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RATIONAL SINGULARITIES, WITH APPLICATIONS TO ALGEBRAIC SURFACES AND UNIQUE FACTORIZATION

by Joseph LIPMAN [1]

CONTENTS

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INTRODUCTION

In two basic papers ([**3**], [**4**]) M. Artin has developed the theory of *rational singularities* of algebraic surfaces. Roughly speaking, these are isolated singularities of a surface whose resolution has no effect on the arithmetic genus of the surface; alternatively, they are singularities which are " cohomologically trivial ". Among these singularities are included all normal points which birationally dominate a regular (i.e. simple) point, and in particular — by abuse of the term " singularity " — the regular points themselves (cf. § 1 for precise statements) (1).

Our purpose is to fill in the theory, and to demonstrate its wide applicability by expanding upon a number of familiar topics in the theory of surfaces:

— Resolution of singularities of surfaces by means of quadratic transformations and normalizations (cf. [**22**]).

— Factorization of birational maps of non-singular surfaces into quadratic transformations (cf. [**24**, § II.1]).

— Complete ideals in two-dimensional regular local rings (cf. [**21**] and [**25**, Appendix 5]; cf. also [**14**] and [**14'**]).

— Factorial henselian two-dimensional local rings (cf. [**7**, § 3]).

— The contractibility criterion of Castelnuovo and M. Artin (cf. [**4**]).

In part I, we show that Zariski's method of desingularization by quadratic transformations and normalizations works for any excellent *surface* (i.e. reduced two-dimensional noetherian scheme). Resolution of singularities for excellent surfaces has been established by Abhyankar and Hironaka ([**2**], [**9**], [**10**]), and we must make use

(1) Rational " double points " have been known and studied for many years; for more historical and bibliographical information cf. [**6**, § 1].

of their result, at least in its weak form of " local uniformization ". Thus it is the *process* — and not the fact — of resolution which is of interest in part I. We also extend the classical theorem on factorization into quadratic transformations to proper birational maps $f : X \to Y$ where X is a regular surface and Y is a surface having only rational singularities. From the point of view of novelty, the *sine qua non* of part I is Proposition (3.1).

In parts II and V we generalize Zariski's theory of complete ideals in two-dimensional regular local rings to two-dimensional normal local rings S *having a rational singularity*. The principal result in part II is to the effect that *any product of complete ideals in such an* S *is again complete* (cf. Theorem (7.1)). This implies, among other things, that rational singularities can be resolved by quadratic transformations alone. In part V we take up the question of *unique* factorization of complete ideals into simple complete ideals. Theorem (20.1) states that *such unique factorization holds for all complete ideals in* S *if and only if the completion (or henselization) of* S *is factorial*. The method used is to resolve the singular point of Spec(S) and to relate the problem of unique factorization to the behaviour of exceptional curves on the resulting regular surface. The necessary preliminaries about curves on surfaces are developed in parts III and IV.

That unique factorization of complete ideals holds when S is *regular* is a central point of Zariski's theory. Our approach yields an alternative proof of this fact, and naturally suggests the question: *which* S — other than the regular ones — *are such that their completion (or henselization) is factorial*? The interest of this question is heightened by the result: *Let* R *be any two-dimensional analytically normal henselian local ring with algebraically closed residue field. If* R *is factorial, then* R *has a rational singularity.* (More generally, R has a rational singularity if and only if R has a finite divisor class group, cf. Theorem (17.4).) In part VI we find that the answer to the above question is: essentially those rings studied by Scheja in [**19**] (cf. § 25 for details). In particular we obtain the following generalization of a theorem proved by Brieskorn [**7**] for local rings over the complex numbers: let R be any non-regular analytically normal two-dimensional henselian local ring with algebraically closed residue field of characteristic $\neq 2, 3, 5$; then R is factorial if and only if the maximal ideal of R is generated by three elements x, y, z satisfying $z^2+y^3+x^5=0$.

To obtain this result we follow the same strategy as Brieskorn: we first show that a ring of the desired type has multiplicity two (§ 22), and then describe explicitly all rational " double points " together with their divisor class groups (§§ 23-24). The rational double points are classified according to the " configuration diagram " of exceptional curves on a minimal desingularization. These diagrams turn out — *a posteriori* — to be precisely the Dynkin diagrams used in the theory of Lie groups and algebras (cf. § 24). This means that the intersection matrix

$$\left(\frac{2(E_i . E_j)}{(E_i . E_i)}\right) \qquad (E_i,\ E_j \text{ exceptional curves})$$

is identical with a " Cartan matrix ", and conversely each Cartan matrix appears as such an intersection matrix for the minimal desingularization of some rational double point. This striking phenomenon was observed by Du Val, who first classified rational double points with algebraically closed residue field, thereby obtaining (in effect) all Dynkin diagrams in which only the integer " 1 " appears (cf. [**7 1/2**]). By allowing arbitrary residue fields, we get the remaining types of diagram. Is there some deeper connection with Lie algebras, or is this all mere coincidence??

The main unanswered question is: *does every complete two-dimensional factorial local ring* R *have a rational singularity*? [1] The answer is affirmative, as we have already noted, if R has an algebraically closed residue field. This restriction on the residue field is entirely due to the same restriction in Complement (11.3). What the question comes down to, in part, is: when does the Picard scheme of a one-dimensional algebraic scheme over an arbitrary field k have just one k-rational point in its connected component?

In the Appendix, we include two basic theorems about surfaces. The first is essentially a well-known theorem of Zariski on the elimination of indeterminacies of rational maps; it is of constant use throughout the paper. The second is a generalization of the contractibility criterion of Castelnuovo and M. Artin; the proof involves most of the theory of rational singularities.

For more details about the contents of the individual parts and sections, we refer to their respective introductory remarks.

There is a certain amount of material of an expository nature included for the usual reasons: " in order to be self-contained " or " for the convenience of the reader ". Generally this consists of facts which are very well known for surfaces over algebraically closed fields, and readily worked out, but not conveniently available, in the context of arbitrary two-dimensional schemes (in which generality they are required, since we work throughout with arbitrary two-dimensional local domains subject only to some restriction of the type " analytically normal "). Similar expositions can be found in [**13**] and [**20**].

I wish to express my appreciation for stimulating conversations with Professors S. S. Abhyankar, H. Hironaka and R. Hartshorne. I am much indebted to the two cited papers of M. Artin without which I could not have begun. Finally, I am dedicating this paper to Professor Oscar Zariski, from whom I have learned so much.

§ 0. Some terminology and notation.

In the absence of explicit indications to the contrary, the following conventions will be in force throughout the paper:

1. All rings and schemes are noetherian. All schemes and maps (=morphisms) are separated.

(1) (Added in proof.) No ! (P. Salmon, *Rend. Lincei*, May 1966). But cf. Proposition (17.5).

2. Whenever we speak of a *birational* map $f: X \to Y$, it is tacitly assumed that both X and Y are reduced schemes.

3. A point x of a scheme X is *regular* if the local ring $\mathcal{O}_{X,x}$ is regular; otherwise x is *singular*. X is *regular*, or *non-singular*, if all of its points are regular. A map $f: X \to Y$ is called a *desingularization* if f is proper and birational and X is regular. Sometimes, by abuse of language, we say that " X is a desingularization of Y " meaning that " there exists a desingularization $f: X \to Y$ ".

4. We say that a map $f: X \to Y$ is a *quadratic transformation* if f is obtained by blowing up a *closed* point (1) of Y. If Y is reduced, then such an f is birational (cf. [**8**, chapter II, § 8.1], or, as we will write from now on [**EGA** II, § 8.1]).

5. The word " *surface* " will mean " *reduced noetherian separated scheme of dimension two* ".

We shall often use, without explicit mention, the following facts:

A) *The normalization* (= integral closure) *of a surface in its total ring of fractions is a surface* (cf. [**EGA** II, (6.3.8)] and [**17**, Theorems (33.2) and (33.12)]).

B) *If* Y *is a surface and* $g: W \to Y$ *is a proper birational map, then* W *is a surface.*

(For, W is clearly noetherian, separated, and of dimension ≤ 2 (dimension formula), and $\dim W \geq 2$ because the inverse image of any non-closed, non-maximal point of Y is a non-empty collection of non-closed, non-maximal points of W.)

6. If X is a scheme and $\mathcal{F}$, $\mathcal{G}$ are $\mathcal{O}_X$-modules, we may write " $\mathcal{F} \otimes \mathcal{G}$ " for " $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ " if no confusion is possible. Similarly, we may denote cohomology groups by " $H^p(\mathcal{F})$ " instead of " $H^p(X, \mathcal{F})$ ".

7. We say that a ring R is *factorial* if R is a Unique Factorization Domain.

I. — APPLICATIONS TO THE BIRATIONAL THEORY OF SURFACES

§ 1. Birational behavior of rational singularities.

We recall the definition of *rational singularity* (cf. [**4**]):

Definition (**1.1**). — *A normal local ring* R *of dimension 2 is said to have a* rational singularity *if there exists a desingularization* $f: X \to \operatorname{Spec}(R)$ *such that* $H^1(X, \mathcal{O}_X) = 0$.

Note that if R is *regular* then R has a rational singularity (take $X = \operatorname{Spec}(R)$).

Two important facts about rational singularities are given in the next proposition.

Proposition (**1.2**). — *Let* R *be a two-dimensional normal local ring having a rational singularity, and let* $g: W \to \operatorname{Spec}(R)$ *be a birational map of finite type.*

1) *If* $w \in W$ *is a normal point of codimension two, then the local ring* $\mathcal{O}_{W,w}$ *has a rational singularity.*

2) *If* W *is* normal *and* g *is* proper *then* $H^1(W, \mathcal{O}_W) = 0$.

---

(1) With *reduced* subscheme structure.

Proof. — The proposition follows from two familiar facts:

A) *If* X *is any regular surface and* $j : Z \to X$ *is a quadratic transformation, then* $H^1(Z, \mathcal{O}_Z) \cong H^1(X, \mathcal{O}_X)$.

B) *Let* $f : X \to \mathrm{Spec}(R)$ *be a desingularization. If g is proper, then there exists a commutative diagram of proper birational maps*

$$\begin{array}{ccc} Z & \xrightarrow{h} & W \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & \mathrm{Spec}(R) \end{array}$$

with j a product *of* (i.e. *succession of*) *quadratic transformations.*

A) is proved in [**20**, pp. 59-61]; for convenience we will give the proof again below. B) is a special case of Theorem (26.1) of the Appendix (" elimination of indeterminacies "). By induction A) holds also when j is a *product* of quadratic transformations; consequently if X and Z are as in B), and we assume, as we may, that $H^1(X, \mathcal{O}_X) = 0$, then also $H^1(Z, \mathcal{O}_Z) = 0$. If furthermore W is *normal*, then $h_*(\mathcal{O}_Z) = \mathcal{O}_W$; since there is a canonical injection

$$H^1(W, h_*(\mathcal{O}_Z)) \hookrightarrow H^1(Z, \mathcal{O}_Z) = 0$$

we conclude that $H^1(W, \mathcal{O}_W) = 0$, proving 2).

To prove 1), we first remark that some affine open neighborhood of w is a dense open subscheme of a scheme W^* which is projective over Spec(R); we may replace W by W^*, i.e. we may assume that g is projective. Let $h : Z \to W$ be as in B) with $H^1(Z, \mathcal{O}_Z) = 0$ as before, and let $V = \mathrm{Spec}(\mathcal{O}_{W,w}) \times_W Z$. V is proper and birational over $\mathrm{Spec}(\mathcal{O}_{W,w})$, V is regular, and since $H^1(Z, \mathcal{O}_Z) = 0$ [**EGA** III, (1.4.15) (especially p. 94)] shows that $H^1(V, \mathcal{O}_V) = 0$ (1). Q.E.D.

We can prove statement A) by showing that $R^1 j_*(\mathcal{O}_Z) = 0$: the desired conclusion then follows from the exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(Z, \mathcal{O}_Z) \to H^0(X, R^1 j_*(\mathcal{O}_Z))$$

of terms of low degree in the spectral sequence for $H^0 j_*$. (Note that $\mathcal{O}_X = j_*(\mathcal{O}_Z)$.)

Since $R^1 j_*(\mathcal{O}_Z)$ is concentrated at the point x which is blown up, we may replace X by an affine neighborhood Spec(T) of x which is such that x corresponds to a maximal ideal in T generated by two elements, say b and c. We have then to show that $H^1(Z, \mathcal{O}_Z) = 0$.

Z is covered by the affine open sets $U_b = \mathrm{Spec}(T[c/b])$, $U_c = \mathrm{Spec}(T[b/c])$. For this covering, the alternating one-cochains with values in $\mathcal{O}_Z$ are the elements of $\Gamma(U_b \cap U_c, \mathcal{O}_Z) = T[c/b, b/c]$. Since

$$\sum_{i,j} t_{ij}\left(\frac{c}{b}\right)^i\left(\frac{b}{c}\right)^j = \sum_{j \leq i} t_{ij}\left(\frac{c}{b}\right)^{i-j} - \sum_{j > i}(-t_{ij})\left(\frac{b}{c}\right)^{j-i}$$

every alternating one-cochain is a coboundary. Thus $H^1(Z, \mathcal{O}_Z) = 0$. Q.E.D.

(Another proof of A) can be obtained from Corollary (23.2).)

(1) We must first show that $R^1 h_*(\mathcal{O}_Z) = 0$! By [**EGA** III, (4.2.2)], $H^2(W, h_*(\mathcal{O}_Z)) = 0$ and $R^1 h_*(\mathcal{O}_Z)$ has support of dimension ≤ 0 ; now use the exact sequence

$$H^1(Z, \mathcal{O}_Z) \to H^0(W, R^1 h_*(\mathcal{O}_Z)) \to H^2(W, h_*(\mathcal{O}_Z))$$

§ 2. Resolution of singularities by quadratic transformations and normalization (method of Zariski).

We show now that the methods of Zariski's original paper [**22**] on the resolution of singularities can be applied to any *excellent* surface (cf. [**EGA** IV, (7.8.5)]) once some form of local uniformization is known. Local uniformization for excellent surfaces has been established through the work of Abhyankar [**2**] and Hironaka [**9**], [**10**]. The following theorem shows therefore that *any excellent surface can be desingularized by a succession of quadratic transformations and normalizations.*

Since the normalization of an excellent surface is a disjoint union of integral excellent surfaces, plus possibly some regular schemes of dimension ≤ 1, we need only consider the integral case. Whenever it is convenient, we will take the point of view of *models*, i.e. given any integral scheme Y, with field of rational functions K, and a birational map $f: X \to Y$, we will regard X as a collection of local rings with field of fractions K (in particular we regard Y in this way), and then f associates to each element S of X the unique element of Y which is dominated by S (cf. [**25**, chapter VI, § 17] and [**EGA** I, § 8]).

If A, B, are subrings of a ring C, then " [A, B] " denotes the least subring of C containing both A and B. C is " essentially of finite type " over A if there are finitely many elements $c_1, c_2, \ldots, c_n$ in C such that C is a ring of fractions of $A[c_1, c_2, \ldots, c_n]$.

Theorem (**2.1**). — *Let* Y *be an integral excellent surface, with field of rational functions* K. *Suppose that each valuation* v *of* K *which dominates a local ring* $R \in Y$ *also dominates a regular local ring* A *whose quotient field is* K, *and which is such that the unique localization of* [R, A] *dominated by* v *is essentially of finite type over both* R *and* A. *Let* Y_1 *be the normalization of* Y *in* K, *and let*

$$(\Sigma) \qquad Y_1 \leftarrow Y_2 \leftarrow Y_3 \leftarrow \ldots$$

be a sequence such that, for $i>1$, Y_i *is the normalization of a surface obtained by blowing up a singular point on* Y_{i-1}. *Then the sequence* (Σ) *is finite.*

Proof. — We follow the line of reasoning in [**22**]. A *normal sequence* in K is defined to be a sequence (finite or infinite)

$$R_1 < R_2 < R_3 < \ldots$$

of normal two-dimensional local rings with quotient field K such that, for $i>1$, R_i belongs to the normalization of the surface obtained by blowing up the maximal ideal of R_{i-1}. Such a normal sequence will be called *singular* if none of its members is regular. We are going to show that there are only finitely many singular normal sequences as above with $R_1 \in Y_1$. Since clearly any local ring which is blown up somewhere in the sequence (Σ) is a member of such a singular normal sequence, Theorem (2.1) will thereby be proved.

The *excellence* of Y will be needed only so that the following statement holds (cf. [**EGA** IV, (7.8.6)]):

For any W *birational and of finite type over* Y, (i) *the normalization* W′ *of* W *in* K *is finite over* W, *and* (ii) W′ *has only finitely many singular points* (1).

Now let $\mathbf{S}_n$ $(n \geq 1)$ be the set of surfaces defined inductively by:

$\mathbf{S}_1 = \{\mathrm{Spec}(R) \mid R \in Y_1,\ R \text{ not regular}\}$;

$\mathbf{S}_{i+1} = \{$normalizations of all those surfaces which can be obtained by blowing up a singular point on a member of $\mathbf{S}_i\}$.

By induction we see that $\mathbf{S}_n$ is a finite set for all n.

Let $\mathbf{T}_n$ $(n \geq 1)$ be the set of all local rings Q such that there exists a singular normal sequence

$$R_1 < R_2 < \ldots < R_n = Q$$

with $R_1 \in Y_1$. Each element of $\mathbf{T}_n$ is a singular point on one of the finitely many members of $\mathbf{S}_n$; hence $\mathbf{T}_n$ is a finite set for all n.

Suppose there were infinitely many singular normal sequences beginning with an element of $\mathbf{T}_1$. Since $\mathbf{T}_1$ is finite, there would be infinitely many beginning with a specific member R_1. Since $\mathbf{T}_2$ is finite, there would be some $R_2 \in \mathbf{T}_2$ such that among those sequences beginning with R_1, infinitely many begin with $R_1 < R_2 < \ldots$ Since $\mathbf{T}_3$ is finite, infinitely many of these latter sequences would begin with $R_1 < R_2 < R_3 < \ldots$ for some fixed $R_3 \in \mathbf{T}_3$. Continuing in this manner, we define R_4, R_5, ..., and so obtain an *infinite* singular normal sequence

$$R_1 < R_2 < R_3 < R_4 < R_5 < \ldots$$

Now $\bigcup_{i \geq 1} R_i$ *is a valuation ring* R_v. (This is proved in [**1**, p. 337] under the assumption that the R_i are regular. The proof in the general case is essentially the same; it can be reconstructed by piecing together the following information in [**25**]: Corollary, p. 21; Proposition 1, p. 330; Corollary 2, p. 339; and the argument in the middle of p. 392.) v dominates $R_1 \in Y_1$, and it follows from the hypotheses of Theorem (2.1) that v dominates a local ring of the form $B_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal in $B = R_1[b_1, b_2, \ldots, b_m]$, such that $B_{\mathfrak{p}}$ is essentially of finite type over a regular local ring A with quotient field K. Since $R_v = \bigcup_i R_i$, R_n contains B for all sufficiently large n, and since R_v dominates R_n, it follows at once that R_n dominates $B_{\mathfrak{p}}$. Then for $m \geq n$, R_m is essentially of finite type over A, whence, by 1) of Proposition (1.2), R_m has a rational singularity.

Let $f: X \to \mathrm{Spec}(R_n)$ be a desingularization. Since v dominates R_n, v dominates

(1) Even if Y is not assumed to be excellent, we find, using theorems of Rees [*J. London Math. Soc.*, **36** (1961), p. 27] and Nagata [**EGA** IV, (6.13.6)] that for (i) to hold it is sufficient that Nor(Y) contain a non-empty open set and that every local ring on Y be analytically unramified. For (ii) to hold it is sufficient that furthermore $\mathrm{Sing}(Y_1)$ be finite (use [**EGA** IV, (6.12.2)]). These conditions on Y and Y_1 certainly must be satisfied if Y can be desingularized (cf. Lemma (16.1)).

some $S \in X$, and then as above we see that R_N dominates S for some $N > n$. We shall show below that:

($\star$) *If* R *has a rational singularity, and* $f : X \to \mathrm{Spec}(R)$ *is a desingularization with* $X \neq \mathrm{Spec}(R)$, *then* X *dominates the quadratic transform* V *of* Spec(R) (i.e. V is the surface obtained by blowing up the maximal ideal of R), *and hence* X *dominates the normalization of* V.

Since no R_i is regular, an easy induction based on ($\star$) shows that S dominates $R_n, R_{n+1}, \ldots, R_N$. Hence $S = R_N$, contradicting the fact that R_N is not regular. This completes the proof, modulo ($\star$).

§ 3. Conclusion of the proof : a key proposition.

($\star$) is a basic point. A proof was given in [**22**, Lemma, p. 686] in the case of surfaces over an algebraically closed ground field of characteristic zero. More generally, ($\star$) follows from Theorem 4 in [**4**], at least if R has an algebraically closed residue field. For our purposes, however, a more direct proof is desirable. We will now prove a generalization of ($\star$) as a separate proposition.

First some notation. Let X be any normal surface and let $\mathscr{R}_X$ be the sheaf of rational functions on X. For any coherent $\mathcal{O}_X$-submodule $\mathscr{J} \neq (0)$ of $\mathscr{R}_X$, let $\mathscr{J}^{-1}$ be the coherent $\mathcal{O}_X$-submodule of $\mathscr{R}_X$ whose sections over any affine open $U \subseteq X$ are given by

$$\mathscr{J}^{-1}(U) = \{ a \in \mathscr{R}_X(U) \mid a\mathscr{J}(U) \subseteq \mathcal{O}_X(U) \}$$

Suppose $\mathscr{J}$ is an $\mathcal{O}_X$-ideal. Then $\mathscr{J}$ is a subsheaf of $(\mathscr{J}^{-1})^{-1}$; locally, $(\mathscr{J}^{-1})^{-1}$ is the intersection of those primary components of $\mathscr{J}$ which belong to height one prime ideals. We say that $\mathscr{J}$ is *divisorial* if $\mathscr{J} = (\mathscr{J}^{-1})^{-1}$.

Proposition (**3.1**). — *Let* R *be a local domain with maximal ideal* $\mathfrak{m} \neq (0)$, *and let* $f : X \to Y = \mathrm{Spec}(R)$ *be a proper map, not an isomorphism, with* $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, *where* X *is a normal surface such that* $H^1(X, \mathcal{O}_X) = 0$. *Then the* $\mathcal{O}_X$*-ideal* $\mathscr{M} = \mathfrak{m}\mathcal{O}_X$ *is divisorial.*

In particular, if the local rings on X *are factorial, then* $\mathfrak{m}\mathcal{O}_X$ *is invertible, so that f factors through the quadratic transform of* Spec(R).

(This proposition generalizes ($\star$) because of (2) of Proposition (1.2).)

Proof. — Write $\mathscr{M}'$ for $(\mathscr{M}^{-1})^{-1}$. Our aim is to show that $\mathscr{M}'/\mathscr{M} = 0$. In the first place, $\mathrm{Supp}(\mathscr{M}'/\mathscr{M})$ is a closed subset of X which clearly contains only points whose local ring on X has dimension > 1. Thus $\mathrm{Supp}(\mathscr{M}'/\mathscr{M})$ is at most zero-dimensional, and so $H^1(X, \mathscr{M}'/\mathscr{M}) = 0$. We have then the exact cohomology sequence

$$0 \to \Gamma(\mathscr{M}'/\mathscr{M}) \to \Gamma(\mathcal{O}_X/\mathscr{M}) \to \Gamma(\mathcal{O}_X/\mathscr{M}') \to H^1(X, \mathscr{M}'/\mathscr{M}) = 0.$$

I claim that $\Gamma(\mathcal{O}_X/\mathscr{M}) = R/\mathfrak{m}$. If this is granted, the sequence shows that one of the R-modules $\Gamma(\mathscr{M}'/\mathscr{M})$, $\Gamma(\mathcal{O}_X/\mathscr{M}')$, must be (0). But $\mathcal{O}_X/\mathscr{M}'$ has a non-zero global section unless $\mathscr{M}' = \mathcal{O}_X$, in which case $\mathcal{O}_X/\mathscr{M}$ has zero-dimensional support, which is impossible (since by assumption f is not an isomorphism) either by some form of Zariski's " main " theorem, for example [**EGA** III, (4.4.1)], or, in view of [**EGA** III, (4.2.2)], by Serre's criterion [**EGA** II, (5.2.2)]. Hence $\Gamma(\mathcal{M}'/\mathcal{M})=0$, and since $\mathcal{M}'/\mathcal{M}$ has at most zero-dimensional support, $\mathcal{M}'/\mathcal{M}=0$ as desired.

Now since $\mathcal{M}=\mathfrak{m}\mathcal{O}_X$, we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_X^n \to \mathcal{M} \to 0$$

for some positive integer n and some coherent $\mathcal{K}$. The fibres of f are of dimension ≤ 1, so $R^2f_*(\mathcal{K})=0$ ([**EGA** III, (4.2.2)]), i.e. $H^2(\mathcal{K})=0$. By hypothesis $H^1(\mathcal{O}_X^n)=0$. Hence $H^1(\mathcal{M})=0$, and therefore

$$\Gamma(\mathcal{O}_X/\mathcal{M})=\Gamma(\mathcal{O}_X)/\Gamma(\mathcal{M}).$$

But clearly

$$\mathfrak{m}\subseteq\Gamma(\mathcal{M})\subsetneqq\Gamma(\mathcal{O}_X)=R$$

so that, indeed, $\Gamma(\mathcal{O}_X/\mathcal{M})=R/\mathfrak{m}$.

This completes the proof of Proposition (3.1) and of Theorem (2.1).

§ 4. Resolution of rational singularities; factorization of proper birational maps into quadratic transformations.

For surfaces having only rational singularities the situation is described in the next theorem.

Theorem **(4.1)**. — *Let* Y *be a normal surface having only finitely many singular points, all of which are rational singularities. Then there exists a unique minimal desingularization* $f: X \to Y$ (i.e. *every desingularization of* Y *factors through* f) (1). *Moreover, any desingularization of* Y *is a product of quadratic transformations.*

Proof. — We shall assume the following result, to be proved later (Proposition (8.1)).

If Y *is a normal surface having only rational singularities, and if* $h: Y' \to Y$ *is a quadratic transformation, then* Y′ *is a normal surface.*

Now, blow up a singular point on Y (if there is one). Then blow up a singular point on the resulting surface. Continue in this way. The preceding result, along with 1) of Proposition (1.2), implies that the surfaces which arise are normal and have only rational singularities. Moreover, *these surfaces have only finitely many singular points.* (For any such point dominates one of the singular points y on Y, and so we need only see that any normal surface W which is proper and birational over $\mathrm{Spec}(\mathcal{O}_y)$ has only finitely many singular points. But such a surface W is dominated by a regular surface Z which is proper over $\mathrm{Spec}(\mathcal{O}_y)$ (cf. B) in the proof of Proposition (1.2)), and the desired conclusion follows from the fact that the local rings on W which are also on Z form an open subset of W which includes all one-dimensional local rings on W. Alternatively, use [**EGA** IV, (6.12.2)].)

As in Theorem (2.1), the preceding process leads eventually to a regular surface.

(1) Cf. also Corollary (27.3).

(The situation here is much simpler, since all the local rings involved already have rational singularities and since, also, normalization is superfluous.) So we have a desingularization $f: X \to Y$, and moreover, because of the way in which this desingularization is obtained, $(\star)$ (§ 2) shows that for every desingularization $f': X' \to Y$, X' dominates X.

It remains to be shown that we can get from X to X' by quadratic transformations alone. This is a classical result [**1**, **24**, **13**]. Actually, since a non-singular surface has only rational singularities we can reach the conclusion by arguing exactly as we have just done, except that we must replace " singular points " by " points which do not dominate X' ". Q.E.D.

Example. — For non-rational singularities the process of successive quadratic transformations and normalizations does not always lead to a *minimal* desingularization. As an example, take the origin on the surface $Z^2 + X^3 + Y^7 = 0$ (over any ground field). The corresponding local ring R is normal, and its maximal ideal $\mathfrak{m}$ has a basis of three elements x, y, z satisfying $z^2 + x^3 + y^7 = 0$.

In the ring $S = R[y^2/x, z/x, xy/z]$ consider the ideal $\mathfrak{p}$ generated by the elements x, $v = z/x$, $w = xy/z$. From the relations

$$x(1 + y(y^2/x)^3) + v^2 = 0$$
$$y = wv \in \mathfrak{p}$$
$$z = xv \in \mathfrak{p}$$
$$y^2/x = w^2v^2/x = -w^2(1 + y(y^2/x)^3) \in \mathfrak{p}$$

we see that $\mathfrak{m} \subseteq \mathfrak{p}$, and that $S/\mathfrak{p} \cong R/\mathfrak{m}$, so that $\mathfrak{p}$ is a maximal ideal and $S_\mathfrak{p}$ is a two-dimensional local domain dominating R. Further we see that $x = v^2.(\text{unit})$ in $S_\mathfrak{p}$, so that $\mathfrak{p}S_\mathfrak{p}$ is generated by the two elements v, w. Thus $S_\mathfrak{p}$ is regular. Now

$$\mathfrak{m}S_\mathfrak{p} = (x, y, z)S_\mathfrak{p} = (v^2, vw, v^3)S_\mathfrak{p}$$
$$= v.\mathfrak{p}S_\mathfrak{p}$$

Hence $\mathfrak{m}S_\mathfrak{p}$ is not principal, and so $S_\mathfrak{p}$ does not dominate the quadratic transform of R.

Actually this is only a local counterexample. To globalize, one can check that the normalization of the surface obtained by blowing up the ideal $(x, y^2, z)R$ is a non-singular surface on which $S_\mathfrak{p}$ is the only point which does not dominate the quadratic transform of R.

II. — COMPLETE AND CONTRACTED IDEALS

§ 5. Complete ideals and projective normalization [1].

A number of results in the sequel have to do with *complete ideals*. This notion can be conveniently described in terms of the integral closure of one graded ring in another. (A valuation-theoretic treatment is also possible, cf. [**25**, Appendix 4] and [**12**].) In this section and the next, we review the salient points and prepare the way for the theorems in § 7.

Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring and let $A = \bigoplus_{n \geq 0} A_n$ be a graded subring of R. Let A' be the integral closure of A in R. A' is a graded subring of R, i.e. $A' = \bigoplus_{n \geq 0} A'_n$ with

(1) In this section only, rings and schemes need not be noetherian (or even separated).

$A'_n = A' \cap R_n$ [**5**, p. 30]. For $d>0$ set $R^{(d)} = \bigoplus_{n\geq 0} R_{nd}$, $A^{(d)} = \bigoplus_{n\geq 0} A_{nd}$, $(A')^{(d)} = \bigoplus_{n\geq 0} A'_{nd}$. One sees easily that:

(i) $(A')^{(d)}$ *is the integral closure of* $A^{(d)}$ *in* $R^{(d)}$.

If M is a multiplicatively closed subset of A consisting of homogeneous elements, among them the element 1, then we denote by $A_{(M)}$ the set of all fractions a/m where $a \in A$ and $m \in M$ are of the same degree. ($A_{(M)}$ is a subring of the usual ring of fractions A_M.) The rings $A'_{(M)}$, $R_{(M)}$ are defined similarly. We have $A_{(M)} \subseteq A'_{(M)} \subseteq R_{(M)}$. It is straightforward to verify that:

(ii) $A'_{(M)}$ *is the integral closure of* $A_{(M)}$ *in* $R_{(M)}$.

Similar considerations apply when A and R are replaced by quasi-coherent graded $\mathcal{O}_X$-algebras $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n \subseteq \mathcal{R} = \bigoplus_{n\geq 0} \mathcal{R}_n$ on a (pre-)scheme X. Let $\mathcal{A}'$ be the integral closure of $\mathcal{A}$ in $\mathcal{R}$ (cf. [**EGA** II, § 6.3]). Over any affine open subset of X, this situation reduces to the previous one. In particular, $\mathcal{A}' = \bigoplus_{n\geq 0} (\mathcal{A}' \cap \mathcal{R}_n)$. In the present context, (ii) becomes:

(iii) *Let* $\mathcal{A}$, $\mathcal{R}$, $\mathcal{A}'$ *be as above, let* $\varphi : \mathcal{A} \to \mathcal{R}$ *be the inclusion map, and let* $\mathrm{Proj}(\varphi) : G(\varphi) \to \mathrm{Proj}(\mathcal{A})$ *be the associated affine morphism, where* $G(\varphi) \subseteq \mathrm{Proj}(\mathcal{R})$ [**EGA** II, (2.8.3)]. *Then the integral closure of* $\mathrm{Proj}(\mathcal{A})$ *in* $G(\varphi)$ *is* $\mathrm{Proj}(\mathcal{A}')$.

Now let X be an integral scheme, with sheaf of rational functions $\mathcal{R}_X$, and let $\mathcal{I}$ be a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{R}_X$. Set $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{I}^n$, $(\mathcal{I}^0 = \mathcal{O}_X)$, and $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{R}_X^n$, $(\mathcal{R}_X^0 = \mathcal{R}_X)$. $\mathcal{A}$ and $\mathcal{R}$ are quasi-coherent graded $\mathcal{O}_X$-algebras. ($\mathcal{R}$ is isomorphic to $\mathcal{R}_X[T]$, T an indeterminate.) Let $\mathcal{A}' = \bigoplus_{n\geq 0} \mathcal{I}_n$ be the integral closure of $\mathcal{A}$ in $\mathcal{R}$. Note that when X is *normal*, $\bigoplus_{n\geq 0} \mathcal{O}_X^n$ is integrally closed, so that *if* $\mathcal{I}$ *is an* $\mathcal{O}_X$*-ideal, then so is* $\mathcal{I}_n$ *for each* n.

Definition (**5.1**). — *Under the preceding circumstances we say that* $\mathcal{I}_1$ *is the* completion *of* $\mathcal{I}$. $\mathcal{I}$ *is* complete *if* $\mathcal{I} = \mathcal{I}_1$.

Remarks. — *a)* $\mathcal{I}_1$ itself is complete. In fact,

$$\mathcal{A} = \oplus \mathcal{I}^n \subseteq \oplus \mathcal{I}_1^n \subseteq \mathcal{A}'$$

so that $\mathcal{A}'$ is also the integral closure of $\oplus \mathcal{I}_1^n$.

b) By (i) above, we see that $\mathcal{I}_d$ is the completion of $\mathcal{I}^d$ for all $d>0$.

