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AN EXTENSION OF WHITNEY'S SPECTRAL THEOREM

by J.-Cl. TOUGERON

1. Notations and results.

For $y \in \mathbf{R}^p$ and $Y \subset \mathbf{R}^p$, $|y|$ denotes the euclidean norm of y and $d(y, Y)$ the euclidean distance from y to Y . If Y is empty, we write $d(y, Y) = 1$.

Let Ω_p denote an open set in \mathbf{R}^p and $\mathcal{E}(\Omega_p)$ the \mathbf{R} -algebra of all C^∞ real-valued functions in Ω_p . When $y \in \Omega_p$, let \mathcal{F}_y^m denote the \mathbf{R} -algebra of Taylor expansions of order m at y of all elements in $\mathcal{E}(\Omega_p)$; if $m < \infty$, \mathcal{F}_y^m is isomorphic to the algebra $\mathcal{F}_p / \mathfrak{m}_p^{m+1}$, where \mathfrak{m}_p denotes the maximal ideal of the formal power series ring $\mathcal{F}_p = \mathbf{R}[[y_1, \dots, y_p]]$; if $m = +\infty$, \mathcal{F}_y^m (simply written \mathcal{F}_y) ⁽¹⁾ is isomorphic to \mathcal{F}_p (by the generalized Borel theorem).

Let $T_y^m : \mathcal{E}(\Omega_p)^q \rightarrow (\mathcal{F}_y^m)^q$ denote the projection associating to each function G its Taylor expansion of order m at y . If Y is a compact set in Ω_p , we write $|G|_m^Y = \sup_{\substack{y \in Y \\ |k| \leq m}} |D^k G(y)|$. We provide $\mathcal{E}(\Omega_p)^q$ with its usual structure of a Fréchet space,

defined by the family of all semi-norms $G \mapsto |G|_m^Y$, where Y ranges over the set of compacts in Ω_p and $m \in \mathbf{N}$.

Let M be a submodule of $\mathcal{E}(\Omega_p)^q$ and let us write

$$\widehat{M} = \{G \in \mathcal{E}(\Omega_p)^q \mid \forall y \in \Omega_p, \exists G' \in M \text{ so that } G - G' \text{ is flat at } y\} = \bigcap_{y \in \Omega_p} (T_y)^{-1}(T_y M).$$

According to a standard result of Whitney (B. Malgrange [1]), \widehat{M} is the closure \overline{M} of M in $\mathcal{E}(\Omega_p)^q$: we propose to extend this theorem.

Let Φ denote a C^∞ function from an open set Ω_n in \mathbf{R}^n to Ω_p . The mapping Φ defines a homomorphism of \mathbf{R} -algebras $\Phi^* : \mathcal{E}(\Omega_p) \ni g \mapsto g \circ \Phi \in \mathcal{E}(\Omega_n)$. Let Ψ be a Φ^* -homomorphism from $\mathcal{E}(\Omega_p)^q$ to $\mathcal{E}(\Omega_n)^r$, i.e. Ψ is a homomorphism of abelian groups and, $\forall G \in \mathcal{E}(\Omega_p)^q$ and $\forall g \in \mathcal{E}(\Omega_p) : \Psi(g \cdot G) = \Phi^*(g) \cdot \Psi(G)$. For $y \in \Omega_p$ and $x \in \Phi^{-1}(y)$, the mapping Ψ induces an \mathbf{R} -linear mapping $\Psi_x^m : (\mathcal{F}_y^m)^q \rightarrow (\mathcal{F}_x^m)^r$, so that $T_x^m \circ \Psi = \Psi_x^m \circ T_y^m$. For $X \subset \Phi^{-1}(y)$, we note Ψ_X^m the \mathbf{R} -linear mapping $(\mathcal{F}_y^m)^q \ni V \mapsto (\Psi_x^m(V))_{x \in X} \in \prod_{x \in X} (\mathcal{F}_x^m)^r$. Finally, let T_X^m be the mapping $\mathcal{E}(\Omega_n)^r \ni F \mapsto (T_x^m F)_{x \in X} \in \prod_{x \in X} (\mathcal{F}_x^m)^r$.

We propose to determine the closure $\overline{\Psi(M)}$ of $\Psi(M)$ in $\mathcal{E}(\Omega_n)^r$. Therefore, let us write

$$\begin{aligned} \widehat{\Psi(M)} &= \{F \in \mathcal{E}(\Omega_n)^r \mid \forall y \in \Omega_p, \exists G \in M \text{ such that } \Psi(G) - F \text{ is flat on } \Psi^{-1}(y)\} \\ &= \bigcap_{y \in \Omega_p} (T_{\Phi^{-1}(y)})^{-1}(\Psi_{\Phi^{-1}(y)} \circ T_y M). \end{aligned}$$

⁽¹⁾ We shall omit afterwards the index m , if $m = +\infty$, and shall write: T_y, Ψ_x, \dots instead of $T_y^\infty, \Psi_x^\infty, \dots$

We shall prove the following result:

Theorem (1.1). — *Let us suppose that Φ verifies the following condition:*

(H) *For all compact sets $X \subset \Omega_n$ and $Y \subset \Omega_p$, there exists a constant $\alpha \geq 0$ such that,*
 $\forall y \in Y:$

$$\Gamma(y) = \sup_{x \in X \setminus \Phi^{-1}(y)} (d(x, \Phi^{-1}(y))^\alpha / |\Phi(x) - y|) < \infty.$$

Then $\overline{\Psi(M)} = \widehat{\Psi(M)}$.

