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# AN INDEX THEOREM FOR SYSTEMS OF DIFFERENCE OPERATORS ON A HALF SPACE

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In our joint paper [3] we presented an index theorem for a certain class of operators analogous to Tœplitz operators. The index which appeared there was real-valued and it involved Breuer's theory [1] of Fredholm operators in a von Neumann algebra of type II — indeed, this result vas of interest precisely because it was the first example of an index theorem for the Breuer index. In the present paper we extend the theorem of [3] to the matrix case, and we apply this result to prove the equivalence of existence and uniqueness of solution for elliptic systems of difference equations, generalizing the result of [6] for a single equation.

Our main result is stated in § 1 and proved in § 2. These sections overlap somewhat with [2] and [3] — our slightly different point of view here allows us to give simpler proofs of the results from these papers which are needed. (See the discussion at the end of § 1.) In § 3 we state without proof the application of the index theorem to difference equations.

## 1. Statement of the main theorem.

For $j \in \mathbf{Z}^n$ a multi-integer let $T_j$ be the translation on $L^2(\mathbf{R}^n)$,

$$(1.1) \qquad T_j v(x) = v(x+j),$$

and let $M_\nu$ be the $\nu \times \nu$ matrix algebra. If $\varphi \in C(\mathbf{T}^n, M_\nu)$ is a continuous, matrix-valued function on the torus with Fourier series $\sum_j m_j e^{i\langle j, \cdot \rangle}$, then the formula $L_\varphi = \sum_j m_j \otimes T_j$ defines a bounded linear operator on $\mathbf{C}^\nu \otimes L^2(\mathbf{R}^n)$. In this paper we study the restriction of such operators to a half space $H = \{x \in \mathbf{R}^n : \langle x, N \rangle \geq 0\}$, where $N$ is a unit vector in $\mathbf{R}^n$. Let $E = \chi_{[0,\infty)}(\langle x, N \rangle)$ be the projection onto $L^2(H)$, and for $\varphi \in C(\mathbf{T}^n, M_\nu)$ let $W_\varphi = E L_\varphi E$.

Let $\mathcal{N}_0$ be the von Neumann algebra of operators on $L^2(\mathbf{R}^n)$ generated by multi-

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integer translations $T_j$ and multiplication by functions of $\langle x, \mathrm{N}\rangle$. Any operator $A \in \mathscr{N}_0$ may be expressed uniquely as a weakly convergent sum:

$$A = \sum_{j \in \mathbf{Z}^n} c_j(\langle x, \mathrm{N}\rangle) T_j$$

where $c_j \in \mathrm{L}^\infty(\mathbf{R}^1)$. Two such operators multiply according to the extension by linearity of the basic product formula

$$(1.2) \qquad c(\langle x, \mathrm{N}\rangle) T_j d(\langle x, \mathrm{N}\rangle) T_k = c(\langle x, \mathrm{N}\rangle) d(\langle x+j, \mathrm{N}\rangle) T_{j+k}.$$

In § 1 of [6] we defined a faithful, normal trace $\tau : \mathscr{N}_0^+ \to [0, \infty]$ such that for non-negative operators

$$(1.3) \qquad \tau(\sum_j c_j(\langle x, \mathrm{N}\rangle) T_j) = \int_{-\infty}^{\infty} c_0(t)\, dt,$$

where $c_0$ is the coefficient of the identity translation in the sum on the left in **(1.3)**. (In particular, for non-negative operators $c_0 \geq 0$ almost everywhere.) Let $\mathscr{N} = \mathrm{M}_\nu \otimes \mathscr{N}_0$; we shall also use $\tau$ for the natural trace on $\mathscr{N}$ defined by

$$(1.4) \qquad \tau(m \otimes A) = \mathrm{tr}(m)\tau(A).$$

Of course $W_\varphi \in \mathscr{N}$ for every $\varphi \in \mathrm{C}(\mathbf{T}^n, \mathrm{M}_\nu)$.

