PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 36 (1969), p. 127-138
<http://www.numdam.org/item?id=PMIHES_1969__36__127_0>

NORMAL CONES IN ANALYTIC WHITNEY STRATIFICATIONS

by HEISUKE HIRONAKA [1]

INTRODUCTION

In this paper, we prove that if $A = \bigcup_\alpha X_\alpha$ is a complex-analytic stratification of a complex-analytic variety A with singularities and if it satisfies Whitney conditions, then A is normally pseudo-flat along each stratum X_α. (See § 2 for the definition of normal pseudo-flatness.) This implies that the normal cone (which is a generalization of normal bundle to the singular case) of A along X_α is flat over the base X_α within an open dense subset of every fibre. This implies, at any rate, that the multiplicity of A is constant along each stratum X_α. This is only a microscopic step forward toward the goal of determining the total scope of differential-geometric descriptions of singularities (notably those of Thom and Whitney) in relation to various algebro-geometrical characterizations of " equisingularity " proposed by Zariski, especially the one given in terms of simultaneous desingularization of all transversal sections.

Zariski took experimentarily the case of a stratum of codimension one in a hypersurface in an affine space, proposed various algebro-geometrical definitions for equisingularity and established the equivalence among them and also between his notion and Whitney condition combined with an additional assumption of equimultiplicity. This paper shows that the equimultiplicity is a consequence of Whitney condition even in the most general case. Incidentally, Zariski's theory is quite satisfactory in the special case and it presents us an attractive model for future exploration into the unknown territory of general equisingularity problems.

From the technical point of view, this paper consists of the following simple observation about bounding the difference between varieties and their tangential cones at a point. For simplicity, take a hypersurface $H : f(x_1, \ldots, x_n) = 0$ in $\mathbf{C}^n$. Let f_0 be the initial form of f, i.e., the sum of those terms of the lowest degree in Taylor expansion of f at the origin. The hypersurface $H_0 : f_0 = 0$ is called the tangential cone of H at the origin 0. Let us assume $0 \in H$. There is a well-known criterion of Euler: $H = H_0$

[1] This research was partially supported by NSF GP-6342.

if and only if the inner product $(x, \text{grad}_x f) = \sum_{i=1}^{n} x_i \partial f / \partial x_i = 0$ on H. It is also known, due to Whitney, that in general

$$\lim_{H \ni x \to 0} (x, \text{grad}_x f) / |x| . |\text{grad}_x f| = 0$$

In fact, one observes a stronger property:

$$|(x, \text{grad}_x f)| \leq \delta |x|^{1+e} . |\text{grad}_x f| \qquad \text{for } x \in H,$$

say within the unit ball B in $\mathbf{C}^n$, where e and δ are positive numbers depending upon H. Now our key lemma says: Given $e > 0$ and $\varepsilon > 0$, we can choose $\delta > 0$ such that for every H satisfying the above inequality with these e and δ, $H \cap B$ lies within the ε-neighborhood of the tangential cone H_0.

1. Whitney stratifications.

Let A be a topological space. Two local imbeddings of A around a point $y_0 \in A$, say $f_i : U_i \to \mathbf{R}^{n_i}$ with an open neighborhood U_i of y_0 in A, are said to be *diffeo-equivalent* if there exists an open neighborhood U of y_0, a pair of non-negative integers (r_1, r_2) and a diffeomorphism h from a neighborhood of $f_1(y_0) \times 0_1$ in $\mathbf{R}^{n_1} \times \mathbf{R}^{r_1}$ to a neighborhood of $f_2(y_0) \times 0_2$ in $\mathbf{R}^{n_2} \times \mathbf{R}^{r_2}$ such that $U \subset U_1 \cap U_2$ and $hg_1(f_1 | U) = g_2(f_2 | U)$ where 0 is the origin of $\mathbf{R}^{r_i}$ and $g_i : \mathbf{R}^{n_i} \to \mathbf{R}^{n_i} \times \mathbf{R}^{r_i}$ is the inclusion by $x \mapsto x \times 0_i$ for $i = 1, 2$. A *differentiably stratified space* is by definition a topological space A furnished with the following two data:

(i) A *diffeo-imbedding type*, i.e., a diffeo-equivalence class $e(x)$ of local imbeddings at each point $x \in A$, such that every point $x_0 \in A$ admits an open neighborhood U and an imbedding $U \to \mathbf{R}^n$ which belongs to $e(x)$ for all $x \in U$. An imbedding of this nature will be called a *permissible imbedding*.