Remark *b)* shows that if all the positive powers of $\mathcal{I}$ are complete, then $\mathcal{A} = \mathcal{A}'$ except in degree zero; in other words $\mathrm{Proj}(\mathcal{A}') = \mathrm{Proj}(\mathcal{A})$. In applying (iii) to the present situation we observe that $\mathrm{Proj}(\mathcal{A})$ is now the scheme obtained by blowing up $\mathcal{I}$, and that $\mathrm{Proj}(\mathcal{R})$ is simply Spec(K) where K is the field of rational functions on X. We conclude:

Lemma (**5.2**). — *Let* X *be an integral scheme and let* $\mathcal{I} \neq 0$ *be a quasi-coherent* $\mathcal{O}_X$*-submodule of* $\mathcal{R}_X$. *If all the positive powers of* $\mathcal{I}$ *are complete, then the scheme obtained by blowing up* $\mathcal{I}$ *is normal.*

c) Once again let $\mathscr{I}$ be any quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{R}_X$. One checks that the completion $\mathscr{I}_1$ can be described as follows: for any open $U \subseteq X$, $\Gamma(U, \mathscr{I}_1)$ consists of those sections $s \in \Gamma(U, \mathscr{R}_X)$ for which there exists an open covering $\{V_\alpha\}$ of U with the property that for each α, if $s_\alpha = s \mid V_\alpha$, there is a relation

$$(s_\alpha)^{n_\alpha} + a_{1\alpha}(s_\alpha)^{n_\alpha - 1} + \ldots + a_{n_\alpha \alpha} = 0$$

with
$$a_{i\alpha} \in \Gamma(V_\alpha, \mathscr{I}^i) \qquad (i = 1, 2, \ldots, n_\alpha)$$

d) Let $\mathscr{I}$ be as in *c)*. If $\mathscr{J}$ is the completion of $\mathscr{I}$, then for any local ring S of a point on X, $\mathscr{J} \otimes S$ is the completion of $\mathscr{I} \otimes S$ on Spec(S).

e) Let $\mathscr{I}$ be as in *c)*, with completion $\mathscr{I}_1$, and let $\mathscr{L}$ be an *invertible* $\mathscr{O}_X$-submodule of $\mathscr{R}_X$. Then the completion of $\mathscr{I}\mathscr{L}$ is $\mathscr{I}_1\mathscr{L}$.

f) Let $f: X \to Y$ be a birational map. Let $\mathscr{J}$ be a quasi-coherent $\mathscr{O}_Y$-submodule of $\mathscr{R}_Y = \mathscr{R}_X$, with completion $\mathscr{J}_1$. Then $\mathscr{J}_1 \mathscr{O}_X \subseteq$ (completion of $\mathscr{J}\mathscr{O}_X$). (*Proof.* — We may assume X and Y to be affine, and apply *c)*.) In particular, if $\mathscr{J} = f_*(\mathscr{I})$ ($\mathscr{I}$ as in *c)*, with completion $\mathscr{I}_1$) then since $f_*(\mathscr{I}).\mathscr{O}_X \subseteq \mathscr{I}$, we have

$$\mathscr{J}_1 \mathscr{O}_X \subseteq (\text{completion of } f_*(\mathscr{I}).\mathscr{O}_X) \subseteq \mathscr{I}_1$$

from which follows

$$\mathscr{J}_1 \subseteq f_*(\mathscr{J}_1 \mathscr{O}_X) \subseteq f_*(\mathscr{I}_1)$$

Hence:

Lemma **(5.3)**. — *Let* X *be an integral scheme with sheaf of rational functions* $\mathscr{R}_X$, *let* $\mathscr{I}$ *be a quasi-coherent* $\mathscr{O}_X$*-submodule of* $\mathscr{R}_X$, *and let* $f: X \to Y$ *be a quasi-compact quasi-separated birational map (so that* $f_*(\mathscr{I})$ *is a quasi-coherent* $\mathscr{O}_Y$*-submodule of* $\mathscr{R}_Y = \mathscr{R}_X$*). If* $\mathscr{I}$ *is complete then so is* $f_*(\mathscr{I})$.

§ 6. Contracted Ideals.

Connected with complete ideals are *contracted* ideals.

Definition **(6.1)**. — *Let* $f: X \to Y$ *be a morphism of schemes such that* $f_*(\mathscr{O}_X) = \mathscr{O}_Y$. *Let* $\mathscr{I}$ *be an* $\mathscr{O}_Y$*-ideal. We say that* $\mathscr{I}$ is contracted for f *if* $\mathscr{I}$ *is of the form* $f_*(\mathscr{J})$ *for some* $\mathscr{O}_X$*-ideal* $\mathscr{J}$.

Suppose $\mathscr{I} = f_*(\mathscr{J})$. The commutative diagram

$$\begin{tikzcd}
f^*f_*(\mathscr{J}) \arrow[r] \arrow[d, equal] & \mathscr{J} \arrow[d, hook] \\
f^*(\mathscr{I}) \arrow[r, "v"] & \mathscr{O}_X
\end{tikzcd}$$

shows that $\mathscr{I}\mathcal{O}_X$ (= image of ν) $\subseteq \mathscr{J}$. Hence $f_*(\mathscr{I}\mathcal{O}_X) \subseteq f_*(\mathscr{J}) = \mathscr{I}$. On the other hand, we always have the commutative diagram

$$\begin{array}{ccccc} \mathscr{I} & \longrightarrow & f_*f^*(\mathscr{I}) & \xrightarrow{f_*(\nu)} & f_*(\mathscr{I}\mathcal{O}_X) \\ \cap\downarrow & & & & \cap\downarrow \\ \mathcal{O}_Y & & \xrightarrow{\quad\approx\quad} & & f_*(\mathcal{O}_X) \end{array}$$

which shows that $\mathscr{I} \subseteq f_*(\mathscr{I}\mathcal{O}_X)$. Thus *$\mathscr{I}$ is contracted for f if and only if $\mathscr{I} = f_*(\mathscr{I}\mathcal{O}_X)$.*

Proposition **(6.2)**. — *Let* Y *be an irreducible normal noetherian scheme. A coherent $\mathcal{O}_Y$-ideal $\mathscr{I}$ is complete if and only if $\mathscr{I}$ is contracted for every proper birational map $f: X \to Y$.*

Proof. — We may assume $\mathscr{I} \neq (0)$. Let $\mathscr{J}$ be the completion of $\mathscr{I}$; $\mathscr{J}$ is an $\mathcal{O}_Y$-ideal. The finite-type $\mathcal{O}_Y$-algebra $\bigoplus_{n\geq 0} \mathscr{J}^n$, being integral over $\bigoplus_{n\geq 0} \mathscr{I}^n$, is actually a finite-type module over $\bigoplus_{n\geq 0} \mathscr{I}^n$, whence, for large N, $\mathscr{I}\mathscr{J}^N = \mathscr{J}^{N+1}$. Let $X = \mathrm{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^n)$ and let $f: X \to Y$ be the structural morphism; f is proper and birational. Also, $\mathscr{J}^N\mathcal{O}_X$ is invertible, and

$$(\mathscr{I}\mathcal{O}_X)(\mathscr{J}^N\mathcal{O}_X) = \mathscr{I}\mathscr{J}^N\mathcal{O}_X = (\mathscr{J}\mathcal{O}_X)(\mathscr{J}^N\mathcal{O}_X)$$

so that
$$\mathscr{I}\mathcal{O}_X = \mathscr{J}\mathcal{O}_X.$$

Hence
$$\mathscr{J} \subseteq f_*(\mathscr{J}\mathcal{O}_X) = f_*(\mathscr{I}\mathcal{O}_X).$$

If $\mathscr{I}$ is contracted for f, then $\mathscr{J} \subseteq \mathscr{I}$, i.e. $\mathscr{I}$ is complete.

Now let $f: X \to Y$ be an arbitrary proper birational map. Let

$$Y' = \mathrm{Spec}(\bigoplus_{n\geq 0} \mathscr{I}^n) \qquad X' = \mathrm{Spec}(\bigoplus_{n\geq 0} (\mathscr{I}^n\mathcal{O}_X)).$$

Then X′ is a closed subscheme of $Y' \times_Y X$ so that the canonical map $X' \to Y'$ is proper. It is also birational. [**EGA** II, (7.3.11)] shows then that $\bigoplus_{n\geq 0} f_*(\mathscr{I}^n\mathcal{O}_X)$ is integral over $\bigoplus_{n\geq 0} \mathscr{I}^n$; in particular, $f_*(\mathscr{I}\mathcal{O}_X)$ is contained in the completion of $\mathscr{I}$. If $\mathscr{I}$ is complete, then consequently $f_*(\mathscr{I}\mathcal{O}_X) = \mathscr{I}$. Q.E.D.

Remark. — Let $f: X \to Y$ be as in the preceding proof. A similar argument shows more generally that *if $\mathscr{K}$ is a coherent $\mathcal{O}_X$-ideal contained in the completion of $\mathscr{I}\mathcal{O}_X$, then $f_*(\mathscr{K})$ is contained in the completion of $\mathscr{I}$.*

In § 8 we will need:

Lemma **(6.3)**. — *Let* Y *and* $\mathscr{I}$ *be as in Proposition* (6.2). *Suppose there exists a proper birational map $f: X \to Y$ with* X normal *such that $\mathscr{I}\mathcal{O}_X$ is invertible. Let* W *be the normalization of the scheme* Z *obtained by blowing up* $\mathscr{I}$. *Then* W *is of finite type over* Y.

Proof. — Let $\mathscr{J}$ be the completion of $\mathscr{I}$. Then $V = \mathrm{Proj}(\bigoplus_{n\geq 0} \mathscr{J}^n)$ is finite and birational over $\mathrm{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^n)$, so that X dominates V, i.e. $\mathscr{J}\mathcal{O}_X$ is invertible. As in

Proposition (6.2), therefore, $\mathscr{J}=f_*(\mathscr{J}\mathcal{O}_X)=f_*(\mathscr{I}\mathcal{O}_X)$. Similarly, for every $n\geq 1$ the completion of $\mathscr{I}^n$ is $\mathscr{I}_n=f_*(\mathscr{I}^n\mathcal{O}_X)$. Thus by (iii) of § 5 and the remark *b)* following it,

$$W = \mathrm{Proj}(\bigoplus_{n\geq 0}\mathscr{I}_n) = \mathrm{Proj}(\bigoplus_{n\geq 0} f_*(\mathscr{I}^n\mathcal{O}_X))$$

Now [**EGA** III, (3.3.1)] shows that W is finite over $\mathrm{Proj}(\bigoplus_{n\geq 0}\mathscr{I}^n)$. Q.E.D. (1).

Corollary **(6.4)**. — *Let* Y *be a normal surface for which there exists a desingularization* $f: X\to Y$. *Let* $g: W\to Y$ *be a birational map of finite type. Then the normalization of* W *is finite over* W (2).

Proof. — The question is local on both Y and W, so we may assume that Y is affine and integral and that g is projective. Then W is obtained by blowing up a coherent ideal $\mathscr{I}$ on Y, and so, as in (6.3), it will be sufficient to find a desingularization Z of Y such that $\mathscr{I}\mathcal{O}_Z$ is invertible. This can be done as in B) of Proposition (1.2) (with Y in place of Spec(R)). Q.E.D.

As another application of (6.2), we generalize Proposition 5 of [**25**, p. 381].

Proposition **(6.5)**. — *Let* $f: X\to Y$ *be a birational proper map of irreducible normal surfaces such that* $R^1f_*(\mathcal{O}_X)=0$. *If* $\mathscr{I}$ *is a complete coherent ideal on* Y, *then* $\mathscr{I}\mathcal{O}_X$ *is a complete ideal on* X.

Proof. — Let $g: W\to X$ be a birational proper map. Then $\mathscr{I}\mathcal{O}_X\subseteq g_*(\mathscr{I}\mathcal{O}_W)$ so that there is an exact sequence

$$0\to \mathscr{I}\mathcal{O}_X\to g_*(\mathscr{I}\mathcal{O}_W)\to \mathscr{K}\to 0.$$

It will be sufficient, by Proposition (6.2), to show that $\mathscr{K}=0$. On X, there is a finite set S of closed points such that g induces an isomorphism from $g^{-1}(X-S)$ to $X-S$; hence the support of $\mathscr{K}$ is contained in S. Thus, if $f_*(\mathscr{K})=0$, then $\mathscr{K}=0$.

Applying f_* to the above sequence, we get an exact sequence

$$0\to f_*(\mathscr{I}\mathcal{O}_X)\xrightarrow{\alpha} f_*g_*(\mathscr{I}\mathcal{O}_W)\to f_*(\mathscr{K})\to R^1f_*(\mathscr{I}\mathcal{O}_X).$$

But $f_*g_*(\mathscr{I}\mathcal{O}_W)=f_*(\mathscr{I}\mathcal{O}_X)=\mathscr{I}$ since $\mathscr{I}$ is complete (Proposition (6.2)), i.e. α is an isomorphism. Therefore, it is enough to show that $R^1f_*(\mathscr{I}\mathcal{O}_X)=0$. For this purpose we may assume Y to be affine, and then $\mathscr{I}\mathcal{O}_X$ is a homomorphic image of $\mathcal{O}_X^n$ for some n. Since R^2f_* vanishes for all coherent sheaves on X, we conclude that $R^1f_*(\mathscr{I}\mathcal{O}_X)$ is a homomorphic image of $R^1f_*(\mathcal{O}_X^n)=0$. Q.E.D.

§ 7. Products of complete and contracted ideals.

Our main purpose in this section is to prove:

Theorem **(7.1)**. — *Let* Y *be a normal irreducible surface having only rational singularities. Then any product of complete coherent* $\mathcal{O}_Y$*-ideals is again complete.*

Theorem (7.1) is a consequence of:

Theorem **(7.2)**. — *Let* Y *be an integral noetherian scheme, let* X *be a normal surface, and let* $f: X\to Y$ *be a proper map with* $f_*(\mathcal{O}_X)=\mathcal{O}_Y$, $R^1f_*(\mathcal{O}_X)=0$. *Then any product of coherent contracted (for f)* $\mathcal{O}_Y$*-ideals is again contracted.*

---

(1) (Added in proof.) It would be simpler to note that X dominates Z, say $g: X\to Z$, and then $W=\mathrm{Spec}(g_*(\mathcal{O}_X))$ is finite over Z.

(2) Corollary (6.4) holds without the assumption that Y is a surface; the proof is indicated in the footnote in § 2.

To deduce Theorem (7.1) from Theorem (7.2), we first remark that it is enough to treat the case $Y = \mathrm{Spec}(S)$, S being a two-dimensional normal local domain with a rational singularity (cf. remark *d)* following Definition (5.1)). Let $\mathcal{I}$, $\mathcal{K}$ be coherent complete $\mathcal{O}_Y$-ideals; it will be sufficient to show that $\mathcal{I}\mathcal{K}$ is complete. Let $\mathcal{J}$ be the completion of $\mathcal{I}\mathcal{K}$ and let W be obtained by blowing up $\mathcal{J}$. There exists a desingularization $f: X \to Y$, and moreover, after applying suitable quadratic transformations, we may assume that X dominates W (cf. B) in the proof of Proposition (1.2)), so that $\mathcal{J}\mathcal{O}_X$ is invertible. Y being affine, the requirement $R^1 f_*(\mathcal{O}_X) = 0$ means simply $H^1(X, \mathcal{O}_X) = 0$, which is certainly satisfied in this case (Proposition (1.2)). So Theorem (7.2) applies: since $\mathcal{I}$ and $\mathcal{K}$ are complete, they are contracted for f (Proposition (6.2)) and so then is $\mathcal{I}\mathcal{K}$. Since $\mathcal{J}\mathcal{O}_X$ is invertible, the first part of the proof of Proposition (6.2) (with $\mathcal{I}$ replaced by $\mathcal{I}\mathcal{K}$) shows that $\mathcal{I}\mathcal{K}$ is complete. Q.E.D.

We begin the *proof of Theorem* (7.2) by collecting together the technical details in:

Lemma (**7.3**). — *Let* X *be any scheme (not necessarily noetherian) for which*:

1) $H^1(\mathcal{O}_X) = 0$, *and*

2) $H^2(\mathcal{L}) = 0$ *for all quasi-coherent* $\mathcal{O}_X$*-modules* $\mathcal{L}$ *such that* $\mathcal{L} \subseteq \mathcal{O}_X^n$, *n finite* (1).

Let $\mathcal{I}$ *and* $\mathcal{J}$ *be two quasi-coherent* $\mathcal{O}_X$*-modules such that*

3) $\Gamma(\mathcal{I})$ *and* $\Gamma(\mathcal{J})$ *are finitely generated* $\Gamma(\mathcal{O}_X)$*-modules, and*

4) $\mathcal{I}$ *and* $\mathcal{J}$ *are generated by their global sections.*

Then the canonical map

$$p : \Gamma(\mathcal{I}) \otimes_{\Gamma(\mathcal{O}_X)} \Gamma(\mathcal{J}) \to \Gamma(\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{J})$$

is surjective.

We will use Lemma (7.3) via:

Corollary. — *Suppose* $\mathcal{I}$ *and* $\mathcal{J}$ *are* $\mathcal{O}_X$*-ideals and let* $\mathcal{I}\mathcal{J}$ *be the image of the natural map* $\mu : \mathcal{I} \otimes \mathcal{J} \to \mathcal{O}_X$. *If, in addition to the conditions of Lemma* (7.3), *we have* $H^1(\text{kernel of } \mu) = 0$, *then*

$$\Gamma(\mathcal{I}) . \Gamma(\mathcal{J}) = \Gamma(\mathcal{I}\mathcal{J}).$$

Indeed, if the Lemma holds, then the composite map

$$\Gamma(\mathcal{I}) \otimes \Gamma(\mathcal{J}) \xrightarrow{p} \Gamma(\mathcal{I} \otimes \mathcal{J}) \xrightarrow{\Gamma(\mu)} \Gamma(\mathcal{I}\mathcal{J})$$

is surjective, and the Corollary results.

Proof of Lemma (**7.3**). — By 3) and 4) there are exact sequences

$$0 \to \mathcal{K}_1 \to \mathcal{O}_X^s \xrightarrow{\alpha} \mathcal{I} \to 0$$

$$0 \to \mathcal{K}_2 \to \mathcal{O}_X^t \xrightarrow{\beta} \mathcal{J} \to 0$$

(1) A simple argument [**EGA** II, top of p. 98] shows that 2) need only be assumed for $n = 1$.

(s, t finite) such that in the derived exact sequences

$$\Gamma(\mathcal{O}_X^s) \xrightarrow{\gamma} \Gamma(\mathcal{I}) \to H^1(\mathcal{K}_1) \to H^1(\mathcal{O}_X^s) = 0$$
$$\Gamma(\mathcal{O}_X^t) \xrightarrow{\delta} \Gamma(\mathcal{J}) \to H^1(\mathcal{K}_2) \to H^1(\mathcal{O}_X^t) = 0$$

the maps γ and δ are surjective. Necessarily, then, $H^1(\mathcal{K}_1) = H^1(\mathcal{K}_2) = 0$.

Let $\mathcal{N}$ be the kernel of $\alpha \otimes \beta$. The exact sequence

$$\mathcal{K} = (\mathcal{K}_1 \otimes \mathcal{O}_X^t) \oplus (\mathcal{K}_2 \otimes \mathcal{O}_X^s) \to \mathcal{O}_X^s \otimes \mathcal{O}_X^t \xrightarrow{\alpha\otimes\beta} \mathcal{I} \otimes \mathcal{J} \to 0$$

gives an exact sequence

$$0 \to \mathcal{L}' \to \mathcal{K} \to \mathcal{N} \to 0.$$

But $H^1(\mathcal{K}) = 0$, and, since $\mathcal{L}' \subseteq \mathcal{K} \subseteq \mathcal{O}_X^{st} \oplus \mathcal{O}_X^{ts}$, 2) gives $H^2(\mathcal{L}') = 0$. Hence $H^1(\mathcal{N}) = 0$ and so the map $\Gamma(\alpha \otimes \beta)$ is surjective. From the commutative diagram

$$\begin{array}{ccc} \Gamma(\mathcal{O}_X^s) \otimes \Gamma(\mathcal{O}_X^t) & \xrightarrow{\approx} & \Gamma(\mathcal{O}_X^s \otimes \mathcal{O}_X^t) \\ \downarrow & & \downarrow {\scriptstyle \Gamma(\alpha\otimes\beta)} \\ \Gamma(\mathcal{I}) \otimes \Gamma(\mathcal{J}) & \xrightarrow{p} & \Gamma(\mathcal{I} \otimes \mathcal{J}) \end{array}$$

we see then that p is surjective. Q.E.D.

Now let $f: X \to Y$ be as in Theorem (7.2), let I, J be coherent $\mathcal{O}_Y$-ideals, and let $\mathcal{I} = I\mathcal{O}_X$, $\mathcal{J} = J\mathcal{O}_X$. Theorem (7.2) is proved if we show that $f_*(\mathcal{I}).f_*(\mathcal{J}) = f_*(\mathcal{I}\mathcal{J})$.

To do so, we may assume that Y is affine and of dimension >0, and then we must show that $\Gamma(\mathcal{I}).\Gamma(\mathcal{J}) = \Gamma(\mathcal{I}\mathcal{J})$. The conditions of Lemma (7.3) are now satisfied. (H^2 vanishes because f is dominant, so the fibres of f are of dimension ≤ 1). As for the kernel of $\mu : \mathcal{I} \otimes \mathcal{J} \to \mathcal{I}\mathcal{J}$, we remark that unless $IJ\mathcal{O}_X = (0)$, both $I\mathcal{O}_{X,x}$ and $J\mathcal{O}_{X,x}$ are invertible whenever $x \in X$ is such that $\dim.\mathcal{O}_{X,x} \leq 1$; thus in any case, the kernel of μ has at most zero-dimensional support, and the Corollary applies. This completes the proof of Theorem (7.2).

In part III we will refer to the following consequence of Lemma (7.3):

Corollary **(7.4)**. — *Let* A *be a ring, let* X *be a quasi-compact separated scheme satisfying* 1) *and* 2) *of Lemma* (7.3) *and let* $g: X \to \mathrm{Spec}(A)$ *be a morphism. Let* $\mathcal{L}$ *be an invertible* $\mathcal{O}_X$*-module such that* $\Gamma(\mathcal{L})$ *is a finitely generated* A*-module. If* $\mathcal{L}$ *is ample and* $\mathcal{L}$ *is generated by its global sections, then* $\mathcal{L}$ *is very ample for* g *(and, of course, conversely).*

Proof. — By [**EGA** II, (4.6.3) and (4.4.3)] it is enough to show that the graded A-algebra $\bigoplus_{n\geq 0} \Gamma(\mathcal{L}^{\otimes n})$ is generated by $\Gamma(\mathcal{L})$ over A, i.e. that

$$\Gamma(\mathcal{L}) \otimes_A \Gamma(\mathcal{L}) \otimes_A \ldots \otimes_A \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L}^{\otimes n})$$

is surjective for each n. But this follows by induction from Lemma (7.3) (with $\mathcal{I} = \mathcal{L}$, $\mathcal{J} = \mathcal{L}^{\otimes k}$ $(k = 1, 2, 3, \ldots)$).

§ 8. Normality of blowing-up and join.

Throughout this section Y *will be a normal irreducible surface having only rational singularities.*

Proposition **(8.1)**. — *Let* Y *be as above and let* $\mathscr{I}$ *be a complete coherent* $\mathcal{O}_{\mathrm{Y}}$*-ideal. Then* $\mathrm{X}=\mathrm{Proj}(\bigoplus_{n\geq 0}\mathscr{I}^n)$ *is normal (whence* X *has only rational singularities). In particular, any quadratic transform of* Y *is normal.*

Proof. — The first assertion follows from Theorem (7.1) and Lemma (5.2). Let y be a closed point of Y, and let $\mathscr{I}$ be the ideal whose stalk at $x\in\mathrm{Y}$ is $\mathcal{O}_x$ if $x\neq y$, and whose stalk at y is the maximal ideal of $\mathcal{O}_y$. The surface obtained by blowing up y is $\mathrm{Proj}(\bigoplus_{n\geq 0}\mathscr{I}^n)$. But it follows easily from the definitions that $\mathscr{I}$ is complete. Q.E.D.

Let K be the field of rational functions on Y. Given two surfaces X_1 and X_2 dominating Y birationally, we say that the closed image X of the canonical map $\mathrm{Spec}(\mathrm{K})\to\mathrm{X}_1\times_{\mathrm{Y}}\mathrm{X}_2$ is the *join* of X_1 and X_2 over Y. A surface Z with function field K dominates X if and only if it dominates both X_1 and X_2.

If X_1 and X_2 are obtained by blowing up $\mathcal{O}_{\mathrm{Y}}$-ideals $\mathscr{I}$, $\mathscr{J}$ respectively, then Z dominates X_1 (resp. X_2) if and only if $\mathscr{I}\mathcal{O}_{\mathrm{Z}}$ (resp. $\mathscr{J}\mathcal{O}_{\mathrm{Z}}$) is an invertible $\mathcal{O}_{\mathrm{Z}}$-ideal; on the other hand, $\mathscr{I}\mathcal{O}_{\mathrm{Z}}$ and $\mathscr{J}\mathcal{O}_{\mathrm{Z}}$ are *both* invertible if and only if $(\mathscr{I}\mathscr{J})\mathcal{O}_{\mathrm{Z}}$ is invertible; it follows at once that X is obtained by blowing up the $\mathcal{O}_{\mathrm{Y}}$-ideal $\mathscr{I}\mathscr{J}$. Now if X_1 is normal, then the considerations of Lemma (6.3) show that X_1 is also obtained by blowing up the completion of $\mathscr{I}$; in other words, we may assume $\mathscr{I}$ to be complete. Similarly, if X_2 is normal, then we may assume $\mathscr{J}$ to be complete. If both $\mathscr{I}$ and $\mathscr{J}$ are complete, then Theorem (7.1) shows that $\mathscr{I}\mathscr{J}$ is complete; thus (Proposition (8.1)), the join of X_1 and X_2 is normal.

More generally, we have:

Proposition **(8.2)**. — *Let* Y *be as above and let* X_1, X_2 *be any normal surfaces which are birational and of finite type over* Y. *Then the join of* X_1 *and* X_2 *over* Y *is also normal.*

Proof. — The question is local on X_1 and X_2, so we may assume X_1 and X_2 to be dense open subschemes of $\overline{\mathrm{X}}_1$, $\overline{\mathrm{X}}_2$ respectively, where $\overline{\mathrm{X}}_1$, $\overline{\mathrm{X}}_2$ are projective over Y. Because of (6.4) we may replace $\overline{\mathrm{X}}_1$, $\overline{\mathrm{X}}_2$ by their normaltiazions (this does not affect X_1 or X_2). In other words, we may assume to begin with that X_1 and X_2 are projective over Y. We may also assume that Y is affine. But then the situation is the one dealt with in the preceding discussion. Q.E.D.

§ 9. Pseudo-rational singularities.

It is conceivable that one might wish to apply Theorem (7.1) and the results in § 8 to a surface without knowing *a priori* that the points of the surface can be desingularized. For this purpose, the definition of rational singularity can be weakened: say that a two-dimensional normal local domain R has a *pseudo-rational singularity* if the following condition holds:

For any projective birational map $g: \mathrm{W} \rightarrow \mathrm{Spec}(\mathrm{R})$ *there exists a normal surface* Z, *proper and birational over* Spec(R), *such that* Z *dominates* W *and* $\mathrm{H}^1(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}}) = 0$.

A) and B) in the proof of Proposition (1.2) show that if R has a rational singularity then R has a pseudo-rational singularity. Conversely, that proof shows that if R has a pseudo-rational singularity and R can be desingularized then R has a rational singularity (use Chow's Lemma). The analytically reducible normal local ring described by Nagata in [**17**; Example 7, p. 209] has a pseudo-rational, but not a rational, singularity. (To prove this, one can use Proposition (23.5).)

Using Chow's Lemma and the fact that projective birational maps into integral affine schemes are obtained by blowing up suitable ideals, one can prove without difficulty the analogue of Proposition (1.2) for pseudo-rational singularities. Theorem (7.1) and the results in § 8 hold for surfaces having only pseudo-rational singularities. The proofs are practically the same.

III. — NUMERICAL THEORY OF RATIONAL EXCEPTIONAL CURVES

The goal of part III is Theorem (12.1), whose statement will be essential later on. In § 10 we review some well known facts about degrees of locally free sheaves on one-dimensional schemes. (These facts are required in §§ 11-12, and also in part IV in connection with intersection theory.) In Proposition (11.1) we see that the numerical characters of an invertible sheaf on a " rational " one-dimensional scheme determine whether the sheaf has enough global sections; this generalizes some of Theorem (1.7) of [**3**]. Theorem (12.1) is a relative version of Proposition (11.1); it is related in part to Theorem 4 of [**4**].

§ 10. Degrees of locally free sheaves on curves.

To begin with, we fix some terminology and notations. By a *curve* we mean a one-dimensional noetherian scheme. A *component* of a curve C is a *one-dimensional* closed subscheme of C which is *integral* (i.e. reduced and irreducible) — the components of C are in one-one correspondence with the (finitely many) non-closed points of C.

We will be dealing mainly with curves C which admit a proper map $f: \mathrm{C} \rightarrow \mathrm{Spec}(\mathrm{A})$ where A is a noetherian ring and the image of f is zero-dimensional, i.e. is a finite set of closed points. For such a C, the cohomology modules of any *coherent* $\mathcal{O}_{\mathrm{C}}$-module $\mathcal{F}$ are of finite length over A; it makes sense therefore to talk about $h^0(\mathcal{F})$ and $h^1(\mathcal{F})$ (the lengths of $\mathrm{H}^0(\mathcal{F})$, $\mathrm{H}^1(\mathcal{F})$ respectively), and about the Euler-Poincaré characteristic $\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F})$. (Of course h^0, h^1, and χ depend on the choice of A and f.) The *degree* of a *locally free* $\mathcal{O}_{\mathrm{C}}$-module $\mathcal{N}$ *of finite rank* n is defined to be the integer

$$\deg_{\mathrm{C}}(\mathcal{N}) = \chi(\mathcal{N}) - \chi(\mathcal{O}_{\mathrm{C}}^n) = \chi(\mathcal{N}) - n\chi(\mathcal{O}_{\mathrm{C}}).$$

Some basic properties of " degree " will follow easily from the next lemma. We say that two coherent $\mathcal{O}_C$-modules $\mathscr{F}$ and $\mathscr{G}$ are *generically isomorphic* if there is an open subset U of C such that $C-U$ is zero-dimensional and such that the restrictions $\mathscr{F}|U$ and $\mathscr{G}|U$ are isomorphic. It is the same thing to say that for each non-closed point x of C, the stalks $\mathscr{F}_x$ and $\mathscr{G}_x$ are isomorphic $\mathcal{O}_x$-modules.

Lemma **(10.1)**. — *Let $f: C \to \operatorname{Spec}(A)$ be as above, and let $\mathscr{F}$ and $\mathscr{G}$ be two coherent $\mathcal{O}_C$-modules which are generically isomorphic. Then for any locally free $\mathcal{O}_C$-module $\mathcal{N}$ of finite rank n, we have*

$$\chi(\mathscr{F} \oplus \mathcal{N}) - \chi(\mathscr{G} \otimes \mathcal{N}) = n(\chi(\mathscr{F}) - \chi(\mathscr{G}))$$

Proof. — By assumption there is an open subset U of C, with inclusion map, say, $i: U \to C$, such that $C-U$ is zero-dimensional and such that $i^*(\mathscr{F})$ and $i^*(\mathscr{G})$ are isomorphic. We have exact sequences

$$\begin{array}{c} 0 \to \mathscr{K}_1 \to \mathscr{F} \to i_*i^*(\mathscr{F}) \to \mathscr{K}_2 \to 0 \\ \| \wr \\ 0 \to \mathscr{K}_3 \to \mathscr{G} \to i_*i^*(\mathscr{G}) \to \mathscr{K}_4 \to 0 \end{array}$$

where, for $i = 1, 2, 3, 4$, $\mathscr{K}_i$ is concentrated on $C-U$ (so that $\chi(\mathscr{K}_i \otimes \mathcal{N}) = n\chi(\mathscr{K}_i)$). The conclusion is obtained by tensoring these exact sequences with $\mathcal{N}$ and taking Euler-Poincaré characteristics. Q.E.D.

Proposition **(10.2)**. — *Let $f: C \to \operatorname{Spec}(A)$ be as above:*

a) *If $\mathcal{M}$ and $\mathcal{N}$ are locally free $\mathcal{O}_C$-modules of finite ranks m, n respectively, then*

$$\deg_C(\mathcal{M} \otimes \mathcal{N}) = n.\deg_C(\mathcal{M}) + m.\deg_C(\mathcal{N}).$$

In particular, if $\mathcal{M}$ and $\mathcal{N}$ are invertible *$\mathcal{O}_C$-modules, then*

$$\deg_C(\mathcal{M} \otimes \mathcal{N}) = \deg_C(\mathcal{M}) + \deg_C(\mathcal{N}).$$

b) *If $h: C' \to C$ is a proper map, with C' a curve, such that $h_*(\mathcal{O}_{C'})$ is generically isomorphic to $\mathcal{O}_C^t$ for some integer $t \geq 0$ (which is always the case, for example, if C is* integral) *then for any locally free $\mathcal{O}_C$-module $\mathcal{N}$ (of rank $n < \infty$)*

$$\deg_{C'}(h^*\mathcal{N}) = t.\deg_C(\mathcal{N})$$

(Here " degree " on C' is relative to the map $f \circ h : C' \to \operatorname{Spec}(A)$.)

Proof. — *a)* By Lemma (10.1) (with $\mathscr{F} = \mathcal{M}$, $\mathscr{G} = \mathcal{O}_C^m$)

$$\chi(\mathcal{M} \otimes \mathcal{N}) - \chi(\mathcal{O}_C^m \otimes \mathcal{N}) = n(\chi(\mathcal{M}) - \chi(\mathcal{O}_C^m))$$

i.e.
$$\deg_C(\mathcal{M} \otimes \mathcal{N}) + mn\chi(\mathcal{O}_C) - m\chi(\mathcal{N}) = n.\deg_C(\mathcal{M})$$

i.e.
$$\deg_C(\mathcal{M} \otimes \mathcal{N}) - m \deg_C(\mathcal{N}) = n.\deg_C(\mathcal{M}).$$

b) The standard exact sequence

$$0 \to H^1(C, h_*h^*\mathcal{N}) \to H^1(C', h^*\mathcal{N}) \to H^0(C, R^1h_*(h^*\mathcal{N})) \to 0$$

along with the isomorphism

$$H^0(C, h_* h^* \mathcal{N}) \xrightarrow{\sim} H^0(C', h^* \mathcal{N})$$

gives
$$\chi(h_* h^* \mathcal{N}) - \chi(h^* \mathcal{N}) = h^0(R^1 h_*(h^* \mathcal{N}))$$
i.e.
$$\chi(h_*(\mathcal{O}_{C'}) \otimes \mathcal{N}) - \chi(h^* \mathcal{N}) = h^0(R^1 h_*(\mathcal{O}_{C'}) \otimes \mathcal{N})$$

(cf. [**EGA** 0_{III}, (12.2.3)]). Since C′ is one-dimensional, at most finitely many of the fibres of h are one-dimensional; consequently $R^1 h_*(\mathcal{O}_{C'})$ has support of dimension ≤ 0, and so

$$\chi(h_*(\mathcal{O}_{C'}) \otimes \mathcal{N}) - \chi(h^* \mathcal{N}) = n \,.\, h^0(R^1 h_*(\mathcal{O}_{C'})) = n \,.\, (\chi(h_*(\mathcal{O}_{C'})) - \chi(\mathcal{O}_{C'}))$$

(Take $\mathcal{N} = \mathcal{O}_C$ to get the last equality.) Thus

$$\begin{aligned} \deg_{C'}(h^* \mathcal{N}) &= \chi(h^* \mathcal{N}) - n\chi(\mathcal{O}_{C'}) \\ &= \chi(h_*(\mathcal{O}_{C'}) \otimes \mathcal{N}) - n\chi(h_*(\mathcal{O}_{C'})) \\ &= \chi(\mathcal{O}_C^t \otimes \mathcal{N}) - n\chi(\mathcal{O}_C^t) \qquad \text{(Lemma (10.1))} \\ &= t(\chi(\mathcal{N}) - n\chi(\mathcal{O}_C)) \\ &= t \,.\, \deg_C(\mathcal{N}). \end{aligned}$$

Q.E.D.