If $P \in \mathscr{N}$ is a projection, we shall call $\tau(P)$ the (relative) dimension of the subspace range $P$. Let $\mathscr{K}$ be the uniformly closed, two sided ideal in $\mathscr{N}$ generated by operators whose range has finite relative dimension. Following Breuer [1] we shall say that $A \in \mathscr{N}$ is (generalized) Fredholm if $A$ is invertible modulo $\mathscr{K}$. It was shown by Breuer that if $A$ is Fredholm, then $\dim \ker A$ and $\dim \ker A^*$ are finite, so that one may define a real-valued index of $A$

$$(1.5) \qquad i(A) = \dim \ker A - \dim \ker A^*.$$

This index has the same algebraic and invariance properties as the ordinary Fredholm index. In particular, even though the index is a real number, it is unchanged by a continuous deformation.

*Theorem 1. — If* $\varphi \in \mathrm{C}(\mathbf{T}^n, \mathrm{M}_\nu)$, $W_\varphi$ *is (generalized) Fredholm if and only if* $\det \varphi$ *is non-vanishing on* $\mathbf{T}^n$, *and in this case the index of* $W_\varphi$ *equals the mean winding number of* $\det \varphi$ *along the line* $\{t\mathrm{N}\}$,

$$\lim_{\mathrm{T} \to \infty} (2\mathrm{T})^{-1}(\arg \det \varphi(\mathrm{TN}) - \arg \det \varphi(-\mathrm{TN})).$$

The above theorem is the main result of this paper. The simplest example to which it applies occurs in the case of a pure shift $S_j = ET_jE$ for $\nu = 1$. Note that according to (1.2)

$$(1.6) \qquad S_j S_k - S_{j+k} = \begin{cases} \chi_{\mathrm{J}}(\langle x, \mathrm{N}\rangle) S_{j+k} & \text{if} \quad \langle j, \mathrm{N}\rangle < 0 \quad \text{and} \quad \langle k, \mathrm{N}\rangle > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\chi_J$ is the characteristic function of the interval $J=[0, -\langle j, N\rangle]$. Thus $S_j$ is Fredholm, since by (1.6) $S_{-j}$ is an inverse of $S_j$ modulo $\mathscr{K}$. Of course $\ker S_j$ and $\ker S_j^*$ may be determined by inspection. For example, if $\langle j, N\rangle \geq 0$ then $\ker S_j^*$ is trivial and $\ker S_j$ equals the range of the projection $\chi_J(\langle x, N\rangle)$ where $J=[0, \langle j, N\rangle]$. Hence for $\langle j, N\rangle \geq 0$ :

$$i(S_j)=\tau(\chi_J(\langle x, N\rangle))-0=\langle j, N\rangle,$$

which is equal to the mean winding number of $e^{i\langle j, \cdot\rangle}$ along $\{tN\}$. The case $\langle j, N\rangle \leq 0$ may be checked similarly.

Let $\mathscr{A}$ be the uniformly closed sub-algebra of $\mathscr{N}$ generated by the operators $\{W_\varphi : \varphi\in C(\mathbf{T}^n, M_\nu)\}$. In § 2 we introduce a symbol calculus for the algebra $\mathscr{A}$, a homomorphism $\sigma : \mathscr{A}\to C(\mathbf{T}^n, M_\nu)$ such that $\ker\sigma=\mathscr{A}\cap\mathscr{K}$ and $\sigma(W_\varphi)=\varphi$. Thus an operator $A\in\mathscr{A}$ is Fredholm if and only if $\sigma(A)$ is invertible in $C(\mathbf{T}^n, M_\nu)$; indeed $A-W_{\sigma(A)}\in\mathscr{K}$, so if $\sigma(A)$ is invertible the index of $A$ equals the mean winding number of $\sigma(A)$. Hence it is a trivial matter to extend theorem 1 to an index theorem for any operator in $\mathscr{A}$.

