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MICHAEL F. ARTIN

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ALGEBRAIC APPROXIMATION OF STRUCTURES OVER COMPLETE LOCAL RINGS

by M. ARTIN [1]

Section 1.

In this paper we generalize to dimensions greater than one a theorem of Greenberg [9], and we apply this generalization to some algebraization problems. Some of our results are announced in [2], and an analytic analogue was treated in [3]. Further applications will appear in [4].

Two general types of question which may sometimes be treated by our methods are the following: We let A be a noetherian ring, and $\mathfrak{m}$ be an ideal of A. Denote by $\hat{A}$ the $\mathfrak{m}$-adic completion of A.

Question (**1.1**). — Let $\overline{S}$ be an " algebraic structure " over $\hat{A}$. Can one approximate it by some structure S over A?

Question (**1.2**). — Let S and S′ be " algebraic structures " over A which induce isomorphic structures over $\hat{A}$. Are S and S′ themselves isomorphic?

In order to pose the problems precisely, we will assume that the structure under consideration is classified by a functor

$$(1.3)\qquad F : (\text{A-algebras}) \to (\text{sets}),$$

so that for every A-algebra B, F(B) is the set of isomorphism classes of structures over B. Then an element $\xi \in F(A)$ or $\overline{\xi} \in F(\hat{A})$ induces by functorality an element of $F(A/\mathfrak{m}^c)$ for each c, and we will say that ξ and $\overline{\xi}$ are *congruent modulo* $\mathfrak{m}^c$ if they induce the same element there. Question (1.1) can thus be rephrased as follows:

Question (**1.4**). — Let F be a functor as above, let c be an integer, and let $\overline{\xi} \in F(\hat{A})$. Does there exist an element $\xi \in F(A)$ such that

$$\xi \equiv \overline{\xi} \quad (\text{modulo } \mathfrak{m}^c)\,?$$

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Moreover, it is clear that most questions of type (1.2) can be reduced to (1.4). In fact, denote by S_B the structure induced by S over an A-algebra B, and by I(B) the set of isomorphisms $S_B \approx S'_B$. Assume that I is a functor on A-algebras. Then question (1.2) is the following: Does $I(A) = \emptyset$ imply $I(\hat{A}) = \emptyset$? This is a weak form of (1.4).

It is natural to put some finiteness condition on the functor F under consideration, and the fundamental one is supplied by the following notion of Grothendieck:

Definition **(1.5)**. — *Let* A *be a ring and let* F *be a functor* (1.3). F *is said to be locally of finite presentation if for every filtering inductive system of* A-*algebras* $\{B_i\}$, *the canonical map*

$$\varinjlim F(B_i) \to F(\varinjlim B_i)$$

is bijective.

This condition is an extremely natural one. It holds for nearly all functors which occur in practice, and is often very easy to verify.

Suppose F locally of finite presentation, let B be an A-algebra, and write B as the direct limit of a filtering system of A-algebras of finite presentation:

$$\varinjlim B_i = B.$$

This can always be done. Then we can apply condition (1.5) to an element $\xi \in F(B)$, to conclude that it is induced functorially from a $\xi_i \in F(B_i)$ for some sufficiently large i. Now choose a finite presentation

$$B_i = A[Y]/(f(Y)),$$

where $Y = (Y_1, \ldots, Y_N)$ and $f = (f_1, \ldots, f_m) \in A[Y]$. Then a homomorphism of A-algebras $B_i \to C$ is given by a solution in C of the system of polynomial equations $f(Y) = 0$. Given such a homomorphism, the element $\xi_i \in F(B_i)$ induces an element of F(C). Thus we conclude the following:

Corollary **(1.6)**. — *Let* F *be a functor* (1.3) *locally of finite presentation, let* B *be an* A-*algebra, and let* $\xi \in F(B)$. *There exists*

(i) *A finite system of polynomial equations*

$$f(Y) = 0,$$

where $Y = (Y_1, \ldots, Y_N)$ *and* $f = (f_1, \ldots, f_m) \in A[Y]$.

(ii) *A functorial rule associating to every solution of this system of equations in an* A-*algebra* C *an element of* F(C).

(iii) *A solution of the system* (i) *in* B, *so that the rule* (ii) *applied to this solution yields* ξ.

It is easily seen, conversely, that if a functor has the property (1.6), then the canonical map of (1.5) is surjective; and the case that it is surjective but not injective seems of little interest. Note that the system of equations and the rule are allowed to depend on the pair (B, ξ). This is an important point, for while (1.6) is a rather weak condition, the existence of a fixed system of equations and rule would be a very strong one.

The above corollary shows that a basic question closely related to all of the questions (1.1), (1.2), (1.4) is

Question **(1.7).** — Let $Y=(Y_1, \ldots, Y_N)$ be variables, let $f=(f_1, \ldots, f_m)\in A[Y]$ be polynomials, and suppose given elements $\bar{y}=(\bar{y}_1, \ldots, \bar{y}_N)\in \hat{A}$ which solve the system of polynomial equations

$$(*) \qquad f(Y)=0.$$

Let c be an integer. Does there exist a solution $y=(y_1, \ldots, y_N)\in A$ of $(*)$ such that

$$y_1 \equiv \bar{y}_i \quad (\text{modulo } \mathfrak{m}^c)?$$

Indeed, we have

Corollary **(1.8).** — *Suppose for the given pair* $(A, \mathfrak{m})$, *that the answer to* (1.7) *is affirmative for each system* $(*)$ *of polynomial equations. Then question* (1.4) *has an affirmative answer for every functor* (1.3) *locally of finite presentation.*

For, given $\bar{\xi}\in F(\hat{A})$, we put $\hat{A}=B$ in (1.6). We obtain a system of equations (i) and a solution (iii), say $\bar{y}$, of this system in $\hat{A}$ which yields $\bar{\xi}$ via the rule (ii). It is clear that to find $\xi\in F(A)$ as in (1.4), it suffices to approximate the solution $\bar{y}$ of (i) by a solution $y\in A$ (modulo $\mathfrak{m}^c$), and to apply the rule (ii) to this approximation.

We now ask for conditions under which (1.7) has an affirmative answer. If the system of equations $(*)$ is linear, then it is sufficient that A be a local ring and $\mathfrak{m} \neq A$ (or, more generally, that $\mathfrak{m}$ be in the Jacobson radical of A), as follows immediately from the faithful flatness of $\hat{A}$ over A in that case. Consequently, (1.4) also has an affirmative answer for functors F which are " sufficiently linear ". As example in the form of (1.2), one has the fact that finite modules M, M′ over a local ring A such that $\hat{M}$ and $\hat{M}'$ are isomorphic are themselves isomorphic (EGA IV, (2.5.8)).

But most structures are not described by linear equations, and so it is natural to study these questions locally for the etale topology. This amounts to assuming that A is a *henselian* local ring. We recall that a local ring is henselian if the following analogue of the implicit function theorem holds (this is roughly EGA IV, (18.5.11), *b*)):

(1.9). — *Let*

$$f(Y)=(f_1(Y), \ldots, f_N(Y))\in A[Y]$$

be polynomials, where $Y=(Y_1, \ldots, Y_N)$. *Let* $y^\circ=(y_1^\circ, \ldots, y_N^\circ)\in k=A/\mathfrak{m}$ *be elements such that*

$$f^\circ(y^\circ)=0$$

and that

$$\det\left(\frac{\partial f_i}{\partial Y_j}\right)^\circ (y^\circ)\neq 0,$$

the symbol $^\circ$ *denoting residue modulo* $\mathfrak{m}$. *Then there are elements* $y=(y_1, \ldots, y_N)\in A$, *with* $y_i\equiv y_i^\circ$ (modulo $\mathfrak{m}$), *such that*

$$f(y)=0.$$

Here $\mathfrak{m}$ denotes the maximal ideal of A. But in fact, when A is a henselian local ring, the maximal ideal may be replaced by any proper ideal.

The henselian condition does not suffice for the general question (1.7). For there exists a henselian discrete valuation ring A whose completion $\hat{A}$ is a purely inseparable algebraic extension ([15], p. 205, ex. 3). Hence there are polynomial equations in this ring A having solutions in $\hat{A}$ but not in A itself. It seems reasonable to conjecture, however, that (1.7) has an affirmative answer when A is an *excellent* (EGA IV, (7.8.3)) hensel ring. This was proved for discrete valuation rings by Greenberg [9] and Raynaud. Various other papers should also be mentioned which treat more special cases ([6], [10], [16], [17]).

Our main result is the following answer to question (1.7):

Theorem **(1.10)**. — *Let* R *be a field or an excellent discrete valuation ring, and let* A *be the henselization* (EGA IV, (18.6)) *of an* R*-algebra of finite type at a prime ideal. Let* $\mathfrak{m}$ *be a proper ideal of* A. *Given an arbitrary system of polynomial equations*

$$f(Y) = 0 \qquad Y = (Y_1, \ldots, Y_N) \tag{1.11}$$

with coefficients in A, *a solution* $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)$ *in the* $\mathfrak{m}$*-adic completion* $\hat{A}$ *of* A, *and an integer* c, *there exists a solution* $y = (y_1, \ldots, y_N) \in A$ *with*

$$y_i \equiv \bar{y}_i \quad (\text{modulo } \mathfrak{m}^c).$$

Applying (1.8), we obtain an affirmative response to question (1.4):

Theorem **(1.12)**. — *With the assumptions of* (1.10), *let* F *be a functor* (1.3) *which is locally of finite presentation. Given any* $\bar{\xi} \in F(\hat{A})$, *there is a* $\xi \in F(A)$ *such that*

$$\xi \equiv \bar{\xi} \quad (\text{modulo } \mathfrak{m}^c).$$

The proof of theorem (1.10) is given in section 5. We use an inductive procedure which is similar to the proof of the analytic analogue [3], except that in the present situation we have to control divisibility by the relevant prime ideal p of R. To do this we have adapted the desingularization theory of Néron [17]. It is reviewed for our set-up in section 4.

Greenberg's result [9] is more precise than (1.10). It asserts the existence of a solution y when $\bar{y}$ is merely a sufficiently near approximate solution. Perhaps an analysis of our method would give this too. Using results of Hermann [12] and Stolzenberg [19], we have carried out such an analysis in section 6, for the case that R is a field.

Another direction in which our result might be improved is towards a more global version. For instance, if $\mathfrak{m}$ is an ideal of a ring A of finite type over R, one can study the corresponding questions in an etale neighborhood of Spec $A/\mathfrak{m}$ in Spec A. We do not know any counterexample to (1.7) in this context.

2. Variant assertions.

This section contains some more or less routine translations of (1.10) and (1.12) into local statements about schemes. In order not to overload the notation, we will carry such translations out only in the case that the ideal $\mathfrak{m}$ is the maximal ideal at the point in question. Throughout the section, S will denote a scheme which is of finite type over a field or over an excellent dedekind domain.

Let s be a point of S. By *etale neighborhood* of s in S we mean an etale map $S' \to S$ together with a rational lifting of s to S':

$$\begin{array}{ccc} s = \operatorname{Spec} k(s) & \longrightarrow & S' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

We will often use the symbol S' to stand for such an etale neighborhood as well as for the underlying scheme.

The henselization $\widetilde{\mathcal{O}}_{S,s}$ of the local ring of S at s is the limit of the rings $\Gamma(S', \mathcal{O}_{S'})$ as S' runs over the (filtering) category of etale neighborhoods. Thus theorem (1.10) translates immediately as

Corollary **(2.1)**. — *Let* $Y = (Y_1, \ldots, Y_N)$ *be variables, and let* $f = (f_1, \ldots, f_m) \in \mathcal{O}_S[Y]$ *be polynomials whose coefficients are global sections of* $\mathcal{O}_S$. *Let* $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_N)$ *be a solution of the system of equations*

$$f(Y) = 0$$

in the complete local ring $\hat{\mathcal{O}}_{S,s}$, *and let* c *be an integer. There exists an etale neighborhood* S' *of* s *in* S, *and a solution* $y = (y_1, \ldots, y_N) \in \Gamma(S', \mathcal{O}_{S'})$ *of the system of equations, such that*

$$y \equiv \overline{y} \quad (\text{modulo } \mathfrak{m}_s^c).$$

Similarly, (1.11) reads

Corollary **(2.2)**. — *Let*

$$F : (\text{schemes}/S)^\circ \to (\text{sets})$$

be a (contravariant) functor locally of finite presentation, and let $\overline{\xi} \in F(\hat{S})$ $(\hat{S} = \operatorname{Spec} \hat{\mathcal{O}}_{S,s})$. *Let* c *be an integer. There is an etale neighborhood* S' *of* s *in* S *and an element* $\xi' \in F(S')$ *such that*

$$\xi' \equiv \overline{\xi} \quad (\text{modulo } \mathfrak{m}_s^c).$$

The congruence notation used is explained in a self-evident way as in section 1. We leave it to the reader to make precise definitions.

A functor F as in (2.2) is said to be locally of finite presentation if it has the property of (1.5) with respect to *affine* S-schemes, viz., let $\{X_i\}$ be a filtering inverse system of $\mathcal{O}_S$-schemes, where $X_i = \operatorname{Spec} B_i$. Then

$$\varinjlim F(X_i) \xrightarrow{\sim} F(\varprojlim X_i).$$

We recall the following fact:

Proposition **(2.3)**. — *Let*

$$\begin{array}{ccc} X & & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

be a diagram of S-*schemes such that* Z *is quasi-compact and quasi-separated, and that* X, Y *are of finite presentation over* Z. *Consider the functor* $Hom_Z(X, Y)$ *which to an* S-*scheme* S′ *associates the set* $\mathrm{Hom}_{Z'}(X', Y')$, *where* $Z' = Z \times_S S'$, *etc. This functor is locally of finite presentation.*

The proposition follows immediately from (EGA IV, (8.8.2)), if one replaces S by Z in that proposition.

