{\partial x_1}.$$

To this singularity we apply the method of § 2:

$$g_2(x_1, x_2) = \left\langle [X_2]_2, x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right\rangle = x_1x_2(3x_1 - 2a_2x_2)$$

$$f_2(x_1, x_2) = \left\langle [X_2]_2, x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right\rangle = -x_1^3 + a_2x_1^2x_2 + 2x_1x_2^2 - a_2x_2^3.$$

There are three invariant lines (lines of zeros of g_1), namely $\{x_1=0\}$, $\{x_2=0\}$ and $\{3x_1=2a_2x_2\}=1$; the sign of g_2 is changing at each of these lines and is positive between the x_1-axis and 1. On the x_1-, resp. x_2-axis, the sign of f_2 is the sign of $-x_1$, resp. $-x_2$ (using $a_2>0$). To determine the sign of f_2 on 1 we substitute $(2a_2\lambda, 3\lambda)$ in f_2 and obtain $f_2(\lambda)=\lambda^3(-8a_2^3+12a_2^3+36a_2-27a_2)=\lambda^3(4a_2^3+9a_2)$; because $a_2>0$, this has the same sign as λ. All this agrees with figure 5 and the statements about the blow ups made there; so our proposition is proved.

IV. — INVARIANT MANIFOLDS

In § 3 and § 4 of the preceeding chapter we studied germs of vector fields "with symmetry" and it turned out that there were certain invariant curves (in the blown up situation stable or unstable manifolds of hyperbolic singularities). Jets with symmetry as in § 3 and § 4 occur as reductions of jets with certain rotations; the invariant curves, found in chapter III, § 3 and 4 should correspond to certain invariant (singular) manifolds in the non-reduced situation. It is the purpose of this chapter to show the actual existence of such invariant (singular) manifolds. The methods we use here are rather close to those of Hirsch and Pugh in [7].

1. The invariant manifold theorem.

We consider a torus, embedded in some manifold, with coordinate functions, defined in a neighbourhood of that torus, $x_1, \ldots, x_n, y_1, \ldots, y_{n'}, z$ ($x_1, \ldots, x_n$ are all defined modulo some constant c); $\{y_1=\ldots=y_{n'}=z=0\}$ is the torus in question. All our considerations will be restricted to the set on which these coordinates are defined. Points will be denoted by (x, y, z), where x, y stand for $x_1, \ldots, x_n$, resp. $y_1, \ldots, y_{n'}$. $\|y\|$ is defined to be $\sum_{j=1}^{n'}|y_j|$. Vectors will be denoted by $(\vec{X}, \vec{Y}, \vec{Z})$, where $\vec{X}$, $\vec{Y}$ and $\vec{Z}$ stand for $\sum_{i=1}^{n} X_i\frac{\partial}{\partial x_i}$, $\sum_{j=1}^{n'} Y_j\frac{\partial}{\partial y_j}$ and $Z\frac{\partial}{\partial z}$; $\|\vec{X}\|$, $\|\vec{Y}\|$ or $\|\vec{Z}\|$ means $\sum_{i=1}^{n}|X_i|$, $\sum_{j=1}^{n'}|Y_j|$ or $|Z|$.

Let φ be a diffeomorphism of a neighbourhood of the torus to a neighbourhood of the torus, at least of class C^{m+1}, such that the sets $\{y=0\}$ and $\{z=0\}$ are invariant and such that if $(0, \vec{Y}, 0)$ is a vector in some point $(x, 0, z)$ then $(\vec{X}', \vec{Y}', \vec{Z}')=\varphi_*(0, \vec{Y}, 0)$ satisfies:

$$\|\vec{Y}'\|\leq(1-C_1|z|^k).\|\vec{Y}\| \tag{1}$$

$$\|\vec{X}'\|+\|\vec{Z}'\|\leq C_2|z|^{\ell}.\|\vec{Y}\| \tag{2}$$

and also such that if $(\vec{X}, 0, \vec{Z})$ is a vector in $(x, 0, z)$ then $(\vec{X}', 0, \vec{Z}') = \varphi_*(\vec{X}, 0, \vec{Z})$ and $(x', 0, z') = \varphi_*(x, 0, z)$ satisfy:

(3) $$|z'|.\|\vec{X}'\| + \|\vec{Z}'\| \geq (1 + C_3|z|^k)(|z|.\|\vec{X}\| + \|\vec{Z}\|)$$

(4) $$(1 + 2C_4|z|^k)|z| \geq |z'| \geq (1 + C_4|z|^k)|z|.$$

$C_1, \ldots, C_4$ above are all positive constants. In view of (3) we define a second " norm ": if $(\vec{X}, \vec{Y}, \vec{Z})$ is a vector in (x, y, z), then $[\vec{X}, \vec{Z}] = |z|.\|\vec{X}\| + \|\vec{Z}\|$. The numbers k, ℓ and m are supposed to be positive integers. Now we suppose we also have positive integers k' and h such that k, k', ℓ, h and m satisfy:

(5) $$h > k - \ell$$

(6) $$k' > k + h + 1$$

(7) $$k' \geq \ell$$

(8) $$m \geq k + k'.$$

Our main purpose in this paragraph is to prove the following:

Theorem **(4.1)**. — *Assume the above situation. Let* $\bar{\varphi}$ *be a* C^{m+1}*-diffeomorphism (from a neighbourhood of the torus to a neighbourhood of the torus) such that in each point of* $z = 0$ *the m-jets of* φ *and* $\bar{\varphi}$ *are the same. Then, for each positive constant* A, *there is an* ε, *such that there is a manifold* $W \subset \mathscr{C}_\varepsilon = \{(x, y, z) \mid \|y\| \leq A.|z|^{k'}, |z| \leq \varepsilon\}$ *of dimension* $n+1$, *which is semi-invariant under* $\bar{\varphi}$, i.e. $\bar{\varphi}(W) \supset W$, *which contains the torus and which is, along the torus, tangent to* $\{y = 0\}$. *The manifold* W *will be of the form* $\{(x, y, z) \mid |z| \leq \varepsilon$ *and* $y = f(x, z)\}$ *where the function* f *is Lipschitz.*

The theorem will follow from a sequence of lemmas. From now on we assume $\bar{\varphi}$ in theorem (4.1) to be fixed.

Lemma **(4.2)**. — *For each* $A > 0$ *there is an* $\varepsilon > 0$ *such that if*:

$(x, y, z) \in \mathscr{C}_\varepsilon = \{(x, y, z) \mid \|y\| \leq A.|z|^{k'}, |z| \leq \varepsilon\}$ *and* $(x', y', z') = \bar{\varphi}(x, y, z)$,

then $|z'| \geq |z|$ *and* $\|y'\| \leq A.|z|^{k'}$.

Proof. — In this proof, as well as in the proofs of the following lemmas, we use the following conventions:

$D_1, D_2 \ldots$ are positive constants (each to be chosen so that the formula in which it first occurs is right);

$\underset{\varepsilon}{\leq}$ means: the inequality holds for ε sufficiently small and $|z| \leq \varepsilon$;

$\overset{(i)}{\leq}$ means: we use formula (i) to obtain this inequality;

$(*)$, $(**)$ means: we explain below how to obtain this inequality.

Let (x, y, z) and $(x', y', z') = \varphi(x, y, z)$ be as in the lemma, i.e. such that:

(9) $$\|y\| \leq A.|z|^{k'}.$$

Then we have:

$$\|y'\| \overset{(1)}{\leq} (1-C_1|z|^k)\|y\| + D_1.\|y\|^2 + D_2|z|^{m+1} \tag{$*$}$$
$$\overset{(9)}{\leq} (1-C_1|z|^k).A.|z|^{k'} + D_3.|z|^{2k'} + D_2|z|^{m+1}$$
$$\underset{\varepsilon}{\overset{(6)(8)}{\leq}} \left(1-\frac{1}{2}C_1|z|^k\right).A.|z|^{k'} \leq A.|z|^{k'},$$

and:

$$|z'| \overset{(4)(2)}{\geq} (1+C_4|z|^k)|z| - C_2|z|^{\ell}.\|y\| - D_4\|y\|^2 - D_5|z|^{m+1} \tag{$**$}$$
$$\overset{(9)}{\geq} (1+C_4|z|^k)|z| - D_6|z|^{\ell+k'} - D_7|z|^{2k'} - D_5|z|^{m+1}$$
$$\underset{\varepsilon}{\overset{(6)(8)}{\geq}} \left(1+\frac{1}{2}C_4|z|^k\right)|z| \geq |z|.$$

The inequalities $(*)$ and $(**)$ were obtained by first replacing φ by " its linear part in $(x, 0, z)$ ", i.e. by a map $\hat{\varphi}$ which is affine in the x, y and z coordinates and which has in $(x, 0, z)$ the same 1-jet as φ; this gives, using (1), (2) and (4), $\|y'\| \leq (1-C_1|z|^k)\|y\|$ and $|z'| \geq (1+C_4|z|^k)|z| - C_2|z|^{\ell}.\|y\|$. The terms $D_1.\|y\|^2$ and $D_4.\|y\|^2$ count for the difference between $\hat{\varphi}$ and φ; the terms $D_2|z|^{m+1}$ and $D_5|z|^{m+1}$ count for the difference between φ and $\bar{\varphi}$.