Let $\mathscr{G}$ be the group of invertible elements in $C(\mathbf{T}^n, M_\nu)$. Both the analytic and topological indices depend only on the symbol, and they are constant on any homotopy class of $\mathscr{G}$. Therefore to prove theorem 1 it would be sufficient to check the index formula on one representative from each homotopy class. In the scalar case $\nu=1$, for any $\varphi\in\mathscr{G}$ we may write :

$$\varphi(\xi)=e^{i\langle k, \xi\rangle}\psi(\xi),$$

where $k\in\mathbf{Z}^n$ is a vector whose $\ell^{th}$ component if the winding number of $\varphi$ around the $\ell^{th}$ factor of $\mathbf{T}^n$ and $\psi$ is homotopic to a constant. Thus the components of $\mathscr{G}$ are classified by the winding numbers, and since $\sigma(S_k)=e^{i\langle k, \cdot\rangle}$, the computation above of the index of a pure shift suffices to prove theorem 1 when $\nu=1$. In the matrix case, however, more invariants are required to classify the components of $\mathbf{T}$. For example note that $SU_2$ is homeomorphic with $S^3$; thus any symbol $\varphi\in C(\mathbf{T}^3, SU_2)$ of non-zero Brouwer degree cannot be homotopic to a constant, although the winding numbers of $\det\varphi$ certainly vanish, since $\det\varphi(\xi)\equiv 1$. In this paper we make no attempt to classify the components of $\mathscr{G}$; instead we prove theorem 1 by computing the analytic index for a wider class of operators.

The original result in this area, theorem (2.2) of [3], was an index theorem for a certain algebra of operators on $L^2(0, \infty)$. Our representation on a half space in $\mathbf{R}^n$ rather than a half line is suggested by the application to difference equations in § 3. Moreover on a half space the analytic index may be computed directly in the von Neumann algebra on $L^2(H)$ generated by $\mathscr{A}$, avoiding the slightly unnatural passage to a representation on $L^2(0, \infty)\otimes\ell^2(\mathbf{R}_d)$ that was required in [3]. In any event, the formulas of § 2 have obvious analogues which apply to the matrix generalization of the algebra considered in [3].

## 2. Proof of the main theorem.

In this paragraph we summarize certain facts about the trace class that will be needed below. (See chap. I, § 6 of Dixmier [4] for proofs.) Let

$$\mathcal{K}_1^+ = \{K \in \mathcal{N}^+ : \tau(K) < \infty\}$$

and let $\mathcal{K}_1$ be the linear span of $\mathcal{K}_1^+$. Then $\mathcal{K}_1$ is a two-sided ideal of $\mathcal{N}$, contained in $\mathcal{K}$, whose closure equals $\mathcal{K}$. The trace $\tau$ extends uniquely to a linear functional on $\mathcal{K}_1$ with the property that $\tau(AK) = \tau(KA)$ for $A \in \mathcal{N}$, $K \in \mathcal{K}_1$. If $K \in \mathcal{K}_1$ let $||K||_1 = \tau(|K|)$ where $|K| = (KK^*)^{1/2}$; then $||\cdot||_1$ is a norm on $\mathcal{K}_1$ such that $|\tau(K)| \leq ||K||_1$ and moreover

$$(2.1) \qquad ||AK||_1 \leq ||A|| \, ||K||_1 \quad \text{and} \quad ||KA||_1 \leq ||A|| \, ||K||_1$$

for any $A \in \mathcal{N}$. Finally if $K = \sum_l K_l$ is a series that converges absolutely with respect to both the operator norm and the trace norm, then $K \in \mathcal{K}_1$.