For a (Cartier) divisor $\mathcal{D}$ on C (cf. [**16**, Lecture 9] or [**EGA** IV, § 21]) we define the degree, $\deg_C(\mathcal{D})$, to be the degree of the corresponding invertible sheaf $\mathcal{O}_C(\mathcal{D})$.

Corollary (**10.3**). — *If $\mathcal{D}$ is an* effective (i.e. positive) *divisor on* C, *then*

$$\deg_C(\mathcal{D}) = h^0(\mathcal{O}_{\mathcal{D}}).$$

Proof. — By definition, there is an exact sequence

$$0 \to \mathcal{O}_C(-\mathcal{D}) \to \mathcal{O}_C \to \mathcal{O}_{\mathcal{D}} \to 0$$

whence
$$\begin{aligned} \deg_C(\mathcal{D}) &= \deg_C(\mathcal{O}_C(\mathcal{D})) \\ &= -\deg_C(\mathcal{O}_C(-\mathcal{D})) \qquad \text{(Proposition (10.2) } a)) \\ &= \chi(\mathcal{O}_C) - \chi(\mathcal{O}_C(-\mathcal{D})) \\ &= \chi(\mathcal{O}_{\mathcal{D}}) \\ &= h^0(\mathcal{O}_{\mathcal{D}}) \end{aligned}$$

Q.E.D.

Some more known facts about degrees of invertible sheaves are collected together in the next proposition.

Proposition (**10.4**). — *Let* $f: C \to \operatorname{Spec}(A)$ *be as above, let* $C_1, C_2, \ldots, C_n$ *be the components of the curve* C, *and let* $\varepsilon_i : C_i \to C$ $(i = 1, 2, \ldots, n)$ *be the corresponding inclusion maps. Let* $\mathcal{L}$ *be an invertible sheaf on* C, *and for* $i = 1, 2, \ldots, n$ *let*

$$\delta_i = \deg_{C_i}(\varepsilon_i^*(\mathcal{L}))$$

(i) *If* $\mathcal{L} \cong \mathcal{O}_C$, *then* $\delta_i = 0$ *for all* i.

(ii) *If* $\mathcal{L}$ *is generated by its sections over* C, *then* $\delta_i \geq 0$ *for all* i.

(iii) $\mathcal{L}$ *is ample if and only if* $\delta_i > 0$ *for all* i.

Proof. — The hard part of the proposition is the implication " $\delta_i > 0$ for all $i \Rightarrow \mathcal{L}$ ample ", which can be proved as in [**11**, p. 318-319: proof that (iv) implies (i)].

For the rest of the proof, we may assume that C is integral (cf. [**EGA** II, (4.6.13) (i *bis*)]). (i) is then obvious. Under the condition of (ii), there must be an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{L} \to \mathcal{K} \to 0$$

where $\mathcal{K}$ has support of dimension ≤ 0, and so

$$\deg_C(\mathcal{L}) = \chi(\mathcal{L}) - \chi(\mathcal{O}_C) = \chi(\mathcal{K}) = h^0(\mathcal{K}) \geq 0.$$

Note that if $\deg_C(\mathcal{L}) = 0$ in this case, then $\mathcal{K} = 0$, i.e. $\mathcal{L} \cong \mathcal{O}_C$.

Now suppose that $\mathcal{L}$ is ample. Because of the additivity of degree, we may replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$ for any $n > 0$; we may therefore assume that $\mathcal{L}$ has non-zero global sections, and then the preceding argument shows that $\deg_C(\mathcal{L}) > 0$. (Here there is strict inequality because, $\Gamma(C, \mathcal{O}_C)$ being artinian, $\mathcal{O}_C$ is not ample [**EGA** II, (5.1.2)]). Q.E.D.

§ 11. Numerical theory of rational curves.

We will be especially interested in a situation in which Proposition (10.4) admits a converse.

Proposition (**11.1**). — *With the notation of Proposition* (10.4), *assume that* $H^1(C, \mathcal{O}_C) = 0$:

(i) *If* $\delta_i = 0$ *for all* i, *then* $\mathcal{L} \cong \mathcal{O}_C$.

(ii) *If* $\delta_i \geq 0$ *for all* i, *then* $\mathcal{L}$ *is generated by its global sections, and* $H^1(\mathcal{L}) = 0$.

(iii) *If* $\delta_i > 0$ *for all* i, *then* $\mathcal{L}$ *is very ample for* f.

Proof. — If $\mathcal{L}$ is generated by its global sections, then $\mathcal{L}$ is a homomorphic image of $\mathcal{O}_C^s$ for some integer $s > 0$, so that $H^1(\mathcal{L})$ is a homomorphic image of $H^1(\mathcal{O}_C^s) = 0$; i.e. $H^1(\mathcal{L}) = 0$. (ii) implies (iii) in view of Proposition (10.4) and Corollary (7.4). Moreover (ii) implies (i) because of the following simple fact:

Let C *be any locally noetherian (pre-)scheme such that* $H^0(C, \mathcal{O}_C)$ *is artinian. Let* $\mathcal{L}$ *be an invertible sheaf on* C *such that both* $\mathcal{L}$ *and* $\mathcal{L}^{-1}$ *are generated by their global sections. Then* $\mathcal{L} \cong \mathcal{O}_C$.

(*Proof.* — Since the connected components of C are open [**EGA** I, (6.1.9)], we may assume that C is connected. Then the image of the canonical map $C \to \mathrm{Spec}(H^0(\mathcal{O}_C))$ is connected, hence consists of one point, and it follows that $H^0(\mathcal{O}_C)$ is a *local* ring. If x is any point of C, the hypotheses imply that there exist global sections λ, λ' of $\mathcal{L}$, $\mathcal{L}^{-1}$ respectively such that $(\lambda \otimes \lambda')_x$ is a unit in $\mathcal{O}_x$; thus $\lambda \otimes \lambda'$ is an element of $H^0(\mathcal{O}_C)$ which is not nilpotent. Since $H^0(\mathcal{O}_C)$ is a local artinian ring, $\lambda \otimes \lambda'$ is a unit in $H^0(\mathcal{O}_C)$, and consequently λ is a nowhere-zero global section of $\mathcal{L}$. Q.E.D.)

It remains therefore to prove the first assertion of (ii), and this will be done in the following roundabout way. Note first that, as in the preceding proof, we may assume C to be connected, so that $H^0(\mathcal{O}_C)$ is an artinian local ring. The effect on degrees of sheaves of replacing A by $H^0(\mathcal{O}_C)$ is simply division by a positive constant, namely the length — as an A-module — of the residue field of $H^0(\mathcal{O}_C)$; hence we may assume that A is an artinian local ring, with maximal ideal, say, $\mathfrak{m}$. By [**EGA** 0_{III}, (10.3.1)] there exists a faithfully flat local A-algebra A′ whose maximal ideal is $\mathfrak{m}A'$, and which is such that $K = A'/\mathfrak{m}A'$ is an algebraic closure of $k = A/\mathfrak{m}$. A′ is artinian since some power of $\mathfrak{m}A'$ vanishes. Let $C' = C \times_A A'$, and let $\pi : C' \to C$ be the projection map. Note that, π being flat, $H^1(C', \mathcal{O}_{C'}) = 0$ [**EGA** III, (1.4.15)]. We will show *first* that in proving (ii) we can replace A by A′, C by C′ and $\mathcal{L}$ by $\pi^*(\mathcal{L})$; in other words we may assume A to have an algebraically closed residue field. *Second* we can remark that, *under the preceding assumption*, (i) is proved, in effect, in [3] or in [**18**]. *Finally* we will see, still assuming A to have an algebraically closed residue field, that (ii) follows from (i).

To begin, then, we show that: *if* D′ *is a component of the curve* C′, *with inclusion map* $\varepsilon : D' \to C'$, *then*

$$\deg_{D'}(\varepsilon^*\pi^*(\mathcal{L})) \geq 0. \tag{1}$$

(Here, of course, " $\deg_{D'}$ " is calculated over A′.) D′ is a component of $\pi^{-1}(C_i)$ for some i, and we have a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[d] & D' \arrow[l] \arrow[r, "\varepsilon"] \arrow[d] & C' \arrow[d, "\pi"] \\ \mathrm{Spec}(k) & C_i \arrow[l] \arrow[r, "\varepsilon_i"'] & C \end{tikzcd}$$

($k = A/\mathfrak{m}$ and $K = A'/\mathfrak{m}A'$). In calculating degrees of invertible sheaves on C_i (resp. D′) we may replace A by k (resp. A′ by K). We may therefore assume, for proving (1), that $C = C_i$, $A = k$, $A' = K$.

Now [**EGA** IV, (4.8.13)] there is a field L with $k \subseteq L \subseteq K$, $[L : k] < \infty$, such that $D' = D \otimes_L K$ for some component D of $C \otimes_k L$. We have then a commutative diagram

$$\begin{tikzcd} D' \arrow[dr, "g"] \arrow[dd, "\pi\circ\varepsilon"'] & \\ & D \arrow[dl, "h"] \\ C & \end{tikzcd}$$

Since $H^p(D', g^*\mathcal{F}) = H^p(D, \mathcal{F}) \otimes_L K$ for any coherent $\mathcal{O}_D$-module $\mathcal{F}$ and any $p \geq 0$ ([**EGA** III, (1.4.15)]) we have

$$\deg_{D'}(\varepsilon^*\pi^*(\mathcal{L})) = \deg_{D'}(g^*h^*(\mathcal{L})) = \deg_D(h^*(\mathcal{L}))$$

where " $\deg_D$ " is calculated over L; it is therefore sufficient to show that $\deg_D(h^*(\mathcal{L})) \geq 0$. But clearly for this purpose we may calculate " $\deg_D$ " over k instead of over L. Then, since $\deg_C(\mathcal{L}) \geq 0$, Proposition (10.2) *b)* (with D in place of C′) gives the desired conclusion.

To complete the first step, we show that: *$\mathscr{L}$ is generated by its sections over* C *if and only if* $\pi^*(\mathscr{L})$ *is generated by its sections over* C$'$. Indeed, if $\mathscr{M}$ is the subsheaf of $\mathscr{L}$ generated by the sections of $\mathscr{L}$ over C, then, since A$'$ is *flat* over A, $H^0(C', \pi^*\mathscr{L}) = H^0(C, \mathscr{L}) \otimes_A A'$, and so $\pi^*(\mathscr{M}) = \mathscr{M} \otimes_A A'$ is the subsheaf of $\pi^*(\mathscr{L}) = \mathscr{L} \otimes_A A'$ generated by the sections of $\pi^*(\mathscr{L})$ over C$'$; and since A$'$ is *faithfully flat* over A the inclusion map $\mathscr{M} \hookrightarrow \mathscr{L}$ is surjective if and only if the corresponding map $\pi^*(\mathscr{M}) \hookrightarrow \pi^*(\mathscr{L})$ is.

We may now assume that the residue field k of A is algebraically closed. Under this assumption (i) is proved by the argument given in [**3**; Lemmas (1.4) and (1.6)] with one small modification, namely in Lemma (1.4) the induction should be carried out with respect to the chain of schemes

$$C_{\mathrm{red}} = C_{(1)} \subseteq C_{(2)} \subseteq \ldots \subseteq C_{(t)} = C$$

where, if $\mathscr{N}$ is the sheaf of nilpotents of $\mathscr{O}_C$, then $C_{(i)}$ is the subscheme of C defined by $\mathscr{N}^i$, and t is such that $\mathscr{N}^{t-1} \neq 0$, $\mathscr{N}^t = 0$. (A similar argument appears in [**18**; Chapter (5.1)].)

To deduce (ii) from (i) we make use of the following description of divisors on curves (cf. [**EGA** IV, § 21.9]).

Lemma (**11.2**). — *Let* $x_1, x_2, \ldots, x_n$ *be closed points on a curve* C, *and for each* $i = 1, 2, \ldots, n$ *let* f_i *be a unit in the total quotient ring of* $\mathscr{O}_{x_i}$. *Then there is a unique divisor* $\mathscr{D}$ *on* C *such that* f_i *is a local equation of* $\mathscr{D}$ *at* x_i $(i = 1, 2, \ldots, n)$ *and* 1 *is a local equation at all* $x \neq x_1, x_2, \ldots, x_n$.

(*Proof.* — It is possible to choose, for each i, an affine neighborhood $U_i = \mathrm{Spec}(R_i)$ of x_i, and a unit g_i in the total quotient ring of R_i such that: *a)* $(g_i)_{x_i} = f_i$; *b)* $(g_i)_x$ is a unit in $\mathscr{O}_x$ for all x in $U_i, x \neq x_i$; *c)* for $j \neq i$, $x_j \notin U_i$. If $U_0 = C - \{x_1, x_2, \ldots, x_n\}$, then the collection $\{(1, U_0), (g_1, U_1), \ldots, (g_n, U_n)\}$ clearly defines the desired $\mathscr{D}$.)

We need the following consequence: let C be a curve, let $C^* = C_{\mathrm{red}}$ be the associated reduced curve, and let $h : C^* \to C$ be the canonical map; for any divisor $\mathscr{D}$ on C, $h^*(\mathscr{D})$ is a well-defined divisor on C*, and from Lemma (11.2), it follows without difficulty that *every divisor* $\mathscr{D}^*$ *on* C* *such that* $h(x)$ *is of depth* 1 *for all* x *in the support of* $\mathscr{D}^*$ *is of the form* $h^*(\mathscr{D})$.

We prove finally that (i) implies (ii) when C is proper over an artinian local ring with algebraically closed residue field k. Each component C_i of C may be regarded as a complete curve over k. Hence we may choose on C_i distinct closed points P_i, Q_i, which are *regular* points of C* and such that $h(P_i)$, $h(Q_i)$ are of depth 1 on C. The effective divisor $\mathscr{D}^* = \sum_i \delta_i P_i$ on C* is of the form $h^*(\mathscr{D})$, where $\mathscr{D}$ is an *effective* divisor on C, and the invertible sheaf $\mathscr{O}_C(\mathscr{D})$ induces the invertible sheaf $\mathscr{O}_{C^*}(\mathscr{D}^*)$ which in turn induces, for each i, an invertible sheaf of degree δ_i on C_i (Corollary (10.3)). By additivity of degree, $\mathscr{L} \otimes \mathscr{O}(-\mathscr{D})$ induces an invertible sheaf of degree 0 on each component C_i, whence, by (i), $\mathscr{L} \otimes \mathscr{O}(-\mathscr{D}) \cong \mathscr{O}_C$, i.e. $\mathscr{L} \cong \mathscr{O}(\mathscr{D})$. Similarly, if $\mathscr{E}^* = \sum_i \delta_i Q_i$ we have $\mathscr{E}^* = h^*(\mathscr{E})$ with $\mathscr{L} \cong \mathscr{O}(\mathscr{E})$.

Since the supports of $\mathscr{D}$ and $\mathscr{E}$ have no point in common, $\mathscr{L}$ is generated by some two of its global sections, and this completes the proof of Proposition (11.1).

In some cases, Proposition (11.1) holds *only if* $H^1(C, \mathcal{O}_C) = 0$.

Complement **(11.3)**. — *With the notation of Proposition* (11.1), *assume that each point in the image of f has an algebraically closed residue field. If either* (i) *or* (ii) *of Proposition* (11.1) *is true (for all invertible sheaves $\mathscr{L}$ on* C) *then* $H^1(C, \mathcal{O}_C) = 0$.

Proof. — As before, we may assume that C is connected and that A is an artinian local ring. Since (ii) implies (i) (cf. beginning of proof of Proposition (11.1)), it is sufficient to show that (i) implies $H^1(C, \mathcal{O}_C) = 0$. This also is done in [**3**; Lemmas (1.4) and (1.6)], Lemma (1.4) being modified as indicated during the proof of Proposition (11.1). Q.E.D.

We will also make use in § 27 of the following simple lemma.

Lemma **(11.4)**. — $f: C \to \mathrm{Spec}(A)$ *being as in* § 10, *assume further that* C *is integral and that* $H^1(C, \mathcal{O}_C) = 0$. *If $\mathscr{L}$ is an invertible $\mathcal{O}_C$-module, then* $H^1(C, \mathscr{L}) = 0$ *if and only if* $\deg_C(\mathscr{L}) \geq -h^0(\mathcal{O}_C)$.

Proof. — If $H^1(\mathscr{L}) = 0$, then

$$\deg_C(\mathscr{L}) = h^0(\mathscr{L}) - h^0(\mathcal{O}_C) \geq -h^0(\mathcal{O}_C).$$

In proving the converse, we may assume that $H^0(\mathscr{L}) = 0$; for, if $H^0(\mathscr{L}) \neq 0$, then, C being integral, we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathscr{L} \to \mathscr{K} \to 0$$

where $\mathscr{K}$ has support of dimension ≤ 0, and so there is a surjection $H^1(\mathcal{O}_C) \to H^1(\mathscr{L})$, i.e. $H^1(\mathscr{L}) = 0$. Now if $\deg_C(\mathscr{L}) \geq -h^0(\mathcal{O}_C)$ and $h^0(\mathscr{L}) = 0$, then

$$\begin{aligned} 0 \leq \deg_C(\mathscr{L}) + h^0(\mathcal{O}_C) &= \deg_C(\mathscr{L}) + \chi(\mathcal{O}_C) \\ &= \chi(\mathscr{L}) \\ &= -h^1(\mathscr{L}) \end{aligned}$$

Thus $h^1(\mathscr{L}) = 0$. Q.E.D.

§ 12. Relativization.

For the applications which we have in mind, it is necessary to give Proposition (11.1) a relativized form.

Let A be a noetherian ring, and let $f: X \to \mathrm{Spec}(A)$ be a map of finite type. If $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module, and C is a closed subscheme of X, defined by a coherent $\mathcal{O}_X$-ideal, say, $\mathscr{I}$, we set

$$\mathscr{F}_C = \mathscr{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathscr{I}) = \mathscr{F}/\mathscr{I}\mathscr{F}$$

Let $i_C: C \to X$ be the inclusion map. There are canonical isomorphisms ([**EGA** III, (1.3.3)])

$$\text{(2)} \qquad H^p(X, \mathscr{F}_C) \xrightarrow{\approx} H^p(C, i_C^*(\mathscr{F})), \qquad p \geq 0,$$

Suppose now that C is a curve on X (by which we mean that C is a closed subscheme of dimension one). We say that C has *exceptional support*, or that C *is an exceptional curve* (relative to f) if the support of C is *proper* over A *and* $f(\mathrm{C})$ is zero-dimensional. Such a C is then of the type considered in §§ 10-11, and so for any invertible sheaf $\mathscr{L}$ on X, we may set

$$(\mathscr{L}.\mathrm{C}) = \deg_{\mathrm{C}}(i_{\mathrm{C}}^{*}(\mathscr{L})).$$

Theorem **(12.1)**. — *Let* A *be a local ring with maximal ideal* $\mathfrak{m}$, *and let* $f: \mathrm{X} \to \mathrm{Spec}(\mathrm{A})$ *be a proper map whose fibres have dimension* ≤ 1. *Assume that* $\mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$. *Let* $\mathscr{L}$ *be an invertible* $\mathcal{O}_{\mathrm{X}}$*-module. Then*:

(i) $(\mathscr{L}.\mathrm{E}) = 0$ *for all integral exceptional curves* E *on* X *if and only if* $\mathscr{L} \cong \mathcal{O}_{\mathrm{X}}$.

(ii) $(\mathscr{L}.\mathrm{E}) \geq 0$ *for all integral exceptional curves* E *on* X *if and only if* $\mathscr{L}$ *is generated by its sections over* X, *and when this is so*, $\mathrm{H}^1(\mathscr{L}) = 0$.

(iii) $(\mathscr{L}.\mathrm{E}) > 0$ *for all integral exceptional curves* E *on* X *if and only if* $\mathscr{L}$ *is ample, and when this is so*, $\mathscr{L}$ *is even very ample for* f.

Proof. — The " if " parts of (i), (ii), (iii) follow at once from Proposition (10.4).

Since the fibres of f have dimension ≤ 1, $\mathrm{H}^2(\mathscr{F}) = 0$ for all coherent $\mathcal{O}_{\mathrm{X}}$-modules $\mathscr{F}$ [**EGA** III, (4.2.2)]. Hence if $\mathscr{L}$ is generated by its sections over X, then $\mathrm{H}^1(\mathscr{L})$ is a homomorphic image of $\mathrm{H}^1(\mathcal{O}_{\mathrm{X}}^{s})$ for some positive integer s, and so $\mathrm{H}^1(\mathscr{L}) = 0$.

If $(\mathscr{L}.\mathrm{E}) > 0$ for all E as in (iii) then $i_{\mathrm{D}}^{*}(\mathscr{L})$ is ample if D is the closed fibre of f (Proposition (10.4)) and consequently [**EGA** III, (4.7.1)] $\mathscr{L}$ is ample. Then (ii) and Corollary (7.4) show that $\mathscr{L}$ is very ample for f. (This last statement is the only part of (iii) in which the hypothesis $\mathrm{H}^1(\mathcal{O}_{\mathrm{X}}) = 0$ is used.)

To complete the proof we need:

Lemma **(12.2)**. — *Let* $\mathscr{F}$ *be a coherent* $\mathcal{O}_{\mathrm{X}}$*-module. Then* $\mathrm{H}^1(\mathscr{F}) = 0$ *if and only if* $\mathrm{H}^1(\mathscr{F}_{\mathrm{C}}) = 0$ *for all exceptional curves* C *on* X.

Proof. — Since H^2 vanishes for all coherent $\mathcal{O}_{\mathrm{X}}$-modules and $\mathscr{F}_{\mathrm{C}}$ is a homomorphic image of $\mathscr{F}$, $\mathrm{H}^1(\mathscr{F}) = 0$ implies $\mathrm{H}^1(\mathscr{F}_{\mathrm{C}}) = 0$.

Conversely, if $\mathrm{H}^1(\mathscr{F}_{\mathrm{C}}) = 0$ for all C, then, letting ^ denote completion with respect to the maximal ideal $\mathfrak{m}$, we have

$$\begin{aligned} \mathrm{H}^1(\mathscr{F})^{\wedge} &= \varprojlim_{k>0} \mathrm{H}^1(\mathscr{F} \otimes (\mathcal{O}_{\mathrm{X}}/\mathfrak{m}^k \mathcal{O}_{\mathrm{X}})) \\ &= \varprojlim_{k>0} (0) \\ &= 0 \end{aligned}$$

(cf. [**EGA** III, (4.1.7)]). Thus $\mathrm{H}^1(\mathscr{F}) = 0$. Q.E.D.

We return to the proof of (ii). Let x be a closed point of X. We have to show that some global section of $\mathscr{L}$ does not vanish at x. Lemma (12.2) implies that $\mathrm{H}^1(\mathrm{C}, \mathcal{O}_{\mathrm{C}}) = 0$ for any exceptional curve C, so Proposition (11.1) shows that $i_{\mathrm{C}}^{*}(\mathscr{L})$

is generated by its sections over C, and the same is obviously true of $i_C^*(\mathfrak{m}\mathcal{O}_X)$. Hence $i_C^*(\mathcal{L}\otimes\mathfrak{m}\mathcal{O}_X)$ is generated by its sections over C so that (cf. beginning of proof of Proposition (11.1), and the isomorphisms (2) above)

$$0=H^1(C, i_C^*(\mathcal{L}\otimes\mathfrak{m}\mathcal{O}_X))=H^1(X, (\mathcal{L}\otimes\mathfrak{m}\mathcal{O}_X)_C).$$

It follows from Lemma (12.2) that $H^1(\mathcal{L}\otimes\mathfrak{m}\mathcal{O}_X)=0$.

The exact cohomology sequence shows then that $H^0(\mathcal{L})\to H^0(\mathcal{L}\otimes(\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X))$ is surjective. For the closed fibre D on X, defined by $\mathfrak{m}\mathcal{O}_X$, we have, as above, that $i_D^*(\mathcal{L})$ is generated by its sections over D (this being obvious if D is zero-dimensional); hence some section of $\mathcal{L}_D=\mathcal{L}\otimes(\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X)$ over X does not vanish at x, and since this section can be lifted to a global section of $\mathcal{L}$, we are done.

The proof of (i) is almost identical, the only difference being that $i_D^*(\mathcal{L})$ — and hence $\mathcal{L}$ — has a global section which does not vanish at *any* closed point x. Q.E.D.

Remarks. — 1. Theorem (12.1) can easily be reformulated so as to apply to the situation where Spec(A) is replaced by an arbitrary locally noetherian scheme.

2. For later use, we set down some simple properties of the " intersection product " $(\mathcal{L}.C)$ defined at the beginning of this section:

a) If C is integral then $(\mathcal{L}.C)$ is an integer multiple of $h^0(C, \mathcal{O}_C)$.

(This is because cohomology groups of coherent sheaves on C are vector spaces over the *field* $H^0(C, \mathcal{O}_C)$).

b) If $\mathcal{M}$ and $\mathcal{N}$ are invertible sheaves on X, then

$$((\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}).C)=(\mathcal{M}.C)+(\mathcal{N}.C)$$

(This follows at once from the additivity of " degree " (Proposition (10.2) *a)*).)

c) If $\mathcal{D}$ is an effective divisor on X whose support contains no associated point of C, then

$$(\mathcal{O}_X(\mathcal{D}).C)=h^0((\mathcal{O}_{\mathcal{D}})_C).$$

(Under our assumptions, $i_{C*}(\mathcal{D})$ is an effective divisor on C, and the assertion follows easily from (2) above and Corollary (10.3).)

IV. — AN EXACT SEQUENCE FOR THE DIVISOR CLASS GROUP

In part IV we continue to lay a proper foundation for the results in parts V and VI. To a large extent, IV is devoted to a systematic presentation of more or less familiar facts in a setting suitable for the subsequent applications. Some of these applications are given in § 17.

Throughout R will be a two-dimensional normal local ring, with maximal ideal $\mathfrak{m}$, admitting a desingularization, say, $f: X \to \mathrm{Spec}(R)$ [1]. We study an exact sequence of abelian groups

$$(3) \qquad 0 \to \mathrm{Pic}^0(R) \to \mathrm{Pic}(U) \to H$$

where $U = \mathrm{Spec}(R) - \{\mathfrak{m}\}$, so that $\mathrm{Pic}(U)$ is the divisor class group of R. The group $\mathrm{Pic}^0(R)$ is the numerically trivial part of $\mathrm{Pic}(X)$, while H is (approximately) the finite abelian group defined by the intersection matrix of the exceptional curves on X. (The requisite intersection theory is reviewed in § 13.) The homomorphisms in (3) are defined in § 14. If R is *henselian*, then $\mathrm{Pic}(U) \to H$ is *surjective* (Proposition (14.4)). In § 15 it is shown that the sequence (3) is actually independent of the choice of the desingularization X (so that the notation " $\mathrm{Pic}^0(R)$ " is justified). In § 16 we examine the relation between (3) and the corresponding sequence for a formally smooth R-algebra.

When R is the local ring of a point on a two-dimensional complex space, Mumford has obtained, by transcendental means, an exact sequence containing (3) ([**15**; Part II]). An immediate consequence, in this case, is that R *has a rational singularity if and only if* R *has a finite divisor class group* (cf. [**7**; Satz (1.5)]). In Theorem (17.4) we reach the same conclusion *for any henselian* R *with algebraically closed residue field.* The treatment given here is purely algebraic.

We also include a result on the factoriality of certain power series rings (Proposition (17.5)).

§ 13. Intersection theory for exceptional curves.

We now review those few facts of intersection theory which we will need in later sections. The results here are all particular instances of the formalism developed by Kleiman in [**11**; Chapter 1].

As in § 12, we deal with an arbitrary map of finite type $f: X \to \mathrm{Spec}(A)$, A being a noetherian ring. The subsequent considerations will be applicable mainly when X is two-dimensional because of the following restriction on our previous terminology: from now on, by a *curve on* X we mean *a one-dimensional closed subscheme of* X *whose defining sheaf of ideals is invertible*, or equivalently, *an effective divisor with one-dimensional support.*

Let D be a divisor on X, and let $\mathcal{O}(D) = \mathcal{O}_X(D)$ be the corresponding invertible sheaf. Let E be a curve on X, *with exceptional support* (cf. § 12). $\mathcal{O}(-E)$ is the sheaf of ideals defining E; let $\mathcal{O}_E = \mathcal{O}_X/\mathcal{O}(-E)$. For any coherent $\mathcal{O}_X$-module $\mathcal{F}$, we set

$$\mathcal{F}(D) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}(D)$$

and as in § 12

$$\mathcal{F}_E = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_E.$$

If $\mathcal{G}$ is an $\mathcal{O}_X$-module which is locally isomorphic to $\mathcal{O}_E$, then the $H^p(X, \mathcal{G})$ are A-modules of finite length, and we set

$$\deg_E(\mathcal{G}) = \chi(\mathcal{G}) - \chi(\mathcal{O}_E) = \deg_E(i_E^*(\mathcal{G}))$$

[1] Cf. Remark (16.2).

where $i_E : E \to X$ is the inclusion map (so that $i_E^*(\mathcal{G})$ is an invertible sheaf on E). This abuse of notation should cause no difficulty. The *intersection number* (D.E) is defined by

$$(D.E) = (\mathcal{O}(D).E) = \deg_E(\mathcal{O}_E(D))$$

(cf. beginning of § 12).

Some basic properties of the intersection number are set out below. They will be used, sometimes tacitly, throughout the sequel.

It will be convenient to write " $h^p(E)$ ", " $\chi(E)$ " in place of " $h^p(\mathcal{O}_E)$ ", " $\chi(\mathcal{O}_E)$ ".

Proposition **(13.1)**. — *Let* D, D_1, D_2 *be divisors on* X, *and let* E, F *be curves on* X *with exceptional support*:

a) *If* E *is* integral, *then* (D.E) *is an integer multiple of* $h^0(E)$.

b)
$$((D_1+D_2).E) = (D_1.E)+(D_2.E)$$
$$(D.(E+F)) = (D.E)+(D.F).$$

c) *If* D *is an* effective *divisor whose support contains no associated point of* E, *then* $(D.E) \geq 0$, *and* $(D.E)=0$ *if and only if the supports of* D *and* E *have no point in common.*

d)
$$(F.E) = \chi(E)+\chi(F)-\chi(E+F) = (E.F).$$

Proof. — *a)*, the first equality in *b)*, and *c)*, all follow easily from *a)*, *b)* and *c)* of Remark 2 at the end of § 12.

Now tensor the exact sequence

$$0 \to \mathcal{O}(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

with the invertible sheaf $\mathcal{O}(-F)$ to obtain an exact sequence

$$0 \to \mathcal{O}(-E-F) \to \mathcal{O}(-F) \to \mathcal{O}_E(-F) \to 0$$

so that the exact sequence

$$0 \to \mathcal{O}(-F)/\mathcal{O}(-E-F) \to \mathcal{O}_X/\mathcal{O}(-E-F) \to \mathcal{O}_X/\mathcal{O}(-F) \to 0$$

can be written as

$$0 \to \mathcal{O}_E(-F) \to \mathcal{O}_{E+F} \to \mathcal{O}_F \to 0$$

Tensoring with an $\mathcal{O}_X$-module $\mathcal{L}$ which is *locally isomorphic to* $\mathcal{O}_{E+F}$, we get an exact sequence

$$0 \to \mathcal{L}_E(-F) \to \mathcal{L} \to \mathcal{L}_F \to 0$$

from which, along with additivity of " degree ", we obtain

$$\begin{aligned}(F.E) &= \deg_E(\mathcal{O}_E(F)) \\ &= \deg_E(\mathcal{L}_E) - \deg_E(\mathcal{L}_E(-F)) \\ &= (\deg_E(\mathcal{L}_E)+\chi(E)) - \chi(\mathcal{L}_E(-F)) \\ &= \chi(\mathcal{L}_E) - (\chi(\mathcal{L}) - \chi(\mathcal{L}_F))\end{aligned}$$

i.e.

(4)
$$(F.E) = \chi(\mathcal{L}_E)+\chi(\mathcal{L}_F)-\chi(\mathcal{L}).$$

In particular, for $\mathscr{L}=\mathscr{O}_{E+F}$, we get

$$(F.E)=\chi(E)+\chi(F)-\chi(E+F) \tag{5}$$

which is the first equality in *d)*. The second follows by interchanging E and F.

Subtracting (5) from (4), we get

$$0=\deg_E(\mathscr{L}_E)+\deg_F(\mathscr{L}_F)-\deg_{E+F}(\mathscr{L}).$$

In particular, if $\mathscr{N}$ is an invertible $\mathscr{O}_X$-module, then

$$\deg_{E+F}(\mathscr{N}_{E+F})=\deg_E(\mathscr{N}_E)+\deg_F(\mathscr{N}_F).$$

For $\mathscr{N}=\mathscr{O}(D)$, this is the second equality in *b)*. Q.E.D.

§ 14. Definition of the sequence.

As before, let R be a two-dimensional normal local ring, with maximal ideal $\mathfrak{m}$, admitting a desingularization $f: X \to \mathrm{Spec}(R)$. Note that every closed point on X is of codimension two (by Zariski's " main theorem " [**EGA** III, (4.4.8)], for example). Since X is regular, it follows that every one-dimensional closed subscheme C of X having no *closed* associated points is defined by an *invertible* $\mathscr{O}_X$-ideal (i.e. C is a curve on X in the sense of § 13), and conversely. Thus all the results of § 13 are applicable.

Let $E_1, E_2, \ldots, E_n$ be the distinct components of the closed fibre, i.e. all the integral curves on X with exceptional support, so that $f^{-1}(\{\mathfrak{m}\})_{\mathrm{red}}=E_1+E_2+\ldots+E_n$. The following lemma of Du Val is important:

Lemma **(14.1)**. — *The intersection matrix* $((E_i.E_j))$ *is negative-definite.*

Proof. — It is sufficient (cf. [**4**; proof of Proposition 2]) to find a curve $C=\sum_i c_iE_i$ $(c_i \geq 0)$ such that α) $(C.E_i) \leq 0$ for all i and β) $(C.C)<0$. In view of Proposition (13.1) *c)*, α) implies that if $c_i=0$ and $c_j>0$ then E_i and E_j do not meet, so that $\bigcup_{c_i=0} E_i$ is both open and closed in $f^{-1}(\{\mathfrak{m}\})$; since $f^{-1}(\{\mathfrak{m}\})$ is connected, it follows that $c_i>0$ for all i, and so β) holds provided $(C.E_i)<0$ for at least one i.

Let r be any non-unit in R, let v_i be the discrete valuation whose center on X is E_i, and let C be the curve $\sum_i v_i(r)E_i$. Then $r\mathscr{O}(C) \subseteq \mathscr{O}_X$, i.e. $r\mathscr{O}(C)=\mathscr{O}(-D)$ where D is an effective divisor *whose support contains no associated point of* C. Moreover there is a discrete valuation v whose center in R is a height one prime ideal containing r; since f is proper, the center of v on X is a one-dimensional integral closed subscheme meeting $\bigcup_i E_i$, and since $v(r)>0$, this center is part of the support of $\mathscr{O}_X/r\mathscr{O}(C)$, i.e. the support of D. Thus D is a curve such that $(D.E_i) \geq 0$ for all i, with strict inequality for at least one i. But

$$0=(r\mathscr{O}_X.E_i)=-(C+D.E_i)=-(C.E_i)-(D.E_i)$$

i.e. $(D.E_i)=-(C.E_i)$, so that C is as desired. Q.E.D.