Lemma **(4.3)**. — *Let* A *be again as in lemma* (4.2). *There exists an* $\varepsilon > 0$ *such that if* $(\vec{X}, \vec{Y}, \vec{Z})$ *is a vector in* $(x, y, z) \in \mathscr{C}_\varepsilon$ *with* $\|\vec{Y}\| \leq A.|z|^h.[\vec{X}, \vec{Z}]$ *then*:

$$(\vec{X}', \vec{Y}', \vec{Z}') = \varphi_*(\vec{X}, \vec{Y}, \vec{Z}) \quad \textit{and} \quad (x', y', z') = \varphi(x, y, z)$$

satisfy:

$$[\vec{X}', \vec{Z}'] \geq [\vec{X}, \vec{Z}]$$

$$|z'| \geq |z| \qquad \textit{(see lemma (4.2))}$$

and: $$\|\vec{Y}'\| \leq A.|z|^h.[\vec{X}, \vec{Z}],$$

and hence: $$\|\vec{Y}'\| \leq A.|z'|^h.[\vec{X}', \vec{Z}'].$$

Proof. — We assume:

(10) $$\|\vec{Y}\| \leq A.|z|^h.[\vec{X}, \vec{Z}]$$

and also (9), i.e. $\|y\| \leq A.|z|^{k'}$. We then have:

$$\|\vec{Y}'\| \overset{(1)}{\leq} (1-C_1|z|^k)\|\vec{Y}\| + D_1(\|y\| + |z|^m)(\|\vec{X}\| + \|\vec{Y}\| + \|\vec{Z}\|) \tag{$*$}$$
$$\underset{\varepsilon}{\overset{(9)}{\leq}} (1-C_1|z|^k)\|\vec{Y}\| + D_2(|z|^{k'} + |z|^m)(\|\vec{Y}\| + |z|^{-1}.[\vec{X}, \vec{Z}])$$
$$\underset{\varepsilon}{\overset{(8)(10)}{\leq}} (1-C_1|z|^k).A.|z|^h.[\vec{X}, \vec{Z}] + D_3.|z|^{k'}.(A.|z|^h + |z|^{-1}).[\vec{X}, \vec{Z}]$$
$$\underset{\varepsilon}{\overset{(6)}{\leq}} \left(1-\frac{1}{2}C_1|z|^k\right).A.|z|^h.[\vec{X}, \vec{Z}] \leq A.|z|^h.[\vec{X}, \vec{Z}],$$

and:

$$[\vec{X}', \vec{Z}'] \underset{\varepsilon}{\overset{(2)(3)}{\geq}} (1+C_3|z|^k)[\vec{X}, \vec{Z}] - C_2|z|^\ell||\vec{Y}|| \\ -D_4(||y||+|z|^m)(||\vec{X}||+||\vec{Y}||+||\vec{Z}||) \qquad (**)$$

$$\overset{(9)(10)}{\geq} (1+C_3|z|^k)[\vec{X}, \vec{Z}] - D_5|z|^{\ell+h}[\vec{X}, \vec{Z}] \\ -D_6(|z|^{k'}+|z|^m)(|z|^h[\vec{X}, \vec{Z}]+|z|^{-1}[\vec{X}, \vec{Z}])$$

$$\underset{\varepsilon}{\overset{(8)}{\geq}} ((1+C_3|z|^k) - D_5|z|^{\ell+h} - D_7|z|^{k'}.|z|^{-1})[\vec{X}, \vec{Z}]$$

$$\underset{\varepsilon}{\overset{(5)(6)}{\geq}} \left(1+\frac{1}{2}C_3|z|^k\right)[\vec{X}, \vec{Z}] \geq [\vec{X}, \vec{Z}].$$

The inequalities $(*)$ and $(**)$ were obtained by first using $(d\varphi)_{x,0,z}$ instead of $(d\bar{\varphi})_{x,y,z}$; this gives, using (1), (2) and (3):

$$||\vec{Y}'|| \leq (1-C_1|z|^k)||\vec{Y}||$$

and: $$[\vec{X}', \vec{Z}'] \geq (1+C_3|z|^k)[\vec{X}, \vec{Z}] - C_2|z|^\ell||\vec{Y}||, \quad \text{for} \quad |z| \leq 1.$$

Then we added terms of the form (const.)$.||y||.$(length of $(\vec{X}, \vec{Y}, \vec{Z})$), resp. const. $|z|^m$.(length of $(\vec{X}, \vec{Y}, \vec{Z})$), counting for the difference between $(d\varphi)_{x,0,z}$ and $(d\varphi)_{x,y,z}$, resp. $(d\varphi)_{x,y,z}$ and $(d\bar{\varphi})_{x,y,z}$.

Lemma **(4.4)**. — *Let* A *be again as in lemma* (4.2). *For any* $B>0$, *there is an* $\varepsilon>0$ *such that if* $(0, \vec{Y}, 0)$ *is a vector in* $(x, y, z) \in \mathscr{C}_\varepsilon$, *then* $(\vec{X}', \vec{Y}', \vec{Z}') = \bar{\varphi}_*(0, \vec{Y}, 0)$ *satisfies*:

$$||\vec{Y}'|| \leq \left(1-\frac{1}{2}C_1|z|^k\right)||\vec{Y}||$$

and: $$||\vec{X}'||+||\vec{Z}'|| \leq B.|z|^{\ell-1}||\vec{Y}||.$$

Proof. — The lemma follows from the following computations (the inequalities $(*)$ and $(**)$ below are obtained just as in the proof of lemma (4.3)):

$$||\vec{Y}'|| \overset{(1)}{\leq} (1-C_1|z|^k)||\vec{Y}|| + D_1(||y||+|z|^m)||\vec{Y}|| \qquad (*)$$

$$\underset{\varepsilon}{\overset{(9)(8)}{\leq}} (1-C_1|z|^k)||\vec{Y}|| + D_2|z|^{k'}||\vec{Y}||$$

$$\underset{\varepsilon}{\overset{(6)}{\leq}} \left(1-\frac{1}{2}C_1|z|^k\right)||\vec{Y}||,$$

$$||\vec{X}'||+||\vec{Z}'|| \overset{(2)}{\leq} C_2|z|^\ell||\vec{Y}|| + D_3(||y||+|z|^m)||\vec{Y}|| \qquad (**)$$

$$\overset{(9)(8)}{\leq} C_2|z|^\ell||\vec{Y}|| + D_4|z|^{k'}||\vec{Y}||$$

$$\underset{\varepsilon}{\overset{(7)}{\leq}} B|z|^{\ell-1}||\vec{Y}||.$$

We now choose the B in lemma (4.4) so small that $(1+2C_4)^h.A.B\leq\frac{1}{4}C_1$ and take $0<\varepsilon<1$ so small that the conclusions of lemmas (4.2), (4.3) and (4.4) are valid.

Definition **(4.5)**. — Let A and ε be as above. $\mathscr{F}$ is the set of C^1-mappings $f(x_1, \ldots, x_n, z)$, defined for $|z|\leq\varepsilon$, with values in $(y_1, \ldots, y_m)$ such that:

1. $\|f(x, z)\|\leq A.|z|^{k'}$ and
2. If $(\vec{X}, \vec{Y}, \vec{Z})$ is a vector, tangent to $\{(x,y,z)\,|\,|z|\leq\varepsilon, y=f(x, z)\}$ in $(x,f(x, z), z)$, then $\|\vec{Y}\|\leq A.|z|^h.[\vec{X}, \vec{Z}]$.

The metric ρ on $\mathscr{F}$ is defined by:

$$\rho(f_1,f_2)=\sup_{\substack{(x,z)\\ 0<|z|\leq\varepsilon}}\frac{\|f_1(x, z)-f_2(x, z)\|}{|z|^k}.$$

$\bar{\mathscr{F}}$ denotes the completion of $\mathscr{F}$ with respect to the metric ρ.

Remark **(4.6)**. — It follows from the lemmas (4.2) and (4.3) that if $f\in\mathscr{F}$, then there is an $f'\in\mathscr{F}$ such that:

$$\bar{\varphi}(\{(x,f(x, z), z)\,|\,|z|\leq\varepsilon\})\cap\{|z|\leq\varepsilon\}=\{(x,f'(x, z), z)\,|\,|z|\leq\varepsilon\}.$$

This f' is of course unique; we define $\Gamma:\mathscr{F}\to\mathscr{F}$ to be the map which associates to each f the f' as above.

Remark **(4.7)**. — For any two $f_1,f_2\in\mathscr{F}$, there is a point (x, z), $|z|\leq\varepsilon$, such that:

$$\rho(f_1,f_2)=\frac{\|f_1(x, z)-f_2(x, z)\|}{|z|^k};$$

this follows from the fact that $k'>k$ (6) and hence:

$$\lim_{|z|\to0}\frac{\|f_1(x, z)-f_2(x, z)\|}{|z|^k}\leq\lim_{|z|\to0}\frac{2A|z|^{k'}}{|z|^k}=0.$$

Lemma **(4.8)**. — *If* $f_1,f_2\in\mathscr{F}$ *and* $\rho(f_1,f_2)\neq0$, *then* $\rho(\Gamma(f_1), \Gamma(f_2))<\rho(f_1,f_2)$.

Proof. — We take some point $(x, 0, z)$, $0<|z|\leq\varepsilon$, and two elements $f_1,f_2\in\mathscr{F}$ and consider the following five points:

$$(x,f_1(x, z), z),\quad (x,f_2(x, z), z),\quad \bar{\varphi}(x,f_1(x, z), z)=(x_1,\Gamma(f_1)(x_1, z_1), z_1),$$
$$\bar{\varphi}(x,f_2(x, z), z)\text{ “}=\text{” }(x_2, \Gamma(f_2)(x_2, z_2), z_2)\quad\text{and}\quad(x_1, \Gamma(f_2)(x_1, z_1), z_1);$$

we assume that all these points are in $\mathscr{C}_\varepsilon$, except perhaps $\bar{\varphi}(x,f_2(x, z), z)$ (in which case “ $=(x_2, \Gamma(f_2)(x_2, z_2), z_2)$ ” does not make sense).

From lemma (4.4) it follows that:

$$\|\Gamma(f_1)(x_1, z_1)-\Gamma(f_2)(x_2, z_2)\|\leq\left(1-\frac{1}{2}C|z|^k\right)\|f_1(x, z)-f_2(x, z)\|$$

and:
$$\|x_1-x_2\|+|z_1-z_2|\leq B.|z|^{\ell-1}.\|f_1(x, z)-f_2(x, z)\|$$

(in case $\bar{\varphi}(x,f_2(x, z), z)\notin\mathscr{C}_\varepsilon$, $\Gamma(f_2)(x_2, z_2)$ has to be replaced by “ the y-coordinate ” of $\bar{\varphi}(x,f_2(x, z), z)$).