*Lemma* **(2.1)**. — *If* $\varphi, \psi \in C(\mathbf{T}^n, M_\nu)$ *then* $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}$. *Indeed if*

$$\varphi = \sum_j m_j e^{i\langle j, \cdot \rangle} \quad \text{and} \quad \psi = \sum_j n_j e^{i\langle j, \cdot \rangle}$$

*are smooth, then* $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}_1$ *and*

$$(2.2) \qquad \tau(W_\varphi W_\psi - W_{\varphi\psi}) = \sum_{\langle j, N\rangle < 0} \langle j, N\rangle \operatorname{tr}(m_j n_{-j})$$

*Proof.* — Suppose that $\varphi, \psi \in C^\infty(\mathbf{T}^n, M_\nu)$ have the indicated Fourier series; then

$$(2.3) \qquad W_\varphi W_\psi - W_{\varphi\psi} = \sum_{j,k} m_j n_k \otimes (S_j S_k - S_{j+k}).$$

Note from (1.6) that for any $m \in M_\nu$:

$$||m \otimes (S_j S_k - S_{j+k})||_1 \leq |\langle j, N\rangle| \operatorname{tr}(|m|)$$

Since $\varphi$ and $\psi$ are smooth, the Fourier coefficients of these functions are rapidly decreasing. Hence the series in (2.3) is absolutely convergent in both the operator and trace norms, so $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}_1$. Moreover the trace of the sum in (2.3) may be evaluated term by term to yield (2.2). Finally it follows by limits that $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}$ for general $\varphi, \psi \in C(\mathbf{T}^n, M_\nu)$. This completes the proof.

*Lemma* **(2.2)**. — *If* $W_\varphi \in \mathcal{K}$ *then* $\varphi = 0$.

*Proof.* — We may assume without loss of generality in the proof of this lemma that $\nu = 1$, for the matrix operator $W_\varphi$ is generalized compact if and only if each of its entries is generalized compact. If $a \geq 0$ let $E_a$ be the projection $\chi_{[0,a)}(\langle x, N\rangle)$. Suppose that $\varphi \in C(\mathbf{T}^n)$ has Fourier series $\sum c_j e^{i\langle j, \cdot\rangle}$. We claim that

$$(2.4) \qquad \lim_{a \to \infty} a^{-1} \tau(S_{-j} W_\varphi E_a) = c_j.$$

It follows from (1.6) that $\tau(S_{-j}S_kE_a)=0$ if $j\neq k$ and that

$$\tau(S_{-j}S_jE_a)=\begin{cases} a & \text{if } \langle j, N\rangle \leq 0 \\ a-\langle j, N\rangle & \text{if } 0<\langle j, N\rangle<a \\ 0 & \text{if } \langle j, N\rangle \geq a. \end{cases}$$

Thus for large $a$

$$a^{-1}\tau(S_{-j}S_kE_a)=\delta_{jk}+O(a^{-1}).$$

Therefore (2.4) holds if $\varphi$ is an exponential polynomial, and by limits it holds in the general case.

Suppose now that $W_\varphi\in\mathscr{K}$ for some $\varphi\in C(\mathbf{T}^n)$. Since $\mathscr{K}_1$ is dense in $\mathscr{K}$, for every $\varepsilon>0$ there exists $K\in\mathscr{K}_1$ such that $\|W_\varphi-K\|\leq\varepsilon$. Thus

$$\begin{aligned}|\tau(S_{-j}W_\varphi E_a)| &\leq |\tau(S_{-j}(W_\varphi-K)\ E_a)|+|\tau(S_{-j}KE_a)| \\ &\leq \varepsilon a+\|K\|_1\end{aligned}$$

If $\varphi=\sum_j c_je^{i\langle j,\cdot\rangle}$, then

$$|c_j|\leq\limsup_{a\to\infty} a^{-1}|\tau(S_{-j}W_\varphi E_a)|\leq\varepsilon.$$

Therefore every Fourier coefficient of $\varphi$ vanishes, so $\varphi$ itself vanishes. This completes the proof.