Let **E** be the additive group of divisors on X with exceptional support, i.e. divisors of the form $\sum_{i=1}^{n} s_iE_i$ with $s_i \in \mathbf{Z}$, the group of rational integers. Since no non-zero principal divisor has exceptional support, the canonical map $\mathbf{E}\to\mathrm{Pic}(X)$ is injective, where $\mathrm{Pic}(X)\cong H^1(X, \mathcal{O}_X^*)$ is the group of divisor classes on X.

The cokernel of this map is easily determined. Let

$$U = X - f^{-1}(\{\mathfrak{m}\}) \cong \mathrm{Spec}(R) - \{\mathfrak{m}\}.$$

Then Pic(U) is nothing but the *divisor class group* of R, i.e. the free group generated by height one prime ideals in R, modulo principal divisors. The restriction map $\rho : \mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is clearly surjective and its kernel consists of classes of divisors D on X which become principal on U. But this condition on D means precisely that D is linearly equivalent on X to a divisor with exceptional support. Thus we have an exact sequence

$$0 \to \mathbf{E} \to \mathrm{Pic}(X) \xrightarrow{\rho} \mathrm{Pic}(U) \to 0$$

Next, for each $i=1, 2, \ldots, n$, let $d_i > 0$ be the greatest common divisor of all the degrees of invertible sheaves on E_i. For each divisor class Δ in Pic(X) we can define $(\Delta.E_i)$ to be $(D.E_i)$ where D is any divisor whose class is Δ. We define a group homomorphism

$$\theta : \mathrm{Pic}(X) \to \mathbf{E}^* = \mathrm{Hom}(\mathbf{E}, \mathbf{Z})$$

by setting

$$(\theta(\Delta))(E_i) = \frac{1}{d_i}(\Delta.E_i) \qquad i=1, 2, \ldots, n$$

for Δ in Pic(X). The kernel of θ is the group of divisor classes whose intersection number with every exceptional curve on X is zero; we call this group $\mathrm{Pic}^0(X)$. Because of the negative-definiteness of $((E_i.E_j))$ (Lemma (14.1)), the restriction of θ to $\mathbf{E}$ is injective, i.e. $\mathbf{E}\cap\mathrm{Pic}^0(X)=(0)$. The cokernel H of this restricted map is seen at once to be the abelian group with generators $e_1, e_2, \ldots, e_n$ subject to the relations

$$\sum_{j=1}^{n} \frac{1}{d_j}(E_i.E_j)e_j = 0 \qquad (i=1, 2, \ldots, n).$$

H is a finite group of order

$$\frac{1}{d_1 d_2 \ldots d_n}\det((E_i.E_j)).$$

Finally, let G be the cokernel of θ itself. We have then a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathbf{E}\cap\mathrm{Pic}^0(X)=0 \arrow[r] \arrow[d] & \mathbf{E} \arrow[r] \arrow[d] & \theta(\mathbf{E}) \arrow[d] & & \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r, "\theta"] \arrow[d, "\rho"] & \mathbf{E}^* \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \rho(\mathrm{Pic}^0(X)) \arrow[r] & \mathrm{Pic}(U) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & G \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

The definitions of the maps in the last row are self-evident, and the verification of exactness is immediate. Thus:

Proposition **(14.2)**. — *With the preceding notation, there is an exact sequence*

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(U) \to H \to G \to 0.$$

We finish § 14 with a sufficient condition for G to vanish.

Lemma **(14.3)** [1]. — *Suppose that for every height one prime ideal* $\mathfrak{p}$ *in* R, *the integral closure of* $R/\mathfrak{p}$ *in its field of fractions is a local ring*, i.e. $R/\mathfrak{p}$ *is unibranch.* (*This condition holds, for example, if* R *is* henselian.) *If* E *is any exceptional curve on* X, *then the restriction map* $\mathrm{Pic}(X) \to \mathrm{Pic}(E)$ *is surjective.*

Before proving this lemma, we deduce:

Proposition **(14.4)**. — *If* R *satisfies the condition of Lemma* (14.3), *then* $G=(0)$, *so that there is an exact sequence*

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(U) \to H \to 0.$$

Proof. — Let $E = E_1 + E_2 + \ldots + E_n$ and let $\varepsilon_i : E_i \to E$ be the inclusion maps. The exact sequence of abelian groups

$$(1) \to \mathcal{O}_E^* \to \prod_{i=1}^{n} \mathcal{O}_{E_i}^*$$

has cokernel with at most zero-dimensional support; consequently there is a surjection

$$H^1(X, \mathcal{O}_E^*) \to H^1(X, \prod_i \mathcal{O}_{E_i}^*) = \prod_i H^1(X, \mathcal{O}_{E_i}^*)$$

from which we conclude easily that the map

$$\mathrm{Pic}(E) = H^1(E, \mathcal{O}_E^*) \xrightarrow{\Pi\varepsilon_i^*} \prod_i H^1(E_i, \mathcal{O}_{E_i}^*) = \prod_i \mathrm{Pic}(E_i)$$

is surjective.

From the definition of d_i, it follows that there is an invertible sheaf of degree d_i on E_i, and therefore there is an invertible sheaf $\mathscr{L}_i$ on E such that $\varepsilon_i^*(\mathscr{L}_i)$ has degree d_i and for $j \neq i$, $\varepsilon_j^*(\mathscr{L}_i)$ has degree zero. Since $\mathscr{L}_i$ is induced by an invertible sheaf on X (Lemma (14.3)), we see that θ is surjective, i.e. $G=(0)$. Q.E.D.

Now we prove Lemma (14.3).

Since E has no embedded associated point, every invertible sheaf on E comes from a divisor on E. Because of Lemma (11.2), it is sufficient to show that if Q is a closed point on E, and $\overline{w}$ is a non-zero-divisor in $\mathcal{O}_{E,Q}$ then there is a divisor D on X whose support meets E only at Q, and whose local equation at Q (on X) induces $\overline{w}$. Let w be an element of $\mathcal{O}_{X,Q}$ whose image in $\mathcal{O}_{E,Q}$ is $\overline{w}$. *We may assume that no* E_i *is a component of the divisor* (w) *on* X. (Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r, \mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_t$ be the prime ideals in $\mathcal{O}_{X,Q}$ corresponding to those E_i passing through Q which are not components of E, the labelling being such that $w \in \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$, $w \notin \mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_t$; if $\mathfrak{q}$ is the kernel of the natural

(1) Cf. [**EGA** IV, (21.9.12)].

surjection $\mathcal{O}_{X,Q} \to \mathcal{O}_{E,Q}$, then we can choose $a \in \mathfrak{q} \cap \mathfrak{p}_{r+1} \cap \ldots \cap \mathfrak{p}_t$, $a \notin \bigcup_{i=1}^{r} \mathfrak{p}_i$, and replace w by $w+a$.) Write $(w) = D + D'$, where no component of D' passes through Q, and every component of D passes through Q. Then D is the desired divisor.

To see this, it is clearly enough to show that *every prime divisor $\mathcal{P}$ on* X *other than* $E_1, E_2, \ldots, E_n$ *meets* E *in at most one point*. Let R′ be the local ring of such a point on X, let $\mathfrak{p}'$ be the height one prime ideal in R′ corresponding to $\mathcal{P}$, and let $\mathfrak{p} = \mathfrak{p}' \cap R$. Then $R' \subseteq R_\mathfrak{p} = R'_{\mathfrak{p}'}$, and if $g : R_\mathfrak{p} \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = K$ is the canonical map, then $g(R) \subseteq g(R') < K$. Since $g(R) \cong R/\mathfrak{p}$ is one-dimensional and unibranch, the theorem of Krull-Akizuki [**17**; § 33.2] shows that the integral closure S of $g(R)$ in K is a discrete valuation ring, and S is contained in, hence equal to, the integral closure of $g(R')$ in K. Thus S dominates $g(R')$. Now if x is a non-unit in $g^{-1}(S)$, then $g(x)$ is a non-unit in S (otherwise x is a unit in $R_\mathfrak{p}$ (since $g(x) \neq 0$) and $g(1/x) = 1/g(x) \in S$, i.e. $1/x \in g^{-1}(S)$); it follows that the sum of two non-units in $g^{-1}(S)$ is a non-unit. We see then that $g^{-1}(S)$ is a local ring which dominates R′. Since S depends only on $\mathcal{P}$, and since there can be no more than one point on X whose local ring is dominated by $g^{-1}(S)$, we are done.

§ 15. Intrinsic nature of the sequence.

In § 14 we have defined an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(U) \to H \to G \to 0$$

of groups associated with a particular desingularization X of a normal two-dimensional local ring R. In this section we will show that the sequence depends only on R and not on X.

To this end, let $g : X' \to X$ be a proper birational map with X′ regular. For any divisor D on X there is a unique divisor $D' = g^*(D)$ on X′ with the property that for any $x' \in X'$, a local equation for D at $g(x')$ is also a local equation for D′ at x'; moreover we have a canonical isomorphism $g^*(\mathcal{O}_X(D)) \xrightarrow{\sim} \mathcal{O}_{X'}(D')$. D′ can be represented uniquely in the form

$$D' = D^\# + F'$$

where $D^\#$ is a formal linear combination of prime divisors whose supports are mapped by g onto curves on X, while the support of F′ is mapped into a zero-dimensional subset of X. $D^\#$ is the *proper transform* of D (by g). It is an easy consequence of Proposition (10.2) *b)* that:

α) *if* C *is an exceptional curve on* X *then*

$$(D'.C^\#) = (D'.C') = (D.C);$$

β) *if* F *is any curve on* X′ *such that* $g(F)$ *is zero-dimensional, then* F *has exceptional support, and*

$$(D'.F) = 0.$$

Let **E**′ be the group of divisors on X′ with exceptional support and let **F** be the subgroup consisting of divisors on X′ whose support is mapped by g into a zero-dimensional subset of X. It is evident that every divisor on X′ can be written *uniquely* in the form $D'+F$ where D is a divisor on X and $F \in \mathbf{F}$; $D'+F$ is principal on X′ if and only if $F=0$ and D is principal on X; and $D'+F \in \mathbf{E}'$ if and only if $D \in \mathbf{E}$. From these facts, we obtain the following commutative diagram of split exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Div}(X) \arrow[r, "g^*"] \arrow[d] & \mathrm{Div}(X') \arrow[r, shift left] \arrow[d] & \mathbf{F} \arrow[r] \arrow[l, shift left] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}(X) \arrow[r, "g^*"] & \mathrm{Pic}(X') \arrow[r, shift left] & \mathbf{F} \arrow[r] \arrow[l, shift left, "i"] & 0 \\
0 \arrow[r] & \mathbf{E} \arrow[r, shift left, "g^*"] \arrow[u, hook] & \mathbf{E}' \arrow[r, shift left] \arrow[l, shift left, "p"] \arrow[u, hook] & \mathbf{F} \arrow[r] \arrow[l, shift left, "i"] \arrow[u, equal] & 0
\end{tikzcd}$$

(where " Div " denotes " Group of divisors on ") and hence the split exact sequence

$$\begin{tikzcd}
0 & \mathbf{E}^* \arrow[l] \arrow[r, shift right, "p^a"'] \arrow[d, equal] & \mathbf{E}'^* \arrow[l, shift right, "g^a"'] \arrow[r, shift right, "i^a"'] \arrow[d, equal] & \mathbf{F}^* \arrow[l, shift right] \arrow[r] \arrow[d, equal] & 0 \\
& \mathrm{Hom}(\mathbf{E}, \mathbf{Z}) & \mathrm{Hom}(\mathbf{E}', \mathbf{Z}) & \mathrm{Hom}(\mathbf{F}, \mathbf{Z}) &
\end{tikzcd}$$

Lemma **(15.1)**. — *Let* $\theta : \mathrm{Pic}(X) \to \mathbf{E}^*$ *be as in* § 14, *and let* $\theta' : \mathrm{Pic}(X') \to \mathbf{E}'^*$ *be similarly defined. Let* $\psi : \mathbf{F} \to \mathbf{F}^*$ *be defined by restricting* θ', i.e. $\psi = i^a \circ \theta' \circ i$. *If* Pic(X′) *and* E'^* *are identified respectively with* $\mathrm{Pic}(X) \oplus \mathbf{F}$, $\mathbf{E}^* \oplus \mathbf{F}^*$ *according to the above splittings, then* θ' *becomes identified with* $\theta \oplus \psi$.

Proof. — The lemma is equivalent to the following four equations:

(i) $g^a \circ \theta' \circ g^* = \theta$.

(ii) $g^a \circ \theta' \circ i = 0$.

(iii) $i^a \circ \theta' \circ g^* = 0$.

(iv) $i^a \circ \theta' \circ i = \psi$.

(iv) is the definition of ψ, (iii) is nothing but the relation $(D'.F)=0$ given in β) above, and (ii) says that $(F.E')=0$ if $F \in \mathbf{F}$, $E \in \mathbf{E}$, which is true since $(F.E')=(E'.F)$.

(i) says that for $D \in \mathrm{Div}(X)$, $E \in \mathbf{E}$, we have

$$(\theta'(D'))(E') = (\theta(D))(E).$$

It is enough to check this for *integral* E. Then $E' = E^\# + \sum_i n_i F_i$, $(F_i \in \mathbf{F})$, and since $(D'.F_i)=0$ for all i we have

$$\begin{aligned}
(\theta'(D'))(E') &= \frac{1}{d^\#}(D'.E^\#) \\
&= \frac{1}{d^\#}(D.E) \qquad \text{(cf. } \alpha\text{) above)} \\
&= \frac{d}{d^\#}(\theta(D))(E)
\end{aligned}$$

where $d>0$ is the greatest common divisor of all degrees of invertible sheaves on E, and $d^{\#}>0$ is defined similarly for $E^{\#}$. But since $E^{\#}$ is birational over E, the canonical map $\mathrm{Div}(E) \to \mathrm{Div}(E^{\#})$ is surjective (as follows from Lemma (11.2) and the fact that every locally principal fractionary ideal of a semi-local domain is principal; or cf. [**EGA** IV, (21.8.5)]) and by Proposition (10.2) *b)*, $d=d^{\#}$. Q.E.D.

We will also need:

Lemma (**15.2**). — *The above map* $\psi : F \to F^*$ *is an isomorphism.*

Proof. — By the Factorization Theorem (cf. Theorem (4.1)) we have $g=g_m=h_m \circ h_{m-1} \circ \ldots \circ h_1$ where for $1 \leq k \leq m$, $h_k : X_k \to X_{k-1}$ $(X_0=X)$ is a quadratic transformation. Let $\mathbf{F}_k$ be the group of divisors on X_k $(1 \leq k \leq m)$ whose support has zero-dimensional image on X, and let $\psi_k : \mathbf{F}_k \to \mathbf{F}_k^*$ be defined as above.

We proceed by induction on m. Just as in Lemma (15.1), we can write $\mathbf{F}_m=\mathbf{F}_{m-1} \oplus \mathbf{F}'$, $\psi_m=\psi_{m-1} \oplus \psi'$ where $\mathbf{F}'$ consists of the multiples of the (unique) integral curve F on X_m whose image on X_{m-1} is a closed point, and $\psi' : \mathbf{F}' \to \mathbf{F}'^*$ is defined by

$$(\psi'(F))(F)=\frac{1}{\delta}(F.F)$$

where δ is the g.c.d. of degrees of invertible sheaves on F. By the inductive hypothesis ψ_{m-1} is an isomorphism; so it is enough to show that ψ' is an isomorphism, i.e. that $(F.F)=\pm\delta$. Since $h^0(F)$ divides δ, this follows from the well-known fact that

$$(F.F)=-h^0(F).$$

(This can be seen — for example — as follows: if x is the point on X_{m-1} which is blown up to give X_m, then F can be identified with the projective line L over the residue field of x, and then if $\mathfrak{n}$ is the maximal ideal of $\mathcal{O}_{X_{m-1},x}$ we have an identification of

$$\mathcal{O}_F(-F)=\mathcal{O}_F \otimes_{\mathcal{O}_{X_m}} \mathfrak{n}\mathcal{O}_{X_m}$$

with $\mathcal{O}_L(1)$. Thus

$$(F.F)=-\deg_L(\mathcal{O}_L(1))=-h^0(L)=-h^0(F)$$

as required. (Alternatively, use Corollary (23.2).) Q.E.D.

We can now prove the main result in this section by piecing together the preceding information in the form of various commutative diagrams.

First:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{E} \arrow[r] \arrow[d, "g^*"'] \arrow[dr, phantom, "\sigma_1"] & \mathrm{Pic}(X) \arrow[r, "\rho"] \arrow[d, "g^*"] \arrow[dr, phantom, "\sigma_2"] & \mathrm{Pic}(U) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \mathbf{E}' \arrow[r] & \mathrm{Pic}(X') \arrow[r, "\rho'"] & \mathrm{Pic}(U) \arrow[r] & 0 \end{tikzcd}$$

The commutative square σ_2 is obtained by applying the functor " Pic " to

$$\begin{tikzcd} g^{-1}(U) \arrow[r, hook] \arrow[d, "\wr"] & X' \arrow[d, "g"] \\ U \arrow[r, hook] & X \end{tikzcd}$$

Second:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\sigma_3"] & \mathrm{Pic}(X) \arrow[r, "\theta"] \arrow[d] \arrow[dr, phantom, "\sigma_4"] & \mathbf{E}^* \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X') \arrow[r] \arrow[d] & \mathrm{Pic}(X')=\mathrm{Pic}(X)\oplus\mathbf{F} \arrow[r, "\theta\oplus\psi"] & \mathbf{E}'^*=\mathbf{E}^*\oplus\mathbf{F}^* \arrow[r] & G' \arrow[r] \arrow[d] & 0 \\
& 0 & & & 0 &
\end{tikzcd}$$

The fact that the commutative square σ_4 induces isomorphisms of the kernels $\mathrm{Pic}^0(X)$, $\mathrm{Pic}^0(X')$, and the cokernels, G, G', follows from Lemmas (15.1) and (15.2).

Third:

$$\begin{tikzcd}
& \mathrm{Pic}(X) \arrow[dr] \arrow[dd] & & 0 \arrow[d] & \\
\mathbf{E} \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbf{E}^* \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\sigma_5"] & H \arrow[r] \arrow[dd] & 0 \\
& \mathrm{Pic}(X') \arrow[dr] & & & \\
\mathbf{E}' \arrow[ur] \arrow[rr] & & \mathbf{E}'^* \arrow[r] & H' \arrow[r] \arrow[d] & 0 \\
& & & 0 &
\end{tikzcd}$$

The square with the $\mathbf{E}$'s is obtained by putting together σ_1 and σ_4 as shown; the map from H to H' is defined to be the unique map making σ_5 commute. This cokernel map $H \to H'$ is an isomorphism for the same reason that $G \to G'$ was.

Fourth:

$$\begin{tikzcd}
& & & \mathrm{Pic}(X') \arrow[rr, dash] \arrow[dd] & & \mathbf{E}'^* \arrow[dd] \arrow[ddrr] & & & \\
& & \mathrm{Pic}(X) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbf{E}^* \arrow[ur] \arrow[dd] \arrow[ddrr] & & & & \\
& \rho'(\mathrm{Pic}^0(X')) \arrow[rr] \arrow[dr, phantom, "\sigma_6"] & & \mathrm{Pic}(U) \arrow[rr] \arrow[dr, phantom, "\sigma_7"] & & H' \arrow[rr] \arrow[dr, phantom, "\sigma_8"] & & G' \arrow[r] & 0 \\
\rho(\mathrm{Pic}^0(X)) \arrow[ur] \arrow[rr] & & \mathrm{Pic}(U) \arrow[ur] \arrow[rr] & & H \arrow[ur] \arrow[rr] & & G \arrow[ur] \arrow[r] & 0 &
\end{tikzcd}$$

Of the six faces in the central cube, all but the bottom one (σ_7) are known to be commutative; since $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is surjective σ_7 is also commutative. Similarly we find that σ_8 is commutative. The commutativity of σ_6 follows at once from that of σ_2 and σ_3.

All this " diagramatic nonsense " plus a few extra trivia is summarized in:

Theorem **(15.3)**. — *The diagram preceding Proposition* (14.2) *is a contravariant functor of* X *as* X *varies in the category of desingularizations of* Spec(R). *If* $g: X' \to X$ *is a map in this category, the corresponding map of diagrams induces an isomorphism of exact sequences*

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(U) & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X') & \longrightarrow & \mathrm{Pic}(U) & \longrightarrow & H' & \longrightarrow & G' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

Thus the groups $\mathrm{Pic}^0(R) = \mathrm{Pic}^0(X)$, $\mathrm{Pic}(U)$, H *and* G *and the exact sequence*

$$0 \to \mathrm{Pic}^0(R) \to \mathrm{Pic}(U) \to H \to G \to 0$$

depend only on R *and not on* X.

Proof. — The only thing left to be said is that in verifying the last assertion, one should recall that any two desingularizations of Spec(R) are both dominated by a third (cf. B) in the proof of Proposition (1.2)).

§ 16. Formally smooth extensions.

Lemma **(16.1)**. — *Let* A *be a reduced local ring, with maximal ideal* $\mathfrak{m}$, *such that there exists a desingularization* $g: Y \to \mathrm{Spec}(A)$. *Let* B *be a local ring, and let* $\varphi: A \to B$ *be a local homomorphism such that, with* A *and* B *topologized by the powers of their respective maximal ideals,* B *is a formally smooth* A*-algebra* [1]. *Let* A′ *be the integral closure of* A *in its total ring of fractions* T_A, *and let* B′, T_B, *be similarly defined. Then*

(i) A′ *is a finitely generated* A*-module.*

(ii) B *is reduced and the projection*

$$g_{(B)}: Z = Y \otimes_A B \to \mathrm{Spec}(B)$$

is a desingularization.

(iii) *With canonical identifications we have*

$$B = A \otimes_A B \subseteq A' \otimes_A B \subseteq T_A \otimes_A B \subseteq T_B$$

and then $A' \otimes_A B = B'$.

(1) For our purposes this can be taken to mean that B is flat over A and that $B/\mathfrak{m}B$ is geometrically regular over $A/\mathfrak{m}$ (cf. [**EGA** 0_{IV}, (19.7.1) and (22.5.8)]).

Proof. — (i) Since g is a desingularization it is easy to see that $A' \cong \Gamma(Y, \mathcal{O}_Y)$, which is a finitely generated A-module.

(ii) To prove that Z is regular it is enough to show that $\mathcal{O}_{Z,z}$ is regular for every closed point $z \in Z$. We have the cartesian diagram

$$\begin{tikzcd} Y \arrow[d, "g"'] & Z \arrow[l, "\sigma"'] \arrow[d, "g_{(B)}"] \\ \mathrm{Spec}(A) & \mathrm{Spec}(B) \arrow[l, "\tau"'] \end{tikzcd}$$

Let $y = \sigma(z)$. Since g, and therefore $g_{(B)}$, is proper, $g(y) = \tau(g_{(B)}(z)) = \mathfrak{m}$, and so the residue field $\mathbf{k}(y)$ is a finitely generated extension of $A/\mathfrak{m}$; since $B/\mathfrak{m}B$ is geometrically regular over $A/\mathfrak{m}$, the fibre

$$\sigma^{-1}(y) = \mathbf{k}(y) \otimes_{A/\mathfrak{m}} B/\mathfrak{m}B$$

is regular; and since $\mathcal{O}_{Y,y}$ is regular and σ is flat, it follows that $\mathcal{O}_{Z,z}$ is regular.

The canonical map

$$\varphi : W = \mathrm{Spec}(T_A) \otimes_A B \to Y \otimes_A B = Z$$

induces an isomorphism $\mathcal{O}_{Z,\varphi(w)} \to \mathcal{O}_{W,w}$ for any $w \in W$, so that $\mathcal{O}_{W,w}$ is regular. Hence $T_A \otimes_A B$ is reduced, and since $B \subseteq T_A \otimes_A B$ (cf. proof of (iii) following) B is reduced. Finally $g_{(B)}$ is birational [**EGA** IV, (6.15.4.1)] and so $g_{(B)}$ is a desingularization, as asserted.

(iii) Since B is flat over A, we have a canonical map $\psi : T_A \to T_B$ which gives rise to an injective map $T_A \otimes_A B \to T_B$; we then obtain the indicated identifications from the inclusion $A \subseteq A' \subseteq T_A$ by tensoring with B. Moreover one checks that when $\Gamma(Y, \mathcal{O}_Y)$ and $\Gamma(Z, \mathcal{O}_Z)$ are naturally identified, as in the proof of (i), with A' and B' respectively, then ψ induces the map

$$\Gamma(\sigma) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Z, \mathcal{O}_Z)$$

corresponding to σ. Finally, since B is flat over A, $\Gamma(\sigma)$ gives (upon extension of scalars) an isomorphism

$$\Gamma(Y, \mathcal{O}_Y) \otimes_A B \xrightarrow{\approx} \Gamma(Z, \mathcal{O}_Z) \qquad [\textbf{EGA III}, (1.4.15)].$$

Thus $A' \otimes_A B = B'$. Q.E.D.

Remark **(16.2)**. — A special case of (16.1) is: *if* A *is a normal local ring admitting a desingularization, then the completion* $\hat{A}$ *is normal.* (Take $B = \hat{A}$ in (iii).) Conversely, if A is a *two-dimensional* local ring such that $\hat{A}$ is normal, then the methods of Hironaka [**10**] show that A admits a desingularization (since the two-dimensional excellent normal local ring $\hat{A}$ can be desingularized by " modification of the closed point ", and such a desingularization " descends " to A).

Proposition **(16.3).** — *Suppose that* A *and* B *satisfy the hypotheses of Lemma* (16.1) *and that furthermore both are two-dimensional* (*so that* $\mathfrak{m}$B *is the maximal ideal of* B *and* B/$\mathfrak{m}$B *is a separable field extension of* A/$\mathfrak{m}$). *If either one of* A *or* B *is normal then so is the other, and there is a natural commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(A) & \longrightarrow & \mathrm{Pic}(U_A) & \longrightarrow & H(A) & \longrightarrow & G(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(B) & \longrightarrow & \mathrm{Pic}(U_B) & \longrightarrow & H(B) & \longrightarrow & G(B) & \longrightarrow & 0 \end{array}$$

Moreover if B/$\mathfrak{m}$B *is a* regular *field extension of* A/$\mathfrak{m}$ *then* γ *is an isomorphism* [1].

(The rows of the diagram are the sequences defined in Theorem (15.3), with A, B, respectively, in place of R.)

Proof. — If A is normal then so is B, by (iii) of Lemma (16.1). Conversely if B is normal then so is A, since, by flatness, $A = B \cap$ (field of fractions of A).

Let $g : Y \to \mathrm{Spec}(A)$, $g_{(B)} : Z = Y_{(B)} \to \mathrm{Spec}(B)$, $\sigma : Z \to Y$, be as in Lemma (16.1), and let $\mathbf{E}_Y$, $\mathbf{E}_Z$ be the groups of exceptional divisors on Y and Z respectively. Since

$$g_{(B)}^{-1}(\{\mathfrak{m}B\}) = g^{-1}(\{\mathfrak{m}\}) \otimes_{A/\mathfrak{m}} B/\mathfrak{m}B$$

each integral curve E with exceptional support on Z dominates a unique such curve $\sigma(E)$ on Y. We define a homomorphism

$$\sigma_1 : \mathbf{E}_Z \to \mathbf{E}_Y$$

by setting, for each integral curve $E \in \mathbf{E}_Z$,

$$\sigma_1(E) = \rho_E \sigma(E)$$

where ρ_E is the integer (depending on E) defined in the following lemma (with $F = \sigma(E)$, $k = A/\mathfrak{m}$ and $K = B/\mathfrak{m}B$).

Lemma **(16.4).** — *Let* F *be an integral curve proper over a field* k. *Let* K *be a field containing* k, *and let* E *be a component of the curve* $F_{(K)}$ *with projection map* $\pi : E \to F$. *Let* L *be a finite algebraic extension of* k *which is a field of definition of* E, *so that* $E = D_{(K)}$, *where* D *is a component of* $F_{(L)}$ *(such* L *exist). Let* $n = [L : k]$ *and let* t *be the degree of the function field of* D *over that of* F. *Finally let* d_F *be the g.c.d. of all the degrees (over* k*) of invertible sheaves on* F *and let* d_E *be the g.c.d. of all the degrees* (*over* K) *of invertible sheaves on* E. *Then*

$$\rho = \frac{t d_F}{n d_E}$$

[1] Cf. correction at the end of this paper.

is an integer, and for every invertible sheaf $\mathscr{M}$ on F *we have*

$$\frac{1}{d_E}\deg_E(\pi^*\mathscr{M}) = \rho.\frac{1}{d_F}\deg_F(\mathscr{M}).$$

Moreover if K *is a* regular *extension of k then $d_E = d_F$ and $\rho = 1$* (1).

Proof. — Let $\pi_1 : D \to F$ be the projection. It follows from Proposition (10.2) *b)* that

$$n.\deg_D(\pi_1^*\mathscr{M}) = t.\deg_F(\mathscr{M})$$

where " $\deg_D$ " is relative to L. Since $E = D_{(K)}$ and degree is defined in terms of cohomology groups, we have

$$\deg_E(\pi^*\mathscr{M}) = \deg_D(\pi_1^*\mathscr{M}).$$

Thus $$\frac{1}{d_E}\deg_E(\pi^*\mathscr{M}) = \frac{n}{nd_E}\deg_D(\pi_1^*\mathscr{M}) = \rho.\frac{1}{d_F}\deg_F(\mathscr{M})$$

as asserted. To see that ρ is an integer, just take $\mathscr{M}$ to be an invertible sheaf of degree d_F.

If K is a regular extension of k, then we can take $L = k$ so that $n = t = 1$. Since ρ is an integer, we need only show now that d_F divides d_E. Let $\overline{F}$ be the normalization of F in the field of functions of F, and let $\overline{E} = \overline{F}_{(K)}$. Then $\overline{E}$ is birational over E [**EGA** IV, (6.15.4.1)] so that $d_F = d_{\overline{F}}$ and $d_E = d_{\overline{E}}$ (cf. end of proof of Lemma (15.1)). Let $\delta_{\overline{F}/k}$ be the g.c.d. of all the field degrees $[k(x):k]$ as x runs through the closed points of $\overline{F}$, and let $\delta_{\overline{E}/K}$ be similarly defined. Since the group of divisors on a curve is generated by effective divisors with one-point support (Lemma (11.2)) and since $\overline{F}$ is regular, we see that $\delta_{\overline{F}/k} = d_{\overline{F}}(= d_F)$ and that $\delta_{\overline{E}/K}$ divides $d_{\overline{E}}(= d_E)$. But since K is a regular field extension of k, we have $\delta_{\overline{F}/k} = \delta_{\overline{E}/K}$. Q.E.D.

Returning to the proof of Proposition (16.3), let σ_1 be as above and let

$$\sigma_1^* : \mathbf{E}_Y^* = \mathrm{Hom}(\mathbf{E}_Y, \mathbf{Z}) \to \mathrm{Hom}(\mathbf{E}_Z, \mathbf{Z}) = \mathbf{E}_Z^*$$

be the adjoint homomorphism. Let $\theta_Y : \mathrm{Pic}(Y) \to \mathbf{E}_Y^*$, $\theta_Z : \mathrm{Pic}(Z) \to \mathbf{E}_Z^*$ be defined as in § 14. *Then the diagram*

$$\begin{array}{ccc} \mathrm{Pic}(Y) & \xrightarrow{\mathrm{Pic}(\sigma)} & \mathrm{Pic}(Z) \\ \Big\downarrow{\scriptstyle\theta_Y} & & \Big\downarrow{\scriptstyle\theta_Z} \\ \mathbf{E}_Y^* & \xrightarrow[\sigma_1^*]{} & \mathbf{E}_Z^* \end{array} \tag{6}$$

is commutative. Indeed the commutativity means precisely that: if E is an integral curve

(1) Cf. correction at the and of this paper.

on Z with exceptional support and $F=\sigma(E)$ is the corresponding curve on Y, then for any invertible sheaf $\mathscr{L}$ on Y, we have

$$\frac{1}{d_E}(\sigma^*(\mathscr{L}).E)=\rho_E.\frac{1}{d_F}(\mathscr{L}.F)$$

and this follows at once from Lemma (16.4).

In view of the definition of the rows (§ 14), it is now completely straightforward to see that a commutative diagram exists as asserted in Proposition (16.3). To see that β (and hence α) is injective, we need only note that if I is an ideal in A such that IB is a principal ideal of B, then I is principal in A (since B is faithfully flat over A). To see that γ is an isomorphism when K is a regular extension of k, it is enough (because of (6)) to check that σ_1 is an isomorphism, and this too follows from Lemma (16.4). This completes the proof.

Remark. — It can be shown that if A is *henselian* and $B=\hat{A}$, then α and β are isomorphisms. (We will not need this result.)

Proposition **(16.5).** — *Let* A, B *be as in Proposition* (16.3). *If either one of* A *or* B *is normal then so is the other, and then* A *has a rational singularity if and only if* B *has a rational singularity.*

Proof. — The assertion about normality is proved at the beginning of the proof of Proposition (16.3). If A has a rational singularity then there is a desingularization $g: Y \to \mathrm{Spec}(A)$ with $H^1(Y, \mathcal{O}_Y)=0$. Then $g_{(B)}: Z=Y\otimes_A B \to \mathrm{Spec}(B)$ is a desingularization of B (Lemma (16.1)) and since B is flat over A,

$$H^1(Z, \mathcal{O}_Z)=H^1(Y, \mathcal{O}_Y)\otimes_A B=0.$$

Thus B has a rational singularity.

Suppose conversely that B has a rational singularity. Assume, for purposes of induction on n, that there exists a cartesian diagram

$$\begin{tikzcd} Y_n \arrow[d, "g_n"'] & Z_n=Y_n\otimes_A B \arrow[l, "\sigma_n"'] \arrow[d, "h_n=g_{n(B)}"] \\ \mathrm{Spec}(A) & \mathrm{Spec}(B) \arrow[l] \end{tikzcd}$$

where g_n is proper and h_n is a product of quadratic transformations. (For $n=0$, we can take $Y_0=\mathrm{Spec}(A)$, $Z_0=\mathrm{Spec}(B)$.) Note that Z_n is normal (Proposition (8.1)). Then

$$H^1(Y_n, \mathcal{O}_{Y_n})\otimes_A B=H^1(Z_n, \mathcal{O}_{Z_n})=0$$

(Proposition (1.2)) and since B is *faithfully* flat over A, $H^1(Y_n, \mathcal{O}_{Y_n})=0$. Also if Z_n is regular, then so is Y_n (because σ_n is faithfully flat, cf. [**EGA** 0_{IV}, (17.3.3) (i)]), and then we are done.