It is easy to find C^∞ mappings Φ which do not satisfy this condition. Nevertheless, we shall prove the following result:

Theorem (1.2). — *An analytic mapping Φ verifies the condition (H).*

Both following paragraphs are devoted to the proofs of these theorems which are independent of each other. In the last paragraph, we give a refinement of the Theorem (1.2), when Φ is a polynomial mapping.

2. Proof of theorem 1.2.

Definition (2.1). — Let \mathfrak{I} be a finitely generated ideal of a subring of the ring of germs at x^0 in \mathbf{R}^n of continuous functions with real values. Let $\varphi_1(x), \dots, \varphi_s(x)$ denote real valued functions, continuous in a neighborhood of x^0 and such that their germs at x^0 generate \mathfrak{I} . Let $V(\mathfrak{I})$ be the set of their zeros.

We say that \mathfrak{I} verifies a *Łojasiewicz inequality of order $\alpha \geq 0$* (or simply that \mathfrak{I} verifies $\mathcal{L}(\alpha)$) if there exist a constant $C > 0$ and a neighborhood V of x^0 such that,

$$\forall x \in V, \sum_{i=1}^s |\varphi_i(x)| \geq C \cdot d(x, V(\mathfrak{I}))^\alpha.$$

Let Ω_p be an open set in \mathbf{R}^p , Ω_n an open set in \mathbf{R}^n , $y = (y_1, \dots, y_p)$ and $x = (x_1, \dots, x_n)$ coordinate systems in Ω_p and Ω_n respectively. Let \mathcal{O} be the sheaf of germs of analytic functions with real values on $\Omega_n \times \Omega_p$; \mathcal{I} a sheaf of ideals, analytic and coherent on $\Omega_n \times \Omega_p$. For $(x^0, y^0) \in \Omega_n \times \Omega_p$, we denote $\mathcal{I}_{(x^0, y^0)}$ the stalk of \mathcal{I} at the point (x^0, y^0) . Let $\varphi_1, \dots, \varphi_s$ be generators of the ideal $\mathcal{I}_{(x^0, y^0)}$; we denote $\mathcal{I}_{(x^0, y^0)}^n$ the ideal generated by $\varphi_1(x, y^0), \dots, \varphi_s(x, y^0)$ in the ring $\mathcal{O}_{(x^0, y^0)}^n$ of germs at (x^0, y^0) in $\Omega_n \times \{y^0\}$ of analytic functions with real values. Permuting x and y , we define similarly the ideal $\mathcal{I}_{(x^0, y^0)}^p$ of $\mathcal{O}_{(x^0, y^0)}^p$. Finally, let $V(\mathcal{I})$ be the set of zeros of \mathcal{I} .

Theorem (1.2) is an easy consequence of the following one (Łojasiewicz inequality with a parameter):

Theorem (2.2). — *Let X be a compact set in Ω_n , Y a compact set in Ω_p . There exists $\alpha \geq 0$ such that the ideal $\mathcal{I}_{(x, y)}^n$ verifies $\mathcal{L}(\alpha)$, $\forall (x, y) \in X \times Y$.*

Indeed, let us suppose this theorem is true, and let Φ be an analytic mapping. Let \mathcal{I} denote the analytic and coherent sheaf generated on $\Omega_n \times \Omega_p$ by $\Phi_1(x) - y_1, \dots, \Phi_p(x) - y_p$. Let X, Y be compact sets in Ω_n, Ω_p respectively. By (2.2) applied to \mathcal{I} , $\forall (x^0, y) \in X \times Y$, there exists a constant $C_{(x^0, y)} > 0$ such that for x in a neighborhood of

$x^0 : |\Phi(x) - y| \geq C_{(\varphi, y)} \cdot d(x, \Phi^{-1}(y))^\alpha$. Hence, the set X being compact, there exists a constant $C_y > 0$ such that, $\forall x \in X$:

$$|\Phi(x) - y| \geq C_y \cdot d(x, \Phi^{-1}(y))^\alpha.$$

Clearly, condition (H) follows.

Proof of (2.2). — Obviously, condition $\mathcal{L}(\alpha)$ is verified, with $\alpha = 0$, for $(x, y) \notin V(\mathcal{I})$. The set $X \times Y$ being compact, it suffices to find, for $(x^0, y^0) \in V(\mathcal{I})$, an $\alpha \geq 0$ such that $\mathcal{I}_{(x, y)}^n$ verifies $\mathcal{L}(\alpha)$ for (x, y) in a neighborhood of (x^0, y^0) . We shall suppose that (x^0, y^0) is the origin of $\mathbf{R}^n \times \mathbf{R}^p$. Now, it is enough to prove the following result:

(2.3) *There exists an $\alpha \geq 0$ such that $\mathcal{I}_{(0, y)}^n$ verifies $\mathcal{L}(\alpha)$ for $(0, y) \in V(\mathcal{I})$ and $|y|$ small enough.*

Indeed, let $\varphi_1(x, y), \dots, \varphi_s(x, y)$ generate \mathcal{I} in a neighborhood of $(0, 0)$, and let us consider the sheaf \mathcal{I} generated on a neighborhood of the origin of $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^p$ by $\varphi_1(x+z, y), \dots, \varphi_s(x+z, y)$. By (2.3) applied to the sheaf \mathcal{I} (with the parameter (z, y) instead of y), there exists an $\alpha \geq 0$ such that $\mathcal{I}_{(0, z, y)}^n = \mathcal{I}_{(z, y)}^n$ verifies $\mathcal{L}(\alpha)$ for (z, y) in a neighborhood of the origin.