One can now combine the assertions (2.2) and (2.3) and specialize them in various ways. We will content ourselves with some illustrative examples:

Corollary **(2.4).** — *Let* X, Y *be* S-*schemes of finite type, and assume* X *proper over* S. *Let* c *be an integer. Suppose given a formal map from* X *to* Y *at* s, i.e., *a compatible system of* S-*maps*

$$\varphi_n : X_n \to Y_n \qquad n = 0, \ldots,$$

where $S_n = \mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1})$, $X_n = X \times_S S_n$, *and* $Y_n = Y \times_S S_n$. *Then there exists an etale neighborhood* S′ *of* s *in* S *and a map*

$$\varphi' : X' \to Y', \qquad X' = X \times_S S', \qquad Y' = Y \times_S S'$$

such that

$$\varphi' \equiv \varphi_c \quad (\text{modulo } \mathfrak{m}_s^{c+1}).$$

For, by (EGA III, (5.4.1)), the formal map φ_n is induced by a map

$$\hat{\varphi} : \hat{X} \to \hat{Y},$$

where ^ denotes the change of base $\hat{S} \to S$ ($\hat{S} = \mathrm{Spec}\, \hat{\mathcal{O}}_{S,s}$). Thus we may set $Z = S$ in (2.3) and apply (2.2).

Corollary **(2.5).** — *Let* X, Y *be* S-*schemes of finite type, and let* x *be a point of* X. *Set* $\hat{X} = \mathrm{Spec}\, \hat{\mathcal{O}}_{X,x}$. *Let* c *be an integer. Given an* S-*map*

$$\bar{\varphi} : \hat{X} \to Y,$$

there is an etale neighborhood X′ *of* x *in* X *and a map*

$$\varphi' : X' \to Y$$

such that

$$\varphi' \equiv \bar{\varphi} \quad (\text{modulo } \mathfrak{m}_x^c).$$

To obtain this corollary, we view $\bar{\varphi}$ as given by its graph, a section of $\hat{X} \times_S Y = \hat{X} \times_X (X \times_S Y)$ over $\hat{X}$. Thus we may apply (2.2) and (2.3) with $X = Z = S$ and with $X \times_S Y$ replacing Y.

Corollary **(2.6).** — *Let* X_1, X_2 *be* S-*schemes of finite type, and let* $x_i \in X_i$ *be points. If the complete local rings* $\hat{\mathcal{O}}_{X_i, x_i}$ $(i = 1, 2)$ *are* $\mathcal{O}_S$-*isomorphic, then* X_1 *and* X_2 *are locally isomorphic for the etale topology. By this we mean that there is a common etale neighborhood* (X', x') *of* x_i *in* X_i, i.e., *a diagram of etale maps*

$$\begin{array}{ccc} & X' & \\ \swarrow & & \searrow \\ X_1 & & X_2 \end{array} \qquad \textit{sending} \qquad \begin{array}{ccc} & x' & \\ \swarrow & & \searrow \\ x_1 & & x_2 \end{array}$$

and inducing isomorphisms of residue fields $\kappa(x_1) \approx \kappa(x') \approx \kappa(x_2)$.

In fact, the isomorphism of the complete local rings yields a map

$$\overline{\varphi} : \hat{X}_1 \rightarrow X_2$$

which can by (2.5) be approximated in an etale neighborhood X′ of x_1 by a map

$$\varphi' : X' \rightarrow X_2,$$

say modulo $\mathfrak{m}^2_{x_1}$. This map φ' induces an isomorphism on the complete local rings $\hat{\mathcal{O}}_{X_2, x_2} \rightarrow \hat{\mathcal{O}}_{X', x'}$ (x' the lifting of x_1) since it agrees with the given isomorphism modulo $\mathfrak{m}^2_{x_1}$ (cf. for instance [3], p. 282). Since φ' is of finite type, it follows that it is an etale map (EGA IV, (17.6.3)). Thus (X', x') is the common etale neighborhood.

For future reference, we include here two lemmas about functors locally of finite presentation:

Lemma **(2.7)**. — *Let* A *be a ring, let* B *be an arbitrary* A-*algebra, and let* G *be a functor locally of finite presentation on* B-*algebras. Define a functor* F *on* A-*algebras by*

$$F(A') = G(A' \otimes_A B).$$

Then F *is locally of finite presentation. An analogous assertion holds for contravariant functors on* B-*schemes.*

This follows immediately from the fact that tensor product commutes with direct limits.

Lemma **(2.8)**. — *Let* A *be a ring, and let* $F \xrightarrow{f} G$ *be a morphism of functors on* A-*algebras, with* G *locally of finite presentation. For an* A-*algebra* A′ *and an element* $\xi \in G(A')$, *denote by* F_ξ *the functor on* A′-*algebras defined by*

$$F_\xi(B') = \{\eta \in F(B') \mid f_{B'}(\eta) = \xi_{B'}\},$$

where $f_{B'} : F(B') \rightarrow G(B')$ *is the map, and where* $\xi_{B'}$ *is induced from* ξ. *If* F_ξ *is locally of finite presentation for every pair* (A', ξ), *then* F *is locally of finite presentation. A similar assertion holds for contravariant functors on* A-*schemes.*

Proof. — Suppose F_ξ locally of finite presentation for all (A', ξ). Let $B = \varinjlim B_i$, where $\{B_i\}$ is a filtering inductive system of A-algebras, and let $\eta \in F(B)$. The element $f(\eta) = \xi \in G(B)$ is induced by a $\xi_i \in G(B_i)$ for suitable i, and η is an element of $F_{\xi_i}(B)$. By assumption, this element is induced by an $\xi_j \in F_{\xi_i}(B_j)$ for suitable j. Thus the map

$$(*) \qquad \varinjlim F(B_i) \rightarrow F(B)$$

is surjective. If ${}_1\eta_i, {}_2\eta_i \in F(B_i)$ have the same image in F(B), then $f({}_1\eta_j) = f({}_2\eta_j) = \xi_j$ in $G(B_j)$ for some j. Hence ${}_1\eta_j$ and ${}_2\eta_j$ are in $F_{\xi_j}(B_j)$, and they represent the same element of $F_{\xi_j}(B)$. Thus they become equal in $F(B_k)$ for some k. This shows injectivity of the map (*).

3. Some applications.

An important application of Theorem (1.12) is to the following result of ([5], exp. XII). Its importance comes from the fact that it is the main tool needed for the proof of the " proper base change theorem " for etale cohomology *(loc. cit.)*. We can now derive it very quickly:

Theorem **(3.1)**. — *Let* S *be the spectrum of a henselian local ring* A, *let* $f: X \to S$ *be a proper, finitely presented map, and let* X_0 *be the closed fibre of* X/S. *Denote by* Et(Z) *the category of schemes finite and etale over a scheme* Z. *The inclusion* $X_0 \to X$ *induces an equivalence of categories*

$$\mathrm{Et}(X) \to \mathrm{Et}(X_0).$$

Proof. — We can write S as a limit of schemes $S_i = \mathrm{Spec}\, A_i$, where A_i is the henselization of a **Z**-algebra of finite type. By (EGA IV, (8.8.2) and (8.10.5)), the proper S-scheme X can be descended to some S_i. Now given a morphism of finite presentation $f: X \to S$ of schemes, the functor associating to an $S' \to S$ the set of isomorphism classes of finite, etale X'-schemes $(X' = X \times_S S')$ is locally of finite presentation. For a finite etale X-scheme is determined by a locally free sheaf $\mathscr{A}$ of $\mathcal{O}_X$-algebras, which is described by a functor locally of finite presentation (EGA IV, 8, in particular, (8.5.2) and (8.5.5)). The supplementary condition on $\mathscr{A}$ to be an etale $\mathcal{O}_X$-algebra is the vanishing of the sheaf of relative differentials, which is a condition locally of finite presentation once $\mathscr{A}$ is given. Hence the assertion follows from (2.8). Similarly, the functor *Hom* of X-maps between two given coverings is locally of finite presentation, by (2.3). This known, one sees immediately that it suffices to prove the theorem in case A is the henselization of a **Z**-algebra of finite type. Thus we may apply Theorem (1.12) to A.

Let X'_0 be a scheme finite and etale over X_0. We want to show that it is induced by an etale X-scheme X'. We adopt the notation

(3.2)
$$S_n = \mathrm{Spec}\, A/\mathfrak{m}^{n+1}, \qquad \hat{S} = \mathrm{Spec}\, \hat{A},$$
$$X_n = X \times_S S_n \qquad \hat{X} = X \times_S \hat{S}.$$

Here $\mathfrak{m} = \max A$, and $\hat{A}$ denotes the ($\mathfrak{m}$-adic) completion of A.

Since nilpotents don't affect etale extensions (SGA I, (8.3)), X'_0 is induced by a compatible system of schemes X'_n finite and etale over X_n, say described by coherent sheaves of $\mathcal{O}_{X_n}$-algebras $\mathscr{A}_n$. By Grothendieck's existence theorem (EGA III, (5.1)), the formal sheaf $\{\mathscr{A}_n\}$ is induced by a coherent sheaf $\hat{\mathscr{A}}$ of $\mathcal{O}_{\hat{X}}$-algebras, and it is immediately seen that $\hat{\mathscr{A}}$ is etale. (We are just repeating Grothendieck's argument (FGA, 182, th. 12) for the case A complete here.) Since, as we saw above, finite etale schemes are classified by a functor locally of finite presentation, we can approximate the sheaf $\hat{\mathscr{A}}$ (modulo $\mathfrak{m}$) by an etale sheaf $\mathscr{A}$ of $\mathcal{O}_X$-algebras, and then $\mathscr{A} \otimes \mathcal{O}_{X_0} \approx \mathscr{A}_0$, as required.

It remains to show that given two finite etale X-schemes X', X'', the inclusion of the closed fibre induces a bijection

$$\mathrm{Hom}_X(X', X'') \to \mathrm{Hom}_{X_0}(X'_0, X''_0).$$

This is shown by a well-known argument:

An X-map $\varphi: X' \to X''$ is described by its graph, which is open and closed in

$X' \times_X X''$ (SGA I, (3.4)). If we assume X' connected and non-empty, then such maps φ are in one-one correspondence with the connected components of $X' \times_X X''$ which are of degree one over X'. The degree of such a component can be measured at any point of X'. Hence if we replace X by a component of $X' \times_X X''$, the bijectivity of the above map reduces immediately to the following assertion:

Lemma **(3.3)**. — *With the notation of* (3.1), X *is connected and non-empty if and only if* X_0 *is.*

Proof. — If X is non-empty, then its image in S must contain the closed point S_0. For, since f is proper, the image is a closed set. Thus X_0 is non-empty. This reasoning applies to any component of X, and shows that X_0 is disconnected if X is.

Suppose X_0 disconnected. Then a connected component C_0 is finite and etale over X_0, and thus is induced by a finite etale X-scheme C. By what has been proved, C is connected. The map $C \to X$ is therefore of degree one at every point of C. Since it is etale and finite, it is an open and closed immersion, i.e., C is a connected component of X. Since C_0 is not all of X_0, C is not all of X, whence X is not connected.

We recall that if X_0 is connected and pointed by a geometric point, and if, say, A is noetherian, then Theorem (3.1) translates immediately as

$$\pi_1(X_0) \xrightarrow{\sim} \pi_1(X), \tag{3.4}$$

where π_1 denotes the pro-finite fundamental group (SGA, exp. V).

A result of a nature similar to (3.1) is the following:

Theorem **(3.5)**. — *Let* R *be a field or an excellent dedekind domain. Let* $S = \operatorname{Spec} A$, *where* A *is a henselization of an* R-*algebra of finite type at a prime ideal. Let* $f: X \to S$ *be a proper map. Then with the notation of* (3.2), *the map*

$$H^1(X, Gl(N)) \to \varprojlim_n H^1(X_n, Gl(N))$$

is injective, and has a dense image. In particular, the map

$$\operatorname{Pic} X \to \varprojlim \operatorname{Pic} X_n$$

is injective and has a dense image.

Here $Gl(N)$ denotes the group scheme of invertible $N \times N$ matrices, and $\operatorname{Pic} Z = H^1(Z, \mathbf{G}_m)$, where $\mathbf{G}_m = Gl(1)$.

Proof. — As is well known, Grothendieck's existence theorem (EGA III, (5.1)) implies that the map

$$H^1(\hat{X}, Gl(N)) \to \varprojlim_n H^1(X_n, Gl(N)) \tag{3.6}$$

is bijective (notation (3.2)). For, a compatible system of elements $a_n \in H^1(X_n, Gl(N))$ is determined by locally free sheaves $\mathscr{L}_n$ and isomorphisms $\mathscr{L}_n \otimes \mathscr{O}_{n-1} \approx \mathscr{L}_{n-1}$, i.e., by a formal sheaf. This formal sheaf is induced by a locally free sheaf $\hat{\mathscr{L}}$ on $\hat{X}$, by Grothendieck's theorem, and $\hat{\mathscr{L}}$ is necessarily locally free. Thus (3.6) is surjective.

Now suppose $\hat{\mathscr{L}}$ is a locally free sheaf on $\hat{X}$ such that $\mathscr{L}_n$ is free for each n. The modules $H^0(X_n, \mathscr{L}_n)$ are of finite length for each n. Hence the images of the maps

$$H^0(X_m, \mathscr{L}_m) \to H^0(X_n, \mathscr{L}_n)$$

$(m \geq n)$ are constant for large m, say equal to $M_n \subset H^0(X_n, \mathscr{L}_n)$. The M_n form an inverse system of modules whose maps $M_n \to M_{n-1}$ are surjective, and clearly (EGA III, (4.1.5))

$$\varprojlim M_n = \varprojlim H^0(X_n, \mathscr{L}_n) \approx H^0(\hat{X}, \hat{\mathscr{L}}).$$

Since $\mathscr{L}_m$ is free, it contains sections $s_m^1, \ldots, s_m^N$ whose determinant is nowhere zero. This being true for all m, the module M_0 must contain sections $s_0^1, \ldots, s_0^N$ of $\mathscr{L}_0$ with nowhere zero determinant. These sections lift successively to M_n for each n, hence to sections $\hat{s}^1, \ldots, \hat{s}^N \in H^0(\hat{X}, \hat{\mathscr{L}})$. Since the determinant of these sections $\hat{s}^i$ is not zero on X_0, it is nowhere zero. Hence (3.6) is injective.