If Z_n is not regular, blow up a non-regular closed point y on Y_n and let Y_{n+1} be the resulting surface. Since σ_n is flat, and since the fibres of σ_n over the closed points of Y_n are regular zero-dimensional schemes (cf. proof of Lemma (16.1)), we see that

$$Z_{n+1} = Y_{n+1} \times_{Y_n} Z_n = Y_{n+1} \otimes_A B$$

is obtained from Z_n by blowing up the finite set of closed points on Z_n which lie over y. None of these points is regular on Z_n, otherwise, as above, y would be regular on Y_n. Continuing in this way, we construct $Y_{n+2}, Z_{n+2}, Y_{n+3}, Z_{n+3}, \ldots$. But for sufficiently large N, Z_N must be regular (cf. proof of Theorem (4.1)). This completes the proof.

§ 17. Applications: finite divisor class groups; factorial henselian rings.

We maintain the notation of § 14.

Proposition **(17.1)**. — *If* R *has a rational singularity then its divisor class group* Pic(U) *is finite. If moreover* R *satisfies the condition of Lemma* (14.3) *(in particular, if* R *is henselian) then* $\mathrm{Pic}(U) \cong H$.

Proof. — Theorem (12.1) (i) (along with Proposition (1.2), 2)) shows that if R has a rational singularity then $\mathrm{Pic}^0(X) = (0)$. Since H is finite, the conclusion follows at once from Propositions (14.2) and (14.4). Q.E.D.

Corollary **(17.2)**. — *The following are equivalent*:

(i) R *has a rational singularity and* $H(R) = (0)$.

(ii) *The henselization* R^* *of* R *has a rational singularity, and* R^* *is factorial.*

(iii) *The completion* $\hat{R}$ *of* R *has a rational singularity, and* $\hat{R}$ *is factorial.*

Proof. — Proposition (16.5) (with $A = R$, $B = R^*$ or $\hat{R}$) shows that either every one or no one of R, R^*, $\hat{R}$, has a rational singularity. Similarly, Proposition (16.3) gives $H(R) = H(R^*) = H(\hat{R})$. Finally, by Proposition (17.1), R^* (respectively $\hat{R}$) is factorial if and only if $H(R^*)$ (respectively $H(\hat{R})$) is trivial. Q.E.D.

(17.1) has a partial converse:

Proposition **(17.3)**. — *Let* R *be as in Lemma* (14.3), *and assume further that* R *has an algebraically closed residue field. If* Pic(U) *is finite then* R *has a rational singularity.*

Proof. — If Pic(U) is finite then so is $\mathrm{Pic}^0(X)$ (Proposition (14.2)). Let C be any curve on X such that $C_{\mathrm{red}} = E_1 + E_2 + \ldots + E_n$. Let $\mathrm{Pic}^0(C)$ be the subgroup of Pic(C) consisting of the classes of those invertible sheaves $\mathscr{L}$ on C such that $\deg_{E_i}(\varepsilon_i^*(\mathscr{L})) = 0$ for all $i = 1, 2, \ldots, n$ (where $\varepsilon_i : E_i \to C$ is the inclusion map). Lemma (14.3) shows that $\mathrm{Pic}^0(C)$ is the image of $\mathrm{Pic}^0(X)$ under the canonical map $\mathrm{Pic}(X) \to \mathrm{Pic}(C)$, and so $\mathrm{Pic}^0(C)$ is finite. The proof of Complement (11.3) works equally well if one assumes only that $\mathrm{Pic}^0(C)$ is finite (instead of $\mathrm{Pic}^0(C) = 0$ as in Complement (11.3)); the conclusion is that $H^1(C, \mathcal{O}_C) = 0$.

For any integer $r > 0$ let X_r be the subscheme of X defined by the ideal $\mathfrak{m}^r \mathcal{O}_X$. Since

$$H^1(X, \mathcal{O}_X)^\wedge = \varprojlim_{r>0} H^1(X, \mathcal{O}_{X_r}) \qquad \textbf{[EGA III, (4.1.7)]}$$

it will be sufficient to show that $H^1(\mathcal{O}_{X_r})=0$ for all r. We cannot use the result of the previous paragraph directly since X_r may not be a " curve on X " in the sense of § 13. However, there *is* such a curve C_r which is a closed subscheme of X_r and which is such that the inclusion $C_r \to X_r$ is an isomorphism outside the (at most zero-dimensional) set of " embedded " associated points of X_r. It follows at once that $H^1(\mathcal{O}_{X_r}) \cong H^1(\mathcal{O}_{C_r})=0$. Q.E.D.

From Propositions (17.1) and (17.3) we obtain:

Theorem **(17.4)**. — *Let* R *be a two-dimensional normal henselian local ring with an algebraically closed residue field, such that there exists a desingularization* $f: X \to \operatorname{Spec}(R)$ (1). *Then the divisor class group of* R *is finite if and only if* R *has a rational singularity. In particular,* R *is factorial if and only if* R *has a rational singularity and the group* H *defined in* § 14 *is trivial.*

As a further application we prove a special case of a conjecture of Samuel.

Proposition **(17.5)**. — *Let* R *be a two-dimensional normal local ring having a rational singularity and such that the group* $H=H(R)$ *is trivial. Let* $\hat{R}$ *be the completion of* R. *Then the power series ring* $\hat{R}[[T_1, T_2, \ldots, T_n]]$ *is factorial for every* $n \geq 0$ (2).

Proof. — We may assume that $R=\hat{R}$ and that R is factorial (Corollary (17.2)). A theorem of Scheja [**19**; Satz 2] states that *if* S *is a complete factorial local ring of depth* ≥ 3, *then any power series ring over* S *is also factorial.* Thus it will be enough for us to show that $R[[T]]$ is factorial with $T=T_1$. Let $\mathfrak{m}$ be the maximal ideal of R. By a theorem of Ramanujam-Samuel [**EGA** IV, (21.14.1)] it is even sufficient to show that the ring $R[[T]]_{\mathfrak{p}}$ is factorial, where $\mathfrak{p}$ is the prime ideal $\mathfrak{m}R[[T]]$.

Now $B=R[[T]]_{\mathfrak{p}}$ is flat over $A=R$, $\mathfrak{m}B$ is the maximal ideal of B, and the residue field $B/\mathfrak{m}B$ is the field of fractions of $(A/\mathfrak{m})[[T]]$, which is a regular field extension of $A/\mathfrak{m}$. Consequently (Proposition (16.3)) $H(B)=H(A)=(0)$. Moreover B has a rational singularity (Proposition (16.5)). Hence, as in (17.1),

$$\operatorname{Pic}(U_B) \subseteq H(B)=0$$

i.e. B is factorial. Q.E.D.

(Actually it will emerge in § 25 that the rings R to which Proposition (17.5) applies are all of the type treated by Scheja in [**19**]; hence — *a posteriori* — the statement of Proposition (17.5) is not new.)

V. — UNIQUE FACTORIZATION OF COMPLETE IDEALS

It is easily seen that any complete ideal in a noetherian normal ring can be expressed as a product of *simple* complete ideals, i.e. of complete ideals which are not themselves the product of two other non-unit ideals. In this part V, we study questions concerning the *uniqueness* of such factorizations in a two-dimensional normal local ring having a

(1) Cf. Remark (16.2).
(2) (Added in proof) Grothendieck has indicated in correspondence that the converse also holds .

rational singularity. The main result (§ 20) is that such uniqueness holds if and only if the completion of the ring is factorial.

In § 21, dropping the assumption of " rational singularity ", we study the condition of unique factorization in the sense of the $*$ product introduced by Krull, namely for any two ideals I, J, $I*J$ is the *completion* of IJ. The results obtained generalize a number of those in [**25**; Appendix 5].

As in IV, R will be a two-dimensional normal local ring with maximal ideal $\mathfrak{m}$, $f: X \to \operatorname{Spec}(R)$ will be a desingularization with $X \neq \operatorname{Spec}(R)$, and $E_1, E_2, \ldots, E_n$ will be the components of the closed fibre $f^{-1}(\{\mathfrak{m}\})$.

§ 18. Correspondence between complete ideals and exceptional curves.

As before we denote by **E** the group of divisors on X of the form $\sum_{i=1}^{n} n_i E_i$ with rational integers n_i. Let $\mathbf{E}^{\#}$ be the set of divisors $D \in \mathbf{E}$, $D \neq 0$, such that $\mathcal{O}(-D)$ is generated by its sections over X. Let $\mathbf{E}^{+}$ denote the set of divisors $D \in \mathbf{E}$, $D \neq 0$, such that

$$(\mathcal{O}(-D)\,.\,E_i) \geq 0 \qquad (i=1, 2, \ldots, n)$$

i.e. $(D.E_i) \leq 0$ for all $i=1, 2, \ldots, n$. For these two sets we have:

(i) $\mathbf{E}^{\#} \subseteq \mathbf{E}^{+}$.

(This follows from the trivial part of Theorem (12.1) (ii).)

(ii) If $D = \sum_i n_i E_i \in \mathbf{E}^{+}$, then $n_i \geq 0$ for all i, i.e. D is a *curve* on X.

(This follows from the negative-definiteness of the intersection matrix (Lemma (14.1)): set $D = A - B$ where A, B are curves without common components; since $((A-B).B) \leq 0$ and $(A.B) \geq 0$, we must have $(B.B) \geq 0$, whence $B = 0$.)

(iii) Both $\mathbf{E}^{\#}$ and $\mathbf{E}^{+}$ are closed under addition: if D_1 and D_2 are in $\mathbf{E}^{\#}$ (respectively $\mathbf{E}^{+}$) then so is $D_1 + D_2$.

Now for any $D = \sum_i n_i E_i$ in $\mathbf{E}^{\#}$, let

$$I_D = \Gamma(X, \mathcal{O}(-D)).$$

Since D is a curve, $\mathcal{O}(-D) \subseteq \mathcal{O}_X$ and hence

$$I_D \subseteq \Gamma(X, \mathcal{O}_X) = R$$

i.e. I_D is an ideal in R. By definition of $\mathbf{E}^{\#}$,

$$I_D \mathcal{O}_X = \mathcal{O}(-D).$$

$I_D \neq R$ since $D \neq 0$. An element r of R is in I_D if and only if $v_i(r) \geq n_i$ for $i = 1, 2, \ldots, n$, where v_i is the discrete valuation corresponding to E_i; thus I_D contains a power of $\mathfrak{m}$. Moreover if x is in the completion of I_D, then we see at once (by remark *c)* of § 5, for example) that $v_i(x) \geq n_i$ for all i, so that $x \in I_D$. In other words, I_D *is an* $\mathfrak{m}$*-primary complete ideal in* R *such that* $I_D \mathcal{O}_X$ *is invertible.*

Conversely, if I is any complete $\mathfrak{m}$-primary ideal in R such that $I\mathcal{O}_X$ is invertible, then

$$I\mathcal{O}_X = \mathcal{O}(-D_I)$$

where $D_I \in \mathbf{E}^\#$, and, by completeness (cf. Proposition (6.2))

$$I = \Gamma(X, \mathcal{O}(-D_I)).$$

Thus *the association of* I_D *to* D *and* D_I *to* I *sets up a one-to-one correspondence between members of* $\mathbf{E}^\#$ *and* $\mathfrak{m}$*-primary complete ideals in* R *which generate invertible* $\mathcal{O}_X$*-ideals.*

For any two ideals I and J in R, we set

$$I * J = \text{completion of } IJ$$

(cf. [**12**]). If I′, J′ are the respective completions of I and J, then

$$I' * J' = I * J.$$

(To see this, we need only prove that $I'J' \subseteq I * J$; this can easily be done directly, or by using the methods of Proposition (6.2). Alternatively, one can use valuation theory as in [**25**; Appendix 4, Proposition 1 *e)*].)

It follows immediately that if K is a third ideal then

$$(I * J) * K = I * (J * K) = \text{completion of } IJK.$$

If I and J are both $\mathfrak{m}$-primary and complete, and such that $I\mathcal{O}_X$, $J\mathcal{O}_X$ are invertible, then the same is true of $I * J$; in fact

$$I * J = \Gamma(X, IJ\mathcal{O}_X) = \Gamma(X, \mathcal{O}(-D_I - D_J))$$

because, being complete, $\Gamma(X, IJ\mathcal{O}_X)$ contains $I * J$, while (Proposition (6.2))

$$I * J = \Gamma(X, (I * J)\mathcal{O}_X) \supseteq \Gamma(X, IJ\mathcal{O}_X).$$

Thus addition *in* $\mathbf{E}^\#$ *corresponds to the* $*$ product *for ideals.*

Let K be an ideal of the form $I * J$ where I and J are proper ideals in R. As above, $K = I' * J'$, so we may assume that I and J are complete. If furthermore $K = I * J$ is $\mathfrak{m}$-primary, then so are both I and J (for otherwise IJ would be contained in a prime ideal $\mathfrak{p} \neq \mathfrak{m}$, and since, clearly, $\mathfrak{p}$ is complete, this would mean $I * J \subset \mathfrak{p}$). Also, if $K\mathcal{O}_X = (I * J)\mathcal{O}_X$ is invertible, then so is $IJ\mathcal{O}_X$ (in fact in this case, as in the proof of Proposition (6.2), $IJ\mathcal{O}_X = (I * J)\mathcal{O}_X$) so that both $I\mathcal{O}_X$ and $J\mathcal{O}_X$ are invertible.

We say that a complete ideal K in R is $*$*-simple* if $K \neq I * J$ for any two proper ideals I, J in R. We say that an element D of $\mathbf{E}^\#$ is *indecomposable* if D cannot be expressed as a sum of two elements of $\mathbf{E}^\#$. It follows from the preceding paragraph that in the above correspondence between members of $\mathbf{E}^\#$ and $\mathfrak{m}$-primary complete ideals, the *indecomposable* elements of $\mathbf{E}^\#$ correspond precisely to the $*$*-simple* $\mathfrak{m}$-primary complete ideals which become invertible on X.

We say that *unique decomposition holds in* $\mathbf{E}^\#$ if every element in $\mathbf{E}^\#$ is in a unique way a sum of indecomposable elements of $\mathbf{E}^{\#}$. The preceding discussion shows that *unique decomposition holds in* $\mathbf{E}^{\#}$ *if and only if each* $\mathfrak{m}$*-primary complete ideal in* R *which generates an invertible* $\mathcal{O}_X$*-ideal, is in a unique way a* $*$ *product of* $*$*-simple complete ideals.*

Finally, let us observe that every ideal I in R is such that $I\mathcal{O}_Z$ is invertible for *some* desingularization $g: Z \to \operatorname{Spec}(R)$. (Choose Z so that Z dominates the surface W obtained by blowing up I, cf. B) of Proposition (1.2).)

We have established:

Proposition **(18.1)**. — *Unique factorization, in the sense of the* $*$ *product, into* $*$*-simple complete ideals, holds for complete* $\mathfrak{m}$*-primary ideals in* R *if and only if unique decomposition holds in* $\mathbf{E}^{\#} = \mathbf{E}^{\#}_X$ *for all desingularizations* $f: X \to \operatorname{Spec}(R)$. *(At least one such* X *is assumed to exist.)*

We can define " unique decomposition " in $\mathbf{E}^{+}$ just as in $\mathbf{E}^{\#}$. Then:

Corollary **(18.2)**. — *If* R *has a rational singularity, then unique factorization into simple complete ideals, in the sense of the usual product of ideals, holds for* $\mathfrak{m}$*-primary complete ideals in* R *if and only if unique decomposition holds in* $\mathbf{E}^{+} = \mathbf{E}^{+}_X$ *for all desingularizations* $f: X \to \operatorname{Spec}(R)$.

Proof. — If R has a rational singularity then the $*$ product for complete ideals is just the usual product (Theorem (7.1)), and $\mathbf{E}^{\#} = \mathbf{E}^{+}$ (Theorem (12.1)). Also, if K is complete and $K = IJ$, then $K = I * J$; and conversely, if $K = I * J$, then $K = I' * J' = I'J'$ where I', J' are the respective completions of I and J. Thus K is $*$-simple if and only if it is simple in the usual sense. Q.E.D.

§ 19. Relation with the group H.

With notation as in § 18 we investigate further the meaning of unique decomposition in $\mathbf{E}^{+}$. The main technical result is:

Proposition **(19.1)**. — *Unique decomposition holds in* $\mathbf{E}^{+} = \mathbf{E}^{+}_X$ *for all desingularizations* $f: X \to \operatorname{Spec}(R)$ *if and only if the group* H *introduced in* § 14 *is trivial.*

Proof. — We first consider a fixed desingularization $f: X \to \operatorname{Spec}(R)$. For each $i = 1, 2, \ldots, n$ let $\delta_i > 0$ be the greatest common divisor of the integers $(E_1 . E_i), (E_2 . E_i), \ldots, (E_n . E_i)$.

Lemma **(19.2)**. — *For the preceding desingularization* $f: X \to \operatorname{Spec}(R)$, *unique decomposition holds in* $\mathbf{E}^{+}$ *if and only if there exist curves* $D_1, D_2, \ldots, D_n$ *with exceptional support on* X *such that, for all* i, j,

$$(D_i . E_j) = -\delta_i \delta_{ij} \qquad (\textit{Kronecker } \delta_{ij})$$

If such curves exist, then they are all the indecomposable members of $\mathbf{E}^{+}$.

Proof of Lemma (19.2). — Suppose that such curves D_i exist; by definition $D_i \in \mathbf{E}^{+}$. For any $D = \sum_j r_j E_j$ in $\mathbf{E}^{+}$, we have, for $i = 1, 2, \ldots, n$,

$$(D . E_i) = -n_i \delta_i \qquad (n_i \geq 0).$$

Hence
$$((D - \sum_j n_j D_j) . E_i) = (D . E_i) - n_i(-\delta_i) = 0$$

so that, by definiteness of the intersection product (Lemma (14.1)),

$$D-\sum_j n_j D_j = 0, \quad \text{i.e. } D=\sum_j n_j D_j.$$

If also $D=\sum_j m_j D_j$, then

$$(D.E_i)=m_i(D_i.E_i) \qquad (i=1, 2, \dots, n)$$

i.e.

$$-n_i\delta_i=-m_i\delta_i$$

so $m_i=n_i$ for all i. Thus unique decomposition holds in $\mathbf{E}^+$, and $D_1, D_2, \dots, D_n$ are all the indecomposable elements in $\mathbf{E}^+$.

Suppose, conversely, that unique decomposition holds in $\mathbf{E}^+$. We first observe that there cannot be more than n indecomposable elements in $\mathbf{E}^+$. Otherwise there would be a relation

$$r_0A_0=r_1A_1+r_2A_2+\dots+r_nA_n$$

(A_i indecomposable, r_i integers, $r_0>0$) and so for $r\geq\max(|r_1|, |r_2|, \dots, |r_n|)$ we would have $r_0A_0+rA_1+\dots+rA_n=(r_1+r)A_1+\dots+(r_n+r)A_n$ contradicting unique decomposition.

Now there exist elements $B_i\in\mathbf{E}^+$ $(i=1, 2, \dots, n)$ such that $(B_i.E_j)=0$ if $i\neq j$; in fact, if (b_{ij}) is the inverse matrix of $((E_i.E_j))$, and N is a *negative* integer such that Nb_{ij} is an integer for all i, j, then setting $B_i=\sum_{k=1}^{n} Nb_{ik}E_k$ we have $(B_i.E_j)=N\delta_{ij}\leq 0$. It follows that there is an indecomposable element D_i of $\mathbf{E}^+$ and a positive integer δ'_i such that

$$(D_i.E_j)=-\delta'_i\delta_{ij}$$

(any indecomposable summand of B_i will do). By the previous paragraph, every element in $\mathbf{E}^+$ must be a linear combination of $D_1, D_2, \dots, D_n$, with integer coefficients, and therefore δ'_i divides all the integers $\{(C.E_i)\,|\,C\in\mathbf{E}^+\}$. We will find a curve $C_i\in\mathbf{E}^+$ such that $(C_i.E_i)=-\delta_i$. Since δ_i divides δ'_i, this will prove that $\delta'_i=\delta_i$.

There exist integers c_{ij} such that $-\delta_i=\sum_j c_{ij}(E_j.E_i)$. If M is any sufficiently large positive integer,

$$C_i=\sum_{j=1}^{n} c_{ij}E_j+M\sum_{j\neq i} D_j$$

is as required, and the lemma is proved.

Returning now to Proposition (19.1), and referring to the definition of H (§ 14) we note that the triviality of H means that for any X as above, there exist elements D'_i of $\mathbf{E}$ $(i=1, 2, \dots, n)$, such that, for all i, j,

$$(D'_i.E_j)=-d_j\delta_{ij}$$

where, as in § 14, d_j is the greatest common divisor of all the degrees of invertible sheaves on E_j. Such D'_i are, by definition, members of $\mathbf{E}^+$. Since obviously d_j divides δ_j, we see by Lemma (19.2) that if H is trivial, then unique decomposition holds in $\mathbf{E}^+$ for all X.

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To prove the converse, it will now be sufficient to show that *there is at least one* X *on which* $d_i = \delta_i$ *for all* $i = 1, 2, \ldots, n$.

To begin with consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & Y \arrow[dl, "f'"] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

of proper birational maps, with X and Y regular, and an integral curve E on X with exceptional support. Let $d(\mathrm{E})$, $\delta(\mathrm{E})$ be defined in the obvious way, i.e. $d(\mathrm{E})$ is the greatest common divisor of all the degrees of invertible sheaves on E, and $\delta(\mathrm{E})$ is the greatest common divisor of all the integers $(\mathrm{D}.\mathrm{E})$, where D is a divisor on X with exceptional support.

Suppose first that $\mathrm{F} = g(\mathrm{E})$ is a *curve* on Y; then E is the proper transform of F on X. As we have seen (cf. proof of Lemma (15.1)) $d(\mathrm{F}) = d(\mathrm{E})$. Furthermore $\delta(\mathrm{E})$ divides $\delta(\mathrm{F})$, because there is a divisor D on Y with exceptional support such that $(\mathrm{D}.\mathrm{F}) = \delta(\mathrm{F})$; and then we have $(\mathrm{D}'.\mathrm{E}) = \delta(\mathrm{F})$ where $\mathrm{D}' = g^*(\mathrm{D})$ (cf. § 15). It follows for example that if $\delta(\mathrm{F}) = d(\mathrm{F})$, then $\delta(\mathrm{E})$ divides $d(\mathrm{E})$ and so $\delta(\mathrm{E}) = d(\mathrm{E})$.

Suppose next that $g(\mathrm{E})$ is a single point on Y. Then $\delta(\mathrm{E}) = d(\mathrm{E}) = h^0(\mathrm{E})$. In fact, by the factorization theorem (Theorem (4.1)) g factors as

$$X \xrightarrow{g_1} Y_1 \xrightarrow{g_2} Y_2 \xrightarrow{g_3} Y$$

where g_1 is such that $g_1(\mathrm{E})$ is a curve on Y_1, *isomorphic to* E, and $g_2(g_1(\mathrm{E}))$ is a single point P on Y_2 such that g_2 is the map obtained by blowing up P (so that $g_1(\mathrm{E}) = g_2^{-1}(\mathrm{P})$). Now, as in the proof of Lemma (15.2), we have

$$(g_1(\mathrm{E}) \,.\, g_1(\mathrm{E})) = -h^0(g_1(\mathrm{E}))$$

and so $\delta(g_1(\mathrm{E}))$ divides $h^0(g_1(\mathrm{E}))$. But clearly $h^0(g_1(\mathrm{E}))$ divides $d(g_1(\mathrm{E}))$. Hence

$$\delta(g_1(\mathrm{E})) = d(g_1(\mathrm{E})) = h^0(g_1(\mathrm{E})).$$

By the preceding remarks (with $\mathrm{F} = g_1(\mathrm{E})$), we conclude that:

$$\delta(\mathrm{E}) = d(\mathrm{E}) = d(g_1(\mathrm{E})) = h^0(g_1(\mathrm{E})) = h^0(\mathrm{E}).$$

Now let $h : \mathrm{Z} \to \mathrm{Spec}(\mathrm{R})$ be some desingularization, and let G be an integral curve on Z with exceptional support.

Lemma (**19.3**). — *Let* Δ *be a divisor on* G. *Then there exists a proper birational map* $h : \mathrm{Y} \to \mathrm{Z}$, *with* Y *regular, such that, if* F *is the proper transform of* G *on* Y *and* $j : \mathrm{F} \to \mathrm{G}$ *is the induced map, then* $j^*(\Delta)$ *is the restriction to* F *of a divisor* D *on* Y *with exceptional support.*

Remark. — The existence of X with $d_i = \delta_i$ for $i = 1, 2, \ldots, n$ can now be shown as follows: let $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_m$ be *all* the integral curves on Z with exceptional support, let $G=G_1$, and choose Δ in Lemma (19.3) so that Δ has degree $d(G_1)$ on G_1; if $Y=Y_1$, $F=F_1$ are as in Lemma (19.3), then $j^*(\Delta)$ has degree $d(G_1)=d(F_1)$ on F_1, and consequently $\delta(F_1)=d(F_1)$; similarly we can find Y_i, F_i with $\delta(F_i)=d(F_i)$ for $i=2, 3, \ldots, m$; then if X is a desingularization of Spec(R) which dominates $Y_1, Y_2, \ldots, Y_m$, the discussion preceding Lemma (19.3) shows that X is as desired. (Such a desingularization X exists, as can be seen by letting W be the *join* of $Y_1, Y_2, \ldots, Y_m$ (cf. § 8) and applying B) of Proposition (1.2)).

Proof of Lemma (19.3). — Let Δ', Δ'' be divisors for which such Y', D', resp. Y'', D'', can be found. Let Y be a desingularization dominating Y' and Y'', let F be the proper transform of G on Y, and let D'_1 (resp. D''_1) be the inverse image of D' (resp. D'') on Y. Then clearly $D'_1+D''_1$ induces on F the inverse image of $\Delta'+\Delta''$.

Hence, in view of Lemma (11.2), we may assume that Δ is an effective divisor with support at a single point P of G. Let $\bar{x}$ be a non-zero element in $\mathcal{O}_{G,P}$ defining Δ, and let x be an element of $\mathcal{O}_{Z,P}$ whose image in $\mathcal{O}_{G,P}$ is $\bar{x}$. The divisor (x) of x on Z has the form

$$(x)=C+C'$$

where each component of C, and no component of C', passes through P. We will choose $h: Y\to Z$ so that the proper transform [C] on Y of the divisor C does not meet F at any point of $h^{-1}(P)$. Then we will have

$$h^*(C)=[C]+D+D'$$

where D is such that the image under h of its support is the point P, while the image under h of the support of D' does not meet P. Since C induces on G a divisor $\Delta+\Delta'$, with $P\notin$support of Δ', $h^*(C)$ induces $j^*(\Delta)+j^*(\Delta')$, the support of $j^*(\Delta)$ being contained in $h^{-1}(P)$, while the support of $j^*(\Delta')$ does not meet $h^{-1}(P)$. It is therefore evident that D is a divisor with exceptional support inducing $j^*(\Delta)$ on F, as desired.

We obtain Y as follows. Let v_0 be the discrete valuation corresponding to G, and let $v_1, v_2, \ldots, v_r$ be the discrete valuations corresponding to the components of C. Note that

$$v_0(x)=0, \qquad v_i(x)>0 \qquad \text{for } i>0.$$

Choose $t\in\mathcal{O}_{Z,P}$ such that

$$v_0(t)>0, \qquad v_i(t)=0 \qquad \text{for } i>0$$

By blowing up, we can find W dominating Z such that the sheaf $(x, t)\mathcal{O}_W$ is invertible. Then we can find a *regular* Y dominating W. What remains to be shown is that for $i>0$ the center of v_0 on Y does not meet the center of v_i on Y, i.e. there is no point y on Y such that v_0 and v_i are both non-negative at $\mathcal{O}_{Y,y}$. But for any y on Y, either x/t or t/x is in $\mathcal{O}_{Y,y}$, and since

$$v_0(x/t)<0, \qquad v_i(t/x)<0 \qquad \text{for } i>0$$

we are done.

§ 20. The main theorem.

Theorem (**20.1**). — *Let* R *be a two-dimensional normal local ring, with maximal ideal* $\mathfrak{m}$, *having a rational singularity. Let* $\hat{R}$ *be the completion of* R, *and let* R^* *be the henselization of* R. *The following conditions are equivalent*:

1) *In* R, *factorization of* $\mathfrak{m}$-*primary complete ideals into simple complete ideals is unique.*

1′) *In* R, *factorization of complete ideals into simple complete ideals is unique.*

2) $\hat{R}$ *is factorial.*

2′) R^* *is factorial.*

Proof. — We have already seen (Corollary (17.2)) that 2) and 2′) are each equivalent to the triviality of H(R), and so is 1) (Corollary (18.2) and Proposition (19.1)). 1′) trivially implies 1). Conversely, the triviality of H(R) implies that R is factorial (since $\mathrm{Pic}(U) \subseteq H$, cf. Proposition (17.1)) from which it is immediate that *every* complete ideal is in a unique way of the form PI, where P is a principal ideal and I is an $\mathfrak{m}$-primary complete ideal; it follows at once that 1) implies 1′). Q.E.D.

Remarks. — 1. Let R be *any* two-dimensional normal local ring. If R has an algebraically closed residue field, then the condition that $\hat{R}$ is factorial *implies* that $\hat{R}$ has a rational singularity. (Theorem (17.4), and note that $\hat{R}$, being excellent, can be desingularized.) Hence also R has a rational singularity (Proposition (16.5)).

2. Any two-dimensional regular local ring R satisfies the conditions of Theorem (20.1). In § 25 we describe quite explicitly the non-regular R which satisfy these conditions.

§ 21. Some consequences of unique $*$-factorization.

With notation as in § 18, we investigate further the condition of unique $*$-factorization of $\mathfrak{m}$-primary complete ideals into $*$-simple complete ideals (cf. Proposition (18.1)).

Lemma (**21.1**). — *For a fixed desingularization* $f: X \to \mathrm{Spec}(R)$, *unique decomposition holds in* $\mathbf{E}^\#$ *if and only if* (i): *unique decomposition holds in* $\mathbf{E}^+$, *and* (ii): $\mathbf{E}^\# = \mathbf{E}^+$.

Proof. — We need only show that *if unique decomposition holds in* $\mathbf{E}^\#$, *then* $\mathbf{E}^+ \subseteq \mathbf{E}^\#$. We first remark that by negative definiteness, there exists D in $\mathbf{E}$ such that $(D.E_i) < 0$ for all i, and then by Theorem (12.1) (iii), $\mathcal{O}(-D)$ is ample. Consequently, given $D' \in \mathbf{E}$, there exists an integer N such that both ND and $D' + ND$ are in $\mathbf{E}^\#$; in other words, $\mathbf{E}^\#$ *generates the group* $\mathbf{E}$. Since $\mathbf{E}$ is a free abelian group of rank n, there must therefore be at least n indecomposable elements in $\mathbf{E}^\#$. But, as in the proof of Lemma (19.2), unique decomposition implies that $\mathbf{E}^\#$ has *at most* n indecomposable elements. Thus there are precisely n such elements, which we may name $D_1, D_2, \ldots, D_n$, and these form a free basis of $\mathbf{E}$.

Now suppose that $A = \sum_{i=1}^{n} t_i D_i$ is such that $-A$ is ample. If N is a suitably large positive integer, then

$$NA - (D_1 + D_2 + \ldots + D_n) \in \mathbf{E}^\#.$$

Since every element in $\mathbf{E}^\#$ is of the form $\sum_i n_i D_i$ with all $n_i \geq 0$, we conclude that $t_1, t_2, \ldots, t_n$ are all >0.

Finally, if $B \in \mathbf{E}^+$, $B = \sum_i s_i D_i$, then

$$((\sum_i (s_i+1)D_i).E_j) = (B.E_j) + (D_1.E_j) + \ldots + (D_n.E_j) < 0$$

because $(B.E_j) \leq 0$ and $(D_i.E_j) \leq 0$ for all i, with $(D_i.E_j) < 0$ for some i (since $(D.E_j) < 0$ (D as above) and $D_1, D_2, \ldots, D_n$ generate $\mathbf{E}$). Hence (Theorem (12.1) (iii)) $-\sum_i (s_i+1)D_i$ is ample, and so, as we have just seen, $s_i+1>0$ for all i. Thus $s_i \geq 0$ for all i, and therefore B is in $\mathbf{E}^\#$. Q.E.D.

We will need the following elementary lemma.

Lemma **(21.2)**. — *Let* A *be a local ring, let* $f : X \to \operatorname{Spec}(A)$ *be a map of finite type, and let* C *be a one-dimensional integral closed subscheme of* X *with exceptional support* (cf. § 12). *Let* $\mathscr{L}$ *be an invertible sheaf on* X, *let* $S = \bigoplus_{n \geq 0} S_n$ *be a graded* A*-algebra, and let* $\psi : S \to \bigoplus_{n \geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$ *be a homomorphism of graded* A*-algebras with associated map* $r : G(\psi) \to \operatorname{Proj}(S)$ (cf. [**EGA** II, § 3.7]) *such that* $G(\psi) \cap C$ *is not empty. Then* $(\mathscr{L}.C) \geq 0$ *and* $(\mathscr{L}.C) = 0$ *if and only if* $C \subseteq G(\psi)$ *and* $r(C)$ *is a single point.*

Proof. — Let Q be a point in $r(C \cap G(\psi))$. For some $n>0$ there is an element $t \in S_n$ such that $Q \in \operatorname{Spec}(S_{(t)})$, so that C meets $r^{-1}(\operatorname{Spec}(S_{(t)})) = X_u$ where $u = \psi(t)$ (cf. [**EGA** II, (3.7.3)]). If $\mathscr{L}' = i_C^*(\mathscr{L}^{\otimes n})$, where $i_C : C \to X$ is the inclusion map, then u induces a section u' of $\mathscr{L}'$ over C such that $C_{u'} = C \cap X_u \neq \emptyset$; consequently we have an exact sequence

$$0 \to \mathscr{O}_C \xrightarrow{u'} \mathscr{L}' \to \mathscr{K} \to 0$$

where $\operatorname{Supp}(\mathscr{K}) = C - C_{u'}$ is of dimension ≤ 0. Thus

$$\begin{aligned} n(\mathscr{L}.C) = \deg_C(\mathscr{L}') &= \chi(\mathscr{L}') - \chi(\mathscr{O}_C) \\ &= \chi(\mathscr{K}) \\ &= h^0(\mathscr{K}) \geq 0. \end{aligned}$$

For $(\mathscr{L}.C) = 0$ it is necessary and sufficient that $\operatorname{Supp}(\mathscr{K})$ be empty, i.e. $C \subseteq X_u$. This is certainly the case if $C \subseteq G(\psi)$ and $r(C)$ is a single point (necessarily Q). Conversely, if $C \subseteq X_u$, then $C \subseteq G(\psi)$, and $r(C)$ is a closed subscheme of the affine scheme $\operatorname{Spec}(S_{(t)})$, with $r(C)$ proper over the closed point of Spec(A); this implies that $r(C)$ is a single point. (Alternatively, if $P \in r(C)$, $P \neq Q$, then, assuming P to be closed, as we may, we could choose t so that $P \notin \operatorname{Spec}(S_{(t)})$, i.e. $C \nsubseteq X_u$.) Q.E.D.

Let R be as usual. For any ideal I in R, denote by W_I the normalization of the scheme obtained by blowing up I, i.e. $W_I = \operatorname{Proj}(\bigoplus_{n \geq 0} I_n)$, where I_n is the completion of I^n. Note that W_I is of finite type over Spec(R) (Corollary (6.4)).