Proof of (2.3). — We proceed by induction on the height k of the ideal $\mathcal{I}_{(0, 0)}$. There exist sheafs of ideals $\mathcal{P}^1, \dots, \mathcal{P}^r$, analytic coherent on a neighborhood of the origin of $\mathbf{R}^n \times \mathbf{R}^p$, such that $\mathcal{P}_{(0, 0)}^1, \dots, \mathcal{P}_{(0, 0)}^r$ are prime ideals of height $\geq k$, and an integer $\beta \geq 1$, such that:

$$\mathcal{I} \supset (\mathcal{P}^1 \cap \dots \cap \mathcal{P}^r)^\beta.$$

Clearly, if $\mathcal{P}_{(0, y)}^i$ verifies $\mathcal{L}(\alpha_i)$ for y small enough, $\mathcal{I}_{(0, y)}$ verifies $\mathcal{L}(\beta \sum_{i=1}^r \alpha_i)$ for y small enough. Hence, we may suppose that $\mathcal{I}_{(0, 0)}$ is prime and its height equals k .

Let $\varphi(y)$ be analytic in a neighborhood of the origin of $0 \times \mathbf{R}^p$ and null in $V(\mathcal{I}) \cap (0 \times \mathbf{R}^p)$ in a neighborhood of the origin. Let \mathcal{J} be the analytic coherent sheaf on a neighborhood of the origin of $\mathbf{R}^n \times \mathbf{R}^p$, generated by \mathcal{I} and φ : obviously, $\mathcal{I}_{(0, y)}^n = \mathcal{J}_{(0, y)}^n$ for y small enough. If $\varphi \notin \mathcal{I}_{(0, 0)}$, we get $\text{ht } \mathcal{I}_{(0, 0)} > k$ and hence the result is proved by the induction hypothesis. Therefore, we may suppose that $\varphi \in \mathcal{I}_{(0, 0)}$, i.e. $\mathcal{I}_{(0, 0)} \supset \mathcal{I}_{(0, 0)}^p$ and $\mathcal{I}_{(0, 0)}^p$ is the ideal of germs $\varphi(y)$ null in $V(\mathcal{I}) \cap (0 \times \mathbf{R}^p)$.

Lemma (2.4). — *With the preceding hypothesis, let $k - \ell$ be the height of the prime ideal $\mathcal{I}_{(0, 0)}^p$. After an eventual permutation on the coordinates x_1, \dots, x_n , there exist*

$$\varphi_1, \dots, \varphi_\ell \in \mathcal{I}_{(0, 0)} \text{ such that } \xi_1 = \frac{D(\varphi_1, \dots, \varphi_\ell)}{D(x_1, \dots, x_\ell)} \notin \mathcal{I}_{(0, 0)}.$$

Proof. — We proceed by induction on the height k of $\mathcal{I}_{(0, 0)}$. Let us suppose that $k > \ell$. There is a sequence $(0, y^i) \in V(\mathcal{I})$, $y^i \rightarrow 0$, such that for each i : $\mathcal{I}_{(0, y^i)}^n \neq 0$ (otherwise $\mathcal{I}_{(0, 0)}$ would be generated by $\mathcal{I}_{(0, 0)}^p$). After an eventual linear change of coordinates on the variables x_1, \dots, x_n , we know (following the analytic preparation theorem, Malgrange [1]) that there exists, for each i , a distinguished polynomial $\Psi_i = x_1^{q_i} + a_{1, i}(x', y) \cdot x_1^{q_i - 1} + \dots + a_{q_i, i}(x', y) \in \mathcal{I}_{(0, y^i)}$ (we write $x' = (x_2, \dots, x_n)$ and the $a_{j, i}$

are analytic functions of (x', y) in a neighborhood of $(0, y^i)$. Besides, we may suppose that $\frac{\partial \Psi_i}{\partial x_1} \notin \mathcal{I}_{(0, y^i)}$. (Indeed, there exists a smaller integer $\beta_i \geq 0$ such that $\frac{\partial^{\beta_i+1} \Psi_i}{\partial x_1^{\beta_i+1}} \notin \mathcal{I}_{(0, y^i)}$; we have only to substitute $\frac{\partial^{\beta_i} \Psi_i}{\partial x_1^{\beta_i}}$ for Ψ_i .) Hence, there exists $\varphi_1 \in \mathcal{I}_{(0, 0)}$ such that $\frac{\partial \varphi_1}{\partial x_1} \notin \mathcal{I}_{(0, 0)}$.