From (4) and definition (4.5) we have, using the above formulas:

$$(1+2C_4)^h . A . B \leq \frac{1}{4} C_1 \quad \text{and} \quad [\vec{X}, \vec{Z}] \leq ||\vec{X}|| + ||\vec{Z}||,$$

$$|| \Gamma(f_2)(x_2, z_2) - \Gamma(f_2)(x_1, z_1) || \leq A . ((1+2C_4 |z|^k) |z|)^h . B . |z|^{\ell-1} || f_1(x, z) - f_2(x, z) ||$$
$$\stackrel{(5)}{\leq} \frac{1}{4} C_1 |z|^k || f_1(x, z) - f_2(x, z) ||.$$

So, using:

$$|| \Gamma(f_1)(x_1, z_1) - \Gamma(f_2)(x_1, z_1) || \leq || \Gamma(f_1)(x_1, z_1) - \Gamma(f_2)(x_2, z_2) || + || \Gamma(f_2)(x_2, z_2) - \Gamma(f_2)(x_1, z_1) ||,$$

we obtain:

$$|| \Gamma(f_1)(x_1, z_1) - \Gamma(f_2)(x_1, z_1) || \leq \left(1 - \frac{1}{4} C_1 |z|^k\right) || f_1(x, z) - f_2(x, z) ||.$$

Now we choose (x, z) so that (x_1, z_1) becomes a point where:

$$\frac{|| \Gamma(f_1)(x_1, z_1) - \Gamma(f_2)(x_1, z_1) ||}{|z_1|^k} = \rho(\Gamma(f_1), \Gamma(f_2)).$$

We then have, because $|z_1| \leq |z|$:

$$\rho(f_1, f_2) \geq \frac{|| f_1(x, z) - f_2(x, z) ||}{|z|^k} \geq \frac{1}{\left(1 - \frac{1}{4} C_1 |z|^k\right)} . \rho(\Gamma(f_1), \Gamma(f_2)) > \rho(\Gamma(f_1), \Gamma(f_2)).$$

This proves the lemma.

Remark **(4.9)**. — Because Γ is a contraction with respect to ρ, it is continuous in the topology defined by ρ and it has a unique continuous extension to $\overline{\mathscr{F}}$; this extension is also denoted by Γ.

Lemma **(4.10)**. — *$(\overline{\mathscr{F}}, \rho)$ is a compact metric space.*

Proof. — Let $f_1, f_2, \ldots \in \mathscr{F}$ be an infinite sequence. It is enough to show that, for any $\delta > 0$, there is an infinite subsequence $f_{i_1}, f_{i_2}, \ldots$ such that for any j, j', $\rho(f_{i_j}, f_{i_{j'}}) < \delta$. We fix such δ and choose δ_1 such that $2 . A . \delta_1^{k'} < \delta . \delta_1^k$; then for any $f_1, f_2 \in \mathscr{F}$ and (x, z) with $0 < |z| < \delta_1$, one has:

$$\frac{|| f_1(x, z) - f_2(x, z) ||}{|| z ||^k} \leq \frac{2 . A . |z|^{k'}}{|z|^k} \leq \frac{2 . A . \delta_1^{k'}}{\delta_1^k} < \delta.$$

So we only have to find a subsequence $\{f_{i_j}\}_{j=1}^{\infty}$ such that for any j, j' we have:

$$\max_{\substack{(x, z) \\ \delta_1 \leq |z| \leq \varepsilon}} \frac{|| f_{i_j}(x, z) - f_{i_{j'}}(x, z) ||}{|| z ||^k} < \delta;$$

the existence of such a subsequence follows from Ascoli's theorem [2].

Lemma **(4.11)**. — *Γ has a unique fixed point in $\overline{\mathscr{F}}$.*

Proof. — Let $K \subset \overline{\mathscr{F}}$ be the set of all ω-limit points of Γ, i.e.:

$$K = \{f \in \overline{\mathscr{F}} \mid \exists f_0 \in \overline{\mathscr{F}}, t_1, t_2, \ldots \to +\infty \quad \text{with} \quad \lim_{i\to\infty} \Gamma^{t_i}(f_0) = f\}.$$

K is clearly invariant and contains all the fixed points of Γ; because $\overline{\mathscr{F}}$ is compact, $\overline{K}$, the closure of K, is non-empty and compact. Let $\rho_0 = \sup_{f_1, f_2 \in \overline{K}} (f_1, f_2)$; because $\overline{K}$ is compact there are $\bar{f}_1, \bar{f}_2 \in \overline{K}$ with $\rho(\bar{f}_1, \bar{f}_2) = \rho_0$. From the definition of $\overline{K}$ it follows that there are $\bar{g}_1, \bar{g}_2 \in \overline{K}$ with $\Gamma(\bar{g}_i) = \bar{f}_i$; by the definition of ρ_0 we have $\rho(\bar{g}_1, \bar{g}_2) \leq \rho_0$; using lemma (4.8) this means that $\rho_0 = 0$. Hence $\overline{K}$ is one point: the unique fixed point.

Remark **(4.12)**. — All the elements of $\overline{\mathscr{F}}$ are clearly Lipschitz functions. If f_0 is the unique fixed point of Γ, then clearly $W = \{(x, y, z) \mid y = f_0(x, z), |z| \leq \varepsilon\}$ is a semi-invariant manifold, so theorem (4.1) is proved. In the following we give two extensions of theorem (4.1).

Proposition **(4.13)**. — *Suppose for some $\delta > 0$ we have: if $(0, \vec{Y}, 0)$ is a vector in (x, y, z) and if $\|y\|, |z| < \delta$, $(\vec{X}', \vec{Y}', \vec{Z}') = \varphi_*(0, \vec{Y}, 0)$ and $(x', y', z') = \varphi(x, y, z)$, then:*

$$(1') \qquad \|\vec{Y}'\| \leq (1 - C_1 |z|^k) \|\vec{Y}\|,$$

$$(2') \qquad \|\vec{X}'\| + \|\vec{Z}'\| \leq C_2 |z|^\ell \|\vec{Y}\|,$$

$$(4') \qquad (1 + 2C_4 |z|^k)|z| \geq |z'| \geq (1 + C_4 |z|^k)|z|.$$

(*Suppose furthermore that all the assumptions in theorem* (4.1) *are satisfied.*)

Then there is a $\delta' > 0$, $\delta' < \varepsilon$ such that the invariant manifold W *in the conclusion of theorem* (4.1) *is such that $W \cap U_{\delta'} = \{q \in \{U_{\delta'}\} \setminus \{|z| = 0\} \mid \varphi^{-n}(q) \in U_{\delta'}$ for $\forall n \geq 0\}$, where $U_{\delta'} = \{(x, y, z) \mid \|y - f_0(x, z)\| < \delta'$ and $|z| \leq \delta'\}$ and f_0 is such that:*

$$W = \{(x, y, z) \mid y = f_0(x, z), |z| \leq \varepsilon\}.$$

Proof. — First of all we take δ' so small that $\delta' + \|f_0(x, z)\| \leq \delta$ for all x and $|z| \leq \delta'$ (below we give more conditions on the smallness of δ'). Let (x, y, z) be some point of $U_{\delta'}$ and $(x', y', z') = \bar{\varphi}(x, y, z)$. For δ' small enough we have, if $|z'| \leq \delta'$, $\|y' - f_0(x', z')\| \leq \left(1 - \frac{1}{4} C_1 |z|^k\right) \cdot \|y - f_0(x, z)\|$; this inequality follows by the same methods as we used in the proofs of lemma (4.4) and lemma (4.8). From the above inequality and (4′) one easily obtains, for δ' small enough:

$$\left(1 - \frac{1}{3} C_4 |z'|^k\right) |z'| \geq |z| \geq (1 - 4C_4 |z'|^k) |z'|$$

and:

$$\|y - f_0(x, z)\| \geq \left(1 + \frac{1}{5} C_1 |z'|^k\right) \|y' - f_0(x', z')\|.$$

Now we choose two positive integers α_1 and α_2 such that for any $|z'|\leq\delta'$:

$$(1-4C_4|z'|^k)^{\alpha_1}\cdot\left(1+\frac{1}{5}C_1|z'|^k\right)^{\alpha_2}\geq 1;$$

it then follows that:

$$|z|^{\alpha_1}\cdot\|y-f_0(x,z)\|^{\alpha_2}\geq|z'|^{\alpha_1}\cdot\|y'-f_0(x',z')\|^{\alpha_2}.$$

This expression can be interpreted as follows:

Define the function $L:U_{\delta'}\to\mathbf{R}$ by $L(x,y,z)=|z|^{\alpha_1}\cdot\|y-f_0(x,y)\|^{\alpha_2}$. Then if $p\in U_{\delta'}$ and $\bar\varphi^{-1}(p)\in U_{\delta'}$ it follows that $L(\varphi^{-1}(p))\geq L(p)$ (so L is a sort of a weak Liapunov function). Now assume that $p\notin W\cap U_{\delta'}$ and $p\notin\{|z|=0\}$, we have to show that there is some positive i such that $\bar\varphi^{-i}(p)\notin U_{\delta'}$. We assume that $\bar\varphi^{-i}(p)\in U_{\delta'}$ for all positive i and derive a contradiction. Let $\ell_0=L(p)$; from the assumptions it follows that $\ell_0>0$. Let $\tilde y^i=y^i-f_0(x^i,z^i)$, where $(x^i,y^i,z^i)=\bar\varphi^{-i}(p)$; from our previous inequalities it follows that $\tilde y^i\geq\tilde y^0$, $\tilde y^0\neq 0$ because $p\notin W\cap U_{\delta'}$. Let $z_0>0$ be such that $L(x,y,z)<\ell_0$ whenever $\|y-f_0(x,z)\|\leqslant\delta'$ and $|z|\leq z_0$ (from the definition of L it is clear that such a positive z_0 exists). From the fact that $L(\varphi^{-i-1}(p))\geq L(\varphi^{-i}(p))$ we conclude that each $|z^i|>z_0$. This implies that, for all positive i, $\|\tilde y^{i+1}\|\geq\left(1+\frac{1}{5}C_1z_0^k\right)\|\tilde y^i\|$ and hence $\|\tilde y^i\|\geq\left(1+\frac{1}{5}C_1z_0^k\right)^i\|\tilde y^0\|$; however, this becomes, for large i, greater than δ'. This is the required contradiction and the proposition is proved.

Corollary **(4.14)**. — *Under the assumptions of proposition* (4.13), *the semi-invariant manifold* W *is unique.*

Proposition **(4.15)**. — *We make the same assumptions as in proposition* (4.13) *but instead of φ and $\bar\varphi$ we assume we have vector fields X and $\bar X$ such that for some $t_1>0$, $\mathscr{D}_{X,t_1}$ satisfies the assumptions for φ, such that $\{\|y\|=0\}$ and $\{|z|=0\}$ are invariant under each $\mathscr{D}_{X,t}$, and such that the m-jets of X and $\bar X$ are the same in each point of $\{|z|=0\}$. Then we obtain a semi-invariant manifold* W *for $\bar X$,* i.e. *such that, for all $t>0$, $\mathscr{D}_{\bar X,t}(W)\supset W$, as in theorem* (4.1) *and proposition* (4.13).