It remains to show that the map

(3.7) $$H^1(X, Gl(N)) \to H^1(\hat{X}, Gl(N))$$

is injective, and with dense image. Now it is clear from (EGA IV, (8.5.2), (8.5.5)) that the functor $H^1(X \times_S \bullet, Gl(N))$ is locally of finite presentation. Thus theorem (1.12) implies that the image is dense. Moreover, if $\mathscr{L}$ is a locally free sheaf on X such that the induced sheaf $\hat{\mathscr{L}}$ is free, then there are sections $\hat{s}^1, \ldots, \hat{s}^N$ of $\hat{\mathscr{L}}$ which have nowhere zero determinant. By (1.12), these sections can be approximated (modulo $\mathfrak{m}$) by global sections $s^1, \ldots, s^N$ of $\mathscr{L}$, and the determinant of s^i will be automatically nowhere zero. Thus $\mathscr{L}$ is free.

We now combine our result with some rigidity theorems of Hironaka ([13], [14]). The first consequence is the fact that isolated singularities are algebraic:

Theorem **(3.8)**. — *Let k be a field, let* $\overline{B} = k[[X]]/(f)$ *be a quotient of the power series ring over k, with* $X = (X_1, \ldots, X_n)$, $f = (f_1, \ldots, f_m)$. *Assume* Spec $\overline{B}$ *formally smooth over* Spec *k outside of its closed point. Then there is an algebraic scheme* S *over k, a point* $s \in S$, *and a k-isomorphism* $\overline{B} \approx \hat{\mathcal{O}}_{S,s}$.

Here Spec $\overline{B}$ is called formally smooth at a point $\mathfrak{p}$ if some $(n-r)$-rowed minor of the jacobian matrix $(\partial f_i / \partial X_j)$ is invertible at $\mathfrak{p}$, where $r = \dim B_{\mathfrak{p}}$.

Note that (S, s) is unique up to local isomorphism for the etale topology, by (2.6). The uniqueness was known previously for an isolated singularity.

This theorem was conjectured by Grauert, and has attracted considerable interest, partly because it resists direct geometric analysis. Various special cases were proved previously by methods which break down in the general case ([1], [13], [14], [18]).

Actually, a good theory of singularities should allow one to *approximate* algebraically an arbitrary formal singularity, say in an " equisingular " way (cf. [22]). One test of such a theory would be to prove that if the given singularity is analytic $(k = \mathbf{C})$, then the approximation has the same topological type in a neighborhood of the origin.

But nilpotent elements should also be taken into account if possible. Then one could hope to derive the theorem on algebraization of formal moduli ([4], (1.6)) as a corollary. We used a rather crude approximation in the proof of that theorem.

Suppose for simplicity that $\operatorname{Spec} \overline{B} = V$ is irreducible and of dimension r. Then Hironaka and Rossi have shown that there is an integer c with the following property: Any ring $\overline{B} = k[[x]]/(f')$ whose truncation (modulo$(x)^c$) is equal to that of $\overline{B}$ and whose dimension is r, is isomorphic to $\overline{B}$. Thus it suffices to choose algebraic series $f'_1, \ldots, f'_m \in k[x]^\sim$ very near $f_1, \ldots, f_m$, and such that the dimension of $\overline{B}'$ is r. (Here $k[x]^\sim$ denotes the henselization of $k[x]$ at the origin.) Of course, a random choice of f'_i will generally cause the dimension to drop.

Here is an *ad hoc* way to control the dimension by means of auxiliary polynomial equations: Adjusting the f_i if necessary, we may suppose that the series $f_1, \ldots, f_{n-r}$ cut out scheme-theoretically a locus of the form $V \cup W$ of dimension r in $\operatorname{Spec} k[[x]]$, where $W \not\supset V$. Choose a series $d \in k[[x]]$ which vanishes on W but not on V. Then $f_1, \ldots, f_{n-r}$ generate the whole ideal $(f_1, \ldots, f_m)$, when d is inverted. Thus we can find series g_{ij} such that for suitable N the equations

$$d^N f_i = \sum_{j=1}^{n-r} g_{ij} f_j$$

hold for $i = n-r+1, \ldots, m$. Now consider the system of equations

$$D^N F_i = \sum_{j=1}^{n-r} G_{ij} F_j$$

in which everything is unknown! Approximate the above solution algebraically to high order. It is easily seen that the resulting locus $f'_1, \ldots, f'_m = 0$ has the correct dimension.

However, the most efficient way to prove theorem (3.8) is by using a recent equivalence theorem of Hironaka ([14], § 2, equivalence theorem I). With a trivial limit argument, theorem (3.8) becomes a special case of the following:

Theorem **(3.9)**. — *Let k be a field, and let* A *be the henselization of the polynomial ring* $k[X]$, $X = (X_1, \ldots, X_n)$ *at a prime ideal. Let* $\mathfrak{a}$ *be a proper ideal of* A. *Given a quotient* $\overline{B} = \hat{A}/(f)$, $f = (f_1, \ldots, f_m)$ *of* $\hat{A}$ *such that* $\operatorname{Spec} \overline{B}$ *is formally smooth over* $\operatorname{Spec} k$ *outside of the locus* $V(\mathfrak{a})$ *in* $\operatorname{Spec} \hat{A}$, *there exists a quotient* B *of* A *and a k-isomorphism* $\hat{B} \approx \overline{B}$.

Here $\hat{}$ denotes $\mathfrak{a}$-adic completion, and formal smoothness at a point $\mathfrak{p}$ of $\operatorname{Spec} \overline{B}$ is defined as above, but with $r = \operatorname{codim}_{\hat{A}} \mathfrak{p}$.

It follows from Hironaka's equivalence theorem, as in ([14], § 3, example 1), that the following holds: Let

$$\hat{L}_2 \xrightarrow{\bar{d}_2} \hat{L}_1 \xrightarrow{\bar{d}_1} \hat{A} \rightarrow \overline{B} \rightarrow 0$$

be a resolution of $\overline{B}$, where L_1, L_2 are free A-modules. Let

$$L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} A$$

be maps satisfying $d_1 d_2 = 0$, $d_i \equiv \overline{d}_i$ (modulo $\mathfrak{a}^c$), $i = 1, 2$.

Let B be the cokernel of d_1. If c is sufficiently large, there is a k-automorphism of $\hat{A}$ carrying $\hat{B}$ to $\overline{B}$.

Now it is clear by Theorem (1.10) that we can approximate the pair $\overline{d}_1$, $\overline{d}_2$ modulo arbitrary powers of $\mathfrak{a}$ by a pair with $d_1 d_2 = 0$. This proves the theorem.

Another immediate consequence of Hironaka [14] is

Theorem **(3.10)**. — *Let* A *be a henselization of an* R*-algebra of finite type at a prime ideal, where* R *is a field or an excellent Dedekind domain. Let* $\mathfrak{a}$ *be a proper ideal of* A. *Given a finite* $\hat{A}$*-module* $\overline{M}$ *which is locally free on* Spec $\hat{A}$ *outside of the locus* $V(\hat{\mathfrak{a}})$, *there is a finite* A*-module* M *such that* $\hat{M}$ *and* $\overline{M}$ *are isormorphic.*

Here again, ^ denotes $\mathfrak{a}$-adic completion. As above, it suffices by ([14], § 3, example II) to approximate a free resolution of $\overline{M}$. Of course, M is unique up to isomorphism, by (1.12).

As a final application, we will derive the following:

Theorem **(3.11)**. — *With the notation of* (3.10), *denote by* C(A) *the category of finite algebras over* A *which are etale outside of the locus* $V(\mathfrak{a})$ *of* Spec A, *and by* $C(\hat{A})$ *the category of* $\hat{A}$*-algebras defined analogously. The extension of scalars* $A \rightarrow \hat{A}$ *induces an equivalence of categories*

$$C(A) \stackrel{\sim}{\rightarrow} C(\hat{A}).$$

This implies in particular that etale coverings of $U = \text{Spec } A - V(\mathfrak{a})$ and of $\hat{U} = \text{Spec } \hat{A} - V(\hat{\mathfrak{a}})$ are in one-one correspondence, which was proved in greater generality in ([1 *a*], II, (2.1)). However, the present proof is more elementary. We went on in ([1 *a*], II, (4.1)) to prove, in the equal characteristic case, that the etale coverings still correspond if $\hat{A}$ is replaced by the completion of A with respect to its maximal ideal, provided $V(\mathfrak{a})$ is everywhere of codimension $\geqq 2$ in Spec A (or that the characteristic is zero). This is connected with some rigidity phenomenon for ramification types in codimension $\geqq 2$ which we do not yet fully understand, and which merits further study.

Now it is clear that the finite algebras satisfying the etaleness condition, and the homomorphisms between two given algebras, are classified by functors locally of finite presentation. Thus we may approximate a given algebra or homomorphism modulo arbitrary powers of $\mathfrak{a}$, by (1.12). It therefore suffices for (3.11) to prove the following rigidity assertions:

Lemma **(3.12)**. — *Let* $\mathfrak{a}$ *be an ideal of a noetherian ring* A, *and assume* A *complete with respect to the* $\mathfrak{a}$*-adic topology. Let* C(A) *denote the category of finite* A*-algebras which are etale over* A *at every point of* $U = \text{Spec } A - V(\mathfrak{a})$.

(i) *Given a map* $\varphi : B_1 \rightarrow B_2$ *in* C(A), *there is an integer* c *such that any map* $\varphi' : B_1 \rightarrow B_2$ *satisfying* $\varphi \equiv \varphi'$ (modulo $\mathfrak{a}^c$) *is equal to* φ.

(ii) *Given* $B \in C(A)$, *there is an integer* c *with the following property*: *Let* $B' \in C(A)$ *be an algebra whose degree over* A *on each component of* U *is the same as that of* B. *(The extra condition on the degrees still gives a functor locally of finite presentation.) If* $B/\mathfrak{a}^c \approx B'/\mathfrak{a}^c$, *then* $B \approx B'$.

Proof. — (i) The customary consideration of the graph of φ given by a map $B_2 \leftarrow B_1 \otimes_A B_2$, and replacement of A by B_2, reduces this problem to the case $B_2 = A$ and, say, $B_1 = B$.

Let A°, B° be the rings obtained from A and B respectively by killing $\mathfrak{a}$-torsion elements. Then $B^\circ \in C(A^\circ)$. A map φ induces $\varphi^\circ : B^\circ \to A^\circ$, and it suffices to prove assertion (i) for φ°. For let c be large enough to work for φ° and so that in addition $T \cap \mathfrak{a}^c = 0$, where T is the ideal of $\mathfrak{a}$-torsion elements of A. Then if $\varphi' \equiv \varphi$ (modulo $\mathfrak{a}^c$), we have $\varphi^\circ(b) = \varphi'^\circ(b)$, i.e.,

$$(\varphi - \varphi')(b) \equiv 0 \quad (\text{modulo } T),$$

and

$$(\varphi - \varphi')(b) \equiv 0 \quad (\text{modulo } \mathfrak{a}^c),$$

whence $\varphi = \varphi'$.

Now if A is $\mathfrak{a}$-torsion free, then the map φ is determined by its restriction to a map $U \to V$, $V = \operatorname{Spec} B - V(\mathfrak{a}B)$. Since this map is etale, it is determined by its underlying set-theoretic map (SGA I, (3.4)). Thus φ is determined set-theoretically, and so it suffices to control the set-theoretic map on each irreducible component of Spec A, which reduces us to the case that A is an integral domain.

Let $V^1, \ldots, V^r$ be the connected components of V, and let B^i be the image of the homomorphism $B \to \Gamma(V^i, \mathcal{O}_{V^i})$.

The map

$$B \to B^* = \prod_i B^i,$$

makes B^* into a finite B-module, and since V is finite and etale over U, its cokernel is annihilated by some power of $\mathfrak{a}$. Because A is an integral domain and U is non-empty, a map $\varphi : B \to A$ extends to $\varphi^* : B^* \to A$, and it is immediately seen that φ^* is given by the projection of B^* onto some factor B^i isomorphic to A. Thus $\varphi^* \equiv \varphi'^*$ (modulo $\mathfrak{a}$) implies that $\varphi^* = \varphi'^*$, whence that $\varphi = \varphi'$. Choose a non-zero element a of A such that $aB^* \subset B$, and c so that $\mathfrak{a}(a) \supset \mathfrak{a}^c \cap (a)$. Then if $\varphi \equiv \varphi'$ (modulo $\mathfrak{a}^c$), we have for all $b \in B$

$$a(\varphi^* - \varphi'^*)(b) = (\varphi - \varphi')(ab) \equiv 0 \quad (\text{modulo } \mathfrak{a}^c).$$

Hence $(\varphi^* - \varphi'^*)(b) \equiv 0$ (modulo $\mathfrak{a}$), as required.

Consider (3.12) (ii): If the ideal $\mathfrak{a}$ is nilpotent, this assertion is trivial. Suppose not. Then U is non-empty. The extension B/A is generated by one element $z \in B$ locally at one of the generic points $u \in U$. Multiplying z by a suitable element invertible at u, we may assume z satisfies a monic equation $g(Z) = 0$ of the right degree with coefficients in A. Let us say that $B \approx B_0 = A[Z]/(g(Z))$ outside of the locus $\{a = 0\}$, where $a \in \mathfrak{a}$ is invertible at u.

By noetherian induction on Spec A, the assertion is true of the ring $A/(a^c)$ for arbitrary c. Hence we may suppose $B/a^cB \approx B'/a^cB'$. Since A is complete with respect to the (a)-adic topology, this means that we may assume $\mathfrak{a}=(a)$.