Let $\mathscr{A}$ be the uniformly closed sub-algebra of $\mathscr{N}$ generated by the operators $\{W_\varphi:\varphi\in C(\mathbf{T}^n, M_\nu)\}$, and let $\pi$ be the canonical projection $\pi:\mathscr{A}\to\mathscr{A}/(\mathscr{A}\cap\mathscr{K})$. By lemma (2.1) the formula $\rho(\varphi)=\pi(W_\varphi)$ defines a norm decreasing $*$-homomorphism of $C(\mathbf{T}^n, M_\nu)$ into $\mathscr{A}/(\mathscr{A}\cap\mathscr{K})$. (It is easily verified that $\rho(\varphi)^*=\rho(\varphi^*)$.) According to lemma (2.2) the kernel of $\rho$ is trivial. The range of $\rho$ contains a generating set for $\mathscr{A}/(\mathscr{A}\cap\mathscr{K})$ ; since the range of a homomorphism of two $C^*$-algebras is closed (corollary 1.8.3 of [5]), $\rho$ defines an isomorphism of $C(\mathbf{T}^n, M_\nu)$ onto $\mathscr{A}/(\mathscr{A}\cap\mathscr{K})$. Therefore if $\varphi\in C(\mathbf{T}^n, M_\nu)$, $W_\varphi$ is invertible mod $\mathscr{K}$ if and only if $\varphi$ is invertible in $C(\mathbf{T}^n, M_\nu)$; in other words, $W_\varphi$ is Fredholm if and only if $\det\varphi$ is non-vanishing on $\mathbf{T}^n$.

If $\varphi$ is an invertible element of $C(\mathbf{T}^n, M_\nu)$, then there exists $\psi\in C(\mathbf{T}^n, M_\nu)$ homotopic to $\varphi$, such that $\psi(\xi)$ belongs to the unitary group $U_\nu$ for each $\xi$. Indeed, for $0\leq t\leq 1$, let :

$$\Phi(t, \xi)=(\varphi(\xi)\ \varphi^*(\xi))^{-t/2}\varphi(\xi);$$

then $\Phi(0, \xi)=\varphi(\xi)$ and $\Phi(1, \xi)\in U_\nu$. Of course both the analytic and topological indices are unchanged by this homotopy. Moreover any function in $C(\mathbf{T}^n, U_\nu)$ may be uniformly approximated by a smooth function. Therefore, to prove theorem 1 it suffices to check the index formula for $\{W_\varphi:\varphi\in C^\infty(\mathbf{T}^n, U_\nu)\}$.

Suppose that $W_\varphi$ is a Fredholm operator. Let $W_\varphi=HV$ be the polar decomposition of $W_\varphi$, where $H=(W_\varphi W_\varphi^*)^{1/2}$ and $V$ is a partial isometry. Then $V^*V=I-P$

and $VV^* = I - P'$, where $P$ and $P'$ are projections onto $\ker W_\varphi$ and $\ker W_\varphi^*$ respectively. Hence $[V, V^*] = P - P'$ belongs to $\mathscr{K}_1$ and $\tau([V, V^*]) = i(W_\varphi)$. However if

$$\varphi \in C^\infty(\mathbf{T}^n, U_\nu),$$

then by lemma (2.1):

$$H^2 - I = W_\varphi W_\varphi^* - W_{\varphi\varphi^*} \in \mathscr{K}_1,$$

so that

$$W_\varphi - V = (H - I)\, V = (H+I)^{-1}(H^2 - I)\, V \in \mathscr{K}_1.$$

Hence $\tau([W_\varphi, W_\varphi^*]) = \tau([V, V^*])$, and we may use (2.2) to evaluate this trace. If $\varphi = \sum_j m_j e^{i\langle j, \cdot\rangle}$ one finds that:

$$\tau([W_\varphi, W_\varphi^*]) = \tau(W_\varphi W_\varphi^* - I) - \tau(W_\varphi^* W_\varphi - I) = \sum_j \langle j, N\rangle \operatorname{tr}(m_j^* m_j).$$