Proof. — We take $\varphi=\mathscr{D}_{\bar X,t}$ and $\bar\varphi=\mathscr{D}_{\bar X,t_1}$. Let W be an invariant manifold for $\bar\varphi$ as in theorem (4.1) and let δ' be as in the conclusion of proposition (4.13). Take some $\delta''>0$ such that for any $t\in(0,t_1)$, $\mathscr{D}_{\bar X,t}(U_{\delta''})\subset U_{\delta'}$. It is clear that:

$$\begin{aligned}W\cap U_{\delta''}&=\{q\in U_{\delta''}\backslash\{|z|=0\}\,|\,\bar\varphi^{-i}(q)\in U_{\delta''}\text{ for }\forall i\geq 0\}\\&=\{q\in U_{\delta''}\backslash\{|z|=0\}\,|\,\bar\varphi^{-i}(q)\in U_{\delta'}\text{ for }\forall i\geq 0\}.\end{aligned}$$

From this it follows that:

$$W\cap U_{\delta''}=\{q\in U_{\delta''}\backslash\{|z|=0\}\,|\,\mathscr{D}_{\bar X,t}(q)\in U_{\delta'},\text{ for }\forall t\leq 0\}.$$

From this last formula it follows that $W\cap U_{\delta''}$ is semi-invariant for $\bar X$.

2. Applications.

Let X be a vector field on $\mathbf{R}^4$ which is at least C^9 and which has a 1-jet X_1 of the form:

$$X_1=\lambda_1\left(y_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial y_1}\right)+\lambda_2\left(y_3\frac{\partial}{\partial y_4}-y_4\frac{\partial}{\partial y_3}\right) \quad \text{with} \quad \alpha_1\lambda_1+\alpha_2\lambda_2\neq 0$$

whenever $\alpha_i\in\mathbf{Z}$ and $1\leq|\alpha_1|+|\alpha_2|\leq 9$.

Modulo changes of coordinates, X can be decomposed as $X=X_n+X_r$ with X_n C^∞ and invariant under the rotations R^i_θ (see proposition (2.6)) and X_r having its 8-jet equal to zero.

We shall denote the vector fields on $S^3\times\mathbf{R}$, obtained by blowing up X, X_n and X_r, by $\widetilde{X}$, $\widetilde{X}_n$ and $\widetilde{X}_r$. $\overline{X}_n$ denotes the vector field on $\mathbf{R}^2$ obtained by reducing X_n (see definition (2.7)). It is clear that the 1-jet of $\overline{X}_n$ is zero and that $\overline{X}_n$ is invariant under the involutions $T_1, T_2 : \mathbf{R}^2\to\mathbf{R}^2$ $(T_1(x_1, x_2)=(-x_1, x_2)$; $T_2(x_1, x_2)=(x_1, -x_2))$; hence also the 2-jet of $\overline{X}_n$ is zero. We now *also assume that the 3-jet of* $\overline{X}_n$ *satisfies condition* II$'$ *e of proposition* (3.10). As we have seen in chapter III, there are four lines invariant under the flow of $\overline{X}_n$, namely the x_1-axis, the x_2-axis and two lines ℓ_1, ℓ_2 of the form

$$\ell_i=\{(x_1, x_2)\,|\,x_2=f_i(x_1)\} \text{ with } f_i(0)=0,\ \left(\frac{d}{dx_1}(f_1)\right)(0)>0,\ \left(\frac{d}{dx_1}(f_2)\right)(0)<0 \text{ and } f_i\ C^\infty.$$

Lemma **(4.16)**. — *In the above situation there is a (germ of a)* C^∞*-diffeomorphism* $\lambda : \mathbf{R}^2\to\mathbf{R}^2$, *which commutes with* T_1 *and* T_2 *and which maps* ℓ_1 *and* ℓ_2 *to straight lines.*

Proof. — Because $\overline{X}_n$ is invariant under T_1 and T_2, $T_i(\ell_1)=\ell_2$ and $T_i(\ell_2)=\ell_1$. Hence $f_2(x_1)=-f_1(x_1)$ and $f_1(-x_1)=-f_1(x_1)$ and there is a C^∞-function $g:\mathbf{R}\to\mathbf{R}$ such that $g(x_1)=g(-x_1)$, $g(0)>0$, $f_1(x_1)=x_1.g(x_1)$ and $f_2(x_1)=-x.g(x_1)$. Now we define λ by $\lambda(x_1, x_2)=(x_1, (g(x_1))^{-1}.x_2)$; this has, at least locally, the required properties.

In view of proposition (2.9) sub 3, we may assume that the invariant lines at $\overline{X}_n$ are all straight, say $\{x_1=0\}$, $\{x_2=0\}$ and $\{x_1=\pm a.x_2\}$. From this it follows that X_n has invariant varieties of the form $\{y_1=y_2=0\}$, $\{y_3=y_4=0\}$ and $\{(y_1^2+y_2^2)=a^2(y_3^2+y_4^2)\}$, the last of them being a cone on a 2-torus. We now want to answer the following basic question: does X have invariant varieties " close " to the above three invariant varieties of X_n?

The answer is yes. We will only prove this for the variety $\{(y_1^2+y_2^2)=a^2(y_3^2+y_4^2)\}$, because all the complications are already available there. At the end of this paragraph we shall give a criterium for the existence of invariant varieties in other cases, namely in those cases where a reduced jet satisfies one of the conditions in proposition (3.8) or (3.10).

We now want to show that the vector fields X_n and X satisfy the assumptions in proposition (4.15), with $k=2$, $\ell=2$, $h=1$, $k'=5$ and $m=7$. First we have to define suitable coordinates on $S^3\times\mathbf{R}$ near the " invariant torus ". Consider coordinate functions θ_1, θ_2, y, z on $S^3\times\mathbf{R}$, θ_i is only defined mod 2π, such that the " blowing down " map $\Phi: S^3\times\mathbf{R}\to\mathbf{R}^4$ takes the form:

$$\Phi(\theta_1,\theta_2,y,z)=$$
$$(z.\cos(y+y_0).\cos\theta_1,\ z.\cos(y+y_0).\sin\theta_1,\ z.\sin(y+y_0).\cos\theta_2,\ z.\sin(y+y_0).\sin\theta_2),$$

where y_0 is such that $\cos^2 y_0=a^2.\sin^2 y_0$. (This is not yet the final coordinate system: θ_1, θ_2 will be replaced by x_1, x_2 with $c.\theta_i\equiv x_i$ (mod $2\pi c$).) We can express $\widetilde{X}_n$ in these coordinates in a neighbourhood of the torus $y=z=0$ and we get:

$$\widetilde{X}_n=a_1\frac{\partial}{\partial\theta_1}+a_2\frac{\partial}{\partial\theta_2}+z^2.f_1(z,y)\frac{\partial}{\partial\theta_1}+z^2.f_2(z,y)\frac{\partial}{\partial\theta_2}+z^3.g(y,z)\frac{\partial}{\partial z}+z^2.h(y,z).y\frac{\partial}{\partial y},$$

with $g(0,0)>0$ and $h(0,0)<0$. Hence the time one integral $\mathscr{D}_{\widetilde{X}_n.1}=\varphi$ in a neighbourhood of $y=z=0$ must be of the form:

$$\varphi(\theta_1,\theta_2,y,z)=(\theta_1+a_1+z^2.\bar{f}_1(z,y),\ \theta_2+a_2+z^2.\bar{f}_2(z,y),\ y+z^2.\bar{h}(y,z).y,\ z+z^3.\bar{g}(y,z)),$$

with $\bar{g}(0,0)>0$ and $\bar{h}(0,0)<0$. Replacing θ_i by x_i with $c.\theta_i\equiv x_i$ (mod $c.2\pi$) gives then:

$$\varphi(x_1,x_2,y,z)=$$
$$(x_1+c.a_1+c.z^2.\bar{f}_1(z,y),\ x_2+c.a_2+c.z^2.\bar{f}_2(z,y),\ y+z^2.\bar{h}(y,z).y,\ z+z^3.\bar{g}(y,z)).$$

We show first that for c small enough (3) (see theorem (4.1)) is satisfied in a neighbourhood of $\{y=z=0\}$ in $\{y=0\}$. Let $(\vec{X}_1,\vec{X}_2,0,\vec{Z})$ be a tangent vector of $\{y=0\}$ in $(x_1,x_2,0,z)$; let $\varphi_*(\vec{X}_1,\vec{X}_2,0,\vec{Z})=(\vec{X}_1',\vec{X}_2',0,\vec{Z}')$ and let $\varphi(x_1,x_2,0,z)$ be $(x_1',x_2',0,z')$. Then, for $|z|\leq\varepsilon$, ε small enough, there is a constant D, independent of c, $(x_1,x_2,0,z)$ and $(\vec{X}_1,\vec{X}_2,0,\vec{Z})$, such that:

$$||\vec{X}_1'||\geq||\vec{X}_1||-c.D.|z|.||\vec{Z}||,$$
$$||\vec{X}_2'||\geq||\vec{X}_2||-c.D.|z|.||\vec{Z}||,$$
$$||\vec{Z}'||\geq(1+D.|z|^2)||\vec{Z}||$$

and:

$$|z'|\geq(1+D.|z|^2)|z|.$$

Using this we obtain:

$$|z'|.(||\vec{X}_1'||+||\vec{X}_2'||)+||\vec{Z}'||\geq$$
$$(1+D.|z|^2).|z|.(||\vec{X}_1||+||\vec{X}_2||-2.c.D.|z|.||\vec{Z}||)+(1+D.|z|^2).||Z||\geq$$
$$(1+D.|z|^2)(|z|.(||\vec{X}_1||+||\vec{X}_2||)+||\vec{Z}||)-2.(1+D.|z|^2)c.D|z|^2.||\vec{Z}||.$$

For c small enough we have that:

$$2.(1+D.|z|^2).c.D.|z|^2<\frac{1}{2}D|z|^2 \quad \text{whenever} \quad |z|\leq\varepsilon;$$

from now on we assume that c is so small that this inequality holds. We then obtain:

$$|z'|.(||\vec{X}_1'||+||\vec{X}_2'||)+||\vec{Z}'||\geq\left(1+\frac{1}{2}D.|z|^2\right)(|z|(||\vec{X}_1||+||\vec{X}_2||)+||\vec{Z}||)$$

and hence condition (3) is satisfied.