Let \mathcal{O}' be the sheaf of germs of analytic functions with real values on $\mathbf{R}^{n-1} \times \mathbf{R}^p = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^p \mid x_1 = 0\}$ and let us write $\mathcal{I}' = \mathcal{I} \cap \mathcal{O}'$. There exists an integer i_0 such that $\text{ht } \mathcal{I}_{(0, y^i)} = k$ for $i \geq i_0$; besides, $\mathcal{O}_{(0, y^i)} |_{\mathcal{I}_{(0, y^i)}}$ is a finitely generated module over $\mathcal{O}'_{(0, y^i)} |_{\mathcal{I}'_{(0, y^i)}}$ and hence their Krull dimensions are equal (by the Cohen-Seidenberg theorem, Malgrange [1]; th. (5.3), chap. III); therefore $\text{ht } \mathcal{I}'_{(0, y^i)} = k - 1$ for $i \geq i_0$. Since $\mathcal{I}'_{(0, 0)}$ is prime, $\text{ht } \mathcal{I}'_{(0, 0)} = \text{ht } \mathcal{I}'_{(0, y^i)}$ for i large enough, so that: $\text{ht } \mathcal{I}'_{(0, 0)} = k - 1$; finally, $\mathcal{I}'_{(0, 0)} \supset \mathcal{I}_{(0, 0)}$. Applying the induction hypothesis to the sheaf \mathcal{I}' (after an eventual permutation on the variables x_2, \dots, x_n), we see that there exist $\varphi_2, \dots, \varphi_\ell \in \mathcal{I}'_{(0, 0)}$ such that $\frac{D(\varphi_2, \dots, \varphi_\ell)}{D(x_2, \dots, x_\ell)} \notin \mathcal{I}'_{(0, 0)}$. Hence:

$$\frac{D(\varphi_1, \dots, \varphi_\ell)}{D(x_1, \dots, x_\ell)} = \frac{\partial \varphi_1}{\partial x_1} \cdot \frac{D(\varphi_2, \dots, \varphi_\ell)}{D(x_2, \dots, x_\ell)} \notin \mathcal{I}_{(0, 0)}.$$

Since $\text{ht } \mathcal{I}_{(0, 0)}^p = k - \ell$ and $\mathcal{I}_{(0, 0)}$ is prime, there exist $\varphi_{\ell+1}, \dots, \varphi_k \in \mathcal{I}_{(0, 0)}^p$ such that, after an eventual permutation on the coordinates y_1, \dots, y_p :

$$\xi_2 = \frac{D(\varphi_{\ell+1}, \dots, \varphi_k)}{D(y_1, \dots, y_{k-\ell})} \notin \mathcal{I}_{(0, 0)}^p; \quad \text{hence} \quad \frac{D(\varphi_1, \dots, \varphi_k)}{D(x_1, \dots, x_\ell, y_1, \dots, y_{k-\ell})} = \xi_1 \cdot \xi_2 \notin \mathcal{I}_{(0, 0)}.$$

By the jacobian criterion for regular points, the localized ring $(\mathcal{O}_{(0, 0)})_{\mathcal{I}_{(0, 0)}}$ is regular of dimension k and its maximal ideal is generated by $\varphi_1, \dots, \varphi_k$. Hence there exists $\xi_3 \in \mathcal{O}_{(0, 0)} \setminus \mathcal{I}_{(0, 0)}$ such that: $\xi_3 \cdot \mathcal{I}_{(0, 0)} \subset (\varphi_1, \dots, \varphi_k)$.

Let ξ be analytic in a neighborhood of $(0, 0) \in \mathbf{R}^{n+p}$ and inducing the germ $\xi_1 \cdot \xi_2 \cdot \xi_3$ at the origin. Let \mathcal{I} be the sheaf of ideals generated, on a neighborhood of the origin of \mathbf{R}^{n+p} , by \mathcal{I} and ξ . For $(0, y) \in V(\mathcal{I})$, y small enough:

— There exists $\alpha \geq 0$ such that $\mathcal{I}_{(0, y)}^n = \mathcal{I}_{(0, y)} + \xi \cdot \mathcal{O}_{(0, y)}^n$ verifies $\mathcal{L}(\alpha)$ (because $\text{ht } \mathcal{I}_{(0, 0)} > \text{ht } \mathcal{I}_{(0, 0)}$ and we apply the induction hypothesis).

— $\xi \cdot \mathcal{I}_{(0, y)}^n$ is contained in the sub-ideal of $\mathcal{I}_{(0, y)}^n$ generated by $\varphi_1, \dots, \varphi_\ell$.

— Finally, ξ belongs to the ideal generated in $\mathcal{O}_{(0, y)}^n$ by the jacobian $\frac{D(\varphi_1, \dots, \varphi_\ell)}{D(x_1, \dots, x_\ell)}$.

So Theorem (2.3) is an immediate consequence of the following lemma (Tougeron and Merrien [2], prop. 3, chap. II):

Lemma (2.5). — *Let \mathfrak{I} be a finitely generated ideal of the ring \mathcal{E}_n of germs at the origin in \mathbf{R}^n of \mathbf{C}^∞ functions with real values. Let $\varphi_1, \dots, \varphi_\ell \in \mathfrak{I}$ and ξ belonging to the ideal generated*

in \mathcal{E}_n by $\varphi_1, \dots, \varphi_\ell$ and all the jacobians $\frac{D(\varphi_1, \dots, \varphi_\ell)}{D(x_{i_1}, \dots, x_{i_\ell})}$, so that $\xi \cdot \mathfrak{J} \subset (\varphi_1, \dots, \varphi_\ell)$. Then if $\mathfrak{J} = \mathfrak{J} + \xi \cdot \mathcal{E}_n$ verifies $\mathcal{L}(\alpha)$, the ideal \mathfrak{J} verifies $\mathcal{L}(\sup(2\alpha, \alpha + 1))$.