Proposition **(21.3)**. — (i) *If unique $*$-factorization holds for* $\mathfrak{m}$*-primary complete ideals in* R, *then for any $*$-simple* $\mathfrak{m}$*-primary complete ideal* I, *the fibre on* W_I *over the closed point of* Spec(R) *is irreducible.*

(ii) *When the hypothesis of* **(i)** *holds, if we denote by* v_{I} *the discrete valuation of the quotient field* K *of* R *whose center on* $\mathrm{W_I}$ *is the reduced closed fibre, then the association of* I *to* v_{I} *sets up a one-one correspondence between* $*$*-simple* $\mathfrak{m}$*-primary complete ideals and valuations of* K *which dominate and are residually transcendental over* R.

Proof. — To say that the closed fibre on $\mathrm{W_I}$ is irreducible for all I as above is to say:

(i)′ *Let* $f: \mathrm{X} \to \mathrm{Spec(R)}$ *be a desingularization, and let* D *be an indecomposable element of* $\mathbf{E}^{\#} = \mathbf{E}^{\#}_{\mathrm{X}}$; *then* $(\mathrm{D}.\mathrm{E}) = 0$ *for all but one of the integral exceptional curves* E *on* X.

(*Proof.* — As in § 18, for any such D we have $\mathcal{O}(-\mathrm{D}) = \mathrm{I}\mathcal{O}_{\mathrm{X}}$ for some I as above, and conversely for any such I, there is an X such that $\mathrm{I}\mathcal{O}_{\mathrm{X}} = \mathcal{O}(-\mathrm{D})$, where D is an indecomposable element of $\mathbf{E}^{\#} = \mathbf{E}^{\#}_{\mathrm{X}}$. Fix a corresponding pair I, D, and set $\mathscr{L} = \mathrm{I}\mathcal{O}_{\mathrm{X}}$. As in Lemma (6.3), let

$$\mathrm{W_I} = \mathrm{Proj}\Big(\bigoplus_{n \geq 0} \Gamma(\mathrm{X}, \mathscr{L}^{\otimes n})\Big).$$

In Lemma (21.2), take $\mathrm{S} = \bigoplus_{n\geq 0} \Gamma(\mathrm{X}, \mathscr{L}^{\otimes n})$, and take ψ to be the identity map, so that $\mathrm{G}(\psi) = \mathrm{X}$, since $\mathscr{L}$ is generated by its sections over X. r is proper and birational, hence surjective, and so the closed fibre on $\mathrm{W_I}$ is $r(f^{-1}\{\mathfrak{m}\})$; also, since $\mathrm{W_I}$ is normal, any one-dimensional integral closed subscheme of $\mathrm{W_I}$ is the image under r of a *unique* integral curve on X. Thus by Lemma (21.2), the closed fibre on $\mathrm{W_I}$ is irreducible if and only if $(\mathscr{L}.\mathrm{E}) = 0$ for all but one E, i.e. $(\mathcal{O}(-\mathrm{D}).\mathrm{E}) = -(\mathrm{D}.\mathrm{E}) = 0$ for all but one E.)

By Proposition (18.1), Lemma (21.1), and Lemma (19.2), unique $*$-factorization implies condition (i)′.

(*Remark.* — Since $\mathbf{E}^{\#}$ generates $\mathbf{E}$, one finds, using Lemma (19.2), that (i)′ implies unique decomposition in $\mathbf{E}^{+}$. Thus if $\mathbf{E}^{+}$ always equals $\mathbf{E}^{\#}$ — which is the case, for example, when R has a rational singularity (Theorem (12.1)) — then (i)′ is *equivalent* to unique $*$-factorization).

(ii) If I and J are distinct $*$-simple $\mathfrak{m}$-primary complete ideals, and $f: \mathrm{X} \to \mathrm{Spec(R)}$ is a desingularization such that $\mathrm{I}\mathcal{O}_{\mathrm{X}}$, $\mathrm{J}\mathcal{O}_{\mathrm{X}}$ are both invertible, then $\mathrm{I}\mathcal{O}_{\mathrm{X}}$, $\mathrm{J}\mathcal{O}_{\mathrm{X}}$ are distinct indecomposable elements of $\mathbf{E}^{\#}_{\mathrm{X}} = \mathbf{E}^{+}_{\mathrm{X}}$; hence if $\mathrm{E_I}$ (resp. $\mathrm{E_J}$) is the unique integral curve with exceptional support on X such that $(\mathrm{I}\mathcal{O}_{\mathrm{X}}.\mathrm{E_I}) \neq 0$ (resp. $(\mathrm{J}\mathcal{O}_{\mathrm{X}}.\mathrm{E_J}) \neq 0$) then $\mathrm{E_I} \neq \mathrm{E_J}$ (cf. Lemma (19.2)). However, from the proof of (i) it is clear that $\mathrm{E_I}$ (resp. $\mathrm{E_J}$) is the center of v_{I} (resp. v_{J}) on X. Thus $v_{\mathrm{I}} \neq v_{\mathrm{J}}$.

Now if v is a valuation dominating R and residually transcendental over R, then the center of v on *some* regular X is an integral curve E; for instance if r_1, r_2 in R are such that $v(r_1/r_2) \geq 0$ and the residue field of r_1/r_2 is transcendental over the residue field of R, then we can choose any X on which the ideal $(r_1, r_2)\mathcal{O}_{\mathrm{X}}$ is invertible. For such an X, there is an indecomposable element of $\mathbf{E}^{\#}_{\mathrm{X}}$, say $\mathrm{I}\mathcal{O}_{\mathrm{X}}$, where I is a $*$-simple $\mathfrak{m}$-primary complete ideal of R, such that $(\mathrm{I}\mathcal{O}_{\mathrm{X}}.\mathrm{E}) \neq 0$. Then, as above, the center of v on $\mathrm{W_I}$ is the reduced closed fibre; in other words $v = v_{\mathrm{I}}$. Q.E.D.

We continue to assume that $*$-factorization in R is unique. The next proposition describes a " reciprocity " relation among $*$-simple $\mathfrak{m}$-primary complete ideals. Let I, v_I, be as in Proposition (21.3). If $f: X \to \mathrm{Spec}(R)$ is a desingularization such that $I\mathcal{O}_X$ is invertible, and E_I is the center of v_I on X, then $I\mathcal{O}_X$ is an indecomposable element of $\mathbf{E}_X^\sharp = \mathbf{E}_X^+$, and § 19 shows that $(I\mathcal{O}_X . E_I) = d_I$, where d_I is the greatest common divisor of degrees of invertible sheaves on E_I. Note that d_I does not depend on the choice of X (cf. proof of Lemma (15.1)).

Proposition **(21.4)**. — *Suppose that unique $*$-factorization holds for $\mathfrak{m}$-primary complete ideals in* R. *Let* I *and* J *be two $*$-simple $\mathfrak{m}$-primary complete ideals. Then, with the notation of the previous paragraph, we have*

$$d_I v_I(J) = d_J v_J(I)$$

Proof. — Let $f: X \to \mathrm{Spec}(R)$ be a desingularization such that both $I\mathcal{O}_X$ and $J\mathcal{O}_X$ are invertible. Let $E_1, E_2, \ldots, E_n$ be the components of $f^{-1}(\{\mathfrak{m}\})$, and let $v_1, v_2, \ldots, v_n$ be the corresponding discrete valuations, the numbering being such that $v_1 = v_I$, $v_2 = v_J$. Let $D_1 = \sum_{i=1}^{n} v_i(I) . E_i$, $D_2 = \sum_{i=1}^{n} v_i(J) . E_i$, so that $I\mathcal{O}_X = \mathcal{O}(-D_1)$, $J\mathcal{O}_X = \mathcal{O}(-D_2)$. Now

$$\begin{aligned}(D_1 . D_2) &= \sum_{i=1}^{n} v_i(J) . (D_1 . E_i)\\ &= v_1(J) . (D_1 . E_1) + \sum_{i=2}^{n} v_i(J) . 0\\ &= -d_I . v_I(J).\end{aligned}$$

Similarly, $(D_1 . D_2) = -d_J . v_J(I)$. Q.E.D.

The final result in this section is to the effect that " the transform of a simple complete ideal is simple ". To be more precise, let $f: X \to \mathrm{Spec}(R)$ be a desingularization, let I be an $\mathfrak{m}$-primary complete ideal in R, and let $\mathscr{I}$ be the *completion* of $I\mathcal{O}_X$. (Note that $\mathscr{I} = I\mathcal{O}_X$ if R has a rational singularity, cf. Proposition (6.5).) The *transform of* I *on* X is defined to be the ideal $\mathscr{I}' = \mathscr{I}\mathscr{I}^{-1}$ (cf. remarks preceding Proposition (3.1)). *$\mathscr{I}'$ is complete* since multiplication by invertible $\mathcal{O}_X$-modules does not affect completeness (Remark *e)*, § 5).

Given two $\mathcal{O}_X$-ideals $\mathscr{J}$, $\mathscr{K}$ we will say that *$\mathscr{J}$ divides $\mathscr{K}$* if there is an $\mathcal{O}_X$-ideal $\mathscr{H}$ such that $\mathscr{J}\mathscr{H} = \mathscr{K}$.

Proposition **(21.5)**. — *Assume that unique $*$-factorization holds for $\mathfrak{m}$-primary ideals in* R. *Let $f: X \to \mathrm{Spec}(R)$ be a desingularization, let* I *be a $*$-simple $\mathfrak{m}$-primary complete ideal in* R *and let $\mathscr{I}' = \mathscr{I}\mathscr{I}^{-1}$ be the transform of* I *on* X (cf. *preceding remarks*). *If $\mathscr{J}$ and $\mathscr{K}$ are complete coherent $\mathcal{O}_X$-ideals such that $\mathscr{I}'$ divides $\mathscr{J}\mathscr{K}$, then either $\mathscr{I}'$ divides $\mathscr{J}$ or $\mathscr{I}'$ divides $\mathscr{K}$.*

Proof. — Let $\mathscr{J}\mathscr{K} = \mathscr{I}'\mathscr{H}$. Let $\mathscr{J}' = \mathscr{J}\mathscr{J}^{-1}$, $\mathscr{K}' = \mathscr{K}\mathscr{K}^{-1}$, $\mathscr{H}' = \mathscr{H}\mathscr{H}^{-1}$. We see easily, since X is regular, that $\mathscr{J}'\mathscr{K}' = \mathscr{I}'\mathscr{H}'$ and consequently we may assume that $\mathscr{J} = \mathscr{J}'$, $\mathscr{K} = \mathscr{K}'$, $\mathscr{H} = \mathscr{H}'$. Then $\mathcal{O}_X/\mathscr{J}$ has zero-dimensional support, so that for sufficiently large n, $\mathfrak{m}^n\mathcal{O}_X \subseteq \mathscr{J}$. Similar remarks apply to $\mathscr{K}$, $\mathscr{H}$, $\mathscr{I}'$. Since $(\mathscr{I}^{-1})^{-1} \supseteq \mathscr{I} \supseteq I\mathcal{O}_X$ and I is $\mathfrak{m}$-primary, we see that also $\mathscr{J} . (\mathscr{I}^{-1})^{-1} \supseteq \mathfrak{m}^n\mathcal{O}_X$ for large n.

Now, as in the proof of Lemma (21.1), there is an $\mathfrak{m}$-primary ideal L such that $\mathscr{L}=\mathrm{L}\mathcal{O}_{\mathrm{X}}$ is an *ample* invertible $\mathcal{O}_{\mathrm{X}}$-ideal. For any $p\geq 0$ we have

$$\mathscr{J}_p\mathscr{K}_p=\mathscr{I}\mathscr{H}_p$$

where

$$\mathscr{J}_p=\mathscr{J}.(\mathscr{I}^{-1})^{-1}.\mathscr{L}^p$$
$$\mathscr{K}_p=\mathscr{K}\mathscr{L}^p$$
$$\mathscr{H}_p=\mathscr{H}\mathscr{L}^{2p}.$$

Note that $\mathscr{J}_p$, $\mathscr{K}_p$, $\mathscr{H}_p$ all contain $\mathfrak{m}^n\mathcal{O}_{\mathrm{X}}$ for sufficiently large n (depending on p). It will be enough for us to find a p such that $\mathscr{I}$ divides either $\mathscr{J}_p$ or $\mathscr{K}_p$, since then by splitting off invertible factors as in the beginning of this proof, we can conclude that $\mathscr{I}'$ divides either $\mathscr{J}$ or $\mathscr{K}$.

We choose p large enough so that $\mathscr{J}_p$, $\mathscr{K}_p$, $\mathscr{H}_p$ are all generated by their global sections. According to the above remarks, $\mathrm{J}=\Gamma(\mathrm{X},\mathscr{J}_p)$ is an $\mathfrak{m}$-primary ideal, and moreover, since $\mathscr{J}_p$ is complete, so is J (Lemma (5.3)). Similar remarks apply to $\mathrm{K}=\Gamma(\mathrm{X},\mathscr{K}_p)$ and $\mathrm{H}=\Gamma(\mathrm{X},\mathscr{H}_p)$. Also since $\mathscr{J}_p\mathscr{K}_p$ is a complete ideal (Theorem (7.1)), $\mathrm{G}=\Gamma(\mathrm{X},\mathscr{J}_p\mathscr{K}_p)=\Gamma(\mathrm{X},\mathscr{I}\mathscr{H}_p)$ is a complete $\mathfrak{m}$-primary ideal in R containing both JK and IH. Since $\mathscr{J}_p\mathscr{K}_p=(\mathrm{JK})\mathcal{O}_{\mathrm{X}}$, and $\mathscr{I}\mathscr{H}_p$ is contained in the completion of $(\mathrm{IH})\mathcal{O}_{\mathrm{X}}$, it now follows from the remark following Proposition (6.2) that

$$\mathrm{G}=\mathrm{J}*\mathrm{K}=\mathrm{I}*\mathrm{H}.$$

By unique $*$-factorization in R, we conclude that I divides either J or K in the sense of the $*$ product, say $\mathrm{J}=\mathrm{I}*\mathrm{M}$. Then

$$\mathscr{J}_p=\mathrm{J}\mathcal{O}_{\mathrm{X}}=(\mathrm{I}*\mathrm{M})\mathcal{O}_{\mathrm{X}}$$

is a complete ideal, which is contained in, and therefore equal to, the completion of $(\mathrm{IM})\mathcal{O}_{\mathrm{X}}$ (Remark *f)*, § 5). Thus, if $\mathscr{M}$ is the completion of $\mathrm{M}\mathcal{O}_{\mathrm{X}}$, we have

$$\mathscr{J}_p=\mathscr{I}*\mathscr{M}=\mathscr{I}\mathscr{M} \qquad \text{(Theorem (7.1)).} \quad \text{Q.E.D.}$$

VI. — PSEUDO-RATIONAL DOUBLE POINTS AND FACTORIALITY

In part VI, the aim is to round out the results of § 17 and § 20 by describing in detail those two-dimensional normal local rings R which have a rational singularity and a trivial group H(R). In § 22 we find that any such R has multiplicity ≤ 2. In § 23 we characterize rational " double points " as being those two-dimensional normal local rings of embedding dimension three whose singularity can be resolved by quadratic transformations only. Using this fact, we describe explicitly all rational double points together with their associated group H (§ 24). It then appears that the only ones with trivial H are essentially those considered by Scheja in [**19**].

Of particular interest is the case when R has an algebraically closed residue field k (cf. Theorem (17.4)). In this case, if k has characteristic $\neq 2, 3, 5$, and R is not regular, then the completion $\hat{R}$ must be of the form $S/(u^2+v^3+w^5)$ where S is a three-dimensional regular local ring with regular parameters u, v, w (cf. Theorem (25.1); also Remark (25.2) for the exceptional characteristics). This result was previously proved by Brieskorn for local rings on two-dimensional complex spaces [7].

§ 22. Trivial H implies multiplicity ≤ 2.

Let A be a noetherian ring and let $f: X \to \mathrm{Spec}(A)$ be a map of finite type. As in § 13, a *curve on* X is understood to be an effective divisor with one-dimensional support.

We begin with some lemmas about exceptional curves on X (relative to f; cf. § 12).

Lemma **(22.1)**. — *Let* $F_1, F_2, \ldots, F_p$ *be integral exceptional curves on* X *and let* $F = \sum_{i=1}^{p} n_i F_i$ *with positive integers* n_i. *Assume that* $H^1(F)=0$. *Then for some j we have*

$$((F-F_j).F_j) \leq h^0(F_j).$$

Proof. — If $F' = \sum_i n'_i F_i$, with $0 \leq n'_i \leq n_i$ $(i=1, 2, \ldots, p)$, then $\mathcal{O}_{F'}$ is a homomorphic image of $\mathcal{O}_F$, whence, F being a curve, $H^1(F')$ is a homomorphic image of $H^1(F)=0$. Thus $\chi(F')=h^0(F')$. If the assertion of Lemma (22.1) were false, then it would follow that for each i,

$$((F-F_i).F_i) \geq 2h^0(F_i)$$

i.e.

$$(F.F_i) \geq (F_i.F_i) + 2\chi(F_i).$$

Now observe that the function

$$K(D) = (D.D) + 2\chi(D)$$

is an *additive* function of curves $D = \sum_i d_i F_i$. Indeed

$$K(D_1+D_2) = (D_1.D_1) + 2(D_1.D_2) + (D_2.D_2) + 2\chi(D_1) + 2\chi(D_2) - 2(D_1.D_2) = K(D_1) + K(D_2).$$

If then, as above $(F.F_i) \geq K(F_i)$ for all i, then

$$\begin{aligned}(F.F) = \sum_i n_i(F.F_i) &\geq \sum n_i K(F_i)\\ &= K(F)\\ &= (F.F) + 2\chi(F)\end{aligned}$$

i.e.

$$0 \geq 2\chi(F) = 2h^0(F)$$

which is absurd. Q.E.D.

Lemma **(22.2)**. — *Let* E, F *be distinct integral exceptional curves on* X *such that* $E \cap F$ *is non-empty. Suppose that* $H^1(E+F)=0$. *Then* $(E.F) = \max\{h^0(E), h^0(F)\}$.

Proof. — As in the proof of (22.1), we have $H^1(E)=H^1(F)=0$. Hence

$$\begin{aligned}0 < h^0(E+F) = \chi(E+F) &= \chi(E) + \chi(F) - (E.F)\\ &= h^0(E) + h^0(F) - (E.F).\end{aligned}$$

Since $(E.F)$ is positive and divisible by both $h^0(E)$ and $h^0(F)$, this inequality can hold only if $(E.F)=\max\{h^0(E), h^0(F)\}$. Q.E.D.

Lemma **(22.3)**. — *Let* P *be a closed point on* X *such that* $\mathcal{O}_{X,P}$ *is a regular two-dimensional local ring, and let* $j: X' \to X$ *be the map obtained by blowing up* P. *Let* C *be an exceptional curve on* X, *and let* C′ *be the curve* $j^{-1}(C)=j^*(C)$. *Then the canonical maps*

$$H^p(X, \mathcal{O}_C) \to H^p(X', \mathcal{O}_{C'})$$

are isomorphisms for all $p \geq 0$.

Proof. — Let $\mathcal{I}=\mathcal{O}_X(-C)$, $\mathcal{I}'=\mathcal{I}\mathcal{O}_{X'}=\mathcal{O}_{X'}(-C')$. One checks that the isomorphism $\mathcal{O}_X \to j_*(\mathcal{O}_{X'})$ induces an isomorphism of $\mathcal{I}$ onto $j_*(\mathcal{I}')$. Hence there is an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to j_*(\mathcal{O}_{C'}) \to R^1 j_*(\mathcal{I}').$$

But A) of Proposition (1.2) shows that $R^1 j_*(\mathcal{O}_{X'})=0$ (by [**EGA** III, (1.4.15)] the question depends only on the local rings of the points on X) and this implies that $R^1 j_*(\mathcal{I}')=0$ (the question is local on X, so we may assume that $\mathcal{I}\cong\mathcal{O}_X$, whence $\mathcal{I}'\cong\mathcal{O}_{X'}$). Thus we have an isomorphism

$$\mathcal{O}_C \xrightarrow{\approx} j_*(\mathcal{O}_{C'}).$$

This proves (22.3) for $p=0$. Also, since $\mathcal{O}_{C'}$ is a homomorphic image of $\mathcal{O}_{X'}$, $R^1 j_*(\mathcal{O}_{C'})=0$. The standard exact sequence

$$0 \to H^1(X, j_*(\mathcal{O}_{C'})) \to H^1(X', \mathcal{O}_{C'}) \to H^0(X, R^1 j_*(\mathcal{O}_{C'}))$$

leads therefore to the isomorphism

$$H^1(X, \mathcal{O}_C) \xrightarrow{\approx} H^1(X', \mathcal{O}_{C'}).$$

This proves (22.3) for $p=1$. For $p>1$, there is nothing to prove. Q.E.D.

Lemma **(22.4)**. — *Let* F *be an integral exceptional curve on* X, *and let* $d(F)$ *be the greatest common divisor of all the degrees of invertible sheaves on* F. *Suppose that* X *is regular at each point through which* F *passes, and that* $H^1(F)=0$. *Then* $d(F)\leq 2h^0(F)$.

Proof. — If F is a regular curve, then the canonical divisors on F have degree $-2h^0(F)$.

Suppose then that there exists a point P of multiplicity $\mu>1$ on F. P is a closed point of X, and the local ring of P on X is two-dimensional and regular. Let $j: X'\to X$ be the map obtained by blowing up P; then $F'=j^*(F)=F_1+\mu F_2$ where F_1 is an integral exceptional curve on X′ (namely F_1 is the *proper transform* of F) and $F_2=j^{-1}(P)_{\text{red}}$.

Let $f(P)=Q\in \operatorname{Spec}(A)$. The residue field $k(P)$ is a finite algebraic extension of the residue field $k(Q)$, of degree, say, δ, and we have

$$(F_2.F_2)=-h^0(F_2)=-\delta$$

(cf. proof of Lemma (15.2)). Moreover

$$((F_1+\mu F_2).F_2)=0$$

(cf. β) in § 15). Hence

$$(F_1.F_2)=\mu\delta.$$

By (22.3), $H^1(F_1+\mu F_2)=H^1(F)=0$, whence $H^1(F_1)=H^1(F_2)=H^1(F_1+F_2)=0$. (22.2) shows therefore that

$$\mu\delta=\max\{h^0(F_1),\, h^0(F_2)\}=\max\{h^0(F_1),\, \delta\}$$

so that

$$h^0(F_1)=\mu\delta.$$

Now, by (22.3),

$$h^0(F)=h^0(F_1+\mu F_2)=\chi(F_1+\mu F_2)=\chi(F_1)+\chi(\mu F_2)-(F_1.\mu F_2).$$

But, by induction on μ, we have

$$\chi(\mu F_2)=\mu\chi(F_2)-\frac{\mu(\mu-1)}{2}(F_2.F_2)$$

Thus
$$0<h^0(F)=h^0(F_1)+\mu h^0(F_2)-\frac{\mu(\mu-1)}{2}(F_2.F_2)-\mu(F_1.F_2)$$

$$=\mu\delta+\mu\delta-\frac{\mu(\mu-1)}{2}(-\delta)-\mu.\mu\delta$$

$$\text{i.e.}\quad 0<\delta\left[2\mu+\frac{\mu(\mu-1)}{2}-\mu^2\right]$$

so that
$$0<3\mu-\mu^2$$

i.e.
$$\mu<3.$$

Since $\mu>1$, we must have $\mu=2$, and

$$h^0(F)=\delta\left[2.2+\frac{2(2-1)}{2}-2^2\right]=\delta.$$

Finally $d(F)=d(F_1)$ (cf. proof of Lemma (15.1)) and

$$d(F_1) \text{ divides } (F_1.F_2)=\mu\delta=2\delta=2h^0(F). \qquad \text{Q.E.D.}$$

We come now to the main result of this section.

Proposition **(22.5)**. — *Let* R *be a two-dimensional normal local ring having a rational singularity. If the group* H(R) *is trivial, then* R *has multiplicity* <2.

Proof. — Let $f: X \to \operatorname{Spec}(R)$ be the *minimal* desingularization of R (cf. Theorem (4.1)). $H^1(X)=0$, and for every exceptional curve C (relative to f), $\chi(C)=h^0(C)$. From the contractibility criterion (§ 27) [1] and negative-definiteness (Lemma (14.1)) we have that:

If E *is any integral exceptional curve on* X *then*

$$(E.E)\leq -2h^0(E).$$

[1] § 27 is independent of § 22. Actually, starting with (19.2), (21.2), and (7.1), one can avoid using the contractibility criterion.

Let $E_1, E_2, \ldots, E_n$ be all the integral exceptional curves on X. We will show in a moment that *for each* i, $K(E_i)=0$ (cf. proof of Lemma (22.1) for the definition and properties of K). Since K is additive, it follows that $K(C)=0$, i.e.

$$(C.C) = -2\chi(C) = -2h^0(C)$$

for every exceptional curve C on X. We may as well assume that R is not regular; then Proposition (3.1) and its proof show that if $\mathfrak{m}$ is the maximal ideal of R, then $\mathfrak{m}\mathcal{O}_X$ is invertible, say $\mathfrak{m}\mathcal{O}_X = \mathcal{O}(-C)$, and $H^0(\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X) = R/\mathfrak{m}$, i.e.

$$h^0(C) = 1.$$

Finally, the proof of Corollary (23.3) (which is completely independent of the considerations of this section) shows that the multiplicity of R is $-(C.C)=2$, as asserted.

The fact that $K(E_i)=0$ is an immediate consequence of Lemma (22.4) and:

Lemma **(22.6)**. — *For each* $i=1, 2, \ldots, n$ *let* $d_i>0$ *be the greatest common divisor of degrees of invertible sheaves on* E_i. *Then, for each* i,

either (i) $$-(E_i.E_i) = d_i > h^0(E_i)$$

or (ii) $$-(E_i.E_i) = 2d_i = 2h^0(E_i).$$

Proof. — The triviality of H(R) implies that for each i there exists an exceptional curve D_i such that

$$(D_i.E_j) = -d_j\delta_{ij} \qquad (j=1, 2, \ldots, n)$$

By Lemma (22.1) we can choose j so that

$$((D_i - E_j).E_j) \leq h^0(E_j).$$

If $j \neq i$, this means that $-(E_j.E_j) \leq h^0(E_j)$, contradicting the assumption that $f: X \rightarrow \mathrm{Spec}(R)$ is the *minimal* desingularization. Hence $j=i$ and

$$-d_i - (E_i.E_i) \leq h^0(E_i).$$

But $d_i = r.h^0(E_i)$ and $-(E_i.E_i) = sd_i$ where r, s are positive integers, and the preceding inequality gives

$$-r + sr \leq 1.$$

Since $-(E_i.E_i) \geq 2h^0(E_i)$ we cannot have $r=s=1$. So the only possibilities are $s=1$, $r>1$, which gives (i), and $s=2$, $r=1$, which gives (ii). This completes the proof.

§ 23. Some special properties of pseudo-rational singularities.

In this section we give some facts about lengths of ideals which will be of further use. We also characterize among " embedded " two-dimensional local rings of multiplicity two those which have pseudo-rational singularities (roughly — those which remain normal under any succession of quadratic transformations) and those which have rational singularities (roughly — those which can be desingularized by quadratic transformations alone) (cf. Proposition (23.5)). This characterization will enable us to give explicit descriptions (§ 24).

Let R be a two-dimensional normal local ring with maximal ideal $\mathfrak{m}$, and let $f: X \to \operatorname{Spec}(R)$ be a proper birational map. Since R is normal, $H^0(\mathcal{O}_X) = R$; also, the support of $H^1(\mathcal{O}_X)$ is contained in the closed point of $\operatorname{Spec}(R)$, i.e. the R-module $H^1(\mathcal{O}_X)$ has finite length which we denote by $h^1(\mathcal{O}_X)$.

For any $\mathfrak{m}$-primary ideal J in R we denote by $\lambda(J)$ the (finite) length of the R-module R/J. If I is an $\mathfrak{m}$-primary ideal in R, and $f: X \to \operatorname{Spec}(R)$ is as above, we have for any $n > 0$

$$I^n \subseteq H^0(I^n \mathcal{O}_X) \subseteq H^0(\mathcal{O}_X) = R$$

so that $H^0(I^n \mathcal{O}_X)$ is also an $\mathfrak{m}$-primary ideal in R, and $\lambda(H^0(I^n \mathcal{O}_X))$ is a well-defined integer. Suppose further that $I\mathcal{O}_X$ is an invertible $\mathcal{O}_X$-ideal, and let C be the curve on X defined by $I\mathcal{O}_X$. Then C has exceptional support relative to f, and so $\chi(C)$, $(C.C)$, are well-defined integers.

Lemma **(23.1)**. — *Let* $f: X \to \operatorname{Spec}(R)$, I, C, *be as in the preceding remarks, and let* $n \geq 0$ *be such that* $H^1(I^n \mathcal{O}_X) = 0$. *Then*

$$\lambda(H^0(I^n \mathcal{O}_X)) = -(C.C)\binom{n}{2} + \chi(C).n + h^1(\mathcal{O}_X).$$

Proof. — From the cohomology sequence associated with the exact sequence

$$0 \to I^n \mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{nC} \to 0$$

we deduce that

$$H^0(nC) \cong H^0(\mathcal{O}_X)/H^0(I^n \mathcal{O}_X) = R/H^0(I^n \mathcal{O}_X)$$

and

$$H^1(nC) \cong H^1(\mathcal{O}_X).$$

Thus
$$\chi(nC) = \lambda(H^0(I^n \mathcal{O}_X)) - h^1(\mathcal{O}_X).$$

An easy induction, based on the relation

$$\chi(C+D) = \chi(C) + \chi(D) - (C.D)$$

(cf. Proposition (13.1)) shows that

$$\chi(nC) = -(C.C)\binom{n}{2} + \chi(C).n$$

and the conclusion follows.

Corollary **(23.2)**. — *Let* R *and* I *be as above, and let* $f: X \to \operatorname{Spec}(R)$ *be the map obtained by blowing up* I (i.e. $X = \operatorname{Proj}(\bigoplus_{n \geq 0} I^n)$). *Then with* C *as above we have, for all sufficiently large* n,

$$\lambda(I^n) = -(C.C)\binom{n}{2} + \chi(C).n + h^1(\mathcal{O}_X).$$

Proof. — Since $I^n \mathcal{O}_X = \mathcal{O}_X(n)$ [**EGA** II, (8.1.7)] we have $H^1(I^n \mathcal{O}_X) = 0$ for all sufficiently large n, and moreover $I^n = H^0(I^n \mathcal{O}_X)$ for all sufficiently large n [**EGA** III, (2.3.4)]. Thus the conclusion follows from Lemma (23.1).

Corollary **(23.3)**. — *Let* R *be a two-dimensional normal local ring, with maximal ideal* $\mathfrak{m}$, *having a pseudo-rational singularity, and let* I *be a complete* $\mathfrak{m}$*-primary ideal in* R, *of multiplicity* μ. *Then for all* $n \geq 0$

$$\lambda(\mathrm{I}^n) = \mu\binom{n}{2} + \lambda(\mathrm{I}).n.$$

Proof. — Let W be the scheme obtained from Spec(R) by blowing up I. By definition of " pseudo-rational " there exists a proper birational map $f: \mathrm{X} \to \mathrm{Spec(R)}$ such that X dominates W (so that $\mathrm{I}\mathcal{O}_\mathrm{X}$ is invertible) and $\mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}) = 0$. For any $n \geq 0$, $\mathrm{I}^n\mathcal{O}_\mathrm{X}$ is a homomorphic image of $\mathcal{O}_\mathrm{X}^t$ for some finite t, and since H^2 vanishes for all coherent sheaves (the fibres of f being of dimension ≤ 1) we have therefore $\mathrm{H}^1(\mathrm{I}^n\mathcal{O}_\mathrm{X}) = 0$. Moreover, since I is complete, so is I^n (Theorem (7.1)), and cf. § 9) and therefore $\mathrm{I}^n = \mathrm{H}^0(\mathrm{I}^n\mathcal{O}_\mathrm{X})$ (Proposition (6.2)). Lemma (23.1) now gives

$$\lambda(\mathrm{I}^n) = -(\mathrm{C}.\mathrm{C})\binom{n}{2} + \chi(\mathrm{C}).n \qquad (n \geq 0).$$

By definition of " multiplicity ", $-(\mathrm{C}.\mathrm{C}) = \mu$. Also, setting $n = 1$, we get $\chi(\mathrm{C}) = \lambda(\mathrm{I})$. Thus

$$\lambda(\mathrm{I}^n) = \mu\binom{n}{2} + \lambda(\mathrm{I}).n. \qquad \text{Q.E.D.}$$

Corollary (23.3) shows that if R is a two-dimensional normal local ring, with maximal ideal $\mathfrak{m}$, *of multiplicity* 2, having a pseudo-rational singularity, then for all $n \geq 0$

$$\lambda(\mathfrak{m}^n) = n^2.$$

Lemma **(23.4)**. — *For a local ring* R *with maximal ideal* $\mathfrak{m}$ *the following conditions are equivalent*:

(i) $\lambda(\mathfrak{m}^n) = n^2$ *for all* $n \geq 0$.

(ii) *The completion* $\hat{\mathrm{R}}$ *is of the form* S/xS *where* S *is a three dimensional regular local ring, with maximal ideal, say,* M, *and* $x \in \mathrm{M}^2$, $x \notin \mathrm{M}^3$.

Proof. — Suppose (ii) holds. The graded ring $\bigoplus_{n \geq 0} \mathrm{M}^n/\mathrm{M}^{n+1}$ is isomorphic to a polynomial ring in three variables over $\mathrm{S}/\mathrm{M} = \mathrm{R}/\mathfrak{m}$ and the initial form of x in this graded ring is a homogeneous polynomial of degree two, which generates the kernel of the natural surjection

$$\bigoplus_{n \geq 0} \mathrm{M}^n/\mathrm{M}^{n+1} \to \bigoplus_{n \geq 0} \hat{\mathfrak{m}}^n/\hat{\mathfrak{m}}^{n+1} \qquad (\hat{\mathfrak{m}} = \mathfrak{m}\hat{\mathrm{R}})$$

It follows easily that the $(\mathrm{R}/\mathfrak{m})$-vector space $\mathfrak{m}^n/\mathfrak{m}^{n+1} (= \hat{\mathfrak{m}}^n/\hat{\mathfrak{m}}^{n+1})$ is of dimension $\binom{n+2}{2} - \binom{n}{2} = 2n+1$ for $n \geq 0$, and so

$$\lambda(\mathfrak{m}^n) = \sum_{i=0}^{n-1}(2i+1) = n^2.$$

Conversely, if (i) holds then $\mathfrak{m}/\mathfrak{m}^2$ has dimension 3 over $R/\mathfrak{m}$, so that by Cohen's structure theorem $\hat{R}$ is of the form S/I, where S is a three-dimensional regular local ring and I is an ideal in S. Also $\mathfrak{m}^2/\mathfrak{m}^3$ has dimension 5 over $R/\mathfrak{m}$, so that I contains an element x such that $x \in M^2$, $x \notin M^3$, M being the maximal ideal of S. I *must be generated by* x; for, as above, the S-module $S/(M^n + xS)$ has length n^2 for all $n \geq 0$, and by hypothesis, $S/(M^n + I)$ has the same length; hence $M^n + xS = M^n + I$, and so

$$I \subseteq \bigcap_{n \geq 0} (M^n + xS) = xS$$

Thus $\hat{R} = S/xS$. Q.E.D.