(ii) A *differentiable stratification*, i.e., an expression $A = \bigcup_\alpha X_\alpha$, a locally finite disjoint union, such that each X_α is a " differentiable submanifold " (connected by definition) of A with respect to local permissible imbeddings and that $\bar{X}_\alpha \cap X_\beta \neq \emptyset$ implies $\bar{X}_\alpha \supset X_\beta$. Each X_α is called a *stratum* of (the stratification of) A, and $X_\alpha \succ X_\beta$ means $\bar{X}_\alpha \supset X_\beta$.

If $A = \bigcup X$ is a differentiably stratified space and Y is a stratum, then for every point $y_0 \in Y$, we can find an open neighborhood U of y_0 in A and a permissible imbedding $U \to E = E_1 \times E_2$ with $E_i = \mathbf{R}^{r_i}$ such that y_0 is mapped to the origin $0 = 0_1 \times 0_2$ and $Y \cap U$ is mapped onto a neighborhood of 0 in $E_1 \times 0_2$. Such an imbedding will be called a (local) Y-*cartesian imbedding* of A around y_0. The tangent space $T_{E,x}$ at every point x is naturally identified with E (as a vector space), and hence, when a Y-cartesian imbedding of A is given as above, $T_{X,x}$ is identified with a vector subspace of E for every $x \in X \cap U$. Furthermore, we can speak of the inner product (z, z') for vectors z and z' in E in the standard euclidian sense. In this sense, the *normal vector space* $N_{E,X,x}$ of X in E at x will be identified with the orthogonal complement of $T_{X,x}$ in E. Let $\pi : E \to E_1 \times 0_2$

denote the projection map $x_1 \times x_2 \mapsto x_1 \times 0_2$. Let us define the following functions on $X \cap \pi^{-1}(Y \cap U)$ for each stratum $X \succ Y$:

$$\alpha_{X,Y}(x) = \max_{u,v}\{|(u, v)| / |u| \cdot |v|\}$$
$$\beta_{X,Y}(x) = \max_u\{|(u, \pi(x)x)| / |u| \cdot |\pi(x)x|\}$$

where u (resp. v) runs through the non-zero vectors in $N_{E,X,x}$ (resp. $T_{Y,\pi(x)}$) and $\pi(x)x$ denotes the vector in E corresponding to the arrow from $\pi(x)$ to x. We call $\alpha_{X,Y}$ (resp. $\beta_{X,Y}$) *Whitney* α- (resp. β-)*function* for the pair (X, Y) (with respect to the Y-cartesian imbedding of A).

Definition (**1.1**). — *A differentiably stratified space* A *is said to satisfy* Whitney conditions (*and its stratification is called* Whitney stratification), *if for every stratum* Y *and every point* $y_0 \in Y$, *there exists a local* Y-*cartesian imbedding of* A *around* y_0 *such that*

$$\lim_{x \to y} \alpha_{X,Y}(x) = \lim_{x \to y} \beta_{X,Y}(x) = 0$$

for every stratum $X \succ Y$ *and every* $y \in Y$ *(whenever the expressions make sense).*

Remark (**1.2**). — The functions $\alpha_{X,Y}$ and $\beta_{X,Y}$ clearly depend upon the choice of a Y-imbedding of A. One can prove, however, that the limit condition of (1.1) for either $\alpha_{X,Y}$ alone or for the combination of the two $\alpha_{X,Y}$ and $\beta_{X,Y}$, is actually independent of the choice with respect to the given $(Y, X, y = y_0)$.

Let A be a *complex*- (resp. *real*-)*analytic space*, not neccessarily reduced. Then the underlying topological space $|A|$ of A has a canonical diffeo-imbedding type, i.e., the one induced by any local complex- (resp. real-)analytic imbedding in terms of the structure sheaf $\mathcal{O}_A$. We shall thus speak of differentiable stratifications of A and obtain various differentiably stratified spaces from A. A differentiable stratification $A = \bigcup X$ is said to be *complex*- (resp. *real*-)*analytic* if every stratum X is a difference of two closed complex- (resp. real-)analytic subspaces of A such that the second subspace contains the singular locus of the first. With a complex- (resp. real-)analytic stratification $A = \bigcup X$ and a point y_0 of a stratum Y, we can speak of a *complex*- (resp. *real*-)*analytic* Y-*cartesian imbedding* of A around y_0 into $E = E_1 \times E_2$ with $E_i = \mathbf{C}^{r_i}$ (resp. $= \mathbf{R}^{r_i}$), in which case the map $U \to E$ is required to be complex- (resp. real-)analytic.