Remark (2.6). — Let $\Phi = (\Phi_1, \dots, \Phi_p)$ be a C^∞ mapping from Ω_n to Ω_p . Let \mathcal{E} be the sheaf of C^∞ functions with real values on Ω_n (or Ω_p); let \mathcal{I} be the sheaf of ideals generated on Ω_n by all the jacobians $\frac{D(\Phi_1, \dots, \Phi_p)}{D(x_{i_1}, \dots, x_{i_p})}$: the set $V(\mathcal{I})$ of zeros of \mathcal{I} is the set of singular points of the mapping Φ .

Let us consider the following condition:

(H') $\forall x \in V(\mathcal{I}), \mathcal{E}_x/\mathcal{I}_x$ is (by Φ) a module of finite type over the ring \mathcal{E}_y (we set $y = \Phi(x)$), i.e. by the Malgrange preparation theorem (Malgrange [1]):

$$(\mathcal{E}_x/\mathcal{I}_x) \otimes_{\mathcal{E}_y} (\mathcal{E}_y/\mathfrak{m}_y) = \mathcal{E}_x/(\mathcal{I}_x + \mathfrak{m}_y \cdot \mathcal{E}_x)$$

is a real vector space of finite dimension (\mathfrak{m}_y : maximal ideal of \mathcal{E}_y).

The condition (H') is a very strong one; nevertheless, it is a generic one, i.e. it is verified on an open dense subset of the space of C^∞ mappings from Ω_n to Ω_p , this space being provided with the Whitney topology. Besides, (H') implies (H).

Indeed, let X and Y be compact sets in Ω_n and Ω_p respectively. By hypothesis, there exists an $\alpha \geq 0$ such that, $\forall (x^0, y^0) \in X \times Y$, the ideal generated by $\Phi_1(x) - y_1, \dots, \Phi_p(x) - y_p$ and all the jacobians $\frac{D(\Phi_1, \dots, \Phi_p)}{D(x_{i_1}, \dots, x_{i_p})}$ in $\mathcal{E}_{(x^0, y^0)}^n$ (ring of germs at (x^0, y^0) in $\mathbf{R}^n \times \{y^0\}$ of C^∞ functions with real values), verifies $\mathcal{L}(\alpha)$. By Lemma (2.5), the ideal generated by $\Phi_1(x) - y_1, \dots, \Phi_p(x) - y_p$ in $\mathcal{E}_{(x^0, y^0)}^n$ verifies $\mathcal{L}(\alpha')$, with an α' independent of the point $(x^0, y^0) \in X \times Y$. Clearly, the condition (H) follows.

3. Proof of theorem 1.1.

With the notations of § 1, we must show that: $\overline{\Psi(M)} = \widehat{\Psi(M)}$.

(3.1) We have: $\overline{\Psi(M)} \subset \widehat{\Psi(M)}$.

Let $F \in \overline{\Psi(M)}$ and let $y \in \Omega_p$. A finite subset X_m of $\Phi^{-1}(y)$ will be called *m-essential* (m is a positive integer), if $\ker \Psi_{\Phi^{-1}(y)}^m = \ker \Psi_{X_m}^m$; clearly, there always exist *m-essential* sets X_m such that $\text{card } X_m \leq \text{card}(\mathcal{F}_y^m)^q$.

Let X be a finite subset of $\Phi^{-1}(y)$ containing such an X_m . By hypothesis, $T_X^m F$ is in the closure of the finite dimensional real space $\Psi_X^m(T_y^m M)$, and therefore belongs to it. So there exist G^m with $G_X^m \in T_y^m M$ such that: $T_X^m F = \Psi_X^m(G_X^m)$ and $T_{X_m}^m F = \Psi_{X_m}^m(G^m)$. Obviously, $G^m - G_X^m \in \ker \Psi_{X_m}^m = \ker \Psi_X^m$; thus: $T_X^m F = \Psi_X^m(G^m)$, and X being arbitrary: $T_{\Phi^{-1}(y)}^m F = \Psi_{\Phi^{-1}(y)}^m(G^m)$.

So, $W^m = (\Psi_{\Phi^{-1}(y)}^m)^{-1}(T_{\Phi^{-1}(y)}^m F) \cap T_y^m M$ is a finite dimensional and non empty affine space. The inverse limit $W = \varprojlim W^m$ is then non empty and contained in $\varprojlim T_y^m M = T_y M$; besides, $T_{\Phi^{-1}(y)} F = \varprojlim T_{\Phi^{-1}(y)}^m F \in \Psi_{\Phi^{-1}(y)}(W)$; hence, we have (3.1).

(3.2) We have $\widehat{\Psi(\mathbf{M})} \subset \overline{\Psi(\mathbf{M})}$.

Let $F \in \widehat{\Psi(\mathbf{M})}$ and let X' be a compact subset of Ω_n . Let X be a compact neighborhood of X' in Ω_n and let us put $Y = \Phi(X)$ and $\Phi_0 = \Phi|_{\overset{\circ}{X}}$. Finally let ε be a number > 0 and μ be a positive integer. We have only to prove the following result:

(3.3) There exist $g \in \mathcal{E}(\Omega_p)$ with $g = 1$ in a neighborhood of Y , and $G \in \mathbf{M}$, such that: $|\Phi^*(g)F - \Psi(G)|_{\mu}^{X'} < \varepsilon$.

This easily results from two lemmas. We first give a definition:

Definition (3.4). — A subset K of Y is (α, m) -elementary if the following conditions are verified:

1) There exists a constant $C > 0$ such that, $\forall x \in X$ and $\forall y \in K$:

$$|\Phi(x) - y| \geq C \cdot d(x, \Phi^{-1}(y))^\alpha.$$

2) The dimension of the real vector space $\Psi_{\Phi^{-1}(y)}^m(\mathbf{T}_y^m \mathbf{M})$ is constant, for $y \in K$.