Then there exists an r so that given B′ as in (ii) and an isomorphism $\theta : B/a^{c+r}B \approx B'/a^{c+r}B'$, there is a map $B_0 \to B'$ compatible with the given map $B_0 \to B$ and with the given isomorphism θ (modulo a^c): To give the map, we have to find a suitable root $z' \in B'$ of the equation $g(Z)=0$. Since B′ is complete (a)-adically, the usual Newton's method yields the required root (cf. [7], p. 93, Cor. 1, or (5.10)). For B_0 is etale over A outside $V(a)$, hence

$$a^m \equiv 0 \quad (\text{modulo } g'(z))$$

in B_0, for some m. Take $r=2m$, and assume θ given. Let $z^\circ \in B'$ be any element representing $\theta(\bar{z})$ in $B'/a^{r+c}B'$ ($^-$ denoting residue modulo a^{r+c}). Then

$$a^m \equiv 0 \quad (\text{modulo } (g'(z^\circ)) + (a^{2m+c}))$$

in B′, whence

$$a^m = g'(z^\circ)x + a^{2m+c}y$$

$$a^m(1-a^{m+c}y) = g'(z^\circ)x.$$

Since $1-a^{m+c}y$ is invertible in B′,

$$a^m \equiv 0 \quad (\text{modulo } g'(z^\circ)).$$

From $g(z)=0$, we obtain $g(z^\circ) \equiv 0$ (modulo a^{2m+c}), hence

$$g(z^\circ) \equiv 0 \quad (\text{modulo } g'(z^\circ)^2 a^c).$$

Thus ([7], p. 93, Cor. 1) implies the existence of a root z' such that

$$z' \equiv z^\circ \quad (\text{modulo } g'(z^\circ)a^c),$$

as required.

It follows that we may assume given the compatible map $B_0 \to B'$, and so we may replace A by B_0, which reduces us to the case that in (ii) the degree of B over A is one at every point of U.

Now consider the canonical map $f : A \to B$. Let I, M be its kernel and cokernel respectively, and define B_1 so as to make the sequences

(3.13)
$$0 \to I \to A \to B_1 \to 0$$
$$0 \to B_1 \to B \to M \to 0$$

exact. Let I′, M′, B_1' be defined in the same way relative to $f' : A \to B'$.

Given an A-isomorphism

(3.14)
$$\theta : B/a^cB \xrightarrow{\sim} B'/a^cB',$$

we obtain an isomorphism

$$M/a^cM \approx M'/a^cM'.$$

Since the degree of B on A is one at every point of U, M is an (a)-torsion module. Thus $M/a^{c-1}M \approx M$ for large c, whence

$$M'/a^cM' \approx M'/a^{c-1}M'.$$

This implies that $M' \approx M'/a^c M'$, and that θ induces a unique isomorphism $M \approx M'$, for large c. We may identify M and M' via this isomorphism.

Now the A-modules $\mathrm{Tor}_1^A(M, A/a^c A)$ form an inverse system, indexed by c, which is essentially zero. This follows immediately from the Artin-Rees lemma applied to an exact sequence $0 \to R \to L \to M \to 0$ with L free. Hence the map ε in the diagram

(3.15)
$$\begin{array}{ccccccccc} \mathrm{Tor}_1(M, A/a^c A) & \longrightarrow & B_1/a^c B_1 & \longrightarrow & B/a^c B & \longrightarrow & M & \to & 0 \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow & & \downarrow & & \| & & \\ \mathrm{Tor}_1(M, A/a^{c-r} A) & \to & B_1/a^{c-r} B_1 & \to & B/a^{c-r} B & \to & M & \to & 0 \end{array}$$

is the zero map for large r and $c' = c - r$. This, together with the similar diagram for B', shows that an isomorphism (3.14) induces an isomorphism

$$B_1/a^{c'} B_1 \xrightarrow{\sim} B'_1/a^{c'} B'_1$$

for such c'.

The ideals I, I' of A are (a)-torsion ideals, and hence are mapped injectively to $A/a^{c'} A$ for large c' (independent of I'). Then the diagram

$$\begin{array}{ccccccccc} 0 & \to & I & \longrightarrow & A/a^{c'} A & \to & B_1/a^{c'} B_1 & \to & 0 \\ & & & & \| & & \downarrow{\scriptstyle\wr} & & \\ 0 & \to & I' & \to & A/a^{c'} A & \to & B'_1/a^{c'} B'_1 & \to & 0 \end{array}$$

shows that $I = I'$. This allows us to replace A by $B_1 = B'_1$, and reduces us to the case that f and f' are injective:

(3.16)
$$0 \to A \to B \to M \to 0$$
$$0 \to A \to B' \to M \to 0.$$

Applying diagram (3.15) again, it follows that the isomorphism (3.14) induces an isomorphism of exact sequences of A-modules

$$\begin{array}{ccccccccc} 0 & \to & A/a^{c'} A & \to & B/a^{c'} A & \longrightarrow & M & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\wr} & & \| & & \\ 0 & \to & A/a^{c'} A & \to & B'/a^{c'} A & \to & M & \to & 0, \end{array}$$

with $c' = c - r$.

Lemma **(3.17)**. — *The map* $\mathrm{Ext}_A^1(M, A) \to \mathrm{Ext}_A^1(M, A/a^c A)$ *is injective for large* c.

Assuming the lemma, it follows that the two sequences (3.16) are isomorphic as sequences of A-modules, if $c \gg 0$. Thus we may assume this to be the case. Now the A-module isomorphism $B \approx B'$ is determined up to an element of $\mathrm{Hom}_A(M, A)$. Since M is (a)-torsion, the inverse system $\mathrm{Hom}_A(M, A/a^c A)$ is essentially equal to $\mathrm{Hom}_A(M, A)$, from which it follows that an isomorphism $\varphi : B \rightarrow B'$ can be found which is compatible with (3.14) modulo $(a^{c'})$, if $c' = c - r$ and r are large. We claim that this is an algebra isomorphism. This amounts to showing that a certain map $B \otimes_A B \rightarrow B' \approx B$ is zero, and it is zero (modulo $(a^{c'})$). Since $B \otimes_A B$ and B are both isomorphic to A on U, this is clear.

Proof of Lemma (3.17). — Let K_c be the kernel of the map in question. To show $K_c = 0$ for $c \gg 0$, it suffices by noetherian induction to show that its support cannot be a constant non-empty set. Suppose $p \in \mathrm{Spec}\, A$ is a generic point of $\mathrm{Supp}\, K_c$ for large c. Since M is (a)-torsion, $p \in V(a)$. Thus we can localize and complete A at this point. The formation of Ext commutes with this process, and we are therefore reduced to the case that K_c is of finite length. Then it suffices to show that a particular extension

$$0 \rightarrow A \rightarrow E \rightarrow M \rightarrow 0$$

which is in K_c for all c is trivial.

Let $F \subset E$ be the (a)-torsion submodule, and let S be such that $a^s M = 0$. If $E/a^c A \leftarrow M$ is a splitting of the sequence

$$0 \rightarrow A/a^c A \rightarrow E/a^c A \rightarrow M \rightarrow 0,$$

then its image in $E/a^c A$ is annihilated by a^s. Thus if $x \in E$ represents an element of this image, we have

$$a^s x = a^c y$$

for some $v \in A$, or
$$a^s (x - a^{c-s} y) = 0,$$

whence
$$x - a^{c-s} y = x' \in F.$$

Thus the image $\overline{F}$ of F in $E/a^{c-s}A$ contains the image of M. If c is large enough, then F is isomorphic to $\overline{F}$. Hence M maps to $F \subset E$ compatibly with the projection map $E \rightarrow M$, i.e., E is the trivial extension.

4. Néron's p-desingularization.

In this section, we consider a pair Λ, Λ' of discrete valuation rings such that Λ' is " unramified " over Λ in the weak sense that a local parameter p of Λ is also a local parameter of Λ'. We make no other restriction on the pair at present.

Put $T = \mathrm{Spec}\, \Lambda$, $T' = \mathrm{Spec}\, \Lambda'$. Let X be a T-scheme of finite type, and let $s' : T' \rightarrow X$ be a point of X with values in T'/T. Suppose X/T smooth at the generic point of s'. (Strictly speaking, we should say the generic point of $s'(T')$. The terminology should not cause confusion.) Then we define, following Néron [17], a measure $l(s')$ of singularity of X at s' as follows: Choose an affine open of X containing s'. Say that this affine is the locus of zeros of $f_1, \ldots, f_m \in \Lambda[y]$ in affine space $\mathbf{E}^N_T$. Let r be the relative dimension of X/T at the generic point of s'. Then for every minor M of rank $N-r$ of the jacobian matrix

$$J=\left(\frac{\partial f_i}{\partial y_j}\right),$$

evaluation on s' yields a matrix with values in Λ', and we define

$$l(s')=\inf_{M}(v'(\det M(s'))) \tag{4.1}$$

as M runs over the minors of rank $N-r$, and where v' denotes the valuation of Λ'.

It is immediately checked that this is independent of the choice of the affine neighborhood and of $\{f_i\}$. Moreover, $l(s')$ is zero if and only if X/S is smooth on s'.

Next, we define Néron's blowing up as follows: Let X be a T-scheme of finite type, and let $Y\subset X$ be a closed subscheme. Let $\mathscr{S}=\mathcal{O}_X\oplus\mathscr{I}\oplus S^2\mathscr{I}\oplus\dots$ be the symmetric algebra on the sheaf of ideals $\mathscr{I}$ of Y in $\mathcal{O}_X$, and consider the (non-homogeneous) ideal $\mathfrak{a}$ of $\mathscr{S}$ generated by sections of the form

$$p[g]-g, \qquad g \text{ a section of } \mathscr{I},$$

where $[g]$ denotes the corresponding element of $\mathscr{S}$ of degree one, and g the element of degree zero (in $\mathcal{O}_X$). Here p is as above the local parameter of Λ. Clearly this defines a quasi-coherent sheaf of ideals of $\mathscr{S}$. Let $\overline{\mathscr{A}}$ be the quasi-coherent sheaf of $\mathcal{O}_X$-algebras obtained by killing p-torsion in $\mathscr{S}/\mathfrak{a}$. Néron's blowing up of Y in X is defined as

$$\overline{X}=Spec_{\mathcal{O}_X}\overline{\mathscr{A}}, \tag{4.2}$$

which is a scheme affine over X.

Suppose $X=\operatorname{Spec}A$ is affine, and let $g_1,\dots,g_r$ generate the ideal of Y. Then it follows from the above description that $\overline{X}=\operatorname{Spec}\overline{A}$, where $\overline{A}$ is the ring obtained by killing p-torsion in

$$A[z_1,\dots,z_r]/\mathfrak{a},$$

$\mathfrak{a}$ being the ideal generated by the relations

$$pz_\nu-g_\nu=0 \qquad \nu=1,\dots,r. \tag{4.3}$$

Note that adding p to the set of generators for the ideal of Y does not affect the blowing up. For it amounts to adding an extra variable z_{r+1} with the relation

$$pz_{r+1}=p,$$

whence, modulo p-torsion, $$z_{r+1}=1.$$

Thus Néron's blowing up of Y in X depends only on the closed fibre Y° of Y over T. Moreover, the map $\overline{X}\to X$ is an isomorphism outside of the locus $\{p=0\}$, as follows again from (4.3).

We revert to the notation of the beginning of this section. Let Y° be the closure in X of the closed point of s', with its reduced structure. We work locally in a neighborhood of s'. With the above notation for $X = \operatorname{Spec} A$, we have

$$g_\nu(s') \equiv 0 \qquad (\text{modulo } p),$$

by definition of Y°. Hence $g_\nu(s')$ is divisible by p in Λ', and so we can find unique elements z_ν in Λ' satisfying the equations (4.3). Since Λ' is p-torsion free, the map $A \to \Lambda'$ defining the section s' extends to $\overline{A}$, and thus the section s' lifts to a section $\overline{s}'$:

$$\begin{array}{ccc} T' & \xrightarrow{\overline{s}'} & \overline{X} \\ & \searrow^{s'} & \downarrow \\ & & X \end{array}$$

Néron's fundamental and beautiful observation is the following:

Theorem **(4.5)**. — *With the above notation, suppose* Y° *generically smooth over the residue field* k *of* Λ, i.e., *that its function field* $k(Y^\circ)$ *is a separable extension of* k. *Then*

$$l(\overline{s}') \leqq l(s'),$$

with equality if and only if $l(s') = 0$, i.e., X/S *is smooth on* s'.

Since $k(Y^\circ)$ is a subfield of the residue field of Λ', it follows that

Corollary **(4.6)**. — *Suppose that the residue field* k' *of* Λ' *is a separable extension of* k, *and let* $s' : T' \to X$ *be a* T-*map such that* X *is smooth at the generic point of* s'. *Consider the operation of replacing* X *by Néron's blowing up* $\overline{X}$ *and* s' *by* $\overline{s}'$, *as in* (4.4). *A finite number of repetitions of this operation results in a situation where* X/T *is smooth on* s'.

The proof of (4.5) consists in a reduction to the case that X is affine and that Y° is the " origin " $\{y_1 = \ldots = y_N = p = 0\}$ in affine y-space $\mathbf{E}^N_T$, and an explicit calculation in that case. We begin with the calculation:

Let X be the locus of zeros of $f_1, \ldots, f_m \in \Lambda[y]$. Then by (4.3) the blowing up is given by the equations

$$pz_i = y_i \qquad i = 1, \ldots, N,$$

and the extra equations needed to kill p-torsion. Thus we may eliminate the variables y_i and view $\overline{X}$ as a locus in affine z-space.

Set

$$f_i(y) = pa_{i0} + \sum_{j=1}^{N} a_{ij} y_j + (\text{degree} \geqq 2).$$

Then
$$f_i(pz) = pa_{i0} + \sum_j pa_{ij} z_j + p^2(\text{degree} \geqq 2),$$

whence
$$f_i(pz) = p\varphi_i(z)$$

with

(4.7)
$$\varphi_i(z) = a_{i0} + \sum_j a_{ij} z_j + p(\text{degree} \geqq 2),$$

and $\overline{X}$ is the locus of zeros of the polynomials $\varphi_i(z)$ together with some extra equations needed to kill p-torsion.

We have
$$p\frac{\partial\varphi_i}{\partial z_j}(z)=\frac{\partial}{\partial z_j}f(pz)=p\frac{\partial f_i}{\partial y_j}(pz),$$

i.e.,

(4.8)
$$\frac{\partial\varphi_i}{\partial z_j}(z)=\frac{\partial f_i}{\partial y_j}(y)\qquad (pz_i=y_i).$$

This shows that $l(\overline{s}')\leqq l(s')$, since the jacobian matrix of X is a submatrix of that of $\overline{X}$ if we use generators $\{f_i\}$, $\{\varphi_i\}$ for the respective ideals.