We can now proceed to a characterization of " pseudo-rational double points ". Let R be a two-dimensional normal local ring having a pseudo-rational singularity. Proposition (8.1) and Proposition (1.2) (and cf. § 9) lead to the following conclusion:

a) If

$$R = R_0 < R_1 < \ldots < R_n \qquad (n \geq 0)$$

is any sequence in which each R_i $(0 < i \leq n)$ is the local ring of a point on the scheme obtained by blowing up the maximal ideal of R_{i-1}, then R_n is normal.

If R has a *rational* singularity, Theorem (4.1) shows that:

b) R can be desingularized by quadratic transformations alone, i.e. there exists a sequence

$$\mathrm{Spec}(R) = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_n$$

of quadratic transformations with X_n regular.

Conversely:

Proposition **(23.5)**. — *Let* R *be a local ring with maximal ideal* $\mathfrak{m}$ *such that for all* $n \geq 0$, $\lambda(\mathfrak{m}^n) = n^2$. *If the preceding condition* **a)** (*respectively* **b)**) *holds, then* R *is a two-dimensional normal local ring of multiplicity* 2 *having a* pseudo-rational (*respectively* rational) *singularity.*

Proof. — Lemma (23.4) shows that $\hat{R}$, and hence R itself, is a two-dimensional Macaulay local ring of multiplicity 2. Certainly R is normal if **a)** holds. If **b)** holds then, since quadratic transformations do not affect non-closed points, we see that for each prime ideal $\mathfrak{p}$ in R other than $\mathfrak{m}$, $R_{\mathfrak{p}}$ is regular; by Serre's criterion [**EGA** IV, (5.8.6)] (or otherwise) we conclude again that R is normal.

Assume that **a)** holds. Let $g : W \to \mathrm{Spec}(R)$ be a projective birational map. By the theorem on elimination of points of indeterminacy by quadratic transformations and normalizations (cf. Appendix), and in view of **a)**, there exists a sequence

$$\mathrm{Spec}(R) = Z_0 \xleftarrow{g_1} Z_1 \xleftarrow{g_2} Z_2 \leftarrow \ldots \xleftarrow{g_q} Z_q$$

of quadratic transformations such that Z_q is normal and dominates W. What we must show is that $H^1(\mathcal{O}_{Z_q}) = 0$. Similarly, if **b)** holds, we must show that $H^1(\mathcal{O}_{X_n}) = 0$.

Corollary (23.2) (with $I=\mathfrak{m}$) shows that $H^1(\mathcal{O}_{Z_1})=0$. It is equivalent to say that $R^1g_{1*}(\mathcal{O}_{Z_1})=0$. We will show in a moment that every two-dimensional local ring R′ on Z_1 either is regular or satisfies the same hypotheses as R. Thus if R″ is the local ring of the point which is blown up to give the map g_2, then we can repeat the argument to show that $R^1g_{2*}(\mathcal{O}_{Z_2})=0$. (Remarks: (i) The sheaf $R^1g_{2*}(\mathcal{O}_{Z_2})$ is concentrated at the point which is blown up; hence to show that this sheaf vanishes, we may first replace Z_1 by Spec(R″) [**EGA** III, (1.4.15)]. (ii) If R″ is *regular*, Lemma (23.2) still applies since $\lambda((\mathfrak{m}''))^n=\binom{n}{2}$ for $n\geq 0$. Also, in this case, all the local rings which appear on Z_2 but not on Z_1 are regular.) Since Z_1 is normal, $g_{2*}(\mathcal{O}_{Z_2})=\mathcal{O}_{Z_1}$, and so the exact sequence

$$\begin{array}{ccccc} 0\to & R^1g_{1*}(g_{2*}\mathcal{O}_{Z_2}) & \to R^1(g_1\circ g_2)_*(\mathcal{O}_{Z_2}) \to & g_{1*}(R^1g_{2*}\mathcal{O}_{Z_2}) \\ & \| & & \| \\ & 0 & & 0 \end{array}$$

(arising from the Leray spectral sequence for $g_1\circ g_2$) shows that $R^1(g_1\circ g_2)_*(\mathcal{O}_{Z_2})=0$. Continuing in this way we conclude ultimately that $R^1(g_1\circ g_2\circ\ldots\circ g_q)_*(\mathcal{O}_{Z_q})=0$, i.e. that $H^1(\mathcal{O}_{Z_q})=0$ as required. In a similar way, we can see that $H^1(\mathcal{O}_{X_n})=0$.

So let R′ be any two-dimensional local ring belonging to a point on Z_1. Then R′ dominates R, and as in the proof of [**EGA** IV, (7.9.3)], there is a unique local ring R^* belonging to the quadratic transform of Spec($\hat{R}$) such that R^* and R′ have the same completion. Let S, M, x be as in Lemma (23.4). Then R^* is of the form S^*/x^*, where S^* is a three dimensional local ring on the quadratic transform of S, and x^* is the transform of x in S^*, i.e. $x^*=xt^{-2}$ where t is a generator of MS^*. If R′ (and hence R^*) is not regular, then $x^*\in(M^*)^2$, $x^*\notin(M^*)^3$ (M^* being the maximal ideal of S^*), and it follows easily that all the hypotheses of Proposition (23.5) which hold for R also hold for R′. This completes the proof.

Remark. — For the case of complex spaces, the preceding characterization of rational double points is given in [**6**; Satz 1].

§ 24. Explicit description of pseudo-rational double points.

Let R be a two-dimensional normal local ring of multiplicity two having a pseudo-rational singularity. Since $\lambda(m^2)=4$ (Corollary (23.3)) every minimal basis of the maximal ideal $\mathfrak{m}$ of R consists of three elements. We shall classify R by studying its behaviour under successive quadratic transformations, and by relating this behaviour to certain conditions involving, more or less explicitly, a suitable basis $\{x, y, z\}$ of $\mathfrak{m}$. We also prove converse statements of the type: "If R is any local ring, with maximal ideal $\mathfrak{m}$ generated by elements x, y, z satisfying... then R is a two-dimensional normal local ring having a (pseudo-)rational singularity." In other words the conditions to be introduced *characterize* (pseudo-)rational "double points".

Basically, the idea is to take a two-dimensional local ring of multiplicity two whose maximal ideal is generated by three elements, say x, y, z, to subject this ring to a succession of quadratic transformations, and to see what conditions on x, y, z, guarantee that the resulting rings are all *normal* (cf. Proposition (23.5)). This approach involves a detailed and rather tedious examination of numerous cases. For orientation, the reader may analyse a ring of the form

$$k[[X, Y, Z]]/(Z^2 - F(X, Y))$$

(k a field) from this point of view.

***

Suppose now that R has a *rational* singularity, and let $f: X \rightarrow \operatorname{Spec}(R)$ be a desingularization. We introduce a notation which conveniently conveys some useful information about exceptional curves on X. A symbol of either of the following types

$$a - b \qquad\qquad \begin{array}{c} a \\ | \\ b \end{array}$$

where a, b are positive integers will stand for a pair of integral exceptional curves E, F, on X such that $h^0(E) = a$, $h^0(F) = b$, and $E \cap F$ is non-empty. We can combine these symbols into diagrams such as

$$\begin{array}{ccccccccccc} & & & & & & & & g & & \\ & & & & & & & & | & & \\ a & - & b & - & c & - & e & - & f & - & h \\ & & & & | & & & & | & & \\ & & & & d & & & & k & & \end{array}$$

which stands for a configuration of nine integral exceptional curves $E_1, E_2, \ldots, E_9$ such that $h^0(E_1) = a$, $h^0(E_2) = b$, $\ldots$, $h^0(E_9) = h$, and such that the non-empty intersections of pairs of E's are those — and only those — indicated by the short straight lines.

We will speak of such diagrams as *configuration diagrams*. When we speak of *the* configuration diagram on X, we mean *the* diagram which contains as many integers as there are exceptional integral curves on X — in other words the *largest* configuration diagram associated with exceptional curves on X.

***

If R has a rational singularity, then, while classifying R as indicated above, we will obtain the configuration diagram on the *minimal* desingularization X of R, as well as the group $H = H(R)$, which is in this case isomorphic to the divisor class group of $\hat{R}$ (cf. (16.3), (17.1)).

We will say that " the exceptional curve on X is of type **C** " if **C** is the configuration diagram on X. It will be found that the following types (and no others) of exceptional curves can occur on X:

A_n	: $1-1-1-\ldots-1$	($n\geq 2$ components). $H=\mathbf{Z}_{n+1}$.
B_n	: $1-2-2-\ldots-2$	($n\geqslant 1$ components, including the first one). H of order $2/d$, $d=1$ or 2 (cf. following discussion).
C_n	: $1-1-1-\ldots-1-2$	($n\geqslant 3$ components, including the last one). $H=\mathbf{Z}_2$.
D_n	: $1-1-1-\ldots-1\begin{smallmatrix}\diagup 1\\ \diagdown 1\end{smallmatrix}$	($n\geqslant 4$ components, including the last two). $H=\mathbf{Z}_2\times\mathbf{Z}_2$ if n is even; $H=\mathbf{Z}_4$ if n is odd.
G_2	: $1-3$	H trivial.
F_4	: $1-1-2-2$	H trivial.
E_6	: $\begin{array}{c} 1 \\ \vert \\ 1-1-1-1-1 \end{array}$	$H=\mathbf{Z}_3$.
E_7	: $\begin{array}{c} 1 \\ \vert \\ 1-1-1-1-1-1 \end{array}$	$H=\mathbf{Z}_2$.
E_8	: $\begin{array}{c} 1 \\ \vert \\ 1-1-1-1-1-1-1 \end{array}$	H trivial.

(These are just the "Dynkin diagrams", cf. for example [N. Jacobson, *Lie Algebras*, Interscience, 1962, pp. 134-135].)

It is always possible to determine the group H once the configuration diagram on the minimal desingularization X is known (except for diagram B_n, cf. below). If $E_1, E_2, \ldots, E_n$ are the integral exceptional curves then, by definition of H, we need to know the intersection matrix $((E_i.E_j))$ and also, for each $i=1, 2, \ldots, n$, the integer $d(E_i)$, which is the greatest common divisor of all the degrees of divisors on E_i.

As in the proof of Proposition (22.5), let C be the curve on X such that $\mathcal{O}(-C)=\mathfrak{m}\mathcal{O}_X$. Since $C=\sum_i n_iE_i$ ($n_i>0$ for all i) and since R has multiplicity two, we have, as in (22.5),

$$0=K(C)=\sum_i n_iK(E_i).$$

Since X is the minimal desingularization $(E_i.E_i)\leq -2h^0(E_i)$, i.e. $K(E_i)\leq 0$ for all i. Hence $K(E_i)=0$ for all i, i.e. $(E_i.E_i)=-2h^0(E_i)$.

The configuration diagram now gives us the intersection matrix because, given two integral curves $E\neq F$ on X such that $E\cap F$ is non-empty, we have

$$(E.F)=\max(h^0(E), h^0(F)) \qquad \text{(Lemma (22.2))}.$$

Furthermore it follows that if, say, $h^0(E) \leq h^0(F)$, then $d(F)$ divides — and hence is equal to — $h^0(F)$. This remark gives us $d(F)$ for all the integral curves represented in the above configuration diagrams except for the one represented by the integer " 1 " in the diagram $1-3$ or in the diagram $1-2-2-\ldots-2$. *In the first case*, if E (resp. F) is the curve represented by " 1 " (resp. " 3 ") then as we have seen

$$(E.E) = -2 \qquad (E.F) = 3$$

Thus $d(E)$ divides both 2 and 3, and so $d(E) = 1$. *In the remaining case*, if E is the curve represented by " 1 " in $1-2-2-\ldots$ then $(E.E) = -2$, so that $d = d(E) = 1$ or 2. Since $H^1(E) = 0$, it is seen at once that any divisor on E of positive degree is linearly equivalent to an effective divisor on E, and it follows easily that $d = 1$ if E has an $(R/\mathfrak{m})$-rational regular point and $d = 2$ otherwise.

It can now be verified by simple computations with generators and relations that in each case $A_n, \ldots, E_8$, H is as specified.

***

We begin the detailed description of pseudo-rational double points R by considering the associated graded ring of R with respect to $\mathfrak{m}$, i.e. the ring $\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Let $k = R/\mathfrak{m}$, let X, Y, Z be indeterminates, and let $\varphi : k[X, Y, Z] \to \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be a surjective homogeneous homomorphism of degree zero. (Such a homomorphism can be determined, for example, by choosing a basis $\{x, y, z\}$ of $\mathfrak{m}$ and setting $\varphi(X)$ (respectively $\varphi(Y)$, $\varphi(Z)$) equal to the image of x (respectively y, z) in $\mathfrak{m}/\mathfrak{m}^2$). Since $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 2n+1$ for all $n \geq 0$ (Corollary (23.3)) we see easily that the kernel of φ is generated by a single form $Q(X, Y, Z)$ of degree two.

For the (unique) quadratic transformation $T_1 \to \mathrm{Spec}(R)$, the closed fibre is $C = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1})$. C may be identified with the projective plane curve (not necessarily reduced) whose homogeneous equation is $Q(X, Y, Z) = 0$. One finds that *the singular locus of* C *is a linear variety of dimension* $2-\tau$, where τ is the least number of linear combinations of X, Y, Z in terms of which Q can be expressed, i.e. τ is the smallest possible dimension of a subspace V of $kX + kY + kZ$ such that Q lies in the subalgebra $k[V]$ of $k[X, Y, Z]$. (If k has characteristic $\neq 2$, this results easily from the fact that Q can be written as a linear combination of τ squares of linear forms. If k has characteristic 2 one may, for example, make use of Zariski's mixed Jacobian criterion for simple points (cf. [**23**]). Details are left to the reader).

If R′ is the local ring of a closed point on T_1, then $R'/\mathfrak{m}R'$ is the local ring of a closed point P′ on C. If P′ is a regular point of C, i.e. if $R'/\mathfrak{m}R'$ is a discrete valuation ring, then since $\mathfrak{m}R'$ is principal, the maximal ideal of R′ is generated by two elements and so R′ itself is regular. Thus, when $\tau = 3$, we have:

CASE I: $C = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1})$ *is a regular curve.*

Here R is completely desingularized by one quadratic transformation, and the exceptional curve C either is a non-degenerate conic, smooth over k, or is defined by an equation of the form $aX^2+bY^2+Z^2=0$ with k of characteristic 2 and $[k^2(a, b) : k^2]=4$. According to the remarks at the beginning of this section, H *is cyclic of order* $2/d$, where $d=1$ or 2 according as C does or does not have a k-rational point.

Suppose *conversely* that R is *any* local ring with maximal ideal $\mathfrak{m}$ such that

$$\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong k[X, Y, Z]/Q \qquad (k=R/\mathfrak{m})$$

where Q is a form of degree 2 such that the projective plane curve $Q(X, Y, Z)=0$ is regular. Then as above R is desingularized by one quadratic transformation, and so by Proposition (23.5) R is in fact a two-dimensional normal local ring having a rational singularity.

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We consider next the case $\tau=2$. Then, as we have seen, C has a unique singular point, and the corresponding point P on the quadratic transform T_1 of Spec(R) is the only possible singular point of T_1.

Over the algebraic closure of k, Q becomes a product of two linear factors. Let K be the splitting field of C, i.e. the least field containing k over which Q splits into two linear factors. Since clearly $Q(X, Y, Z)$ is determined by R up to a k-linear change of variables, K depends only on R. If Q is a product of linear factors over k then $K=k$. Otherwise Q is irreducible and assuming, as we may, that

$$Q(X, Y, Z)=aX^2+bXY+cY^2 \qquad (a, b, c\in k)$$

we have $a\neq 0$, and K is obtained from k by adjoining a root of the equation $aX^2+bX+c=0$; thus $[K:k]=2$.

Now we examine the behaviour of Q and K when R undergoes a quadratic transformation. Assuming always that $Q=aX^2+bXY+cY^2$ as above we have, with suitable generators x, y, z of $\mathfrak{m}$, and elements α, β, γ of R whose residues mod. $\mathfrak{m}$ are a, b, c respectively,

$$\alpha x^2+\beta xy+\gamma y^2\in\mathfrak{m}^3.$$

Let R′ be the local ring of the point on T_1 corresponding to the unique singular point of C. We know that R′ also has a pseudo-rational singularity. When C is identified as before with the plane curve defined by $Q(X, Y, Z)=0$, the singular point has co-ordinates $(0, 0, 1)$; hence R′ has the same residue field k as R, $\mathfrak{m}R'=zR'$, and the maximal ideal $\mathfrak{m}'$ of R′ is generated by $x'=x/z, y'=y/z, z'=z$. Since $\mathfrak{m}^3R'=z^3R'=(z')^3R'$, division of the above relation by z^2 gives

$$\alpha(x')^2+\beta(x'y')+\gamma(y')^2\in z'R'.$$

If R′ is regular, there is nothing more to be done. If R′ is not regular, then R′ is again a pseudo-rational double point (cf. proof of (23.5), for example), and

$$\alpha(x')^2+\beta(x'y')+\gamma(y')^2\in z'\mathfrak{m}'$$

(otherwise $z' \in (x', y')R'$ and so $\mathfrak{m}' = (x', y')R'$). If we replace R, x, y, z by R', x', y', z' in the above discussion about graded rings, we see then that the corresponding Q' is of the form

$$Q'(X, Y, Z) = aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 \quad (a, b, c \textit{ as before}; \ d, e, f \in k).$$

Setting $Z = 0$ we see that $\tau' \geq 2$. If $\tau' = 3$ we have achieved a reduction to Case I. Suppose that $\tau' = 2$. We can then write

$$Q'(X, Y, Z) = (p_1X + q_1Y + r_1Z)(p_2X + q_2Y + r_2Z) \quad (p_i, q_i, r_i \text{ algebraic over } k)$$

If Q splits over a field $L \supseteq k$, then setting $Z = 0$, we see that p_1, q_1, p_2, q_2 may be assumed to lie in L, and then comparison of the coefficients of XZ, YZ in the two expressions for Q' shows that r_1, r_2 also are in L; in other words Q' splits over L. Conversely if Q' splits over L, then we may assume that $p_i, q_i, r_i \in L$, and again setting $Z = 0$, we see that Q splits over L. So *Q and Q' have the same (least) splitting field*, namely K.

The next step, if R' is not already regular, is to blow up R' so that we have a sequence of quadratic transformations

$$f: T_2 \to T_1 \to \operatorname{Spec}(R).$$

We are interested in the closed fibre $f^{-1}(\{\mathfrak{m}\})$ on T_2. The irreducible components of this fibre are of two kinds, namely those belonging to the inverse image C' of $\{\mathfrak{m}'\}$ and those belonging to the proper transform C^* of C. C^* can be identified with the curve obtained by blowing up the singular point on C.

Lemma **(24.1)**. — *Assume that* R' *is not regular, and let* C', C^*, K *be as above.*

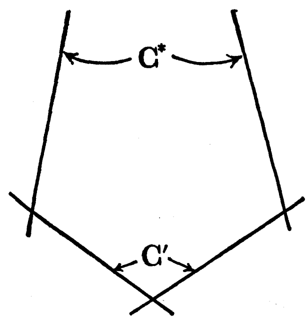
(i) *If* $\tau' = 3$, *then, as in* Case I, C' *is regular and* T_2 *is regular. In this case, if* $K = k$ *then* C' *has a k-rational point, so that* $C' \cong \mathbf{P}^1_k$.

If $\tau' = 2$, *then* C' *has a unique singular point, and* T_2 *is regular outside this point. In this case, if* $K = k$ *then* C' *is a pair of projective lines over k meeting at the singular point of* C'; *if* $K \neq k$ *then* C' *has no k-rational regular point.*

(ii) *If* $K \neq k$, *then* C^* *is k-isomorphic to the projective line* $\mathbf{P}^1_K$. *If* $K = k$, *then* C^* *is a pair of disjoint projective lines over k.*

(iii) *Each point of* $C^* \cap C'$ *has residue field* K *and is regular on both* C^* *and* C' (*hence also on* T_2). *The intersections of* C^* *and* C' *on* T_2 *are all transversal.*

(iv) *If* $K = k$ *and* $\tau' = 2$, *then each irreducible component of* C^* *meets precisely one irreducible component of* C' *(just once, transversally) and vice-versa.*



Granting Lemma (24.1) for the moment, suppose further that R has a *rational* singularity. Then R can be desingularized by successive quadratic transformations, so we can deduce, *by repeated application of Lemma* (24.1), a complete description of the case $\tau=2$, as follows.

CASE II a: $C=\mathrm{Proj}(\bigoplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1})$ *is a reduced curve with two distinct components.*

In this case $K=k$ and the exceptional curve on a minimal desingularization of R will be of type $1-1-1-1-\ldots-1$, each component being isomorphic to $\mathbf{P}^1_k$.

CASE II b: $C=\mathrm{Proj}(\bigoplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1})$ *is reduced and irreducible, and has precisely one singular point.*

Here $K\neq k$ and the exceptional curve on a minimal desingularization will be of type $1-2-2-2-\ldots-2$. The components for which $h^0=2$ are k-isomorphic to $\mathbf{P}^1_K$, and the component — call it C'' — for which $h^0=1$ is just like C' in (i) of Lemma (24.1). H has order $2/d$, where $d=1$ if C'' has a k-rational regular point (in which case C'' is k-isomorphic to $\mathbf{P}^1_k$) and $d=2$ otherwise.

We return now to the *proof of Lemma* (24.1).

We begin with (iii). Let S be the local ring on T_2 of a point through which both C' and C^* pass, and let $\mathfrak{m}'S=tS$, so that $t=0$ is the " local equation " of C'. Then S/tS is the local ring of a point on the plane curve $Q'(X, Y, Z)=0$, and since z'/t vanishes along C^*, $S/(t, z'/t)S$ is the local ring of a point on the scheme

$$\mathrm{Proj}(k[X, Y, Z]/(Q'(X, Y, Z), Z))\cong\mathrm{Proj}(k[X, Y]/Q(X, Y))$$

which is a reduced zero-dimensional scheme, all of whose points have residue field K. Thus t, z'/t are regular parameters in S (and in particular $z'/t=0$ must be the local equation of C^*). This proves (iii).

It is now clear that C^* is a regular curve. When $K\neq k$, C and C^* are irreducible; in this case, to see that $C^*\cong\mathbf{P}^1_K$, we need only note that *the field of functions* $k(C^*)=k(C)$ *is a purely transcendental extension of* K. Indeed, if (u, v, w) is a generic point of the plane curve C, then u/v satisfies the irreducible equation

$$a(u/v)^2+b(u/v)+c=0$$

so that $K=k(u/v)$. Hence

$$k(C)=k(u/v, w/v)=K(w/v)$$

and since $k(C)$ cannot be algebraic over k, w/v is transcendental over K, as required.

The (straightforward) proofs of the remaining assertions of Lemma (24.1) are left to the reader.

Finally we examine the *converse* situation, namely let R be *any* local ring with maximal ideal $\mathfrak{m}$ such that

$$\bigoplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}\cong k[X, Y, Z]/Q \qquad (k=R/\mathfrak{m})$$

where Q is a form of degree 2 such that the projective plane curve $Q(X, Y, Z)=0$ has just one singular point. As above we find that this condition is " stable " under quadratic transformations, namely if R' is the local ring of a closed point on the quadratic transform of R, and if $\mathfrak{m}'$ is the maximal ideal of R', then R' is " at least as good as R " in the sense that either R' is regular or

$$\bigoplus_{n\geq 0}(\mathfrak{m}')^n/(\mathfrak{m}')^{n+1}\cong k[X, Y, Z]/Q'$$

where Q' is a form of degree 2 such that the curve $Q'=0$ has at most one singular point. (We have tacitly made use here of the fact that the condition " $\lambda(\mathfrak{m}^n)=n^2$ for $n\geq 0$ " is stable, cf. proof of (23.5).)

Moreover, *if* R *is normal, then so is* R'. For R' is a Macaulay ring, so we need only check that $R'_{\mathfrak{p}}$ is a discrete valuation ring for every height one prime ideal $\mathfrak{p}$ in R'. Now $R'_{\mathfrak{p}}=R_{\mathfrak{p}\cap R}$ unless $\mathfrak{p}\cap R=\mathfrak{m}$; so we need only check those $\mathfrak{p}$ which contain $\mathfrak{m}$. For such $\mathfrak{p}$, since $R'/\mathfrak{m}R'$ is the local ring of a point on the curve $Q(X, Y, Z)=0$, which is a *reduced* curve, we see immediately that $\mathfrak{p}R'_{\mathfrak{p}}=\mathfrak{m}R'_{\mathfrak{p}}$. Thus $\mathfrak{p}R'_{\mathfrak{p}}$ is principal and so $R'_{\mathfrak{p}}$ is a discrete valuation ring.

It follows now by (23.5) and (16.2) that if R is normal, then R has a pseudo-rational singularity, and if the completion $\hat{R}$ is normal, then R has a rational singularity. Actually in specific examples it may be possible to check, without first assuming R to be normal, that R can be desingularized by quadratic transformations. Then again (Proposition (23.5)) we can conclude that R has a rational singularity.

Examples. — Let k be a field, and let a be an element of k which is not a square in k. Let $b\neq 0$ be an element of k, and let n be a positive integer:

(i) $$R=k[[X, Y, Z]]/(X^2-aY^2+bZ^{2n+1}).$$

We find easily that R is desingularized by n quadratic transformations and that H is trivial.

(ii) $$R=k[[X, Y, Z]]/(X^2-aY^2+bZ^{2n+2})$$

(with $[k^2(a, b):k^2]=4$ if k has characteristic 2).

After n quadratic transformations, the " local equation " $X^2-aY^2+bZ^{2n+2}=0$ becomes $X^2-aY^2+bZ^2=0$, and then one further quadratic transformation gives a desingularization (cf. Case I). In the total exceptional curve on the desingularization, the component C'' (cf. description of Case II b) is the regular projective plane curve whose equation is

$$X^2-aY^2+bZ^2=0.$$

The group H is trivial if this curve has no k-rational point. Otherwise H is of order two.

(iii) A more complicated — in appearance — example along these lines is the ring discussed by Scheja in [**19**; Satz 6].

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We turn now to the case $\tau = 1$. We may assume that $Q(X, Y, Z) = Z^2$, so that with a suitable choice of generators x, y, z of $\mathfrak{m}$ we have $z^2 \in \mathfrak{m}^3$. If R′ is the local ring of a closed point on the quadratic transform of R, then $\mathfrak{m}R'$ is principal, say $\mathfrak{m}R' = tR'$. From the fact that $R'/\mathfrak{m}R'$ is the local ring of a point on the two-fold line $Z^2 = 0$, we find that the image of z/t in $R'/\mathfrak{m}R'$ is a *non-zero* element whose square vanishes. In particular, z/t is a non-unit in R′, so that either $\mathfrak{m}R' = xR'$ or $\mathfrak{m}R' = yR'$. We may therefore assume, for definiteness, that $t = x$. Then $R'/(x, z/x)R'$ is the local ring of a point on the line $Z = 0$, and it follows that the maximal ideal $\mathfrak{m}'$ of R′ is generated by $x' = x$, $z' = z/x$, and $y' = F(y/x)$ where $F(T) \in R[T]$ (T an indeterminate) is a monic polynomial of lowest possible degree such that $F(y/x) \in \mathfrak{m}'$. Clearly the degree of F is also the degree $[R'/\mathfrak{m}' : R/\mathfrak{m}]$.

The relation $z^2 \in \mathfrak{m}^3$ can be written in the form

$$z^2 - G(x, y) \in z\mathfrak{m}^2$$

where $G(U, V) \in R[U, V]$ (U, V indeterminates) is a homogeneous form of degree 3. R′ being as above, with $\mathfrak{m}R' = xR'$, we obtain upon dividing by x^2,

$$(z/x)^2 - xG(1, y/x) \in x(z/x)R' \tag{1}$$

Now R′ *is regular if and only if* $G(1, y/x)$ *is a unit in* R′. For if $G(1, y/x)$ is a unit, then (1) shows that $x \in (z/x)R'$, so that $\mathfrak{m}'$ is generated by the two elements z/x and y', and R′ is regular. Conversely, if R′ is regular, then $\mathfrak{m}'$ is generated by two of the three elements $x, z/x, y'$. But x cannot be one of these generators, since R'/xR' contains a non-zero nilpotent element, as we have seen. Hence $x \in (\mathfrak{m}')^2$, and $z/x, y'$ are regular parameters for R′. It follows therefore from (1) that $G(1, y/x)$ is a unit (otherwise $(z/x)^2 \in (\mathfrak{m}')^3$).

Let $\overline{G}(U, V) \in k[U, V]$ ($k = R/\mathfrak{m}$) be the form obtained from $G(U, V)$ by reducing the coefficients modulo $\mathfrak{m}$. $\overline{G}(U, V)$ *is not identically zero*. For, if all the coefficients of $G(U, V)$ were in $\mathfrak{m}R' = xR'$, we could divide (1) by x^2 to obtain an equation of integral dependence for z/x^2 over R′; but R′ is normal (since R is assumed to have a pseudo-rational singularity) and so we would have $z/x^2 \in R'$, i.e.

$$z/x \in xR' = \mathfrak{m}R'$$

which is not true, as we have remarked. It follows from the preceding paragraph that there is a one-one correspondence between the set of non-regular R′, and the set of irreducible factors of $\overline{G}(U, V)$ over k. In particular, there are at most three such R′.

We assume now that R′ is not regular, i.e. that $G(1, y/x) \in \mathfrak{m}'$. If F(T) is as above, then there is a polynomial $P(T) \in R[T]$ such that

$$G(1, T) - F(T)P(T) \in \mathfrak{m}R[T]$$

(because, " modulo $\mathfrak{m}$ ", $F(T)$ is the *minimum* polynomial for y/x over the field $R/\mathfrak{m}$). It follows from (1) that

$$(z/x)^2 - xF(y/x)P(y/x) \in x(z/x)R' + x\mathfrak{m}R'$$

i.e.

$$(z')^2 - x'y'P(y/x) \in x'z'R' + (x')^2R', \tag{2}$$

The situation is very simple if $P(y/x)$ is a unit in R', because then the graded ring of R' with respect to $\mathfrak{m}'$ is isomorphic to

$$k'[X, Y, Z]/(Z^2 - aXY - bXZ - cX^2)$$

($k' = R'/\mathfrak{m}'$; $a, b, c \in k'$; $a \neq 0$) and this is seen at once to be the homogeneous coordinate ring of a smooth plane conic having a k'-rational point. Thus (Case I above) R' will be completely desingularized by one quadratic transformation.

$P(y/x)$ is certainly a unit if $\overline{G}(U, V)$ has no multiple factors over k. From the foregoing considerations, we now obtain quite simply the following cases. (Details are left to the reader.)

CASE III a. — $\overline{G}(U, V)$ *is irreducible over* k. R has a rational singularity, and the total exceptional curve on a minimal desingularization of R is of type $1-3$. One component is isomorphic to the projective line over k, while the other is (k-)isomorphic to the projective line over the splitting field of $\overline{G}$.

CASE III b. — $\overline{G}(U, V)$ *is the product of a linear and an irreducible quadratic factor over* k. R has a rational singularity, and the total exceptional curve on a minimal desingularization of R is of type $1-1-2$. Two of the components are isomorphic to projective lines over k, while the third is isomorphic to a projective line over the splitting field of $\overline{G}$.

CASE III c. — $\overline{G}(U, V)$ *is a product of* distinct *linear factors over* k. R has a rational singularity, and the total exceptional curve on a minimal desingularization of R is of type

$$\begin{array}{ccc} & & 1 \\ & & \diagup \\ 1 - 1 & & \\ & & \diagdown \\ & & 1 \end{array}$$

all components being isomorphic to projective lines over k.

Conversely the preceding arguments show that if R is *any* local ring with maximal ideal $\mathfrak{m}$ such that

$$\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong k[X, Y, Z]/Z^2 \qquad (k = R/\mathfrak{m}).$$

and such that $\mathfrak{m}$ is generated by three elements x, y, z satisfying a relation of the type

$$z^2 - G(x, y) \in z\mathfrak{m}^2$$

where $G(U, V) \in R[U, V]$ is a form of degree three such that $\overline{G}(U, V)$ is non-zero and has no multiple factors over k, then R can be desingularized by quadratic transformations, and consequently (Proposition (23.5)) R has a rational singularity.

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There remains to be considered the possibility that $\overline{G}(U, V)$ has multiple factors. With a suitable choice of x, y, we may assume that either $\overline{G}(U, V)=UV^2$ or $\overline{G}(U, V)=aV^3$ $(0 \neq a \in k)$.

We first examine the case $\overline{G}(U, V)=UV^2$. According to our previous considerations there are precisely two non-regular points on the quadratic transform of R, namely those in whose local ring $y, x/y, z/y$ (respectively $x, y/x, z/x$) generate the maximal ideal. The first of these will be desingularized by one quadratic transformation. The second — call it R′ — is more interesting. For this R′, equation (2) becomes

$$(z')^2-(x')(y')^2 \in x'z'R'+(x')^2R' \tag{3}$$

$(x'=x, y'=y/x, z'=z/x)$. This shows first of all that R′ either is of a previously considered type (Case II *a*, II *b*, III *b* or III *c*) or is again of the type under discussion at this moment. In other words we have a situation which is " stable " under quadratic transformations.

To complete the description we must examine the behaviour of exceptional curves when R′ is blown up. Let C, C*, C′ have the same meaning as in the discussion of Case II (but relative to the rings R, R′ which we are now considering). Let R″ be the local ring of a closed point on the surface obtained by blowing up R′, through which C* passes. In R″ there is then a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \cap R'$ contains x', but $\mathfrak{p} \cap R \neq \mathfrak{m}'$. By (3) $\mathfrak{p} \cap R'$ also contains z'. Hence $\mathfrak{m}'R''=y'R''$ and $x''=x'/y'$, $z''=z'/y'$ are non-units in R″. These conditions determine R″ uniquely; the maximal ideal of R″ is generated by $x'', y''=y'$, and z''.

From (3) we obtain, in R″,

$$(z'')^2-x''y'' \in x''z''R''+(x'')^2R''. \tag{4}$$

As a consequence, we find that R″ *is not regular*. To see this, note that the associated graded ring of R′ with respect to $\mathfrak{m}'$ is of the form

$$k'[X, Y, Z]/Q'(X, Z) \qquad (k'=R'/\mathfrak{m}')$$

Q′ being a form of degree 2 (cf. (3)); it follows that $R''/\mathfrak{m}'R''$ is not regular, since it is the local ring of the singular point (0, 1, 0) on the curve $Q'(X, Z)=0$. Hence if R″ is regular, then $y'' \in (\mathfrak{m}'')^2$ and x'', z'' are regular parameters. But (4) shows that, with suitable $\alpha, \beta \in R''$,

$$(z'')^2+\alpha x''z''+\beta(x'')^2=x''y'' \in (\mathfrak{m}'')^3$$

which cannot be if x'', z'' are regular parameters. So R″ is not regular. However (4) also shows that R″ is completely desingularized by one quadratic transformation, the inverse image C″ of $\{\mathfrak{m}''\}$ being isomorphic to $\mathbf{P}^1_k$ (cf. Case I).