Lemma (3.5). — Let us suppose that Φ verifies the condition (H) and let Z be a compact and non empty subset of Y . Then, there exists a closed set $E(Z) \subsetneq Z$ such that each compact set in $Z - E(Z)$ is (α, m) -elementary (m is an arbitrary integer, but α is the real number associated to X and Y by the condition (H)).

Proof. — With the notations of (1.1), the function: $Y \ni y \mapsto \Gamma(y)$ is lower semi-continuous (because, for a fixed x , the mapping $Y \ni y \mapsto d(x, \Phi^{-1}(y))$ is lower semi-continuous). So there exists an open dense set Z_0 in Z , such that this function is bounded on each compact set in Z_0 .

Let $y^0 \in Z_0$: if x^0 belongs to the fiber $\Phi_0^{-1}(y^0)$, we have: $\lim_{\substack{y \rightarrow y^0 \\ y \in Z_0}} d(x^0, \Phi_0^{-1}(y)) = 0$.

(Indeed, by hypothesis, there exists a constant $C > 0$ such that, for each $y \in Z_0$ in a neighborhood of y^0 , we have $|y^0 - y| \geq C \cdot d(x^0, \Phi^{-1}(y))^\alpha$.)

Let $X(y^0) = \{x^1(y^0), \dots, x^s(y^0)\}$ be an m -essential subset of the fiber $\Phi_0^{-1}(y^0)$ for $y^0 \in Z_0$. We can associate to each $y \in Z_0$ a subset $X(y) = \{x^1(y), \dots, x^s(y)\}$ of $\Phi_0^{-1}(y)$, so that $\lim_{y \rightarrow y^0} x^i(y) = x^i(y^0)$ for $i = 1, \dots, s$. Clearly, we have the following inequalities, for $|y - y^0|$ small enough:

$$\dim_{\mathbf{R}} \Psi_{\Phi_0^{-1}(y)}^m(\mathbf{T}_y^m \mathbf{M}) \geq \dim_{\mathbf{R}} \Psi_{X(y)}^m(\mathbf{T}_y^m \mathbf{M}) \geq \dim_{\mathbf{R}} \Psi_{X(y^0)}^m(\mathbf{T}_{y^0}^m \mathbf{M}) = \dim_{\mathbf{R}} \Psi_{\Phi_0^{-1}(y^0)}^m(\mathbf{T}_{y^0}^m \mathbf{M}).$$

So the function $Z_0 \ni y \mapsto \dim_{\mathbf{R}} \Psi_{\Phi_0^{-1}(y)}^m(\mathbf{T}_y^m \mathbf{M})$ is lower semi-continuous, bounded with integer values. Therefore, there exists an open and non empty subset Z_1 of Z_0 in which this function is constant. Then it suffices to put $E(Z) = Z - Z_1$.

Lemma (3.6). — Let K be a compact and (α, m) -elementary subset of Y , and let us suppose that $m \geq \mu\alpha$. Then we can find $g \in \mathcal{E}(\Omega_p)$ with $g = 1$ in a neighborhood of K , and $G \in \mathbf{M}$, such that:

$$|\Phi^*(g)F - \Psi(G)|_{\mu}^{X'} < \varepsilon.$$

Proof. — The following proof takes inspiration from the proof of the spectral theorem (B. Malgrange [1], lemma (1.4), chap. II).

Let $y^0 \in K$. By hypothesis, there exists a neighborhood V_{y^0} of y^0 and G_1, \dots, G_k in M such that for $y \in V_{y^0} \cap K$, $\Psi_{\Phi_0^{-1}(y)}^m(T_y^m G_1), \dots, \Psi_{\Phi_0^{-1}(y)}^m(T_y^m G_k)$ is a basis of the real vector space $\Psi_{\Phi_0^{-1}(y)}^m(T_y^m M)$. Hence there exist continuous functions $\lambda_1, \dots, \lambda_k$ on $V_{y^0} \cap K$, such that:

$$T_{\Phi_0^{-1}(y)}^m F = \Psi_{\Phi_0^{-1}(y)}^m \left(\sum_{i=1}^k \lambda_i(y) \cdot T_y^m G_i \right)$$

for all $y \in V_{y^0} \cap K$. Using a partition of unity, we can find $G_1, \dots, G_\ell \in M$, continuous functions $\lambda_1, \dots, \lambda_\ell$ on K , and a constant C , such that, for all $y \in K$:

$$T_{\Phi_0^{-1}(y)}^m F = \Psi_{\Phi_0^{-1}(y)}^m \left(\sum_{i=1}^{\ell} \lambda_i(y) \cdot T_y^m G_i \right)$$

and

$$\sup_{\substack{1 \leq i \leq \ell \\ y \in K}} |\lambda_i(y)| \leq C.$$

Let us put $G_y = \sum_{i=1}^{\ell} \lambda_i(y) G_i$; clearly, $F - \Psi(G_y)$ is m -flat on $\overline{\Phi_0^{-1}(y)}$. Let ω be a modulus of continuity on the compact set X for $F, \Psi(G_1), \dots, \Psi(G_\ell)$: there exists a constant $C_1 > 0$ such that $C_1 \cdot \omega$ is a modulus of continuity on X for all functions $F - \Psi(G_y), y \in K$.