Denote by a symbol $^\circ$ the residue (modulo p). Then by (4.7), the polynomials φ_i° are linear, and the jacobian matrix

$$J^\circ=\frac{\partial\varphi_i^\circ}{\partial z_j}(z)=\frac{\partial f_i^\circ}{\partial y_j}(0)$$

is just the constant matrix (a_{ij}°), whence $J(s')^\circ=(a_{ij}^\circ)$. If we assume that X is not smooth on s', then the rank of (a_{ij}°) is $<N-r$.

Suppose that the infimum in (4.1) is taken on for a minor of the form $\frac{\partial f_i}{\partial y_j}$ where i, j run from 1 to $N-r$. Since the rank of (a_{ij}°) is less than $N-r$, we may make an invertible linear transformation of $f_1, \ldots, f_{N-r}$ with coefficients in Λ so that, say, $a_{1j}^\circ=0$ for all j. Then f_1 has the form

$$f_1(y)=pa_{10}+\sum_j p\alpha_{1j}y_j\ +(\text{degree}\geqq 2).$$

Hence
$$f_1(pz)=pa_{10}+p^2(\sum_j\alpha_{1j}y_j+(\text{degree}\geqq 2)).$$

Since $f_1(pz)$ vanishes on the section $\overline{s}'$, it follows that

$$a_{10}\equiv 0\quad(\text{modulo } p),$$

whence
$$\varphi_1(z)\equiv 0\quad(\text{modulo } p).$$

Thus $p^{-1}\varphi_1$ is a polynomial vanishing on $\overline{X}$, and if we replace φ_1 by this polynomial in the jacobian matrix (4.8), the value of the subdeterminants involving $i=1$ is decreased by one, whence $l(\overline{s}')\leqq l(s')-1$, as required.

It remains to reduce the general case to the above one. The problem is local on X in a neighborhood of s', hence we may assume $X=\operatorname{Spec} A$ affine. Let $d=\dim Y$. Since Y is generically smooth over Spec k, it is generically etale and finite over $(y_1, \ldots, y_d)$-space $\mathbf{E}_T^d$. Let $\widetilde{\Lambda}$ be the local ring of $\mathbf{E}_T^d$ at the generic point of its closed fibre. This ring $\widetilde{\Lambda}$ is a discrete valuation ring with local parameter p. Since s' maps the closed point of T' to the generic point of Y, the induced map $T' \to \mathbf{E}^d_T$ carries T' to $\operatorname{Spec} \widetilde{\Lambda} = \widetilde{T}$. We have a cartesian diagram

$$\begin{array}{ccc} \mathbf{E}^{N-d}_{\widetilde{T}} & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathbf{E}^N_T & \longrightarrow & \mathbf{E}^d_T \end{array}$$

Let $\widetilde{X}$, $\widetilde{Y}^\circ$ be the subschemes of $\mathbf{E}^{N-d}_{\widetilde{T}}$ induced by X, Y° respectively. Then s' lifts to a $\widetilde{T}$-map $\widetilde{s}' : T' \to \widetilde{X}$, and the image of the closed point of T' is $\widetilde{Y}^\circ$, which is a closed point of $\widetilde{X}^\circ$. With the above notation, the schemes $\widetilde{X}$, $\widetilde{Y}^\circ$ are defined respectively by the equations $\{f_i = 0\}$, $\{g_\nu = 0\}$, where these elements are viewed as polynomials in $\widetilde{\Lambda}[y_{d+1}, \ldots, y_N]$. It therefore follows from equations (4.3) that Néron's blowing up $\overline{X}$ of Y° in X induces the blowing up of $\widetilde{Y}^\circ$ in $\widetilde{X}$. Moreover, it is clear that $l(s') = l(\widetilde{s}')$, provided at least that the coordinates $y_1, \ldots, y_N$ are chosen " generically ", and that $l(\overline{s}') \leq l(\overline{\widetilde{s}}')$ in any case. (Here the work " generically " means so that the infimum in (4.1) is taken on for a minor of $J = (\partial f_i / \partial y_j)$ in which j runs over indices $> d$.) Hence we may replace (T, X, s') by $(\widetilde{T}, \widetilde{X}, \widetilde{s}')$, which reduces us to the case that Y° is a closed point of X°, with residue field separable over k.

Finally, the integer $l(s')$ does not change if Λ' is replaced by any larger discrete valuation ring Λ'_1 having p as local parameter. Moreover, it is clear that Néron's blowing up commutes with etale extensions $\Lambda \to \Lambda_1$, where Λ_1 is a discrete valuation ring. An appropriate choice of Λ_1, Λ'_1 followed by suitable localization reduces us to the case that Y° is a rational point of X° over k, whence by translation to the case that Y° is the point $\{y = 0\}$.

5. Proof of theorem (1.10).

We begin with some preliminary reductions. First of all, it is enough to treat the case that $\mathfrak{m}$ is the maximal ideal of A. For, suppose that the theorem has been proved in that case, and let $\mathfrak{m}$ be any ideal. Let $m_1, \ldots, m_r$ be generators for the ideal $\mathfrak{m}^c$. Under the assumptions of (1.10), the elements $\overline{y}_i$ are in the $\mathfrak{m}$-adic completion $\overline{A}$ of A. Thus there are elements $y'_i \in A$ such that

$$y'_i \equiv \overline{y} \pmod{\mathfrak{m}^c \overline{A}},$$

whence

$$\overline{y}_i = y'_i + \sum_j \overline{a}_{ij} m_j$$

for suitable $\overline{a}_{ij}$ in $\overline{A}$. The elements $\{\overline{y}_i, \overline{a}_{ij}\} \in \overline{A}$ are thus a solution of the larger system of equations

$$f_\nu(Y_1, \ldots, Y_N) = 0 \qquad \nu = 1, \ldots, m$$
$$Y_i - y'_i - \sum_j A_{ij} m_j = 0 \qquad i = 1, \ldots, N, \qquad j = 1, \ldots, r,$$

and we can view these elements as lying in the completion $\hat{A}$ of A with respect to its maximal ideal. The theorem in the known case implies the existence of a solution $\{y_i, a_{ij}\} \in A$, and $y_i \equiv y'_i \equiv \bar{y}_i \pmod{\mathfrak{m}^c}$, as required.

Next, we may assume that R is a discrete valuation ring and that the maximal ideal $\mathfrak{m}$ of A lies over the closed point of Spec R. For, if R is a discrete valuation ring but $\mathfrak{m}$ lies over the generic point, then we replace R by its field of fractions, and if R is a field, then we replace it by the power series ring $R[[t]]$, where t acts on A as zero. Since $R[[t]]$ is excellent, this is permissible.

Moreover, we may assume that $K = A/\mathfrak{m}$ is finite over the residue field k of R. For, since A is the henselization of an R-algebra of finite type at a prime ideal, K is a field extension of k of finite type. Let d be its transcendence degree. Then we can find elements $z_1, \ldots, z_d \in A$ so that K is finite over $k(z_1^\circ, \ldots, z_d^\circ)$, where $^\circ$ denotes the residue modulo $\mathfrak{m}$. Consider the map $R[Z] \to A$ sending $Z_i \to z_i$. The inverse image of $\mathfrak{m}$ is the prime ideal of $R[Z]$ generated by the local parameter p of R. Since A is local, this map factors through the localization R' of $R[Z]$ at this prime ideal, which is an excellent (EGA IV, (7.8.6) (i) and (7.8.3) (ii)) discrete valuation ring. Clearly, A is the henselization of an R'-algebra of finite type. Thus we may replace R by the ring R', whose residue field is $k(Z)$.

Finally, say that A is the henselization of the R-algebra of finite type A_0 at a maximal ideal which we will denote also by $\mathfrak{m}$. Then we can make A_0 into a finite algebra over a polynomial ring $R[X]$ $(X = (X_1, \ldots, X_n))$ in such a way that $\mathfrak{m}$ lies over the "origin" $(p, X_1, \ldots, X_n)$ of Spec $R[X]$. This is clear: Write A_0 as a quotient of some $R[Z_1, \ldots, Z_n]$. Let $g_i^\circ(Z_i) \in k[Z_i]$ be a monic polynomial satisfied by the residue of Z_i in $A_0/\mathfrak{m}$. Choose a monic polynomial $g_i(Z_i) \in R[Z_i]$ representing g_i°, and set $X_i = g_i(Z_i)$. Then the resulting map $R[X] \to A_0$ is as required.

Now let $R[X]^\sim$ denote the henselization of $R[X]$ at the origin (p, X). The $R[X]^\sim$-algebra $\tilde{A}_0$ obtained from A_0 by extension of scalars is a product of local rings (EGA IV, (18.5.11) *a)*) which are the henselization of A_0 at the various points lying over the origin. Our ring A is among them, hence is a finite $R[X]^\sim$-algebra.

We claim that it is enough to prove theorem (1.10) for the ring $R[X]^\sim$ itself. Indeed, suppose the theorem proved in that case, let A be any finite local $R[X]^\sim$-algebra, and

(5.1) $$f(Y) = 0, \qquad Y = (Y_1, \ldots, Y_N), f = (f_1, \ldots, f_m)$$

a system of polynomial equations with coefficients in A. Let F be the functor which to an $R[x]^\sim$-algebra B associates the set of solutions of (5.1) in the ring $A \otimes B$ (the tensor product being over the ring $R[X]^\sim$). This functor is locally of finite presentation, by lemma (2.7). Thus we may apply corollary (1.8). Since A is finite over $R[X]^\sim$, we have

$$\hat{A} \approx A \otimes R[X]^\wedge,$$

the symbol $\wedge$ denoting completion of a local ring with respect to its maximal ideal.

Hence a solution $\bar{y}$ of (5.1) in $\hat{A}$ yields an element $\bar{\xi}$ of $F(R[X]^\wedge)$, which we may approximate (p, X)-adically to arbitrary order by a $\xi \in F(R[X]^\sim)$, to obtain the required solution y in A.

We are therefore reduced to the case that $A = R[X]^\sim$ $(X = (X_1, \ldots, X_n))$, and that $\mathfrak{m} = (p, X)$. Proceeding by induction on the number $n \geqq 0$ of variables X_i, we may fix n and assume the theorem true for fewer than n variables.

Lemma **(5.2)**. — *It suffices to treat the case that the given polynomials* $f = (f_1, \ldots, f_m) \in A[Y]$ *have coefficients in the polynomial ring* $R[X]$.

Proof. — Suppose the theorem proved for such polynomials, and let f be arbitrary. Consider the homomorphism

$$A[Y] \to R[X]^\wedge \tag{5.3}$$

defined by the substitution of $\bar{y}$ for Y. Since $R[X]^\wedge = \hat{R}[[X]]$ is an integral domain, its kernel is a prime ideal $\mathfrak{p}$, and $\mathfrak{p}$ contains $(f_1, \ldots, f_m)$. Clearly, it is permissible to add extra equations to the system so that $(f_1, \ldots, f_m)$ generate the whole ideal $\mathfrak{p}$. Let $\mathfrak{p}_0 = \mathfrak{p} \cap R[X, Y]$. Since A is the henselization of $R[X]$, the ring $A[Y]$ is a limit of etale extensions of $R[X, Y]$. Therefore $A[Y]/\mathfrak{p}_0 A[Y]$ is reduced (SGA I, (9.2)), i.e., $\mathfrak{p}_0 A[Y]$ is an intersection of prime ideals. The ring A being algebraic over $R[X]$, it is easily seen for reasons of dimension that $\mathfrak{p}$ is one of these prime ideals. If $\mathfrak{p} = p_0 A[Y]$ we are through. Otherwise we have

$$p_0 A[Y] = \mathfrak{p} \cap \mathfrak{q} \supset \mathfrak{p}\mathfrak{q}$$

for some ideal $\mathfrak{q}$ not containing $\mathfrak{p}$. Since $\mathfrak{p}$ is the kernel of the map (5.3), it follows that there is an element $g \in \mathfrak{q}$ with $g(\bar{y}) \neq 0$.

Let $\varphi = (\varphi_1, \ldots, \varphi_l) \in R[X, Y]$ be generators for $\mathfrak{p}_0$. Applying the theorem to the solution $\bar{y}$ of the system of equations

$$\varphi(Y) = 0,$$

which is possible by assumption, we can find $y = (y_1, \ldots, y_n) \in A$ arbitrarily close to $\bar{y}$, $\mathfrak{m}$-adically, so that $\varphi(y) = 0$. Since $g(\bar{y}) \neq 0$, it follows that $g(y) \neq 0$ if y is sufficiently near $\bar{y}$. But since $\mathfrak{p}\mathfrak{q} \subset \mathfrak{p}_0 A[Y]$, we have

$$f_i(y) g(y) = 0,$$

hence

$$f_i(y) = 0$$

for all i. Thus y is the required solution of (5.1).

We now suppose $f = (f_1, \ldots, f_m) \in R[X, Y]$. Let $S = \operatorname{Spec} R[X]$, $\tilde{S} = \operatorname{Spec} R[X]^\sim$ and $\hat{S} = \operatorname{Spec} R[X]^\wedge$. It is permissible to assume that the elements f_i generate the whole kernel of the map

$$R[X, Y] \to R[X]^\wedge \tag{5.4}$$

defined by the substitution of $\bar{y}$ for Y, which is a prime ideal $\mathfrak{p}$ of $R[X, Y]$. Then the map

$$\operatorname{Spec} R[X, Y]/(f) = V \to S$$

is generically smooth. For, since R is excellent, so is R[X] (EGA IV, (7.8.6), (i)). Therefore (EGA IV, (7.8.2), (ii)) the map $\hat{S} \to S$ is regular, and so fract(R[X]^) is a separable extension of fract(R[X]). (We write fract(D) for the field of fractions of an integral domain D.) Since R[X, Y]/$\mathfrak{p}$ is a subring of R[X]^, fract(R[X, Y]/$\mathfrak{p}$) is a separable extension of fract(R[X]), which proves the assertion.