By definition C* passes through R″. Also every component of C′ passes through R″ (since every component of the curve $Q'(X, Z)=0$ passes through (0, 1, 0)). Hence C″

is met by the proper transforms of C^* and of the components of C'. It is simple to check that through any point of C'' there passes at most one integral exceptional curve other than C''.

If R has a rational singularity, it will be desingularized by quadratic transformations. By repeated application of the foregoing considerations, it is now straightforward to deduce the following:

CASE IV. — $\overline{G}(U, V)$ *is the product over k of a linear factor and the square of another* (distinct) *linear factor*. The total exceptional curve on a minimal desingularization of R is of one of two types:

a)
$$\underbrace{1 - 1 - 1 - \dots - 1 - 1}_{r \text{ components}} \begin{matrix} \diagup\, 1 \\ \diagdown\, 1 \end{matrix}$$

$(r \geq 3)$

b)
$$\underbrace{1 - 1 - 1 - \dots - 1 - 1}_{r \text{ components}} - 2$$

The components are all isomorphic to $\mathbf{P}^1_k$ except for the component for which $h^0 = 2$, and this component is k-isomorphic to the projective line over a quadratic extension of k.

We leave to the reader the formulation and proof of a suitable converse. (*Note:* if $\mathfrak{p}$ is the prime ideal $(x', z')R'$, then $\mathfrak{p}$ is the only prime ideal in R' containing $\mathfrak{m}R'$, and (3) shows that $\mathfrak{p}R'_{\mathfrak{p}}$ is principal, namely $\mathfrak{p}R'_{\mathfrak{p}} = (z')R'_{\mathfrak{p}}$).

* * *

We deal finally with the case $\overline{G}(U, V) = aV^3$ $(a \neq 0)$. For suitable x, y, z generating $\mathfrak{m}$, we have then

$$\eta z^2 + \alpha y^3 + \beta z x^2 + \gamma x^4 \in (x^3y, x^2y^2, xyz, y^2z)R \tag{5}$$

with $\eta, \alpha, \beta, \gamma \in R$, $\overline{\eta} = 1$, $\overline{\alpha} = a$ (where " — " denotes " residue mod. $\mathfrak{m}$ "). The behaviour of R will depend on the nature of the form

$$P(X, Z) = Z^2 + \overline{\beta} ZX + \overline{\gamma} X^2 \in k[X, Z] \qquad (k = R/\mathfrak{m}).$$

If $P(X, Z)$ is not a square in $k[X, Z]$ we leave (5) as it is. When $P(X, Z)$ is a square, there is an element δ in R such that

$$\eta z^2 + \beta z x^2 + \gamma x^4 \equiv (z + \delta x^2)^2 \bmod (z, x^2)^2\mathfrak{m}.$$

Setting $w = z + \delta x^2$ (so that $(z, x^2)R = (w, x^2)R$) we obtain a relation

$$\eta' w^2 + \alpha y^3 + \beta' w x^2 + \gamma' x^4 \in (x^3y, x^2y^2, xyw, y^2w)R$$

where $\eta' \equiv 1$ (mod. $\mathfrak{m}$) and β', γ' are non-units. Hence we obtain

$$\eta'' w^2 + \alpha y^3 + \rho x^3 y + \sigma x^5 \in (x^3w, x^2y^2, xyw, y^2w)R$$

with $\eta'' \equiv 1 \pmod{\mathfrak{m}}$. We may as well assume that $\eta''=1$; also we may as well write z for w; in other words if $P(X, Z)$ is a square, then there are generators x, y, z of $\mathfrak{m}$ with

$$z^2+\alpha y^3+\rho x^3 y+\sigma x^5 \in (x^3 z, x^2 y^2, xyz, y^2 z)R \tag{5'}$$

By performing quadratic transformations, and with arguments of the type we have already used, we now obtain the following classification. Details are left to the reader.

CASE V. — $\bar{G}(U, V)$ *is a constant multiple of the cube of a linear form over* k. R has a rational singularity; the total exceptional curve on a minimal desingularization is as indicated below under the appropriate conditions on β, γ, ρ, σ. *Conversely*, if R is *any* local ring with maximal ideal $\mathfrak{m}$ such that

$$\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong k[X, Y, Z]/Z^2$$

and such that $\mathfrak{m}$ has a basis x, y, z satisfying a relation of the form (5) or (5'), with $\eta \equiv 1 \pmod{\mathfrak{m}}$, α a unit, and β, γ, ρ, σ subject to one of the following conditions, then R has a rational singularity.

CASE V *a*. — *The form* $P(X, Z)$ *is irreducible in* $k[X, Z]$:

$$1 - 1 - 2 - 2$$

CASE V *b*. — $P(X, Z)=(Z+pX)(Z+qX)$ *with* p, q *in* k, $p \neq q$:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & | & & & & \\ 1 & - & 1 & - & 1 & - & 1 & - & 1 \end{array}$$

CASE V *c*. — $P(X, Z)$ *is a square in* $k[X, Z]$ *and* ρ *is a unit in* R:

$$\begin{array}{ccccccccccc} & & & & & & 1 & & & & \\ & & & & & & | & & & & \\ 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \end{array}$$

CASE V *d*. — $P(X, Z)$ *is a square in* $k[X, Z]$, ρ *is a non-unit in* R *and* σ *is a unit in* R:

$$\begin{array}{ccccccccccccc} & & & & & & & & 1 & & & & \\ & & & & & & & & | & & & & \\ 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \end{array}$$

This completes the classification of rational and pseudo-rational double points.

§ 25. Rational factorial rings.

In this final section we give necessary and sufficient conditions for R to have a rational singularity and a trivial group H. The conditions are expressed in the form of relations satisfied by suitable generators x, y, z of the maximal ideal $\mathfrak{m}$. Because of

Proposition (22.5) and the classification in § 24, we already have such relations; the idea now is to choose x, y, z so that the relations become as simple as possible. In particular we characterize *all* complete two-dimensional factorial local rings with *algebraically closed* residue field (Theorem (25.1), Remark (25.2)).

Theorem **(25.1)**. — *Let* R *be a two-dimensional local ring with maximal ideal* $\mathfrak{m}$ *such that* $R/\mathfrak{m}$ *is an algebraically closed field of characteristic* $\neq 2, 3, 5$. *Assume that* R *is not regular. The following conditions are equivalent*:

(i) *The completion* $\hat{R}$ *is factorial.*

(i)′ $\hat{R}$ *is normal, and the henselization* R^* *of* R *is factorial.*

(ii) *There exists a basis* $\{x, y, z\}$ *of* $\mathfrak{m}$ *and units* α, β *in* R *such that*

$$z^2 + \alpha y^3 + \beta x^5 = 0.$$

(iii) *There exists a basis* $\{x^*, y^*, z^*\}$ *of the maximal ideal* $\mathfrak{m}^*$ *of* R^* *such that*

$$(z^*)^2 + (y^*)^3 + (x^*)^5 = 0.$$

(iv) *There exists a three-dimensional regular local ring* S *with regular parameters* u, v, w *such that*

$$\hat{R} \cong S/(u^2 + v^3 + w^5).$$

Proof (i)$\Leftrightarrow$(i)′. — R can be desingularized if its completion $\hat{R}$ is normal (Remark (16.2)); hence the equivalence of (i) and (i)′ is given by (17.3) and (17.2).

(i)$\Rightarrow$(ii). — By (17.3) and (17.2), (i) implies that R has a rational singularity and that H(R) is trivial. Proposition (22.5) shows then that R has multiplicity two. Since $R/\mathfrak{m}$ is algebraically closed, " 1 " is the only integer which can appear in the configuration diagram on a minimal desingularization of R. The only possible diagram, then, is E_8 (cf. earlier part of § 24) and we must therefore be in Case V *d* of § 24, so that for a suitable basis $\{x, y, z\}$ of $\mathfrak{m}$ there is a relation

$$z^2 + \alpha y^3 + t x^4 y + \beta x^5 + p x^3 z + q x^2 y^2 + r x y z + s y^2 z = 0 \tag{6}$$

where α and β are units in R, and p, q, r, s, t are in R (this is derived from equation (5′) in § 24, where ρ is a *non-unit* i.e. $\rho \in (x, y, z)R$; also we have put β in place of σ).

Setting

$$z' = z + (p/2)x^3 + (r/2)xy + (s/2)y^2$$

we have

$$(z')^2 + \alpha' y^3 + t' x^4 y + \beta' x^5 + q' x^2 y^2 = 0$$

for suitable α', t', β', q', with $\alpha' \equiv \alpha$, $\beta' \equiv \beta$ (mod. $\mathfrak{m}$). In other words, for suitable x, y, z *we may assume in* (6) *that* $p = r = s = 0$.

Next, setting

$$y' = y + (q/3\alpha)x^2 - (u/3\alpha)x^3 \qquad (u \text{ arbitrary})$$

we find similarly that in (6) *we may assume further that* $q = ux$.

Finally, setting

$$x' = x + vy$$

and choosing suitable values for u, v we find that in (6) we can take $p=q=r=s=t=0$, proving (ii).

(ii) $\Rightarrow$ (iii). — Since $R^*/m^* = R/m$ is algebraically closed of characteristic $\neq 3, 5$ and R^* is henselian, α is a cube in R^* and β is a fifth power. Since $\mathfrak{m}^* = \mathfrak{m}R^*$ it is now clear that (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). — Since $\hat{R}$ is also the completion of R^*, it follows at once from (iii), in view of the Cohen structure theorem, that $\hat{R}$ is a homomorphic image of $\overline{R} = S/(u^2 + v^3 + w^5)$. But in Case V *d* of § 24 we have seen that $\overline{R}$ is factorial; since $\dim.\hat{R} = \dim.\overline{R} = 2$, we must therefore have $\hat{R} = \overline{R}$. Q.E.D.

Remark **(25.2)**. — If $R/\mathfrak{m}$ has characteristic 2, 3, or 5, we must change Theorem (25.1) somewhat. (i) and (i)′ remain the same, but for the relation in (ii) we have two or more possibilities, as indicated below, with α and β units in R. (iii) will now state that α and β can both be assumed to be 1 if R is henselian. The corresponding change in (iv) is obvious.

Characteristic 5:

$$\alpha z^2 + \beta y^3 + x^5 = \gamma x^4$$

where γ is one of:

a) 0;

b) y.

Characteristic 3:

$$\alpha z^2 + y^3 + \beta x^5 = \gamma y^2$$

where γ is one of:

a) 0;

b) x^3;

c) x^2.

Characteristic 2:

$$z^2 + \alpha y^3 + \beta x^5 = \gamma z$$

where γ is one of:

a) 0;

b) x^3;

c) x^3y;

d) x^2y;

e) xy.

Proofs are omitted. As in (ii) of Theorem (25.1), they are computational (though somewhat more involved, especially for characteristic two).

* * *

Remark **(25.3).** — Let R be a two-dimensional normal local ring with maximal ideal $\mathfrak{m}$. Without any further assumption on $R/\mathfrak{m}$, we have actually shown that R has a rational singularity, a trivial group H, and the configuration diagram E_8 *if and only if* R satisfies (ii) of Theorem (25.1) (cf. also Remark (25.2)).

The other types of rational singular points with trivial group H can be discussed similarly. The results are given below. For simplicity, we assume that $R/\mathfrak{m}$ has characteristic $\neq 2$. Once again proofs are computational and are omitted.

Configuration Diagram	*Relation on suitable generators x, y, z of $\mathfrak{m}$.*
G_2 : 1—3	$z^2+\alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3=0$ ($\alpha, \beta, \gamma, \delta \in R$, and if $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$, are the respective residues mod. $\mathfrak{m}$, the form $\bar{\alpha}X^3+\bar{\beta}X^2Y+\bar{\gamma}XY^2+\bar{\delta}Y^3$ is irreducible over $R/\mathfrak{m}$).
F_4 : 1—1—2—2	$\alpha z^2+y^3+\beta x^4=0$ ($-\bar{\alpha}\bar{\beta}$ not a square in $R/\mathfrak{m}$; $R/\mathfrak{m}$ of characteristic $\neq 3$) If $R/\mathfrak{m}$ has characteristic 3 $\alpha z^2+y^3+\beta x^4=\gamma x^2y^2$ with α, β as above, $\gamma=0$ or 1.
B_n : 1—2—2—...—2 (n components)	$z^{2n+1}+\alpha y^2+\beta x^2=0$ ($-\bar{\alpha}\bar{\beta}$ not a square in $R/\mathfrak{m}$). Or: $z^{2n}+\alpha y^2+\beta x^2=0$ where α, β are such that the curve $Z^2+\bar{\alpha}Y^2+\bar{\beta}X^2=0$ has no rational point over $R/\mathfrak{m}$.

Example **(25.4).** — Let R be henselian and suppose that $R/\mathfrak{m}$ is the field of real numbers (or, more generally, any real closed field). From the preceding, we see that R has a rational singularity and trivial H if and only if $\mathfrak{m}$ is generated by x, y, z satisfying one of the following relations [1]:

$$z^2+y^3+x^5=0 \qquad (E_8)$$
$$z^2+y^3+x^4=0 \qquad (F_4)$$
$$z^{2n}+y^2+x^2=0 \qquad (B_n)$$
$$z^{2n+1}+y^2+x^2=0 \qquad (B_n) \qquad (n \geq 1)$$

APPENDIX: TWO FUNDAMENTAL THEOREMS ON SURFACES

§ 26. Elimination of indeterminacies by quadratic transformations and normalization.

Throughout this section S will be an arbitrary scheme (not necessarily separated), and $\varphi : X \rightarrow Y$ will be an S-rational transformation (= " S-application rationnelle " [**EGA** I, § 7.1]) of S-schemes X, Y, where X is a *surface* and Y is *separated* and *of finite type* over S. Associated with φ is the diagram

$$\begin{tikzcd} & G \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

where G is the graph of φ [**EGA** IV, (20.4.2), (20.2.7)]. A *point of indeterminacy* of φ is a point of $p(G)$ at which φ is not defined.

Suppose now that X is integral, with field of rational functions K, and let $f : X' \rightarrow X$ be a separated birational map (X′ integral). We have a commutative diagram

$$\begin{tikzcd} & \mathrm{Spec}(K) \arrow[dl, "\text{(birational)}"'] \arrow[dr] & \\ X' \arrow[rr, "\psi"] \arrow[d, "f"'] & & X' \times_S Y \arrow[d, "\pi_2"] \\ X \arrow[rr, "\varphi"'] & & Y \end{tikzcd}$$

where ψ is a rational section whose domain of definition is the same as that of $\varphi \circ f$. The graph G′ of $\varphi \circ f$ is the (reduced) closure of the image of Spec(K) in $X' \times_S Y$; G′ is birational and of finite type over X′. Identifying $X' \times_S Y$ with $X' \times_X (X \times_S Y)$,

---

[1] We assume that R is not regular.

we see at once that $\operatorname{Spec}(K) \to X' \times_S Y$ factors through $X' \times_X G \subseteq X' \times_S Y$; thus G' is a closed subscheme of $X' \times_X G$. (In fact G' is just the *join* of X' and G over X.) We may regard ψ as a rational section of G' over X'; then if Γ is the graph of ψ we have a commutative diagram

$$\begin{tikzcd} \Gamma \arrow[rr, "\approx"] \arrow[dr] & & G' \arrow[dl] \\ & X' & \end{tikzcd}$$

from which we conclude that ψ and $\varphi \circ f$ have the same points of indeterminacy.

We may — and, for simplicity of language, we shall — regard X and X' as *models*, i.e. as collections of local domains with quotient field K; then f becomes the map which associates to each local ring $R' \in X'$ the unique local ring $R \in X$ such that R' dominates R. When we consider G' in this way, the domain of definition of $\varphi \circ f$ (i.e. that of ψ) consists of those $R' \in X'$ which dominate an element of G' (equivalently $R' \in G'$) (cf. [**EGA** I, § 6.5]); thus *the points of indeterminacy of* $\varphi \circ f$ *are those* $R' \in X'$ *which are dominated by, but not equal to, some element of* G'.

***

We say that a valuation v of K has *center* R' on X' if v dominates R'. We say that v is *exceptional* (for φ) if it has a center on G which is not a closed point of G, while its center on X is two-dimensional. G, being birational and of finite type over X, is of dimension ≤ 2; thus the center on G of an exceptional v must be one-dimensional, so that v is discrete, of rank one. *There are at most finitely many exceptional v.* For X and G have identical dense open subsets U_X, U_G, and so the closure of the center on G of an exceptional v must be an irreducible component of $G - U_G$; thus there are at most finitely many possible centers, and each such center, being one-dimensional, is the center of at most finitely many v.

The proof of the main theorem in this section will depend on the following property of points of indeterminacy:

With the preceding notation (X being integral), *assume that* X' *is* normal. *If* $R' \in X'$ *is a point of indeterminacy for* $\varphi \circ f$ *then* R' *is the center on* X' *of a valuation which is exceptional for* φ (1).

Proof. — Suppose that $R' \in X'$ is dominated by, but not equal to, some $Q' \in G'$. By Zariski's " Main Theorem " [**EGA** III, (4.4.8)] there is such a Q' which is residually transcendental over R'. Let Q (resp. R) be the unique local ring on G (resp. X) dominated by Q' (resp. R'). Since G' is a subscheme of $X' \times_X G$, the residue field of Q' is generated over that of R' by the canonical image of the residue field of Q; hence Q is residually transcendental over R. It follows at once that any valuation of K dominating Q' (and hence also R') is exceptional for φ.

(1) The converse is also true provided that R' is two-dimensional.

* * *

Theorem **(26.1)** (Zariski). — *Let* $\varphi : X \to Y$ *be an* S-*rational transformation, where* X, Y *and* S *are as in the beginning of this section. Then there exists a birational closed map* $f : X' \to X$ *which is obtained as a succession of normalizations and quadratic transformations such that the* S-*rational transformation* $\varphi \circ f$ *has no points of indeterminacy.*

Proof. — Since the normalization of X is a disjoint union of normal surfaces, we may as well assume that X is integral and normal. The preceding discussion shows then that there are at most finitely many points of indeterminacy of φ, all of codimension two on X. Let $g_1 : Z_1 \to X$ be obtained by blowing up a point of indeterminacy of φ, and let $h_1 : X_1 \to Z_1$ be the normalization of Z_1. If $\varphi \circ g_1 \circ h_1$ has no points of indeterminacy, we are done. Otherwise, repeat the process with $(X_1, \varphi \circ g_1 \circ h_1)$ in place of (X, φ). Continue in this manner. If the process ever stops, the theorem is proved. If not, there is obtained an infinite sequence $X \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$ of normal surfaces. (In order to be canonical, we could have defined g_1 to be the map obtained by blowing up simultaneously *all* the points of indeterminacy of φ. $g_2, g_3, \ldots$ would be defined similarly). In any case, we are led to the following statement, which is somewhat stronger than Theorem (26.1):

Theorem **(26.2)**. — *Let* S, $\varphi : X \to Y$, *be as above,* X *being normal and integral, with field of functions* K. *Let*

$$(\Sigma): \qquad X = X_0 \underset{f_1}{\leftarrow} X_1 \underset{f_2}{\leftarrow} X_2 \leftarrow \ldots$$

be a sequence of normal *surfaces and birational maps with the following property:*

(P) *For each* $i \geq 0$ *there is a point* $x_i \in X_i$ *such that* x_i *is a point of indeterminacy of* $\varphi \circ f_1 \circ f_2 \circ \ldots \circ f_i$, *and such that for all* $y \in f_{i+1}^{-1}(x_i)$ *the maximal ideal of* $\mathcal{O}_{X_i, x_i}$ *is contained in a proper principal ideal of* $\mathcal{O}_{X_{i+1}, y}$.

Then the sequence (Σ) *is finite.*

Proof. — Assume that (Σ) is infinite. For each $i = 0, 1, 2, \ldots$ choose a point $x_i \in X_i$ such that x_i satisfies condition (P). Each such x_i being the center of an exceptional v, of which v there are only finitely many, *some exceptional* v *must dominate infinitely many* x_i. There results an infinite sequence of two-dimensional local rings

$$R_1 < R_2 < R_3 < \ldots$$

with field of fractions K such that (i): the maximal ideal of each R_i is contained in a proper principal ideal of R_{i+1}, and such that (ii): all the R_i are dominated by a single discrete rank one valuation v which is *residually transcendental* over them. This is impossible, because (i) implies that the valuation ring R_v must be equal to $\bigcup_{i \geq 0} R_i$ (cf. argument in middle of p. 392 of [**25**]). Q.E.D.

§ 27. Rational contraction of one-dimensional effective divisors.

Let A be a noetherian ring and let $f : X \to \mathrm{Spec}(A)$ be a map of finite type. As in § 13, a *curve on* X will be an effective divisor with one-dimensional support. Let

$E_1, E_2, \ldots, E_n$ be distinct integral curves on X with exceptional support (cf. § 12) such that X is *normal* at every point $x \in \bigcup_i E_i$. We say that $\bigcup_i E_i$ *contracts to a point* (over the ground ring A) if there is a (separated) scheme Y of finite type over A and a proper Spec(A)-morphism $h : X \to Y$ such that $h(\bigcup_i E_i)$ is a single *normal* point P and such that h induces an isomorphism of $X - \bigcup_i E_i$ onto $Y - P$.

(*Remarks.* — Such an h is easily seen to be birational (even if X and Y are not reduced). Moreover, the local ring S of P on Y is necessarily two-dimensional, and the condition that P be normal is equivalent to the condition that $h_*(\mathcal{O}_X) = \mathcal{O}_Y$; for the proof, replace h by the projection $h' : X' = X \times_Y \operatorname{Spec}(S) \to \operatorname{Spec}(S)$, note that h' is proper and birational, and that X′ is a normal integral surface. Observe further that for $x \in \bigcup_i E_i$, $\mathcal{O}_{X',x} = \mathcal{O}_{X,x}$ and that, by normality, $S \cong \bigcap_{x \in \cup_i E_i} \mathcal{O}_{X',x}$; it follows that Y and h are unique (up to isomorphism).)

We say that $\bigcup_i E_i$ is *rationally contractible* if there exists h as above with $R^1 h_*(\mathcal{O}_X) = 0$.

Suppose now that h contracts $\bigcup_i E_i$ to P. Then $\bigcup_i E_i = h^{-1}(P)$ is connected ([**EGA** III, (4.3.3)]). Furthermore, the intersection matrix $((E_i . E_j))$ is negative-definite. (The proof of (14.1), with S in place of R, applies to the present situation.) As in ([**4**, p. 131-132]) there exists among the curves $C = \sum_i c_i E_i$ such that $(C . E_i) \leq 0$ for all i a unique *smallest* one, which is called the *fundamental curve* for $\bigcup_i E_i$.

We are now prepared for the generalization of M. Artin's contractibility criterion. We reiterate that " curve on X " is to be construed as in the beginning of this section.

Theorem **(27.1)**. — *Let* A *be a noetherian ring and let* $f : X \to \operatorname{Spec}(A)$ *be a* projective *map. Let* $E_1, E_2, \ldots, E_n$ *be distinct integral curves on* X *with exceptional support (relative to f) such that* X *is normal at every point of* $\bigcup_i E_i$; *assume further that* $\bigcup_i E_i$ *is connected and that the intersection matrix* $((E_i . E_j))$ *is negative-definite. Let* C *be the fundamental curve for* $\bigcup_i E_i$. *Then there exists* $h : X \to Y$ *contracting* $\bigcup_i E_i$ *rationally over* A *to a point* P *if and only if* $\chi(C) > 0$. *When this condition holds,* Y *is projective over* A, *the multiplicity of* P *on* Y *is* $-(C^2)/h^0(C)$, *and* Y *is regular at* P *if and only if* $(C^2) = -h^0(C)$.

Proof. — We first prove necessity. Let $h : X \to Y$ contract $\bigcup_i E_i$ rationally to the point P, let S be the local ring of P on Y and let $X' = X \times_Y \operatorname{Spec}(S)$. X′ is a normal surface and the inverse image D′ on X′ of any curve $D = \sum_i d_i E_i$ is a curve on X′ which is *isomorphic* to D. The cohomology groups of D′ can be considered as finite-length modules over S, and the residue field of S is a finite algebraic extension of that of the point $Q = f(\bigcup_i E_i) \in \operatorname{Spec}(A)$ (since P is a closed point of the fibre on Y over Q); thus if we replace X by X′ and A by S, the effect is merely to divide all the integers involved in the theorem by the residual degree of P over Q. We may therefore assume to begin with that $A = S$ is a two-dimensional normal local domain (with maximal ideal, say, $\mathfrak{m}$), that X is a normal surface with $H^1(X, \mathcal{O}_X) = 0$, that $f = h$ is a proper birational map and that $\bigcup_i E_i$ is the support of the closed fibre $f^{-1}(\{\mathfrak{m}\})$.

By Proposition (3.1), the ideal $\mathfrak{m}\mathcal{O}_X$ is divisorial. But the irreducible components E_i of the subscheme defined by $\mathfrak{m}\mathcal{O}_X$ are defined by *invertible* $\mathcal{O}_X$-ideals; hence $\mathfrak{m}\mathcal{O}_X$ is *invertible* and so defines a curve C'. By (ii) of Theorem (12.1), C' is the fundamental curve C of $\bigcup_i E_i$. Since $H^1(\mathcal{O}_C)$ vanishes ($\mathcal{O}_C$ being a homomorphic image of $\mathcal{O}_X$), and since $H^0(\mathcal{O}_C)=S/\mathfrak{m}$ (cf. proof of (3.1)), we have $\chi(C)=h^0(C)=1>0$.

Moreover, the powers of $\mathfrak{m}$ are contracted for f (Theorem (7.2)), i.e. $H^0(\mathfrak{m}^k\mathcal{O}_X)=\mathfrak{m}^k$ for all $k\geq 0$; also $H^1(\mathfrak{m}^k\mathcal{O}_X)=0$ since $\mathfrak{m}^k\mathcal{O}_X$ is a homomorphic image of $\mathcal{O}_X^t$ for some finite t; hence by (23.1),

$$\lambda(\mathfrak{m}^k)=-(C.C)\binom{k}{2}+k.$$

This shows that S has multiplicity $-(C^2)=-(C^2)/h^0(C)$, and that S is regular (i.e. $\lambda(\mathfrak{m}^2)=3$) if and only if $(C^2)=-1=-h^0(C)$.

For the remaining assertions, the proof of Theorem (2.3) of [**3**] can be imitated; we indicate a bare outline, leaving the details to the interested reader. First of all, the proof of Theorem 3 of [**4**], suitably modified, shows that if $\chi(C)>0$ for the fundamental curve of $\bigcup_i E_i$ then $\chi(D)>0$ for *every* curve $D=\sum_i d_iE_i$. (In making the indicated modification, all statements about the arithmetic genus $p(D)$ of a curve D are to be replaced by statements about $\chi(D)$; in particular interpret $p(D)\leq 0$ to mean $\chi(D)>0$.) Lemmas (11.4) and (22.1) then enable the argument *c)* ⇒ *a)* of Theorem (1.7) of [**3**] to be carried out, the conclusion being that $H^1(D)=0$ for all $D=\Sigma d_iE_i$.

Now our Proposition (11.1) is applicable. As in Theorem (2.3) of [**3**] we can therefore find a very ample invertible sheaf $\mathcal{H}$ on X and a curve $D=\sum_i d_iE_i$ such that, with $\mathcal{L}=\mathcal{H}(D)$, we have:

a) $(\mathcal{L}.E_i)=0$ for all i.

b) The canonical map h of X into $Y=\operatorname{Proj}(\bigoplus_{n\geq 0}\Gamma(X,\mathcal{L}^{\otimes n}))$ is everywhere defined, and is an isomorphism outside $\bigcup_i E_i$.

(In proving this last statement, note that the injection $\mathcal{H}\to\mathcal{L}$ is an isomorphism outside $\bigcup_i E_i$ and observe the proof of the last assertion of [**EGA** II, (4.5.2)].) Lemma (21.2) shows, since $(\mathcal{L}.E_i)=0$, that $h(E_i)$ is a single point of Y for each i, and since $\bigcup_i E_i$ is connected, $h(\bigcup_i E_i)$ is a single point. From [**EGA** II, (3.7.3)] and [**EGA** I, (9.3.2)] it follows that $h_*(\mathcal{O}_X)=\mathcal{O}_Y$; in particular h is dominant, and since X is projective over A, h is surjective and projective. To see that Y is projective it is enough to show that $\bigoplus_{n\geq 0}\Gamma(X,\mathcal{L}^{\otimes n})$ is a finitely generated A-algebra. The construction of $\mathcal{L}$ is such that $\mathcal{L}$ is generated by its sections over X; consequently if $\mathcal{S}=\mathcal{S}(f_*(\mathcal{L}))$ is the symmetric algebra on the coherent Spec(A)-module $f_*(\mathcal{L})$, then $\bigoplus_{n\geq 0}\mathcal{L}^{\otimes n}$ is a homomorphic image of $f^*(\mathcal{S})$, and so $\bigoplus_{n\geq 0}f_*(\mathcal{L}^{\otimes n})$ is a finitely generated $\mathcal{S}$-module [**EGA** III, (3.3.1)]. The conclusion follows.

It remains to be shown that $R^1h_*(\mathcal{O}_X)=0$. For this purpose, we can replace Y by Spec(S), and X by the normal surface $X\times_Y \mathrm{Spec}(S)$, where S is the local ring of P on Y. Let $\mathfrak{m}$ be the maximal ideal of S. As in Lemma (12.2) it is sufficient to show that $H^1(\mathcal{O}_X/\mathfrak{m}^k\mathcal{O}_X)$ vanishes for $k\geq 0$. Since, as we have seen, $H^1(D)=0$ for all $D=\sum_i d_i E_i$ it is enough to show that $\mathfrak{m}\mathcal{O}_X$ contains a power of the ideal $\mathscr{M}=((\mathfrak{m}\mathcal{O}_X)^{-1})^{-1}$, because $\mathscr{M}$ defines a curve like D. Since X is quasi-compact, this is a purely local question, and the affirmative answer results from the fact that for any $x\in X$, $(\mathfrak{m}\mathcal{O}_X)_x$ contains $\mathscr{M}_x\cap$ (some power of the maximal ideal of $\mathcal{O}_x$). This completes the proof.

Corollary **(27.2)**. — *Let* Y *be a normal surface having only finitely many singular points, all of which are rational singularities. If* Y *is proper over a noetherian ring* A, *then* Y *is projective over* A.

Proof. — Let $\varphi: Y\to \mathrm{Spec}(A)$ be a proper map; we wish to show that φ is projective. Arguing as in Corollary (2.5) of [**3**], we may assume that Y is regular (cf. Theorem (4.1)). Note that if $h: X\to Y$ is a quadratic transformation then φ is projective if and only if $\varphi\circ h$ is, because of Theorem (27.1) (and the *uniqueness* of contractions, cf. remarks precedings (27.1)). The same holds true, by induction, if h is a *product* of quadratic transformations; it will therefore be sufficient to find such an h with $\varphi\circ h$ projective.

Chow's Lemma [**EGA** II, (5.6.2)] gives the existence of a proper birational map $g: W\to Y$ such that $\varphi\circ g$ is projective. By Theorem (26.1), there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & W \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

with h a product of quadratic transformations. f is projective, since h is, and so $\varphi\circ h=\varphi\circ g\circ f$ is projective. Q.E.D.

Corollary **(27.3)**. — (Cf. [**6**]; Lemma (1.6).) *Let* Y *be a surface which admits a desingularization* $g: Z\to Y$. *Then* Y *has a unique minimal desingularization* $f: X\to Y$ (i.e. *every desingularization of* Y *factors through* f). $Z=X$ *if and only if*

(M): $(E^2)\leq -2\chi(E)$ *for every exceptional integral curve* E *on* Z.

(*Remark.* — The terminology of (M) needs a word of explanation: E is " exceptional " if $g(E)$ is a single point Q of Y, and then (E^2) and $\chi(E)$ are calculated over some affine neighbourhood Spec(A) of Q.)

Proof. — After normalizing, we may assume that Y is integral and normal. If $g: Z\to Y$ is a desingularization, then Z carries only finitely many exceptional curves (relative to g) and therefore it is clear that Z dominates a *relatively* minimal desingularization $f: X\to Y$. Because of the Factorization Theorem (cf. (4.1)), $Z=X$ if (M) holds. Conversely if $Z=X$, then (27.1) and (27.2) show that no integral exceptional curve E on Z satisfies simultaneously $\chi(E)>0$ (i.e. $h^1(E)=0$) and $(E^2)=-h^0(E)$;

thus either $\chi(E) \leq 0$, in which case (M) holds because $(E^2) < 0$, or $\chi(E) = h^0(E) > 0$ and $(E^2) \leq -2h^0(E) = -2\chi(E)$.

Let $g_1 : Z_1 \to Y$ be a desingularization. We wish to show that Z_1 dominates X. Starting from (26.1), for example, we can find a relatively minimal desingularization W of the *join* of X and Z_1. Z_1 dominates X if and only if $W = Z_1$. Suppose $W \neq Z_1$. By the Factorization Theorem, we have a diagram

$$\begin{tikzcd} W \arrow[rr, "h"] \arrow[dr] & & W_1 \arrow[dl] \\ & Z_1 & \end{tikzcd}$$

where h is a quadratic transformation with center, say, P. The image of $F = h^{-1}(P)$ on X is a *curve* E (otherwise W_1 dominates X, contradicting the minimality of W). Note that E is an exceptional curve (relative to Y) and that F is the proper transform of E on W. Also

$$(F.F) + \chi(F) = 0.$$

We leave as an exercise the following fact (Lemma (22.3) is useful in the proof):

Let E be an integral curve on Z with exceptional support, let $j : Z' \to Z$ be obtained by blowing up a closed point x whose multiplicity on E is $\nu \geq 0$. Let E^* be the proper transform of E on Z', let $F' = j^{-1}(x)$, and let $E' = j^{-1}(E) = E^* + \nu F'$. Then

$$\begin{aligned}(E.E) + \chi(E) = (E'.E') + \chi(E') &= (E^*.E^*) + \chi(E^*) + \chi(\nu F') \\ &\geq (E^*.E^*) + \chi(E^*).\end{aligned}$$

Since there is a sequence of quadratic transformations

$$Z = Z^{(0)} \leftarrow Z^{(1)} \leftarrow \ldots \leftarrow Z^{(n)} = W \qquad (n \geq 0)$$

repeated application of the preceding fact shows that

$$(E^2) + \chi(E) \geq 0$$

contradicting (M). Q.E.D.

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Correction (added in proof). The last statement in Proposition (16.3), concerning regular extensions, is not true in general, and similarly for the last statement in Lemma (16·4). The trouble lies in the (incorrect) equality $\delta_{\overline{F}/k} = \delta_{\overline{E}/K}$ at the very end of the proof of (16.4). This equality does hold, however, for the two cases in which (16.3) is applied later on, namely $K = k$ (obviously) and $K = k((T))$ (= fraction field of the power series ring $k[[T]]$). In the latter case, since F has divisors which are of degree > 0, and which are therefore ample, we can fix a projective embedding of F/k, and the corresponding embedding for E/K; it is enough to show, for a closed point $x \in E$ and its reduction $\overline{x} \in F$ (with respect to the unique discrete valuation ring R of $K(x)$ extending $k[[T]]$) that $n = [K(x) : K]$ is divisible by $\overline{n} = [k(\overline{x}) : k]$; but this is clear because if e is the ramification and f is the residue field degree of R over $k[[T]]$, then $ef = n$ and $\overline{n}$ divides f.