Let $x \in X'$ and $a \in \overline{\Phi_0^{-1}(y)}$ such that $d(x, \Phi_0^{-1}(y)) = d(x, a)$. The function $F - \Psi(G_y)$ being m -flat at a , we have:

$$|D^k F(x) - D^k \Psi(G_y)(x)| = |(R_a^m(F - \Psi(G_y)))^k(x)| \leq C_1 \cdot d(x, \Phi_0^{-1}(y))^{m-|k|} \cdot \omega(d(x, \Phi_0^{-1}(y))).$$

Clearly, there exists a constant C'_1 such that $d(x, \Phi_0^{-1}(y)) \leq C'_1 \cdot d(x, \Phi^{-1}(y))$ for all $x \in X'$ and $y \in K$. Hence, the compact K being (α, m) -elementary and $m \geq \mu\alpha$, we see that there exist a constant C_2 and a modulus of continuity ω' such that:

$$(3.6.1) \quad |D^k F(x) - D^k \Psi(G_y)(x)| \leq C_2 |\Phi(x) - y|^{\mu-|k|} \cdot \omega'(|\Phi(x) - y|)$$

for all n -integers k such that $|k| \leq \mu$, all $x \in X'$ and all $y \in K$.

Let d be a real number > 0 . The open cubes of side $2d$, centered at the points $(j_1 d, \dots, j_p d)$ (j_1, \dots, j_p are integers) constitute an open covering \mathfrak{S} of \mathbf{R}^p . Let $g_i (i \in \mathfrak{S})$ be a partition of unity subordinate to \mathfrak{S} such that, for $|k| \leq \mu$,

$$(3.6.2) \quad \sum_{i \in \mathfrak{S}} |D^k g_i(y)| \leq \frac{C_3}{d^{|k|}} \quad \text{for all } y \in \mathbf{R}^p$$

(C_3 is a constant only depending on μ and p). Let \mathfrak{S}' be the finite family of those cubes L in \mathfrak{S} which meet K . For $L \in \mathfrak{S}'$, let y_L be a point in $L \cap K$. Let us put:

$$g = \sum_{L \in \mathfrak{S}'} g_L, \quad G = \sum_{L \in \mathfrak{S}'} g_L \cdot G_{y_L}.$$

Obviously, $g=1$ in a neighborhood of K and:

$$|\Phi^*(g)F - \Psi(G)|_{\mu}^{X'} \leq \sum_{L \in \mathcal{S}'} \sup_{\substack{x \in X' \\ |k| \leq \mu}} |D^k(\Phi^*(g_L)(F - \Psi(G_{y_L})))|_x|$$

and so, by Leibniz's formula and (3.6.1), (3.6.2):

$$|\Phi^*(g)F - \Psi(G)|_{\mu}^{X'} \leq C_4 \cdot \omega'(d)$$

where C_4 is independent of d . Hence if we choose d sufficiently small, the lemma follows.

Proof of (3.3). — First let us decompose the compact set Y with the help of Lemma (3.5). Let α be the real number associated to X and Y by the condition (H) and let m be an integer $\geq \mu\alpha$.

Let T be a well ordered set. We construct, by transfinite induction, a mapping $T \ni \tau \mapsto Y_{\tau}$ with values in the set of compact subsets of Y . If 1 denotes the first element of T , we put $Y_1 = Y$. Suppose the mapping is defined in the interval $[1, \tau_1[$: we put $Y_{\tau_1} = \bigcap_{\tau < \tau_1} Y_{\tau}$, if τ_1 has no predecessor; on the other hand, if $\tau_1 = \tau + 1$, we put: $Y_{\tau+1} = E(Y_{\tau})$ if $Y_{\tau} \neq \emptyset$ and $Y_{\tau+1} = \emptyset$ if $Y_{\tau} = \emptyset$.

If the cardinal of T is sufficiently large, there exist some τ such that $Y_{\tau} = \emptyset$. Let ν_1 be the smallest element τ of T such that $Y_{\tau} = \emptyset$: we have $\nu_1 = \nu + 1$ for a $\nu \in T$ (otherwise, we should have $\bigcap_{\tau < \nu_1} Y_{\tau} = \emptyset$, which is absurd, because the Y_{τ} , $\tau < \nu_1$, are compact and non empty sets such that $Y_{\tau+1} \subset Y_{\tau}$ for each τ). Let us consider the following assertion:

(H_{τ}) *There exist g_{τ} in $\mathcal{E}(\Omega_p)$ with $g_{\tau}=1$ in a neighborhood V_{τ} of Y_{τ} , and G_{τ} in M , such that $|\Phi^*(g_{\tau})F - \Psi(G_{\tau})|_{\mu}^{X'} < \varepsilon$.*

The set of all τ such that (H_{τ}) is true is non empty: Indeed, by (3.6), it contains ν (because Y_{ν} is a compact and (α, m) -elementary set). Let τ_1 be the smallest element of this set: we have to show that $\tau_1 = 1$.

Indeed, suppose that $\tau_1 > 1$. Necessarily, $\tau_1 = \tau + 1$ for an element $\tau \in T$ (otherwise, we should have $Y_{\tau_1} = \bigcap_{\tau < \tau_1} Y_{\tau}$ and therefore $Y_{\tau} \subset V_{\tau_1}$, hence (H_{τ}), for a $\tau < \tau_1$, which is absurd).