Let Λ be the localization of R[X] at the prime ideal P generated by the local parameter p of R, and similarly let Λ' be the localization of R[X]^ at $\hat{P} = p.R[X]^\wedge$. These rings Λ, Λ' are discrete valuation rings, and p is a local parameter for them both. Moreover, the residue field of Λ' is a separable extension of that of Λ. This is because these fields are fract(k[[X]]) and fract(k[X]) respectively, and k[X] is excellent (EGA IV, (7.8.6), (i)), $k = R/p$. Thus we may apply corollary (4.6) to this pair.

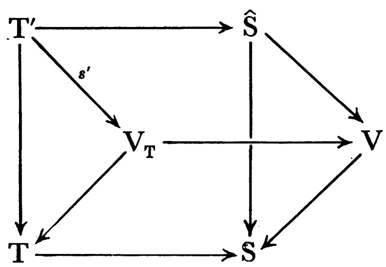
Write $T = \operatorname{Spec} \Lambda$, $T' = \operatorname{Spec} \Lambda'$, $V_T = V \times_S T$. The solution $\bar{y}$ yields an S-map

$$\sigma : \hat{S} \to V,$$

which induces a map

$$s' : T' \to V_T$$

making a commutative diagram



We want to reduce ourselves to the case that V_T is smooth over T on s'. Since T, T' are obtained by localization from S, $\hat{S}$ respectively, this is equivalent with the assertion that V be smooth over S at the point $\sigma(\hat{P})$, where $\hat{P} = p.R[X]^\wedge$. To do this, let $W^\circ \subset V$ be the closure of $\sigma(\hat{P})$ with its reduced structure. Since V_T is a localization of V, the scheme $W^\circ_T = W^\circ \times_S T$ is the closure in V_T of the closed point of s'. Thus we can induce Néron's blowing up of W°_T in V_T by blowing up W° in V: Let $(g_1, \ldots, g_r) \in R[X, Y]$ generate the ideal of W°. Then we blow up W° in V by killing p-torsion in the ring $R[X, Y, Z]/\mathfrak{a}$, $Z = (Z_1, \ldots, Z_r)$, where $\mathfrak{a}$ is the ideal generated by the relations

(5.5)
$$pZ_\nu - g_\nu = 0 \quad \nu = 1, \ldots, r$$
$$f_i = 0 \quad i = 1, \ldots, m.$$

Let the blow up be the spectrum of this ring, say $\bar{V}$. Clearly $\bar{V}_T = \bar{V} \times_S T$ is Néron's blowing up of W°_T in V_T. Moreover, since $\sigma(\hat{P})$ lies in W°, it follows that

$$g_\nu(\bar{y}) \equiv 0 \quad (\text{modulo } p.R[X]^\wedge).$$

Hence we can find $\bar{z}_\nu \in R[X]^\wedge$ satisfying the equations

$$p\bar{z}_\nu - g_\nu(\bar{y}) = 0,$$

whence since $R[X]^{\wedge}$ is p-torsion free, σ lifts to an S-map

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\bar{\sigma}} & \bar{V} \\ & \searrow^{\sigma} \quad \swarrow & \\ & V & \end{array}$$

and $\bar{\sigma}$ induces the lifting of s' to $\bar{s}' : T' \to \bar{V}_T$.

Now our problem is to approximate $\mathfrak{m}$-adically the map $\sigma : \hat{S} \to V$ given by the solution $\bar{y}$ of (5.1) by a map $\tilde{S} \to V$, and it is clearly sufficient to approximate the map $\bar{\sigma}$ instead, i.e., to solve the system of equations given by (5.5) together with the additional equations needed to kill p-torsion. Since $\bar{V}$ is reduced and irreducible because V is, we may replace V by $\bar{V}$ and σ by $\bar{\sigma}$. Since $\bar{V}$ and V are isomorphic outside of the locus $\{p=0\}$ (cf. section 4), the elements $(f_1, \ldots, f_m)$ still generate the whole kernel of (5.4). By Corollary (4.6), a finite number of repetitions of this process results in a situation where V is smooth over S at $\sigma(\hat{P})$. We have therefore proved

Lemma **(5.6).** — *It suffices to treat the case that* $f=(f_1, \ldots, f_m) \in R[X, Y]$, *and that* $V = \operatorname{Spec} R[X, Y]/(f)$ *is smooth at the point* $\sigma(\hat{P})$, *where*

$$\sigma : \hat{S} \to V$$

is the map defined by the substitution of $\bar{y}$ *for* Y, *and where* $\hat{P} = p . R[X]^{\wedge}$.

Assume the conditions of the lemma hold. Let r be the relative dimension of V/S at $\sigma(\hat{P})$. Then by the jacobian criterion for smoothness, there is an $(N-r)$-rowed minor M of the jacobian matrix $(\partial f_i/\partial Y_j)$ such that

$$\delta = \det M \in R[X, Y]$$

has the property
$$\delta(X, \bar{y}) \not\equiv 0 \ (\text{modulo } p)$$

in $R[X]^{\wedge}$. We may suppose that M is the minor $1 \leq i, j \leq N-r$. Let $V' \subset \operatorname{Spec} R[X, Y]$ be the locus of zeros of $f_1, \ldots, f_{N-r}$. Then V′ and V are equal locally at the point $\sigma(\hat{P})$, hence we can write, set theoretically,

$$V' = V \cup W$$

for some closed set $W \subset \operatorname{Spec} R[X, Y]$ which does not contain the image $\sigma(\hat{S})$. Thus there is an element $g \in R[X, Y]$ vanishing on W such that

$$g(X, \bar{y}) \neq 0.$$

Let $y = (y_1, \ldots, y_N) \in R[X]^{\sim}$ be any solution of the system of equations

(5.7)
$$f_1(Y) = \ldots = f_{N-r}(Y) = 0.$$

If $y \equiv \bar{y}$ modulo a sufficiently high power of the maximal ideal of $R[X]^{\wedge}$, then it follows that $g(X, y) \neq 0$, hence that the image of $\operatorname{Spec} R[X]^{\sim}$ in $\operatorname{Spec} R[X, Y]$ under the map defined by the substitution of y for Y does not lie in W. Since $\operatorname{Spec} R[X]^{\sim}$ is reduced and irreducible, and since the image lies in V′, it follows that it lies in the subscheme V. Hence

$$f_i(y)=0$$

for all $i=1, \dots, m$. Thus it suffices to treat the system of equations (5.7), whence

Lemma (**5.9**). — *With the notation of* (5.6), *we may assume in addition that* $m=N-r$, *where r is the relative dimension of* V/S *at* $\sigma(\hat{P})$, *and that the determinant* δ *of the minor* $(\partial f_i/\partial Y_j)$, $1\leqq i, j\leqq m$, *satisfies*

$$\delta(X, \bar{y}) \not\equiv 0 \quad (\text{modulo } p).$$

We can now complete the proof as in the analytic case [3]: Recall the following

Lemma (**5.10**). — *Let* A *be a ring and* $\mathfrak{a}$ *an ideal of* A *such that the pair* (A, $\mathfrak{a}$) *satisfies the implicit function theorem* (1.9) *with* $\mathfrak{a}=\mathfrak{m}$. *For instance,* A *may be a henselian local ring and* $\mathfrak{a}$ *any proper ideal. Let* $f=(f_1, \dots, f_m)\in A[Y]$ *be polynomials in the variables* $Y=(Y_1, \dots, Y_N)$, *let M be the matrix* $(\partial f_i/\partial Y_j)$, $1\leqq i, j\leqq m$, *and let* $\delta=\det M$. *Suppose given elements* $y^\circ=(y_1^\circ, \dots, y_N^\circ)\in A$ *such that*

$$f(y^\circ)\equiv 0 \quad (\text{modulo } \delta^2(y^\circ).\mathfrak{a}).$$

Then there are elements $y=(y_1, \dots, y_N)\in A$ *such that*

$$f(y)=0$$

and that

$$y\equiv y^\circ \ (\text{modulo } \delta(y^\circ).\mathfrak{a}).$$

We will give a proof of the following stronger assertion, due to Tougeron [20]:

Lemma (**5.11**). — *Let* A, $\mathfrak{a}$, *f be as in* (5.10). *Let J be the jacobian matrix* $(\partial f_i/\partial Y_j)$, $i=1, \dots, m$; $j=1, \dots, N$. *Suppose given elements* $y^\circ=(y_1^\circ, \dots, y_N^\circ)\in A$ *such that*

$$f(y^\circ)\equiv 0 \quad (\text{modulo } \Delta^2.\mathfrak{a}),$$

where Δ *is the annihilator ideal of the* A*-module* C *presented by the relation matrix* $J(y^\circ)$ (i.e., C *is the cokernel of the homomorphism* $A^N\to A^m$ *whose matrix is* $J(y^\circ)$). *Then there is a solution* $y=(y_1, \dots, y_N)$ *of the system of equations* $f(Y)=0$ *with*

$$y\equiv y^\circ \quad (\text{modulo } \Delta.\mathfrak{a}).$$

Proof. — Let $\delta_1, \dots, \delta_r$ generate the annihilator Δ. This means that, writing $J=J(y^\circ)$, there are $N\times m$ matrices N_i with

$$JN_i=\delta_i I,$$

where I is the $m\times m$ identity matrix. Write

$$f(y^\circ)=\sum_{i,j}\delta_i\delta_j\varepsilon_{ij}$$

for suitable vectors $\varepsilon_{ij}=(\varepsilon_{ij1}, \dots, e_{ijm})$ with $\varepsilon_{ij\nu}\in\mathfrak{a}$. We try to solve the equations

$$f(y^\circ+\sum_{i=1}^{r}\delta_i U_i)=0$$

for elements $U_i = (U_{i1}, \ldots, U_{iN})$ of A^N. Expansion by Taylor's formula in vector notation yields

$$\begin{aligned} 0 &= f(y^\circ) \quad + J \,.\, \sum_i \delta_i U_i + \sum_{i,j} \delta_i \delta_j Q_{ij} \\ &= J \,.\, \sum_i \delta_i U_i + \sum_{i,j} \delta_i \delta_j (Q_{ij} + \varepsilon_{ij}) \\ &= \sum_i \delta_i J \,.\, U_i + \sum_i \delta_i J \,.\, (\sum_j N_j \,.\, (Q_{ij} + \varepsilon_{ij})), \end{aligned}$$

where Q_{ij} are vectors of polynomials in the U_i all of whose terms are of degree $\geqslant 2$. Thus it suffices to solve the r vector equations

$$0 = U_i + \sum N_j \,.\, (Q_{ij} + \varepsilon_{ij}),$$

which give Nr equations in the Nr unknowns $\{U_{i\nu}\}$. The jacobian of this system of equations is the identity matrix at $U = 0$. Thus the implicit function theorem implies the existence of a solution $u_i = (u_{i1}, \ldots, u_{iN})$ with $u_{i\nu} \equiv 0$ (modulo $\mathfrak{a}$), and $y = y^\circ + \Sigma \delta_i u_i$ is the required solution of $f(Y) = 0$.

If we apply the above lemma with $\mathfrak{a} = \mathfrak{m}^c$ to our situation, it follows that, in the notation of (5.10), it suffices to find $y^\circ = (y_1^\circ, \ldots, y_N^\circ) \in A = R[X]^\sim$ such that

$$y^\circ \equiv \bar{y} \quad (\text{mod } \mathfrak{m}^c)$$

and that
$$f(y^\circ) \equiv 0 \ (\text{modulo } \delta^2(X, y^\circ) \,.\, \mathfrak{m}^c).$$

For then the lemma implies the existence of the required $y \in A$.

Now since $\bar{y} \in \hat{A}$ is a solution of (5.1), we have trivially

$$f(\bar{y}) \equiv 0 \quad (\text{modulo } \delta^2(X, \bar{y}) \,.\, \mathfrak{m}^c).$$

Thus, setting $g = \delta^2$, we may apply the following lemma to complete the proof.

Lemma **(5.12)**. — *Suppose the theorem proved for the ring* $R[X]^\sim$ *when there are fewer than n variables* X, *and let* $A = R[X_1, \ldots, X_n]^\sim$ $(n \geq 0)$. *Let* $g, f_1, \ldots, f_m \in R[X, Y]$ *be polynomials and let* $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)$ *be elements of* $\hat{A}$ *such that*

$$g(X, \bar{y}) \not\equiv 0 \quad (\text{modulo } p)$$

and that
$$f_i(X, \bar{y}) \equiv 0 \ (\text{modulo } g(X, \bar{y}))$$

for $i = 1, \ldots, m$. *Then there are elements* $y = (y_1, \ldots, y_N) \in A$ *such that*

$$f_i(X, y) \equiv 0 \quad (\text{mod } g(X, y))$$

for all i and that
$$y \equiv \bar{y} \ (\text{modulo } \mathfrak{m}^c).$$

Proof. — If $g(X, \bar{y})$ is invertible, $g(X, y)$ will be invertible for all $y \equiv \bar{y}$ (modulo $\mathfrak{m}$). Then the desired divisibility is trivial. We may thus assume $g(X, y)$ not invertible. This completes the proof in the case $n = 0$, since $g(X, \bar{y}) \not\equiv 0$ (modulo p) just means that it is an invertible element in that case.