The verification of the conditions (1') (2') and (4') for φ is trivial, using the above explicit formula for φ. It is also clear that $k=2$, $\ell=2$, $h=1$, $k'=5$ and $m=7$ satisfy the conditions (5), (6), (7) and (8). Using the results in chapter III it follows that the 7-jets of X_n and X are equal in each point of $\{|z|=0\}$ and that $\widetilde{X}_n$ and $\widetilde{X}$ are C^8. So all the assumptions in proposition (4.15) are satisfied, hence $\widetilde{X}$ has an invariant manifold close to $\{y=0\}$ and X has a singular invariant manifold close to:

$$\{(y_1^2+y_2^2)=a^2(y_3^2+y_4^2)\}.$$

Thus we have proved one case of the following proposition; the other cases can be proved by the same method (in some cases X has to be replaced by $-X$).

Proposition **(4.17)**. — *Let X be a vector field on $\mathbf{R}^4$ which is at least C^9 and whose 1-jet X_1 is of the form:*

$$X_1=\lambda_1\left(y_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial y_1}\right)+\lambda_2\left(y_3\frac{\partial}{\partial y_4}-y_4\frac{\partial}{\partial y_3}\right) \quad \textit{with} \quad \alpha_1\lambda_1+\alpha_2\lambda_2\neq 0$$

whenever $\alpha_i\in\mathbf{Z}$ and $1\leq|\alpha_1|+|\alpha_2|\leq 9$, and whose 3-jet is in normal form. Let $\overline{X}_3$ be the reduced 3-jet of X and let Z be some C^∞-representative of $\overline{X}_3$ on $\mathbf{R}^2$, invariant under T_1 and T_2. Let $\widetilde{Z}$ be the vector field on $S^1\times\mathbf{R}$ obtained by blowing up Z and "dividing by r^2" (see chapter III, § 2).

If $(\bar{x}_1=0,\ \bar{x}_2=1,\ r=0)\in S^1\times\mathbf{R}$ is a hyperbolic singularity of $\widetilde{Z}$ with one expanding and one contracting eigenvalue, then X has an invariant manifold which is close to $\{y_1=y_2=0\}$.

If $(\bar{x}_1=1,\ \bar{x}_2=0,\ r=0)\in S^1\times\mathbf{R}$ is a hyperbolic singularity of $\widetilde{Z}$ with one expanding and one contracting eigenvalue, then X has an invariant manifold which is close to $\{y_3=y_4=0\}$.

If $(\bar{x}_1=a,\ \bar{x}_2=b,\ r=0)\in S^1\times\mathbf{R}$, $a,b\neq 0$, $a^2+b^2=1$, is a hyperbolic singularity of $\widetilde{Z}$ with one expanding and one contracting eigenvalue, then X has an invariant variety which is close to $\{b^2(y_1^2+y_2^2)=a^2(y_3^2+y_4^2)\}$.

Modulo a few modifications (to be indicated below) the following proposition is proved in the same way:

Proposition **(4.18)**. — *Let X be a vector field on $\mathbf{R}^3$ which is at least C^7 and whose 1-jet X_1 is of the form $X_1=\lambda\left(y_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial y_1}\right)$, $\lambda\neq 0$, and whose 2-jet is in normal form. Let $\overline{X}_2$ be the reduced 2-jet of X and let Z be some C^∞-representative of $\overline{X}_2$ on $\mathbf{R}^2$, invariant under T_1. Let $\widetilde{Z}$ be the vector field on $S^1\times\mathbf{R}$ obtained by blowing up Z and dividing by r.*

If $(\bar{x}_1=0,\ \bar{x}_2=1,\ r=0)\in S^1\times\mathbf{R}$ is a hyperbolic fixed singularity of $\widetilde{Z}$ with one expanding and one contracting eigenvalue, then X has an invariant line close to $\{y_2=y_3=0\}$.

If $(\bar{x}_1=a,\ \bar{x}_2=b,\ r=0)\in S^1\times\mathbf{R}$, $a\neq 0$, $a^2+b^2=1$, is a hyperbolic singularity of $\widetilde{Z}$ with one expanding and one contracting eigenvalue, then X has a (singular) variety close to $\{b^2(y_1^2+y_2^2)=a^2.y_3^2\}$.

In the proof of this last proposition one has to apply proposition (4.15) with $k=1$, $\ell=1$, $h=1$, $k'=4$ and $m=5$. Also, in the application of proposition (4.15) to the vector field $\widetilde{X}$, obtained by blowing up X, on $S^2\times\mathbf{R}$, one has to restrict oneself to

$S^2\times[0,\infty)$ or $S^2\times(-\infty,0]$, because if in $S^2\times[0,\infty)$ the flow is " going away from $S^2\times\{0\}$ " then in $S^2\times(-\infty,0]$ the flow is " tending towards $S^2\times\{0\}$ ". Also a zero-dimensional torus has to be interpreted as one point.

V. — NON-STABILIZABLE JETS; PROOF OF THEOREM (1.19)

The non-stabilizable jets we shall construct, have a non-stability of the same sort as in [18]; here we also show that they occur with codimension 3.

1. A special singularity in $\mathbf{R}^3$.

In $\mathbf{R}^3$ we take the vector field $X=(x_1^2+x_2^2-x_3^2)\frac{\partial}{\partial x_3}-2x_1x_3\frac{\partial}{\partial x_1}+2x_2x_3\frac{\partial}{\partial x_2}$ and investigate some of its properties.

Property (5.1). — X is invariant under T_{1*} and T_{2*}; $T_1(x_1,x_2,x_3)=(-x_1,x_2,x_3)$ and $T_2(x_1,x_2,x_3)=(x_1,-x_2,x_3)$ (the proof of this property is trivial).

Property (5.2). — Let $\widetilde{X}$ be the vector field on $S^2\times\mathbf{R}$ obtained by blowing up X and let $\overline{X}=\frac{1}{r}.\widetilde{X}$. Then the singularities of $\overline{X}\,|\,S^2\times\{0\}$ are $(\bar{x}_1=0,\ \bar{x}_2=0,\ \bar{x}_3=\pm1)$ and $\left(\bar{x}_1=0,\ \bar{x}_2=\pm\frac{1}{2}\sqrt{3},\ \bar{x}_3=\pm\frac{1}{2}\right)$.

Proof. — As we have seen in chapter II the singularities of $\overline{X}\,|\,S^2\times\{0\}$ correspond (in the case where the coefficient functions of X are homogeneous polynomials of degree 2) to invariant lines in $\mathbf{R}^3$ or to lines where:

$$\langle X, V_{ij}\rangle=0 \quad \text{for} \quad 1\leq i,j\leq 3 \quad \text{with} \quad V_{ij}=\frac{1}{2}\left(x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}\right).$$

By a short calculation we get:

$$\langle X, V_{12}\rangle=2.x_1.x_2.x_3$$

$$\langle X, V_{13}\rangle=\frac{1}{2}x_1(x_1^2+x_2^2+x_3^2)$$

$$\langle X, V_{23}\rangle=\frac{1}{2}x_2(x_1^2+x_2^2-3x_3^2).$$

The set of points where these functions are simultaneously zero is $\{x_1=x_2=0\}$ and $\{x_1=0,\ x_2^2=3x_3^2\}$. The set of points in $S^2\times\mathbf{R}$ which are mapped by Φ on this set is:

$$\{\bar{x}_1=0,\ \bar{x}_2=0,\ \bar{x}_3=\pm1,\ r\text{ arbitrary}\}\cup\left\{\bar{x}_1=0,\ \bar{x}_2=\pm\frac{1}{2}\sqrt{3},\ \bar{x}_3=\pm\frac{1}{2},\ r\text{ arbitrary}\right\}.$$

This proves property 2.

Property (5.3). — All the singularities of $\overline{X}|S^2\times\{o\}$ are hyperbolic; in $(\overline{x}_1=o, \overline{x}_2=o, \overline{x}_3=\pm 1)$ there are saddles, in $\left(\overline{x}_1=o, \overline{x}_2=\pm\frac{1}{2}\sqrt{3}, \overline{x}_3=+\frac{1}{2}\right)$ there are sinks and in $\left(\overline{x}_1=o, \overline{x}_2=\pm\frac{1}{2}\sqrt{3}, \overline{x}_3=-\frac{1}{2}\right)$ there are sources.

Proof. — First one should notice that $X|\{x_2=o\}$, resp. $X|\{x_1=o\}$, satisfies condition II, resp. IV, of proposition (3.8). From this the statement about the points $(\overline{x}_1=o, \overline{x}_2=o, \overline{x}_3=\pm 1)$ follows immediately. From the fact that $X|\{x_1=o\}$ satisfies condition IV in proposition (3.8), it follows that for $\overline{X}|\{\overline{x}_1=r=o\}$ the points $\left(\overline{x}_1=o, \overline{x}_2=\pm\frac{1}{2}\sqrt{3}, \overline{x}_3=\frac{1}{2}\right)$, resp. $\left(\overline{x}_1=o, \overline{x}_2=\pm\frac{1}{2}\sqrt{3}, \overline{x}_3=-\frac{1}{2}\right)$, are sinks, resp. sources. The same fact for $\overline{X}|\{r=o\}$ instead of $\overline{X}|\{\overline{x}_1=r=o\}$ follows from the fact that in each point of $\{x_1=o\}\backslash\{o\}$, $\frac{\partial}{\partial x_1}\langle X, V_{13}\rangle>o$.

Property (5.4). — The only recurrent points of $\overline{X}|S^2\times\{o\}$ are the points in which $\overline{X}$ is zero.

Proof. — Because of the symmetries we may restrict our attention to:

$$S^2\times\{o\}\cap\{\overline{x}_1\geq o\}\cap\{\overline{x}_2\geq o\}.$$

The boundary of this set is invariant under the flow of $\overline{X}$ and does not contain any recurrent points of $\overline{X}$ other than the zeros of $\overline{X}$. If $S^2\times\{o\}\cap\{\overline{x}_1>o\}\cap\{\overline{x}_2>o\}=W$ would contain any recurrent point, then, because $\overline{X}$ is nowhere zero there, it would have to contain a closed integral curve $\gamma : S^1\rightarrow W$ or a closed embedded curve $\gamma' : S^1\rightarrow W$ which is everywhere transversal to $\overline{X}$ [13; proposition (7.1)]. From the existence of a closed embedded curve as γ or γ' it would follow that $\overline{X}$ is somewhere zero in W; this is a contradiction and the property is proved.