Remark (**1.3**). — Let $A = \bigcup X$ be a complex-analytic stratification of a complex-analytic space A. Let us pick a stratum $Y, y_0 \in Y$ and a complex-analytic Y-cartesian imbedding of A around y_0 into $\mathbf{C}^r = \mathbf{C}^{r_1} \times \mathbf{C}^{r_2}$. Then, by means of the standard hermitian form $((z, z'))$ in the complex vector space $\mathbf{C}^r$, we can define

$$\alpha_{X,Y}((x)) = \max_{u,v}\{|((u, v))| / |u| \cdot |v|\}$$
$$\beta_{X,Y}((x)) = \max_u\{|((u, \pi(x)x))| / |u| \cdot |\pi(x)x|\}$$

in the same way as before, but using complex normal (resp. complex tangent) vectors u (resp. v) to the complex manifold X (resp. Y). On the other hand, by means of the standard identification of $\mathbf{C}^r$ with $\mathbf{R}^{2r}$, i.e., $(z_1, \ldots, z_r) \leftrightarrow (x_1, y_1, \ldots, x_r, y_r)$ where $z_i = x_i + \sqrt{-1}\, y_i$, we obtain the derived real-analytic (hence, differentiable) stratification of the derived real-analytic space, $A_{\text{real}} = \bigcup X_{\text{real}}$, and the derived real-analytic Y_{real}-cartesian imbedding of A_{real} into $\mathbf{R}^{2r} = \mathbf{R}^{2r_1} \times \mathbf{R}^{2r_2}$. In this sense, we also have the previously defined Whitney functions $\alpha_{X,Y}$ and $\beta_{X,Y}$. One can then prove $\alpha_{X,Y}((x)) = \alpha_{X,Y}(x)$ and $\beta_{X,Y}((x)) = \beta_{X,Y}(x)$ for all $X \succ Y$ and all $x \in X$.

2. Normal cone and blowing-up.

Let $A = \bigcup X$ be a complex- (resp. real-)analytic stratification of a complex- (resp. real-)analytic space A. For the ideal sheaf $\mathscr{I}$ of a stratum X in A, the direct sum $\bigoplus_{m=0}^{\infty} (\mathscr{I}^m/\mathscr{I}^{m+1})$ can be viewed in a natural way as a graded $\mathcal{O}_X$-algebra, which we denote by $\mathrm{gr}_X(A)$. The homogeneous part of degree m of $\mathrm{gr}_X(A)$, denoted by $\mathrm{gr}_X^m(A)$, is the restriction of the sheaf $\mathscr{I}^m/\mathscr{I}^{m+1}$ to X. One can then prove that $\mathrm{gr}_X(A)$ is of finite presentation as $\mathcal{O}_X$-algebra and hence defines a complex (resp. real) fibre space $C_{A,X} = \mathrm{Specan}(\mathrm{gr}_X(A))$ over X (cf. [1]).

Definition **(2.1)**. — *This* $C_{A,X}$ *together with the projection* $p : C_{A,X} \to X$ *is called the* (intrinsic) normal cone *of* X *in* A.

Due to the graded structure of $\mathrm{gr}_X(A)$, we have the homothety in the fibres $C_{A,X,x}$ of $p : C_{A,X} \to X$, in terms of which we can speak of *lines* in the fibres. The notion of lines can be made somewhat more explicit as follows. Restricting our attention to local phenomena, let us assume that A is imbedded in a complex (resp. real) manifold E. Then we have a canonical imbedding of $C_{A,X}$ into the normal bundle $N_{E,X}(= C_{E,X})$ of X in E. This imbedding induces an imbedding of fibres $C_{A,X,x} \to N_{E,X,x}$ over each point x of X in such a way that the image is defined by homogeneous polynomial equations in the vector space $N_{E,X,x}$. Hence $C_{A,X,x}$ is point-set-theoretically a union of complex (resp. real) lines in the vector space $N_{E,X,x}$.