We have $|\Phi^*(g_{\tau_1})F - \Psi(G_{\tau_1})|_{\mu}^{X'} \leq \varepsilon' < \varepsilon$, with $g_{\tau_1} = 1$ in an open neighborhood V_{τ_1} of Y_{τ_1} . Let us put $K = Y_{\tau} - V_{\tau_1}$: K is a compact and (α, m) -elementary subset of Ω_p . By (3.6), applied to $\Phi^*(1 - g_{\tau_1})F$ instead of F , there exist $h \in \mathcal{E}(\Omega_p)$ with $h = 1$ in a neighborhood of K , and $G \in M$, such that:

$$|\Phi^*(h(1 - g_{\tau_1})) \cdot F - \Psi(G)|_{\mu}^{X'} < \varepsilon - \varepsilon'.$$

Let us put $g_{\tau} = g_{\tau_1} + h - h \cdot g_{\tau_1}$ and $G_{\tau} = G + G_{\tau_1}$. Clearly, $g_{\tau} \in \mathcal{E}(\Omega_p)$, $g_{\tau} = 1$ in a neighborhood of Y_{τ} , $G_{\tau} \in M$ and $|\Phi^*(g_{\tau}) \cdot F - \Psi(G_{\tau})|_{\mu}^{X'} < \varepsilon$. Hence condition (H_{τ}) is fulfilled, which is absurd.

Remark (3.7). — I do not know if Theorem (1.1) is always true without the hypothesis (H): unfortunately, I have no counter-example.

4. A refinement of theorem 1.2 when Φ is polynomial.

Let us recall the following definition: a set in \mathbf{R}^n is *semi-algebraic* if it is a finite union of subsets X_i , each X_i being defined by a finite number of polynomial equalities or inequalities.

The image of a semi-algebraic set by a polynomial mapping $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^p$ is semi-algebraic (this is a fundamental result of Seidenberg and Tarski, cf. [3]); if X and Y are semi-algebraic sets in \mathbf{R}^n and if $X \subset Y$, the closure of X in Y and $Y \setminus X$ are semi-algebraic. Finally, it is obvious that finite unions or finite intersections of semi-algebraic sets are semi-algebraic.

Let Φ be a polynomial mapping from $\Omega_n = \mathbf{R}^n$ to $\Omega_p = \mathbf{R}^p$ and let X and Y be compact and semi-algebraic sets in \mathbf{R}^n and $\Phi(\mathbf{R}^n)$ respectively. The following theorem improves (1.2):

Theorem (4.1). — There exists a closed and semi-algebraic set $D(Y)$ in Y , such that $Y \setminus D(Y)$ is dense in Y , and constants $C > 0$, $\alpha \geq 0$, $\beta \geq 0$ such that, for all $x \in X$ and $y \in Y$:

$$(4.1.1) \quad |\Phi(x) - y| \geq C \cdot d(x, \Phi^{-1}(y))^\alpha \cdot d(y, D(Y))^\beta.$$

Proof. — By (1.2), there exists an $\alpha \geq 0$ (we suppose that α is an integer, which is always possible) such that, $\forall y \in Y$:

$$\Gamma(y) = \sup_{x \in X \setminus \Phi^{-1}(y)} (d(x, \Phi^{-1}(y))^\alpha / |\Phi(x) - y|) < \infty.$$

Let us put

$$D(Y) = \{y \in Y \mid \Gamma \text{ is not bounded in every neighborhood of } y\}.$$

Clearly, $D(Y)$ is closed and $Y \setminus D(Y)$ is dense in Y (because the mapping $Y \ni y \mapsto \Gamma(y)$ is lower semi-continuous). Let us verify that $D(Y)$ is semi-algebraic.

First, the set

$$A_1 = \{(x, y, \tau) \in X \times Y \times \mathbf{R}^+ \mid |\Phi(x) - y| > \tau \cdot d(x, \Phi^{-1}(y))^\alpha\}$$

is semi-algebraic. Indeed, A_1 is the image of the semi-algebraic set

$$A_0 = \{(x, x', y, \tau) \in X \times \mathbf{R}^n \times Y \times \mathbf{R}^+ \mid \Phi(x') = y \text{ and } |\Phi(x) - y| > \tau \cdot |x - x'|^\alpha\}$$

by the projection: $X \times \mathbf{R}^n \times Y \times \mathbf{R}^+ \rightarrow X \times Y \times \mathbf{R}^+$. Now the set

$$A_2 = \{(y, \tau) \in Y \times \mathbf{R}^+ \mid \exists x \in X \text{ such that } |\Phi(x) - y| \leq \tau \cdot d(x, \Phi^{-1}(y))^\alpha\}$$

is semi-algebraic, because it is the image of $(X \times Y \times \mathbf{R}^+) \setminus A_1$ by the projection: $X \times Y \times \mathbf{R}^+ \rightarrow Y \times \mathbf{R}^+$. Clearly, we have

$$D(Y) \times \{0\} = \overline{A_2} \cap Y \times \{0\},$$

and therefore $D(Y)$ is semi-algebraic.

Let us prove inequality (4.1.1) (the proof is similar to that of Lemma 1 in [4]).

Let us put:

$$B_1 = \{(y, \delta, \tau) \in Y \times \mathbf{R}^+ \times \mathbf{R}^+ \mid d(y, D(Y)) \geq \delta\}$$

$$B_2 = \{(y, \delta, \tau) \in B_1 \mid \forall x \in X, |\Phi(x) - y| > \tau d(x, \Phi^{-1}(y))^\alpha\}.$$

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