Remark (5.5). — From the above properties it follows that:

$$\overline{X}|S^2\times\{o\}\cap\{\overline{x}_1\geq o\}\cap\{\overline{x}_2\geq o\}$$

must be topologically equivalent to the flow indicated below.

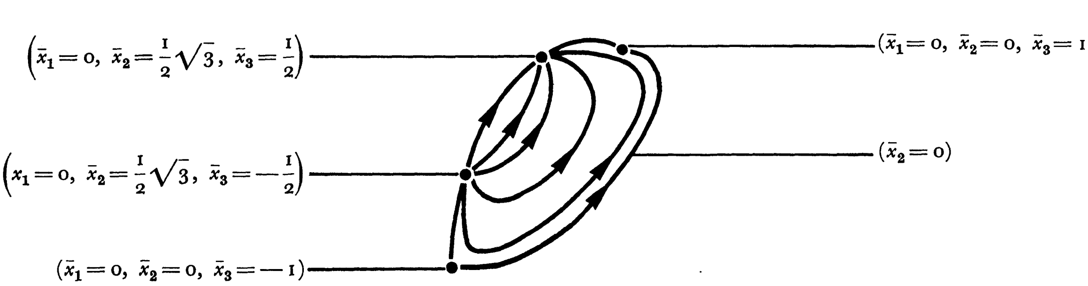

Fig. 10

Remark (5.6). — Because $X|\{x_2=0\}$, resp. $X|\{x_1=0\}$, satisfies condition II, resp. IV, in proposition (3.8), all the points on $S^2\times\{0\}$ where $\overline{X}$ is zero are hyperbolic singularities of X (this time not restricted to $S^2\times\{0\}$). In the figure below it is indicated how the flow of X looks in $\{x_1\geq 0, x_2\geq 0\}$ (compare also figure 3).

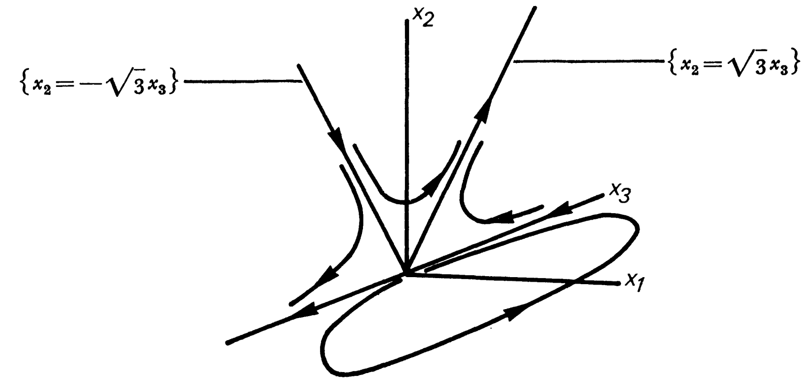

FIG. 11

x_1, x_2 and x_3 are placed at the positive side of their axis.

Remark (5.7). — Let U be some bounded open neighbourhood of $\mathbf{R}^3$ containing 0. Then the set of points $p\in U$ such that $L_{\alpha, X, U}(p)$ is the origin (see definition (1.4)) consists of two half open pieces of line:

$$\{x_1=0, x_2=+\sqrt{3}.x_3, x_2\geq 0\} \quad \text{and} \quad \{x_1=0, x_2=-\sqrt{3}.x_3, x_2\leq 0\}$$

and one open piece of $\{x_2=0\}$. An analogous statement holds for those points $p\in U$ with $L_{\omega, X, U}(p)=0$. The set of those points $p\in U$ for which both $L_{\alpha, X, U}(p)$ and $L_{\omega, X, U}(p)$ is the origin, is an open set of $\{x_2=0\}$, containing a non-empty neighbourhood of 0 in the x_1-axis.

Remark (5.8). — Given any bounded neighbourhood U of 0 in $\mathbf{R}^3$, there is a sequence of points $\{p_i\}_{i=1}^{\infty}$ in $U\cap\{x_2=0\}$ converging to 0 and a sequence of (3-dimensional) neighbourhoods $\{U_i\}_{i=1}^{\infty}$ of these points, i.e. $p_i\in U_i$, such that:

1. $U_i\subset U\cap\{x_1\neq 0\}$, $U_i\cap U_j=\emptyset$ for all $i\neq j$.
2. For each $q\in U_i$ there are positive real numbers t_q^+ and t_q^- such that:

$$\mathscr{D}_X(q, (-t_q^-, +t_q^+))\subset U_i$$

and such that for each $t>t_q^+$, resp. $t<-t_q^-$, either $\mathscr{D}_X(q, t)\notin\bigcup_{i=1}^{\infty}U_i$ or there is a $t'\in(t_q^+, t)$, resp. $t'\in(t, -t_q^-)$, with $\mathscr{D}_X(q, t')\notin U$.

3. For each $q'\in U\cap\{x_2=0\}$, $L_{\alpha, X, U}(q')$ and $L_{\omega, X, U}(q')$ are both the origin.

The existence of sequences $\{p_i\}_{i=1}^{\infty}$ and $\{U_i\}_{i=1}^{\infty}$ as above follows easily from the following considerations:

In the half-plane $\{x_3=0, x_1\geq 0\}$ X is non-zero, except in the origin, and is pointing in the direction of the positive x_3-axis. Because of this, and the fact that the planes $\{x_1=0\}$ and $\{x_2=0\}$ are invariant under the flow of X, an integral curve, starting in a point of $\{x_3=0, x_1\geq 0, x_1^2+x_2^2>0\}$, will never come back to the half-plane $\{x_3=0, x_1\geq 0\}$. If we now choose the points p_i on the positive x_1-axis and the sets U_i as small flow-boxes, it is clear that the above three conditions will be satisfied. The role played by U is completely inessential, but introducing the U here makes the following remark easier to formulate.

Remark **(5.9)**. — All the properties and remarks concerning X, stated in this paragraph, also hold for any C^∞-vector field X' which is invariant under T_{1*} and T_{2*}, the 1-jet of which is zero in the origin and the 2-jet of which is close to the 2-jet of X; only the following modifications must be made:

a) The points $\left(\bar{x}_1=0, \bar{x}_2=\pm\frac{1}{2}\sqrt{3}, \bar{x}_3=\pm\frac{1}{2}\right)$ in property 2 and further may be slightly different for X', but they will still be in $\bar{x}_1=0$.

b) Instead of the lines $\{x_1=0, x_2=\pm\sqrt{3}\,.x_3\}$ other lines, C^1-close to them, and also lying in $\{x_1=0\}$, will be invariant under the flow of X' (see also chapter II).

c) The neighbourhood U in remark (5.7) and remark (5.8) must be chosen sufficiently small (depending on X').

2. The proof of theorem (1.19).

Proposition **(5.10)**. — *Let Y be a C^∞-vector field on $\mathbf{R}^5$ which has a 1-jet Y_1 of the form* $Y_1=\lambda_1\left(y_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial y_1}\right)+\lambda_2\left(y_3\frac{\partial}{\partial y_4}-y_4\frac{\partial}{\partial y_3}\right)$ *with λ_1, λ_2 independent over the rationals. Let X_2 be the 2-jet obtained by reducing the 2-jet of Y (with respect to some coordinates in which Y is in normal form). Let X_2 be so close to the 2-jet of X in § 1 that for any representative X' of X_2, which is invariant under T_{1*} and T_{2*}, all the properties and remarks, in § 1, are valid.*

Then there are two C^∞-vector fields Y' and Y'' on $\mathbf{R}^5$, both having the same infinite-jet as Y, such that their singularities in 0 are not weakly-C^0-equivalent.

Proof. — Let the ∞-jet of Y be in normal form with respect to the coordinates $y_1, \ldots, y_5$; i.e. $[Y]_\infty$ is invariant under the rotations R_θ^i (see proposition (2.6)), $i=1, 2$, $\theta\in[0, 2\pi)$. We choose the vector field Y' so that it is invariant under these same rotations R_θ^i; let X' be the vector field on $\mathbf{R}^3$ obtained by reducing Y' (this goes in the same way as reducing a germ, see definition (2.7)). Following remarks (5.7), (5.8) and (5.9) we can choose, for any sufficiently small neighbourhood U of $0\in\mathbf{R}^3$, a sequence of points $\{p_i\}_{i=1}^{\infty}$ in U converging to 0 and neighbourhoods $\{U_i\}_{i=1}^{\infty}$ of these points as in remark (5.8). We want to obtain something analogous for the vector field Y'.

First we observe that the " map ":

$$\Psi(y_1, \ldots, y_5) = (\pm\sqrt{y_1^2+y_2^2}, \pm\sqrt{y_3^2+y_4^2}, y_5)$$

maps integral curves of Y' to integral curves of X' (and hence also preserves α- and ω-limits). We take $U' = \Psi^{-1}(U)$, p_i' is chosen so that $\Psi(p_i') = p_i$ and U_i' is a small flow-box for the flow Y', contained in $\Psi^{-1}(U_i)$.

The vector field Y'' will be obtained by changing Y' in each of the sets U_i'; this change, to be described below, can be made C^∞ and arbitrarily C^k-small for any k so the total change can be made so that Y'' is still C^∞ (and hence will have the same ∞-jet as Y' and Y).

The fact that U_i' is a flow-box means that there is an open (bounded) 4-dimensional submanifold W_i, containing p_i' and transversal to Y', and positive numbers ε_i^+ and ε_i^- such that $U_i' = \mathscr{D}_{Y'}(W_i, (-\varepsilon_i^-, +\varepsilon_i^+))$. For each, sufficiently small, vector field Z with support contained in U_i', there is a unique map $P_Z : W_i \to W_i$ such that for each $w \in W_i$, $\mathscr{D}_{Y'}(w, -\varepsilon_i^-)$ and $\mathscr{D}_{Y'}(P_Z(w), +\varepsilon_i^+)$ are on the same integral curve of $Y'+Z$ (in U_i'); clearly $P_0 = \mathrm{id}$. We now take Z so that $P_Z(W_i \cap \{y_3 = y_4 = 0\})$ and $W_i \cap \{y_3 = y_4 = 0\}$ have some isolated points of intersection (possible because $\dim(W_i) = 4$ and $\dim(W_i \cap \{y_3 = y_4 = 0\}) = 2$); we then take $Y'' \mid U_i = (Y'+Z) \mid U_i$.