Suppose $n>0$. Since $g(X,\bar{y})\not\equiv 0$ (modulo p), we may adjust the coordinates X via an automorphism of R[X] of the type

$$X'_i = X_i + X_n^{e_i} \quad i=1, \ldots, n-1$$
$$X'_n = X_n$$

in such a way that $g(X,\bar{y})\not\equiv 0$ (modulo $(p, X_1, \ldots, X_{n-1})$).
Then $\hat{B} = R[X]^{\wedge}/(g(X,\bar{y}))$ is a finite algebra over the ring

$$\hat{R}[[X_1, \ldots, X_{n-1}]] = R[X_1, \ldots, X_{n-1}]^{\wedge}$$

([21], p. 259, Cor. 1), from which it follows immediately that the Weierstrass Preparation Theorem holds for $g(X,\bar{y})$ with respect to the variable X_n, i.e., that

(5.13) $$g(X,\bar{y}) = \bar{a}(X_n)\,.\,(\text{unit})$$

where $$\bar{a}(X_n) = X_n^r + \bar{a}_{r-1}X_n^{r-1} + \ldots + \bar{a}_1 X_n + \bar{a}_0$$

is a monic polynomial with coefficients $\bar{a}_\nu$ which are non-units in $R[X_1, \ldots, X_{n-1}]^{\wedge}$ (this can be seen as in ([21], p. 261)). Then

(5.14) $$\hat{B} \approx \hat{R}[[X_1, \ldots, X_{n-1}]][X_n]/(\bar{a}(X_n)).$$

We make the substitution

(5.15) $$Y^*_\nu = \sum_{j=0}^{r-1} Y_{\nu j} X_n^j \quad \nu = 1, \ldots, N$$

for Y_ν into the polynomials g, f_i, where $Y_{\nu j}$ are variables. Dividing by the variable polynomial

(5.16) $$A(X_n) = X_n^r + A_{r-1}X_n^{r-1} + \ldots + A_1 X_n + A_0,$$

we obtain

(5.17)
$$g(X, Y^*) = A(X_n)Q + \sum_{j=0}^{r-1} G_j X_n^j$$
$$f_i(X, Y^*) = A(X_n)Q'_i + \sum_{j=0}^{r-1} F_{ij} X_n^j,$$

where Q, Q'_i, G_j, F_{ij} are polynomials in the variables $\{X_\nu, Y_{\nu\mu}, A_\nu\}$ with coefficients in R, and where G_j, F_{ij} do not involve the variable X_n.

Next, divide $\bar{y}_\nu$ by $\bar{a}(X_n)$ (which is possible by (5.14)):

(5.18) $$\bar{y}_\nu = \bar{a}(X_n)\bar{z}_\nu + \sum_{j=0}^{r-1} \bar{y}_{\nu j} X_n^j$$

with $\bar{z}_\nu \in R[X]^{\wedge}$ and $\bar{y}_{\nu j} \in R[X_1, \ldots, X_{n-1}]^{\wedge}$. Set

(5.19) $$\bar{y}^*_\nu = \sum_{j=0}^{r-1} \bar{y}_{\nu j} X_n^j.$$

Since $$\bar{y} \equiv \bar{y}^* \ (\text{modulo } \bar{a}(X_n)),$$

it follows by Taylor's formula that

$$g(X, \bar{y}^*) \equiv g(X, \bar{y})$$

and that $$f_i(X, \bar{y}^*) \equiv f_i(X, \bar{y}) \quad (\text{modulo } \bar{a}(X_n)),$$

whence by (5.13)

(5.20) $$\begin{aligned} g(X, \bar{y}^*) &\equiv 0 \\ f_i(X, \bar{y}^*) &\equiv 0 \quad (\text{modulo } \bar{a}(X_n)) \end{aligned}$$

for $i = 1, \ldots, m$.

Substitute $\bar{y}_{\nu\mu}$, $\bar{a}_\nu$ for $Y_{\nu\mu}$, A_ν in (5.17). The congruence (5.20) shows that

$$\begin{aligned} G_j(X_1, \ldots, X_{n-1}, \{\bar{y}_{\nu\mu}\}, \{\bar{a}_\nu\}) &= 0 \\ F_{ij}(X_1, \ldots, X_{n-1}, \{\bar{y}_{\nu\mu}\}, \{\bar{a}_\nu\}) &= 0 \end{aligned}$$

for all relevant i, j. Thus $\{\bar{y}_{\nu\mu}, \bar{a}_\nu\}$ is a solution in $R[X_1, \ldots, X_{n-1}]^\wedge$ of the system of equations

(5.21) $$\begin{aligned} G_i &= 0 \\ F_{ij} &= 0. \end{aligned}$$

By the induction hypothesis, there are elements $\{y_{\nu\mu}, a_\nu\} \in R[X_1, \ldots, X_{n-1}]^\sim$ solving the system (5.21), with

$$\begin{aligned} y_{\nu\mu} &\equiv \bar{y}_{\nu\mu} \\ a_\nu &= \bar{a}_\nu \quad (\text{modulo } (p, X_1, \ldots, X_{n-1})^c) \end{aligned}$$

for arbitrarily large c.

Choose $$z_\nu \equiv \bar{z}_\nu \quad (\text{modulo } \mathfrak{m}^c)$$

where $\bar{z}$ is as in (5.18), and set

$$\begin{aligned} a(X_n) &= X_n^r + a_{r-1}X_n^{r-1} + \ldots + a_1 X_n + a_0 \\ y_\nu^* &= \sum_{j=0}^{r-1} y_{\nu j} X_n^j \\ y_\nu &= a(X_n) z_\nu + y_\nu^*, \end{aligned}$$

so that

(5.22) $$y_\nu \equiv \bar{y}_\nu \quad (\text{modulo } \mathfrak{m}^c).$$

Then since $\{y_{\nu\mu}, a_\nu\}$ is a solution of (5.21), we have by (5.17)

$$\begin{aligned} g(X, y^*) &\equiv 0 \\ f_i(X, y^*) &\equiv 0 \quad (\text{modulo } a(X_n)), \end{aligned}$$

whence by Taylor's formula

(5.23) $$\begin{aligned} g(X, y) &\equiv 0 \\ f_i(X, y) &\equiv 0 \quad (\text{modulo } a(X_n)), \ i = 1, \ldots, m. \end{aligned}$$

Now it is clear that if we write by Weierstrass

$$g(X, y) = b(X_n) \, . \, (\text{unit})$$

where $b(X_n)$ is a monic polynomial whose coefficients are non-units of $R[X_1, \ldots, X_{n-1}]^\wedge$, then the degree of b will be r (5.13), provided c (5.22) is chosen sufficiently large. Since $a(X_n)$ divides $b(X_n)$ by (5.23), and since these polynomials have the same degree, it follows that they are equal. Thus (5.23) implies that

$$f_i(X, y) \equiv 0 \quad (\text{modulo } g(X, y))$$

for all i. This completes the proof of Lemma (5.12) and of Theorem (1.10).

6. The case of a ground field.

Let $f = (f_1, \ldots, f_m)$ be polynomials with coefficients in some field. By *degree* of f we mean the sum of the degrees of the f_i. The degree of an ideal $\mathfrak{a}$ of a polynomial ring is the minimum among the degrees of generating sets for $\mathfrak{a}$, and the degree of a closed subscheme V of affine space is the degree of its ideal $\mathscr{I}(V)$.

This section is devoted to the following result:

Theorem **(6.1)**. — *There is an integer valued function* $\beta = \beta(n, N, d, \alpha)$, *defined for non-negative integer values of* n, N, d, α, *with the following property*:

Let k be a field, let $f = (f_1, \ldots, f_m)$ *be polynomials in* $k[X, Y]$, *where* $X = (X_1, \ldots, X_n)$ *and* $Y = (Y_1, \ldots, Y_N)$, *with* $\text{degree}(f) \leqq d$. *Suppose given polynomials* $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N) \in k[X]$ *such that*

$$f(X, \bar{y}) \equiv 0 \quad (\text{modulo } (X)^\beta).$$

Then there are elements $y = (y_1, \ldots, y_N) \in k[X]^\sim$ *solving the system of equations*

$$f(X, Y) = 0 \tag{6.2}$$

and such that

$$y \equiv \bar{y} \ (\text{modulo } (X)^\alpha).$$

Here as before, $k[X]^\sim$ denotes the henselization of $k[X]$ at the maximal ideal $(X) = (X_1, \ldots, X_n)$.

This result implies easily the strong form of Greenberg's theorem [9] for the ring $k[X]^\sim$ ($n = 1$). One also obtains corollaries of the following type:

Corollary **(6.3)**. — *Let* $f : U \to V$ *be a morphism of schemes of finite type over* k. *Given* α, *there is a* $\beta \geqq \alpha$ *with the following property*:

For every rational point $v \in V$ *and every* V*-map* $\varphi : \text{Spec}(\mathcal{O}_{V,v}/\mathfrak{m}_v^\beta) \to U$, *there is an etale neighborhood* V′ *of* v *in* V, *and a* V*-map* $\varphi' : V' \to U$, *such that*

$$\varphi' \equiv \varphi \quad (\text{modulo } \mathfrak{m}_v^\alpha).$$

We leave the verification of this corollary to the reader. Note that we have to assume v a *rational* point. We do not know how to get a uniform bound β for all closed points v of V.

It is remarkable that while applications of theorem (1.10) leap to the mind, there do not seem to be so many applications of the stronger assertion (6.1). (Perhaps this can be explained as lack of insight on my part.) The applications of (6.1) I know of require only the following weaker result:

Corollary **(6.4)**. — *Let* $f=(f_1, \ldots, f_m)\in k[X, Y]$. *If for every* α *there are elements* $\bar{y}\in k[X]$ *such that*

$$f(X, \bar{y}) \equiv 0 \quad (\text{modulo } (X)^\alpha),$$

then there is a solution of the system of equations (6.2) *in the ring* $k[X]^\sim$.

We have needed such an assertion (cf. [11], § 3). But (6.4) is hardly stronger than (1.10). In fact, one can deduce (6.4) from (1.10) if either k is a finite field, or if it is algebraically closed and uncountable. For, in those cases a kind of compactness argument using Greenberg's functor [8] (which we leave to the reader) shows, under the hypotheses of (6.4), that the system (6.2) has a solution in the power series ring $k[[X]]$, whence in $k[X]^\sim$, by (1.10).

To prove (6.1), we will need the following result. Stolzenberg [19] has shown that the methods of Hermann [12] yield:

Theorem **(6.5)** (Hermann-Stolzenberg). — *There exists an integer valued function* $\gamma(n, d)$, *defined for non-negative integer values of* n, d, *with the following property*:

Let k *be a field, and let* $\mathfrak{a}$ *be an ideal of the polynomial ring* $k[x]$, $x=(x_1, \ldots, x_n)$, *of degree* $\leqq d$. *For a suitable primary decomposition of the ideal* $\mathfrak{a}$:

$$\mathfrak{a}=\mathfrak{q}_1\cap\ldots\cap\mathfrak{q}_s,$$

the following integers are bounded by $\gamma(n, d)$:

(i) *The number* s.

(ii) *The degree of each primary ideal* $\mathfrak{q}_i$.

(iii) *The degree of the associated prime ideal* $\mathfrak{p}_i=\sqrt{\mathfrak{q}_i}$.

(iv) *Exponents* e_i *such that* $\mathfrak{p}_i^{e_i}\subset\mathfrak{q}_i$.

Moreover, the degree of $\mathfrak{a}\cap\mathfrak{b}$ *is bounded in terms of the degrees of* $\mathfrak{a}$ *and* $\mathfrak{b}$.

Using this theorem, we can obtain (6.1) by an analysis of the proof of (1.10) given in section 5, as follows:

The proof is by induction on n, the theorem being trivial for $n=0$. Next, we apply induction on the Krull dimension of $V=\operatorname{Spec} k[X, Y]/(f)$. This is permissible since the dimension is bounded by $n+N$.

We will use the following terminology: Given polynomials $\bar{y}=(\bar{y}_1, \ldots, \bar{y}_N)\in k[X]$, the notation

$$(X, \bar{y})\in V \quad (\text{modulo } (X)^c)$$

will mean

$$f(X, \bar{y}) \equiv 0 \quad (\text{modulo } (X)^c).$$

We may assume V reduced and irreducible. For let $\bar{V}=V_{\text{red}}$. The degree of $\bar{V}$ may be bounded *a priori*, by (6.5). So may an exponent e such that $\mathscr{I}(\bar{V})^e\subset(f)$. Now if $\bar{y}\in k[X]$ satisfies

$$(X, \bar{y})\in V \quad (\text{modulo } (X)^{ec}),$$

then

$$(X, \bar{y})\in \bar{V} \quad (\text{modulo } (X)^c).$$

Thus if $\bar{\beta}$ works for $\bar{V}$, $e\bar{\beta}=\beta$ works for V.

Assume V reduced, and let V^i be its irreducible components. Their number and degrees may be bounded *a priori*, again by (6.5). Let $I^i = \mathcal{I}(V^i)$. We have

$$(f) = I^1 \cap \ldots \cap I^r \supset I^1 . I^2 \ldots I^r.$$

Therefore, if $\overline{y}$ satisfies $\qquad (X, \overline{y}) \in V \quad (\text{modulo } (X)^{rc}),$

then for some i $\qquad (X, \overline{y}) \in V^i \quad (\text{modulo } (X)^c).$

Thus, assuming the result known for the V^i, we can find the required solution y lying on one of the V^i, provided c is sufficiently large.

Now suppose the map $V \to S = \operatorname{Spec} k[X]$ is not generically surjective. Then its image lies in a proper closed set C of S. An elementary consideration of the universal family of varieties V defined by polynomials of fixed degree shows that the degree s of a non-zero polynomial vanishing on C can be bounded as a function of degree(f) alone. Then there is no $\overline{y} \in k[X]$ with

$$f(X, \overline{y}) \equiv 0 \qquad (\text{modulo } (X)^{s+1}),$$

and so the constant $\beta = s$ works trivially for such V.

Thus we may suppose V reduced and irreducible and that the map $V \to S = \operatorname{Spec} k[X]$ is generically surjective. We may moreover assume that V is generically smooth over $k[X]$. This is seen by the following argument due to Raynaud ([9], p. 61, case 2). Suppose it is not. Then the characteristic of k is $p \leqslant \deg(f)$, and there is a purely inseparable extension k' of k, with $k'^p \subset k$, such that $V' = V \times_S S'$ is not reduced at its generic point. (Here $S' = \operatorname{Spec} k'[X']$ and $X'^p_i = X_i$.) The degree $[k' : k]$ of such an extension may be bounded as a function of degree(f) alone. For the polynomials f_i have coefficients in a subfield k_0 of k whose transcendence degree over the prime field is bounded by the number of coefficients of f, and we may take for k' the join of $k_0^{1/p}$ and k.

Set $Z = V'_{\text{red}}$. Then deg Z is bounded, since deg V' = deg V. Denote by π_* the functor $\prod_{S'/S}$ of Grothendieck (FGA, 195-12) (Weil's restriction of the ground ring). It is adjoint to the base change $\bullet \times_S S'$. Hence we have a canonical map

$$V \to \pi_* V'.$$

Now π_* carries closed immersions to closed immersions. Hence if W is the object making the solid diagram below cartesian, then W is a closed subscheme of V.