In the real-analytic case, we consider various real-analytic maps $\varphi : D \to A$, where $D = (-1, 1)$, such that $\varphi(0) \in X$ and $\varphi(t) \in A - X$ for all $t \neq 0$, where X is a stratum of A. Given such φ, the ideal $\mathscr{I}$ of X in A induces an ideal in $\mathcal{O}_D$ which has a certain order $e = e_\varphi$ at $t = 0$, i.e., $\varphi^{-1}(\mathscr{I})\mathcal{O}_{D,0} = (t^e)\mathcal{O}_{D,0}$ and hence $\varphi^{-1}(\mathscr{I}^m)\mathcal{O}_{D,0} = (t^{me})\mathcal{O}_{D,0}$ for all $m \geq 0$. Thus φ induces an epimorphism of graded algebras $h = h_\varphi : \mathrm{gr}_X(A)_{\varphi(0)} \to \mathbf{R}[u]$, the polynomial ring, such that if $w \in I^m$ represents $\omega \in \mathrm{gr}_X^m(A)$ then $h(\omega) = (w^*/t^{me})(0)u^m$ where w^* denotes the function on D induced by w in terms of φ. Let H_φ denote the real half-line in $C_{A,X,\varphi(0)}$ which corresponds to $u \geq 0$ by means of h_φ. Let $-H_\varphi$ denote the complementary half-line of H_φ, i.e., the one corresponding to $u \leq 0$. If $\psi(t) = \varphi(-t)$, then H_ψ is equal to either H_φ or $-H_\varphi$ depending upon whether e_φ is even or odd. We will denote by $C_{A,X}^0$ the union of all those half-lines H_φ in $C_{A,X}$ (and the vertices).

In the complex-analytic case, we take the induced real-analytic stratification $A' = \bigcup X'$ corresponding to the given complex-analytic stratification $A = \bigcup X$. Then $C_{A',X'}$ in the real-analytic sense is canonically isomorphic to the induced real-analytic space of $C_{A,X}$ in the complex-analytic sense. By means of this isomorphism, we shall view $C^0_{A',X'}$ as a subset of $C_{A,X}$ and denote it by $C^0_{A,X}$.

Definition (**2.2**). — *The topological subspace* $C^0_{A,X}$ *of* $C_{A,X}$ *is called the* extrinsic normal cone *of* X *in* A.

Let $g: A^* \to A$ be the blowing-up with center X, i.e., the composition of the blowing-up $A^* \to A$ — (boundary of X) with the closed center X and the inclusion into A. (The blowing-up in the real-analytic case is obtained by taking the " real part " of the complex blowing-up $\tilde{g}: \tilde{A}^* \to \tilde{A}$ with center $\tilde{X}$, where $\tilde{A}$ (resp. $\tilde{X}$) is a complexification of A (resp. X). The term " real part " makes sense with respect to local imbeddings of $\tilde{A}^*$ into products $\mathbf{C}^n \times \mathbf{P}^m_{\mathbf{C}}$ associated with local imbeddings of A into $\mathbf{R}^n$ and local ideal bases $(h_0, h_1, \ldots, h_m)$ of X in A.) Now the proper fibre space $g^{-1}(X) \to X$, induced by g, is related to the intrinsic normal cone $C_{A,X}$ as follows: $C_{A,X} = \mathrm{Specan}(\mathrm{gr}_X(A))$ and $g^{-1}(X) = \mathrm{Projan}(\mathrm{gr}_X(A))$. This shows for instance a one-to-one correspondence between the set of points of $g^{-1}(X)$ and the set of complex (resp. real) lines (through vertices) in $C_{A,X}$. In the complex-analytic case, $g^{-1}(A - X)$ is dense in A^*. In the real-analytic case, however, the closure of $g^{-1}(A - X)$ need not be the entire set A^* (nor even real-analytic in general).