We now investigate what the difference is between Y' and Y''. For Y' we have: if $q \in U_i'$, then $\mathscr{D}_{Y'}(q, t)$ leaves U for both positive and negative time if $q \notin \{y_3 = y_4 = 0\}$ and $\mathscr{D}_{Y'}(q, t)$ stays in U and tends to the origin for both $t \to +\infty$ and $t \to -\infty$ if $q \in \{y_3 = y_4 = 0\}$; this follows from the properties of $\{U_i\}_{i=1}^\infty$ in remark (5.8) and the construction of $\{U_i'\}_{i=1}^\infty$. For Y'' we see that the set of points q in U_i' with:

$$L_{\alpha, Y'', U'}(q) = L_{\omega, Y'', U'}(q) = 0$$

contains isolated lines (because of remark (5.8) sub 2, the changes in the different U_i''s do not interfere). Hence we have:

For any small enough bounded neighbourhood U'' of 0 in $\mathbf{R}^5$, the set of points $q \in U''$ with $L_{\alpha, Y', U''}(q) = L_{\alpha, Y', U''}(q) = 0$ is an open 3-manifold; the set of points $q \in U''$ with $L_{\alpha, Y'', U''}(q) = L_{\alpha, Y'', U''}(q) = 0$ contains locally isolated (1-dimensional) lines. From this it follows that the germs of Y' and Y'' are not weakly-C^0-equivalent.

The proof of theorem (1.19). — Let $W_1 \subset J_2^5$ be the set of 2-jets of those vector fields X on $\mathbf{R}^5$ for which the matrix $\left(\dfrac{\partial X_i}{\partial x_j}\right)$ has eigenvalues 0, $\pm\lambda_1 i$ and $\pm\lambda_2 i$ with $\alpha_1\lambda_1 + \alpha_2\lambda_2 \neq 0$ whenever $1 \leq |\alpha_1| + |\alpha_2| \leq 3$. W_1 is clearly an open codimension 3 submanifold of J_2^5. According to proposition (2.5) any element of W_1 can be transformed (by a diffeomorphism) into normal form. We take $W_2 \subset W_1$ to be the subset of those $\alpha \in W_1$ whose 2-jet is in normal form and whose reduced 2-jet α' (on $\mathbf{R}^3$) is so close to the 2-jet of X in § 1 that for any representative X' of α' which is invariant under T_{1*}

and T_{2*}, all properties and remarks in § 1 are valid. It is clear that the set $W_3 \subset J_2^5$, consisting of jets which can, by a diffeomorphism, be transformed into W_2, is a subset of W_1 and contains an open subset of W_1. We choose W to be an open subset of W_1, contained in W_3. The residual subset $P \subset W_3$ is the set consisting of those $\alpha \in W_3$ for which the eigenvalues of the " linear part ", $0, \pm\lambda_1 i, \pm\lambda_2 i$, satisfy $\alpha_1\lambda_1+\alpha_2\lambda_2 \neq 0$ whenever $\alpha_1, \alpha_2 \in \mathbf{Z}$ and $1 \leq |\alpha_1|+|\alpha_2|$. It should be remarked that P consists of jets, all satisfying the conditions imposed upon the 2-jet of Y in proposition (5.10). From this proposition it now follows that if $\alpha \in P$ and $\alpha' \in J_k^5$, $k \geq 2$, is such that $\pi_2(\alpha')=\alpha$, then α' has two representatives whose germs are not weakly-C^0-equivalent (taking a k-jet instead of an ∞-jet does not change the situation), and the proof of proposition (5.10) shows that it has two representatives satisfying the condition in the conclusion of theorem (1.19). This proves the theorem.

VI. — THE PROOF OF THEOREMS (1.15) AND (1.16)

1. The proof of theorem (1.15).

Case 1: the subsets of W_1^1. — Every germ in W_1^1 can be represented by a vector field of the form:

$$X = a_2 x^2 \frac{\partial}{\partial x} + a_3 x^3 \frac{\partial}{\partial x} + \text{higher order terms};$$

a_2 and a_3 are determined already by the 3-jet. We define $V_{1,2}$, resp. $V_{1,3}$, to be the subset of those germs α in W_1^1 for which $\pi_2(\alpha)=0$, resp. $\pi_3(\alpha)=0$; $V_{1,2}$ and $V_{1,3}$ are clearly closed (in $W_1^1 = V_{1,1}$) and (semi-)algebraic and have the right codimension. Let now α, α' be arbitrary germs in $V_{1,1} \setminus V_{1,2}$. Then representatives X, X' of α, α' will be of the form $X = x^2.f(x).\frac{\partial}{\partial x}$ and $X' = x^2.f'(x)\frac{\partial}{\partial x}$ with $f(0)$ and $f'(0)$ non-zero. We now prove that if $f(0)$ and $f'(0)$ have the same sign, then the germs α and α' are C^0-equivalent. By continuity of f and f' there is an $\varepsilon > 0$ such that $|f(x)|$, $|f'(x)| > 0$ whenever $|x| \leq \varepsilon$. Then both X and X', restricted to $U = \left\{ x \in \mathbf{R} \,\middle|\, |x| < \frac{1}{2}\varepsilon \right\}$, have only three different orbits, namely:

$$\mathcal{O}_1 = \left\{ x \in \mathbf{R} \,\middle|\, -\frac{1}{2}\varepsilon < x < 0 \right\}, \qquad \mathcal{O}_2 = \{0\} \quad \text{and} \quad \mathcal{O}_3 = \left\{ x \in \mathbf{R} \,\middle|\, 0 < x < \frac{1}{2}\varepsilon \right\}.$$

So the identity on U maps integral curves of X to integral curves of X'; because $f(0)$ and $f'(0)$ have the same sign the X- and X'-orbits have the same sense. Also the germs of the vector fields $x^2\frac{\partial}{\partial x}$ and $-x^2\frac{\partial}{\partial x}$ are C^0-equivalent (because if $\varphi : \mathbf{R} \to \mathbf{R}$ is defined by $\varphi(x)=-x$, then $\varphi_*\left(x^2\frac{\partial}{\partial x}\right)=-x^2\frac{\partial}{\partial x}$). Hence it follows that any two germs $\alpha, \alpha' \in V_{1,1}\backslash V_{1,2}$ are C^0-equivalent and hence any $\alpha \in V_{1,1}\backslash V_{1,2}$ is $V_{1,1}$-(weakly-)C^0-stable.

For α, α' two germs in $V_{1,2}\backslash V_{1,3}$ we can proceed in the same way: let X, X' be representatives, $X=x^3.f(x)\frac{\partial}{\partial x}$ and $X'=x^3.f'(x)\frac{\partial}{\partial x}$ with $f(0), f'(0)$ non-zero. As above, if $f(0)$ and $f'(0)$ have the same sign, then α and α' are C^0-equivalent. On the other hand, $x^3\frac{\partial}{\partial x}$ and $-x^3\frac{\partial}{\partial x}$ are not (weakly-)C^0-equivalent, the first being a source and the second being a sink. If, however, α and α' are close enough they will be in the same C^0-equivalence class; hence any $\alpha \in V_{1,2}\backslash V_{1,3}$ is $V_{1,2}$-(weakly-)C^0-stable.

The sets $V_{1,1}$, $V_{1,2}$, $V_{1,3}$ were defined in terms of 3-jets and the proof works also for C^3-germs, hence $V_{1,1}$, $V_{1,2}$ and $V_{1,3}$ may be considered as subvarieties in $\mathscr{G}^{1,3}$ (C^3-germs on $\mathbf{R}^1$).

Case 2: the subsets of W_2^2. — We first observe that if α is a germ, belonging to W_2^2, then we can bring its 5-jet in normal form by a coordinate transformation. Such a 5-jet in normal form is of the following type (see proposition (2.3)):

$$X_5=(\lambda+a_1(x_1^2+x_2^2)+a_2(x_1^2+x_2^2)^2)\left(x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right)$$
$$+(b_1(x_1^2+x_2^2)+b_2(x_1^2+x_2^2)^2)\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right), \qquad \lambda \neq 0.$$

From now on we shall use the following notation: if $\alpha \in W_2^2$ and its 5-jet is in normal form, then the coefficients in the above expression, which are determined by α, are denoted by $\lambda(\alpha)$, $a_i(\alpha)$, $b_i(\alpha)$. Note that for every $\alpha \in W_2^2$, with 5-jet in normal form, $\lambda(\alpha) \neq 0$.

We define $\widetilde{V}_{2,2}$ to be the set of those germs $\alpha \in W_2^2$ whose 5-jet is in normal form and for which $b_1(\alpha)=0$; we define $\widetilde{V}_{2,3}$ to be the set of those germs $\alpha \in W_2^2$ whose 5-jet is in normal form and for which $b_1(\alpha)=b_2(\alpha)=0$. The sets $V_{2,i}$, $i=2, 3$, are now defined by:

$$V_{2,i}=\{\alpha \in W_2^2 \mid \exists\, \varphi : \mathbf{R}^2 \to \mathbf{R}^2 \quad \text{such that} \quad \varphi_*(\alpha) \in \widetilde{V}_{2,i}\}.$$

Using the Seidenberg-Tarski theorem (1.18) as in § 1 it easily follows that $V_{2,i}$ is a closed semi-algebraic subset of W_2^2 with the right codimension for $i=1, 2, 3$ ($V_{2,1}=W_2^2$).

To show that any $\alpha \in V_{2,i}\backslash V_{2,i+1}$, is $V_{2,i}$-weakly-C^0-stable, it is clearly enough to show this using $\widetilde{V}_{2,i}$, $\widetilde{V}_{2,i+1}$ instead of $V_{2,i}$ and $V_{2,i+1}$. Let now $\alpha, \alpha' \in V_{2,1}\backslash V_{2,2}$; if α' is in a small neighbourhood of α, then $b_1(\alpha)$ and $b_1(\alpha')$ will have the same sign and hence α and α' are both sinks (if $b_1(\alpha), b_1(\alpha')<0$) or both sources (if $b_1(\alpha), b_2(\alpha')>0$). Hence α and α' are (weakly-)C^0-equivalent. A similar argument works when:

$$\alpha, \alpha' \in V_{2,2}\backslash V_{2,3}.$$

Finally, the sets $V_{2,i}$ are defined in terms of 5-jets, and for the proof it is enough to assume that all germs are C^5, so $V_{2,1}$, $V_{2,2}$ and $V_{2,3}$ may be considered as submanifolds of $\mathscr{G}^{2,5}$.