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & \swarrow & \downarrow \\ \pi_* Z & \longrightarrow & \pi_* V' \end{array}$$

To say $W = V$ means that there is a dotted arrow as above making a commutative diagram. By the adjointness property, we obtain

$$\begin{array}{ccc} & \swarrow & V' \\ & & \| \\ Z & \longrightarrow & V' \end{array}$$

which contradicts the assumption that V' is not reduced. Thus $W \neq V$, and so $\dim W < \dim V$.

Now a bound for $\deg W$ is given in terms of bounds for $\deg V$ and $\deg Z$, as is mmediately seen. Since by induction the theorem is known for dimensions $< \dim V$, it suffices to show that if $\overline{y} \in k[X]$ and

$$\textbf{(6.6)} \qquad f(X, \overline{y}) \equiv 0 \quad (\text{modulo } (X)^{dec}),$$

where e is an exponent with $\mathscr{I}(Z)^e \subset I(V')$ and where d is a sufficiently large fixed integer, we have

$$(X, \overline{y}) \in W \quad (\text{modulo } (X)^c).$$

For then if c is sufficiently large we can find $y \in k[X]^{\sim}$ with $(X, \overline{y}) \in W$, whence since $W \subset V$, $f(X, \overline{y}) = 0$.

Let $S_l = \operatorname{Spec} k[X]/(X)^l$. Then by (6.6), $\overline{y}$ defines an S-morphism

$$\varphi : S_l \to V, \qquad l = dec.$$

By base change, we have

$$\varphi \times_S S' : S_l \times_S S' \to V',$$

whence a map $S'_l \to V'$, where $S'_l = \operatorname{Spec} k'[X']/(X')^l$. This map sends $S'_{dc} \to Z$. Since (X) generates an (X')-primary ideal in $k'[X']$, it follows that $S_c \times_S S'$ is a closed subscheme of S'_{dc} for d sufficiently large. Hence we obtain

$$S_c \times_S S' \to Z,$$

or

$$S_c \to \pi_* Z.$$

Taking into account some functorial identities, this implies that φ sends $S_c \to W \subset V$, as required.

Thus we may assume V irreducible and generically smooth over S. Say that the relative dimension of V/S at the generic point is r. Then with suitable labeling, the polynomial

$$\textbf{(6.7)} \qquad \delta = \det(\partial f_i / \partial Y_j) \qquad (i, j = 1, \ldots, r)$$

is not identically zero on V. A primary decomposition for the ideal $(f_1, \ldots, f_r)$ yields

$$(f_1, \ldots, f_r) = \mathscr{I}(V) \cap J$$

where degrees of generating sets for these ideals are bounded and where $\mathscr{I}(V) \not\supset J$.

Set $W = \operatorname{Spec} k[X, Y]/J$. The locus $V(\delta) \cap V$ contains $W \cap V$. Hence $\delta^e \in \mathscr{I}(V) + J$ for a suitable e which is bounded. Thus

(6.8) $$\delta(X, y) \not\equiv 0 \quad (\text{modulo } (X)^e)$$

implies $$(X, y) \notin W \cap V \quad (\text{modulo } (X)^{ec}).$$

It suffices to prove the following lemma:

Lemma **(6.9)**. — *Fix d, n, N. Then for all integers α, γ, there is a $\beta \geqq \gamma$ such that if $(f_1, \ldots, f_r)$ has degree $\leqq d$, and if $\bar{y} \in k[X]$ are polynomials with the property*

$$f_i(X, \bar{y}) \equiv 0 \quad (\text{modulo } (X)^\beta) \qquad i = 1, \ldots, r$$
$$\delta(X, \bar{y}) \not\equiv 0 \quad (\text{modulo } (X)^\gamma),$$

then there are $y \in k[X]^\sim$ with $y \equiv \bar{y}$ (modulo $(X)^\alpha$) and $f_i(X, y) = 0$ for $i = 1, \ldots, r$.

For suppose this shown, and let $f = (f_1, \ldots, f_m)$, α be as given. Choose $\gamma \geqq \alpha$ large enough so that

$$f(X, \bar{y}) \equiv 0 \quad (\text{modulo } (X)^\gamma)$$

and $$\delta(X, y) \equiv 0 \quad (\text{modulo } (X)^\gamma)$$

imply the existence of $y \equiv \bar{y}$ (modulo $(X)^\alpha$) solving the system of equations $f = \delta = 0$. This is possible by the induction hypothesis on dim V. Now apply the lemma with α replaced by $\alpha' = \max(\alpha, \gamma e)$ to find β. If $\delta(X, \bar{y}) \equiv 0$ (modulo $(X)^\gamma$) we are through by the choice of γ, and if not, the lemma implies the existence of $v \equiv \bar{y}$ (modulo $(X)^{\alpha'}$) such that

$$f_i(X, y) = 0 \quad \text{for} \quad i = 1, \ldots, r,$$

i.e., $(x, y) \in W \cup V$. Then

$$\delta(X, y) \not\equiv 0 \quad (\text{modulo } (X)^\gamma),$$

hence by (6.8) $$(X, y) \notin W \cap V \quad (\text{modulo } (X)^{\gamma e}).$$

But $$(X, y) \in V \quad (\text{modulo } (X)^{\alpha'}),$$

therefore $$(X, y) \notin W \quad (\text{modulo } (X)^{\gamma e}).$$

Whence $(X, y) \in V$, as required.

Now to prove lemma (6.9), it suffices to prove

Lemma **(6.10)**. — *Assume Theorem* (6.1) *true for fewer than n variables* X. *For all integers α, γ, D, there is a β with the following property:*

Let $f = (f_1, \ldots, f_m)$ and g be polynomials in $k[X, Y]$ of degree $\leqq$ D. If $\bar{y} \in k[X]$ satisfies

$$f(X, \bar{y}) \equiv 0 \quad (\text{modulo } (g(X, \bar{y})) + (X)^\beta)$$

but $$g(X, \bar{y}) \not\equiv 0 \quad (\text{modulo } (X)^\gamma),$$

then there are elements $y \in k[X]^\sim$ such that $y \equiv \bar{y}$ (modulo $(X)^\alpha$) and

$$f(X, y) \equiv 0 \quad (\text{modulo } g(X, y)).$$

For we let $g=\delta^2$, $D=\max(d, \deg g)$, $r=m$, and substitute $\gamma'=2\gamma$ and $\alpha'=\alpha+\gamma'$ into (6.10), to find β'. We choose $\beta'\geqq\alpha'$. Then $\beta=\beta'$ works for Lemma (6.9): If

$$f(X, \overline{y}) \equiv 0 \quad (\text{modulo } (X)^{\beta})$$

then the hypotheses of Lemma (6.10) are verified with α', γ'. Hence the lemma implies the existence of $y^\circ \in k[X]^\sim$ with $y^\circ \equiv \overline{y}$ (modulo $(X)^{\alpha'}$) and

$$f_i(X, y^\circ) = g(X, y^\circ) h_i(X) \qquad i=1, \ldots, r.$$

Since $f(X, y^\circ) \equiv f(X, \overline{y}) \equiv 0$ (modulo $(X)^{\alpha'}$), it follows that $h(X) \equiv 0$ (modulo $(X)^{\alpha}$). Thus

$$f(X, y^\circ) \equiv 0 \qquad (\text{modulo } \delta^2(X, y^\circ).(X)^{\alpha}),$$

and so we may apply Lemma (5.10) to complete the proof.

Proof of Lemma (6.10). — We want to apply the Weierstrass Preparation Theorem to the polynomial $g(X, \overline{y})$, and we need to bound the degree of the resulting monic polynomial:

Lemma **(6.11)**. — *Given γ, there is an r with the following property:*

For every field k and every power series $\varphi(x) \in k[[X]]$ such that

$$\varphi(X) \not\equiv 0 \qquad (\text{modulo } (X)^{\gamma}),$$

there exists a k-automorphism θ of the polynomial ring $k[X]$ sending the origin $\{X=0\}$ to itself, such that the transformed series $\theta(\varphi)=\varphi'$ satisfies

$$\varphi'(0, \ldots, 0, X_n) \not\equiv 0 \qquad (\text{modulo } (X_n^r)).$$

Proof. — Let the order of vanishing of φ at the origin be $s \leqq \gamma$. To say $\varphi'(0, \ldots, 0, X_n) \not\equiv 0$ (modulo $(X_n)^r$) is equivalent with the assertion that the homogeneous component φ_s' of φ of degree s does not vanish at the point $(0, \ldots, 0, 1)$ of the projective space $\mathbf{P}_k^{n-1}$. Provided that the field k has at least $\gamma-1 \geqq s-1$ elements, there will be some rational point of $\mathbf{P}_k^{n-1}$ at which φ_s is not zero. Then a homogeneous linear transformation of the variables X_i results in a situation where $\varphi'(0, \ldots, 0, X_n) \not\equiv 0$ (modulo $(X_n)^s$). Thus only fields with fewer elements have to be considered, which reduces the problem to a finite number of exceptional finite fields k. We may consider them separately.

Consider coordinate changes of the form

$$X_i' = X_i + X_n^{e_i} \qquad i=1, \ldots, n-1$$
$$X_n' = X_n.$$

No given power series φ can be in the ideal $(X_1', \ldots, X_{n-1}')$ for all integer values of $e_1, \ldots, e_{n-1}$. This is because the branches $\{X_1' = \ldots = X_n' = 0\}$ are dense in Spec $k[[X]]$. Suppose that the minimum order of vanishing, as $\{e_1, \ldots, e_{n-1}\}$ vary, of

$$\varphi'(0, \ldots, 0, X_n') = \varphi(-X_n'^{e_1}, \ldots, -X_n'^{e_{n-1}}, X_n')$$

is unbounded, and choose a sequence $\{\varphi_\nu\}$ of polynomials such that the minimum order for φ_ν is at least ν. Since the field k is finite, this sequence has a subsequence which converges in $k[[X]]$ to a series φ. From

$$\varphi'(0, \ldots, 0, X'_n) \not\equiv 0 \quad (\text{modulo } (X)^c)$$

we get

$$\varphi'_\nu(0, \ldots, 0, X'_n) \not\equiv 0 \quad (\text{modulo } (X)^c)$$

for ν sufficiently large. This is a contradiction, which proves the lemma.

Returning to the notation of Lemma (6.10), we may make a change of variables X as in (6.11), so that (5.13) holds with

(6.12) $$\overline{a}(X_n) = X_n^r + \overline{a}_{r-1}X_n^{r-1} + \ldots + \overline{a}_1 X_n + \overline{a}_0,$$

the $\overline{a}_\nu$ being non-units in the ring $k[[X_1, \ldots, X_{n-1}]]$ (they are actually in $k[X_1, \ldots, X_{n-1}]^\sim$). By (6.11), only a finite number of degrees r need be considered, and so we may treat each r separately, i.e., fix r.

With the notation of (5.15), (5.16), we may divide as in (5.17), to obtain polynomials Q, Q'_i, G_j, F_{ij} in the variables $X_\nu, Y_{\nu\mu}, A_\nu$ with coefficients in k, and where G_i, F_{ij} do not involve X_n. Moreover the degrees of these polynomials are bounded *a priori*.

Lemma **(6.13)**. — *Let* $\varphi(X) \in k[[X]]$ *be any series, and divide by* $\overline{a}$ (6.12):

$$\varphi(X) = \overline{a}(X_n)\pi + \sum_{j=0}^{r-1} \rho_j X_n^j,$$

where $\pi \in k[[X]]$ *and* $\rho_j \in k[[X_1, \ldots, X_{n-1}]]$. *Then*

$$\varphi(X) \equiv 0 \quad (\text{modulo } (X)^{r(c+1)})$$

implies that

$$\pi \equiv 0$$

and

$$\rho_j \equiv 0 \quad (\text{modulo } (X)^c) \quad \textit{for } i = 0, \ldots, r-1.$$

We leave this calculation to the reader.

Proceeding as in (5.18), (5.19), we have

$$g(X, \overline{y}^*) \equiv 0 \quad (\text{modulo } \overline{a}(X_n)).$$

Thus by (5.17)

(6.14) $$G_j(X_1, \ldots, X_{n-1}, \{\overline{y}_{\nu\mu}\}, \{\overline{a}_\nu\}) = 0, \quad j = 0, \ldots, r-1.$$

Moreover, $$f_i(X, \overline{y}) \equiv 0 \quad (\text{modulo } (g(X, \overline{y})) + (X)^\beta)$$

implies that $$f_i(X, \overline{y}) \equiv 0 \quad (\text{modulo } (\overline{a}(X_n)) + (X)^\beta).$$

Therefore we can write $$f_i(X, \overline{y}^*) = \overline{a}(X_n)h_i(X) + \varphi_i(X)$$

with $h_i, \varphi_i \in k[[X]]$, and where

$$\varphi_i(X) \equiv 0 \quad (\text{modulo } (X)^\beta).$$

Divide by $\overline{a}(X_n)$: $$\varphi_i(X) = \overline{a}(X_n)\pi_i + \sum_{j=0}^{r-1} \rho_{ij} X_n^j.$$

Then by uniqueness of division and (5.17),

$$\rho_{ij} = F_{ij}(X_1, \ldots, X_{n-1}, \{\overline{y}_{\nu\mu}\}, \{\overline{a}_\nu\}).$$

Thus lemma (6.13) implies that

$$\textbf{(6.15)} \qquad F_{ij}(X_1, \ldots, X_{n-1}, \{\bar{y}_{\nu\mu}\}, \{\bar{a}_\nu\}) \equiv 0 \quad (\text{modulo } (X)^{[\beta/r-1]}).$$

By the induction hypothesis, (6.14) and (6.15) imply the existence of a solution of the system of equations $G_i = F_{ij} = 0$ in $k[X_1, \ldots, X_{n-1}]^\sim$ if β is sufficiently large — the fact that $\bar{y}_{\nu\mu}$, $\bar{a}_\nu$ are power series instead of polynomials does not matter, since we can replace them by their truncations at high order. Thus the proof may be completed as in section 5, taking for c the maximum of α and r.

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