Remark (**2.3**). — In the complex-analytic case, $C^0_{A,X}$ is the underlying topological space of $C_{A,X}$. In fact, let ω be any point of $C_{A,X}$ — (vertices). Then ω corresponds to a point w of $g^{-1}(X)$. Since $g^{-1}(X)$ is in the closure of $g^{-1}(A - X)$, we can find a holomorphic map $f: B \to A^*$, where $B = \{z \in \mathbf{C} : |z| < 1\}$, such that $f(0) = w$ and $f(B - \{0\}) \subset g^{-1}(A - X)$. Let $\lambda: (-1, 1) \to B$ be the inclusion of the real part. Then for each $\alpha \in \mathbf{C}^* = \mathbf{C} - \{0\}$, $|\alpha| \leq 1$, we obtain a real-analytic map $\varphi_\alpha: (-1, 1) \to A$ by $\varphi_\alpha(t) = g(f(\alpha\lambda(t)))$. The order $e = e_{\varphi_\alpha}$ is equal to the order of the complex ideal $(gf)^{-1}(\mathscr{I})\mathcal{O}_{B,0}$ which is independent of α. Now, if ω' is any point of the half-line H_{φ_1}, other than the vertex, then ω' corresponds to the same point w as ω. Hence we have $\alpha \in \mathbf{C}^*$ such that $\omega = \alpha^e \omega'$ with respect to the canonical action of $\mathbf{C}^*$ on the cone $C_{A,X}$. Then ω is a point of H_{φ_β} with $\beta = \alpha / |\alpha|$.

Definition (**2.4**). — *We say that* A *is* normally pseudo-flat *along a stratum* X, *if the projection* $p_0: C^0_{A,X} \to X$ *is universally open*, i.e., *for every real-analytic map* $K \to X$, *the base extension* $p_0 \times_X K : C^0_{A,X} \times_X K \to K$ *is an open map of topological spaces.*

Remark (**2.5**). — Let us consider the complex-analytic case and assume that A is equidimensional. Then the following conditions are equivalent to one another:

(i) A is normally pseudo-flat along X.

(ii) $p: C_{A,X} \to X$ is universally open with respect to complex-analytic base extensions.

(iii) For every holomorphic map $B \to X$, where $B = \{z \in \mathbf{C} : |z| < 1\}$, the induced map $(p \times_X B)_{\mathrm{red}} : (C_{A,X} \times_X B)_{\mathrm{red}} \to B$ is flat, where red indicates the reduced structure.

(iii)* For the blowing-up $g: A^* \to A$ with center X and for every $B \to X$ as in (iii), the induced morphism $(g \times_X B)_{\mathrm{red}}$ is flat.

(iv) $\dim C_{A,X,x}$ is independent of $x \in X$.
(iv)* $\dim g^{-1}(x)$ is independent of $x \in X$.

Moreover, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iii)* $\Rightarrow$ (iv) $\Leftrightarrow$ (iv)* do not require the equidimensionality assumption on A. In fact, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. In view of the spec-proj relation between $C_{A,X}$ and $g^{-1}(X)$, we see (iii) $\Leftrightarrow$ (iii)* and (iv) $\Leftrightarrow$ (iv)*. (iii)* $\Rightarrow$ (iv)* because g is proper and X is connected. Let us prove (iv) $\Rightarrow$ (i). First of all, the ideal sheaf of $g^{-1}(X)$ in A^* is invertible as $\mathcal{O}_{A^*}$-module. Hence the equidimensionality assumption on A implies the same of $g^{-1}(X)$. Since $C_{A,X}$—(vertices) is a $\mathbf{C}^*$-bundle over $g^{-1}(X)$, $C_{A,X}$ is also equidimensional. Thanks to this fact, (iv)* implies that given any point $w \in C_{A,X}$, we can find an irreducible complex subvariety W of $C_{A,X}$ through the point w, such that $\dim W = \dim X$ and $p^{-1}(x) \cap W$ is finite (possibly empty) for all $x \in X$. It is easy to see that p induces a universally open map $W \to X$ even with respect to all real-analytic base extensions. The point w being arbitrary, p itself must be universally open in this sense. (i) follows, because $C^0_{A,X}$ is the underlying topological space of $C_{A,X}$.