On the other hand, by a trivial computation with Fourier series one sees that if $\varphi = \sum_j m_j e^{i\langle j, \cdot\rangle}$ belongs to $C^\infty(\mathbf{T}^n, M_\nu)$, then

$$(2\pi)^{-n}\int_{\mathbf{T}^n} \operatorname{tr}(\varphi^*(\xi)\, \nabla_N \varphi(\xi))\, d\xi = \sum_j \langle j, N\rangle \operatorname{tr}(m_j^* m_j),$$

where $\nabla_N = \frac{1}{i}\sum_{\ell=1}^{n} N_\ell \frac{\partial}{\partial \xi_\ell}$. Of course if $\varphi \in C^\infty(\mathbf{T}^n, U_\nu)$ then $\varphi^*(\xi) = \varphi^{-1}(\xi)$. Observe that

$$\operatorname{tr}(\varphi^{-1}(\xi)\, \nabla_N \varphi(\xi)) = \nabla_N(\log\det\varphi(\xi)).$$

Now we may write:

$$\log\det\varphi(\xi) = i\langle k, \xi\rangle + \psi(\xi), \tag{2.5}$$

where $\psi \in C^\infty(\mathbf{T}^n)$ and the $\ell^{th}$ component of $k \in \mathbf{Z}^n$ is the winding number of $\det\varphi$ around the $\ell^{th}$ factor of $\mathbf{T}^n$. Note that by (2.5), the mean winding number of $\det\varphi$ along $\{tN\}$ equals $\langle k, N\rangle$. But

$$i(W_\varphi) = \tau([W_\varphi, W_\varphi^*]) = (2\pi)^{-n}\int_{\mathbf{T}^n} \nabla_N(\log\det\varphi(\xi))\, d\xi = \langle k, N\rangle,$$

since $\int \nabla_N \psi\, d\xi$ vanishes by an integration by parts. This completes the proof of theorem 1.

## 3. Application to elliptic difference equations.

In this section we assume the reader is familiar with the result of [6]. Suppose

$$Q_{\alpha\beta}(D) = \sum_{j\in\mathbf{Z}^n} c_j^{\alpha\beta}\, T_j, \quad \alpha, \beta = 1, \ldots, \nu$$

is a system of difference operators on $\mathbf{R}^n$. Let $m_j$ be the $\nu\times\nu$ matrix with entries $c_j^{\alpha\beta}$, and let

$$Q(\xi) = \sum_j m_j e^{i\langle j, \xi\rangle}.$$

We whall call $Q_{\alpha\beta}(D)$ properly elliptic if $\det Q(\xi) \neq 0$ for $\xi \in \mathbf{R}^n$ and arg det $Q$ is periodic. Consider a boundary value problem for an elliptic system:

$$\sum_{\beta=1}^{\nu} Q_{\alpha\beta}(D) v_\beta(x) = 0 \qquad \text{for} \quad \langle x, N\rangle \geq a_\alpha,\ \alpha = 1, \ldots, \nu \tag{3.1}$$

$$\sum_{\beta=1} q_{\gamma\beta}(x, D) v_\beta(x) = g_\gamma(x) \qquad \text{for} \quad 0 \leq \langle x, N\rangle < b_\gamma,\ \gamma = 1, \ldots, M. \tag{3.2}$$

Here the unknown functions $v_\beta$ belong to $L^2(H)$; we shall require that $c_j^{\alpha\beta} = 0$ for $\langle j, N\rangle < -a_\alpha$ so that in (3.1) no attempt to evaluate $v_\beta$ in $\mathbf{R}^n - H$ is made. As in [6] we assume the boundary operators $q_{\gamma\beta}(x, D)$ depend on $x$ only through $\langle x, N\rangle$. We regard the boundary data $g$ as an element of $\bigoplus_{\gamma=1}^{M} L^2([0, b_\gamma] \times \partial H)$. Finally we suppose that

$$\sum_{\alpha=1}^{\nu} a_\alpha = \sum_{\gamma=1}^{M} b_\gamma.$$

*Theorem 2. — Under the above hypotheses, the following two statements are equivalent.*

(i) *Equation* (3.1) *with homogeneous boundary conditions has the unique solution zero in* $\bigoplus_{1}^{\nu} L^2(H)$.

(ii) *For a dense set of boundary data g there is at least one solution of* (3.1), (3.2) *in* $\bigoplus_{1}^{\nu} L^2(H)$.

We omit the proof of this theorem, as it is completely analogous to that of [6].

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