2. The proof of theorem (1.16).

Case 1: the subsets of W_3^2. — We define $V_{3,3}$ to be the set of those $\alpha \in W_3^2$ for which there is no diffeomorphism φ with the property that the 2-jet of $\varphi_*(\alpha)$ is of the form $(\varphi_*(\alpha))_2 = x_1\frac{\partial}{\partial x_2} + a_2 x_2^2 \frac{\partial}{\partial x_1} + b_2 x_2^2 \frac{\partial}{\partial x_2}$ with $a_2 \neq 0$. By the same arguments as before, $V_{3,3}$ is a closed semi-algebraic subset of W_3^2; by proposition (2.2), $V_{3,3}$ has codimension 1 in W_3^2 and by proposition (3.12), every $\alpha \in V_{3,2} \setminus V_{3,3}$ is $V_{3,2}$-(weakly-)C^0-stable ($V_{3,2} = W_3^2$). The set $V_{3,3}$ is defined in terms of the 3-jet, but in the proof of proposition (3.12) we need the vector field to be C^5. Hence $V_{3,2}$, $V_{3,3}$ may be considered as subsets of $\mathscr{G}^{2,5}$.

Case 2: the subsets of W_4^3. — We define the subset $V_{4,3}$ as follows: $\alpha \in V_{4,3}$ if the reduction of the 2-jet of α (with respect to any system of coordinates which brings the 2-jet of α in normal form) does not satisfy the assumptions in proposition (3.8). As in case 1 it is clear that $V_{4,3}$ is a closed semi-algebraic subset of $W_4^3 = V_{4,2}$ with codimension 1.

The only non-trivial thing to prove is that each $\alpha \in V_{4,2} \setminus V_{4,3}$ is $V_{4,2}$-weakly-C^0-stable. To prove this it is enough to show that if $\alpha, \alpha' \in V_{4,2}$, if the 2-jets of both α and α' are in normal form and if the reduced 2-jets α_r, α'_r satisfy both the same of the conditions I, II, III, IV and V in proposition (3.8), then α and α' are weakly-C^0-equivalent. To prove this last statement we have to distinguish between the five cases.

Case 2, I (α_r, α'_r *satisfy condition* I *in proposition* (3.8)).

Let X and X' be representatives of α and α'. By propositions (3.8) and (4.8) there are invariant manifolds (lines) ℓ and ℓ' for X and X' along which the flow is contracting at one side and expanding at the other side. Using the same reasoning as in the proof of proposition (4.13) it follows that for sufficiently small neighbourhoods U of 0 in $\mathbf{R}^3$, all points $q \in U$ with $L_{\alpha,X,U}(q)$ or $L_{\omega,X,U}(q)$ non-zero are on $\ell \cap U$; a similar statement holds for X' and ℓ'. Hence a weak-C^0-equivalence between X and X' only has to map orbits of X to orbits of X' as far as they are in ℓ, resp. ℓ'. This is easy to construct.

Case 2, II. — Let X and X' be again representatives of α and α'. Let α_r be the reduced 2-jet of X and let X_r be the vector field on $\mathbf{R}^2$ representing α_r and having, as coefficient functions, polynomials of degree 2.

In $\mathbf{R}^2$ we want to make two subsets Q_1 and Q_2 such that:

(i) ∂Q_i is a smooth manifold and $T_1(Q_i)=Q_i$ (X is also invariant under T_{1*});

(ii) $\mathrm{int}(Q_1)\cap\mathrm{int}(Q_2)=\emptyset$;

(iii) $\mathrm{int}(Q_1\cup Q_2)\ni o$;

(iv) in each point $q\in Q_i$, the component of $X_r(q)$ normal to ∂Q_i has length $\geq C.(\rho(q,o))^2$ for some positive constant C; $\rho(q,o)$ denotes the distance from q to the origin; X_r points at ∂Q_1 to the inside of Q_1 and at ∂Q_2 to the outside of Q_2.

The following figure makes clear that such Q_1 and Q_2 exist:

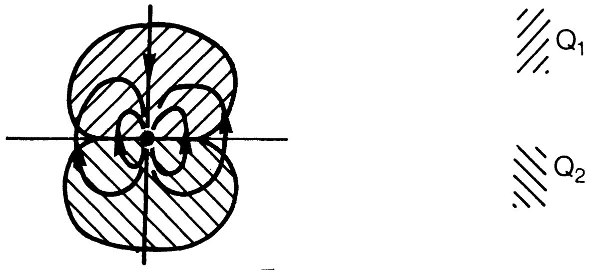

FIG. 12

We define $t.Q_i$ to be the set of points $(t.x_1, t.x_2)$ with $(x_1,x_2)\in Q_i$. Now we consider the map $\Phi:\mathbf{R}^3\to\mathbf{R}^2$ defined by $\Phi(y_1,y_2,y_3)=(\sqrt{y_1^2+y_2^2},y_3)$; modulo terms of order ≥ 3, Φ maps X equivariantly to X_r. Hence, for t small enough, say $t\leq t_0$, $t.\hat{Q}_i=\Phi^{-1}(t.Q_i)$ has the following properties:

(i') $\partial(t.\hat{Q}_i)$ is smooth;

(ii') $\mathrm{int}(t.\hat{Q}_1)\cap\mathrm{int}(t.\hat{Q}_2)=\emptyset$;

(iii') $\mathrm{int}(t.\hat{Q}_1\cup t.\hat{Q}_2)\ni o$;

(iv') in each point $q\in\partial(t.\hat{Q}_i)$, the component of $X(q)$, normal to $\partial(t.\hat{Q}_i)$ has length $\geq\frac{1}{2}C.(\rho(q,o))^2$ for the same positive constant as above in (iv); X points at $\partial(t.\hat{Q}_1)$ to the inside of $t.\hat{Q}_1$ and at $\partial(t.\hat{Q}_2)$ to the outside of $t.\hat{Q}_2$.

Let $t.\hat{Q}'_i$, $t\leq t'_0$ be a family of subsets of $\mathbf{R}^3$ having the same properties with respect to X'. Take a $t_1\leq t'_0; t_0$ and a homeomorphism:

$$\varphi:(\partial(t_1.\hat{Q}_1)\cup\partial(t_1.\hat{Q}_2))\to(\partial(t_1.\hat{Q}'_1)\cup\partial(t_1.\hat{Q}'_2));$$

we may, and do, assume that such a homeomorphism exists. It is now easy to see that there is a unique extension $\tilde{\varphi}:(t.\hat{Q}_1\cup t.\hat{Q}_2)\to(t.\hat{Q}'_1)\cup(t.\hat{Q}'_2)$ of φ which maps integral curves of X, parameter preserving, to integral curves of X'. φ hence realizes a C^0-equivalence between α and α'.

Case 2, III. — Let X and X' be again representatives of α and α'. By proposition (4.18) X has an invariant variety W close to $\{(y_1^2+y_2^2)=a.y_3^2\}$ for some $a>0$. From chapter IV, § 1 it follows that for r small $S_r\cap W$ is Lipschitz close to:

$$S_r\cap\{(y_1^2+y_2^2)=a.y_3^2\};\qquad S_r=\{(y_1,y_2,y_3)\,|\,y_1^2+y_2^2+y_3^2=r^2\}.$$

Hence, for r small enough, $S_r \backslash W$ consists of three components, one containing $(0, 0, -r)$ (the closure of this component will be called $R_{r,-}$) and homeomorphic with an open disc, one containing $S_r \cap \{y_3 = 0\}$ homeomorphic to $S^1 \times (0, 1)$, and one containing $(0, 0, +r)$ (the closure of this component will be called $R_{r,+}$) homeomorphic with an open disc. Again for r small enough X will be transversal to S_r in all points of $R_{r,+}$ and $R_{r,-}$, pointing inside S_r in the points of $R_{r,-}$ and pointing outside S_r in the points of $R_{r,+}$. Also for any $q \in S_r \backslash (R_{r,+} \cup R_{r,-})$, neither the positive nor the negative X-orbit through q stays for all time in $D_r = \{(y_1, y_2, y_3) \mid y_1^2 + y_2^2 + y_3^2 \leq r\}$ (this last statement is proved by the methods used in the proof of proposition (4.13)).

For X' we take the analogous sets $S_{r'}$, $R'_{r',+}$ and $R'_{r',-}$. We now take homeomorphisms φ_+ and φ_-, $\varphi_\pm : R_{r,\pm} \to R'_{r',\pm}$ and extend them to a homeomorphism $\varphi : D_r \to D_{r'}$ in such a way that for any $q \in R_+$, resp. $q \in R_-$, the negative, resp. positive, X integral curve through q is mapped, parameter preserving, to the corresponding X' integral curve through $\varphi_\pm(q)$. Such an extension realizes a weak-C^0-equivalence between α and α'.

Cases 2, IV and V. — These cases can be handled by the same methods as were used in the previous three cases.

Finally, the subsets $V_{4,2}$ and $V_{4,3}$ may be considered as subsets of $\mathcal{G}^{3,7}$ because in all our arguments we used only differentiability up to order 7 (see also proposition (4.18)).

Case 3: the subsets of W_5^4. — This is completely analogous to case 2, but based this time on propositions (3.10) and (4.17); the sets $V_{5,2}$ and $V_{5,3}$ may be considered as subsets of $\mathcal{G}^{4,9}$ (see proposition (4.17)); $V_{5,2}$ is defined by:

If $\alpha \in V_{5,2}$, then $\alpha \notin V_{5,3}$ if and only if:

a) the eigenvalues $\pm\lambda_1 i$, $\pm\lambda_2 i$ satisfy:

$$n_1\lambda_1 + n_2\lambda_2 \neq 0$$

whenever $n_i \in \mathbf{Z}$ and $1 \leq |n_1| + |n_2| \leq 9$;

b) α or $-\alpha$ has a reduced 3-jet which satisfies one of the conditions I *a*, ..., *d*, II′ *a*, ..., *e* in proposition (3.10).

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Manuscrit reçu le 1er juin 1972.