3. Equimultiplicity along a stratum.

Let $A = \bigcup X$ be a complex- (resp. real-)analytic stratification. Let us pick a stratum X and an X-cartesian imbedding of A around a point x_0 of X into $E = E_1 \times E_2$. As we are interested in local phenomena, we localize the situation around the given point x_0 and assume that the entire A is imbedded in E and $X = (E_1 \times o_2) \cap A$. For each $d > 0$, let $S_d = \{(u, v) \in E : |v| = d\}$ and $\lambda : E - (E_1 \times o_2) \to S_1$ the map defined by $(u, v) \mapsto (u, v/|v|)$. $C_{A,X}$ is imbedded in the normal bundle $N_{E,X}$, and $N_{E,X}$ is naturally identified with $\pi^{-1}(X) = X_1 \times E_2$, where $X = X_1 \times o_2$.

Remark (3.1). — $C^0_{A,X} \cap S_1 = \pi^{-1}(X) \cap \lim_{d \to 0} \lambda(A \cap S_d)$. In fact, take the real case and let $\varphi : D \to A$ be as before. The ideal $\mathscr{I}$ of X in A is generated by the coordinate functions $(v_1, v_2, \ldots, v_{r_2})$ of E_2. With $e = e_\varphi$, we let $v_i\varphi - a_i t^e \in (t^{e+1})\mathcal{O}_{D,0}$ where $a_i \in \mathbf{R}$. Then $H_\varphi \cap S_1 = (\varphi(o)_1, a/|a|)$ with $\varphi(o) = (\varphi(o)_1, \varphi(o)_2)$ and $a = (a_1, \ldots, a_{r_2})$. It is clearly contained in $\lim_{d \to 0} \lambda(\varphi((o, 1)) \cap S_d)$ and hence in $\lim_{d \to 0} \lambda(A \cap S_d)$. Conversely, let $G = \lim_{d \to 0} \lambda(A \cap S_d) \cap \pi^{-1}(X)$ and $G^* = g^{-1}(X) \cap \overline{g^{-1}(A - X)}$, where $g : A^* \to A$ is the blowing-up with center X. We have a natural surjection $s : G \to G^*$. Let $L_\omega = \{(u, v) \in E : (v, \omega_2) > o\}$ for each $\omega = (\omega_1, \omega_2) \in G$. We then have $s(\omega) \in \overline{g^{-1}(A \cap L_\omega)}$. Since A^* is analytic and $g^{-1}(A \cap L_\omega)$ is open and semi-analytic in A^*, there exists an analytic map $\psi : D \to A^*$ such that $\psi(o) = s(\omega)$ and $\psi(t) \in g^{-1}(A \cap L_\omega)$ for all $t > o$. Then for $\varphi = g\psi$, we get $\omega \in H_\varphi \cap S_1 \subset C^0_{A,X} \cap S_1$.

Remark (3.2). — In the complex-analytic case, if A is normally pseudo-flat, then A_{red} is *equimultiple* along X, i.e., the multiplicity $\nu_x(A_{red})$ of the reduced complex space A_{red} at the point x is independent of $x \in X$.

In fact, the normal pseudo-flatness of A along X implies that $\dim C_{A,X,x}$ ($= c$, say) is independent of $x \in X$. This condition is unaffected if A is replaced by the union of those irreducible components of A_{red}, whose dimensions are $c + \dim X$ and which contain X. So is the equimultiplicity of A along X. Thus we assume:

(3.2.1) A *is reduced, equidimensional, and* X *is contained in every irreducible component of* A.

Let P_2 be the projective space of dimension $r_2 - 1$ associated with the vector space E_2. Then P_2 carries a complex line bundle $F_2 \to P_2$ whose fibres correspond in a one-to-one fashion to the complex lines through the origin o_2 in E_2. We shall identify P_2 with the zero section. Let $\sigma_2 : F_2 \to E_2$ be the natural map, and $\sigma = \mathrm{id}_{E_1} \times \sigma_2 : F \to E$ where $F = E_1 \times F_2$. This σ is the blowing-up with center $E_1 \times o_2$. Hence there is a canonical imbedding of A^* into F such that σ induces $g : A^* \to A$. Since $\dim g^{-1}(x) = \dim C_{A,X,x} - 1 = c - 1$ for all $x \in X$, by localizing the situation around any given point x_0 of X, we may assume that there exists a nonempty open set U in the grassmannian $\mathbf{Grass}(r_2 - c, r_2)$ such that if L_β is any vector subspace of dimension $r_2 - c$ of E_2 which corresponds to a point $\beta \in U$, then we have $A^* \cap (E_1 \times L_\beta^*) = \emptyset$ where L_β^* denotes the linear subspace of P_2 which corresponds to L_β. Let $v_1 = v_2 = \ldots = v_c = 0$ be the linear equations for L_β in E_2, and let $\tau_2 : E_2 \to E_2'$ be a homomorphism to a vector space E_2' of dimension c defined by $(v_1, \ldots, v_c)$. Then $\tau = \mathrm{id}_{E_1} \times \tau_2 : E \to E' = E_1 \times E_2'$ induces a morphism $h : A \to E'$. We then have:

(3.2.2) $X = h^{-1}(E_1 \times o_2')$ *and h is a finite morphism in a neighborhood of* X.

Let $E_x' = x_1 \times E_2'$, $A_x = A \cap (x_1 \times E_2) = h^{-1}(E_x')$ and $h_x : A_x \to E_x'$ the morphism induced by h, where $x \in X$ and $h(x) = x_1 \times o_2'$. Note that A_x does not depend upon the choice of $\beta \in U$, while h_x does. By choosing $\beta \in U$, we may assume: $\nu_x(A_x) = \deg h_x$ for all x near x_0, where (and below) $\deg h_x$ denotes the degree of the map h_x in a sufficiently small neighborhood of x in A_x. Now, for almost all $x \in X$, A is normally flat along X at x so that $\mathrm{gr}_x(A)$ is isomorphic to $\mathrm{gr}_x(A_x) \otimes_{\mathbf{C}} \mathrm{gr}_x(X)$. This isomorphism implies $\nu_x(A) = \nu_x(A_x)$. On the other hand, (3.2.1) and (3.2.2) imply that $\deg h_x$ is independent of $x \in X$. Thus $\nu_x(A) = \nu_x(A_x) \geqslant \deg h_x = \deg h_{x_0} = \nu_{x_0}(A_{x_0}) \geqslant \nu_{x_0}(A)$ for almost all $x \in X$ near x_0. Hence we get $\nu_x(A) = \nu_{x_0}(A)$ for such x by the upper-semicontinuity of multiplicity. Since x_0 is arbitrary, A must be equimultiple along X.

4. A key lemma.

Let us fix $E = \mathbf{R}^r$ and let V be the closed unit ball about the origin o. We are interested in various closed subsets H of V furnished with stratification $H = \bigcup Y$ such that each stratum Y is of the form $V \cap \tilde{Y}$ with a differentiable submanifold $\tilde{Y}$ of E. We define $H(d) = \lambda(H \cap S_d)$ for $d > 0$ and $H(0) = \lim_{d \to 0} H(d)$, where $S_d = \{v \in E : |v| = d\}$ and $\lambda : E - \{0\} \to S_1$ is defined by $v \mapsto v/|v|$. As was seen before, if H is real-analytic then $H(0) = C^0_{H,0} \cap S_1$ with the extrinsic tangential cone $C^0_{H,0}$ of H. We define the following function on $H - \{0\}$:

$$\beta_H(y) = \max_u \{|(u, oy)| / |u| . |oy|\}$$

where u runs through the non-zero vectors in the orthogonal complement $N_{E,Y,y}$ of $T_{Y,y}$ with the stratum Y containing y. (Here and below, $T_{Y,y}$ means $T_{\tilde{Y},y}$ when $y \in Y \cap S_1$.) We shall measure the difference between two closed subsets A and B of S_1 by

$$\mathrm{dif}(A, B) = \max_{x \in A \cup B} (\min_{b \in B} |bx| + \min_{a \in A} |ax|).$$

Lemma **(4.1)**. — *Given* $e > 0$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that the following statement is true*: *For any* $H = \bigcup Y$ *as above, if* $\beta_H(y) < \delta |oy|^e$ *for all* $y \in H - \{o\}$, *then* $\mathrm{dif}(H(o), H(d)) < \varepsilon d^e$ *for all* d *with* $0 \leq d \leq 1$.

Proof. — To begin with, we assume $\delta < 1$ so that $\beta_H(y) < 1$ for all y. We define a " vector field " ω on $H - \{o\}$ as follows: For each $y \in H - \{o\}$, $\omega(y) = |oy| v/(v, oy)$ with the orthogonal projection v of oy to the tangent space $T_{Y,y}$, where Y is the stratum of H containing y. By a *complete integral curve* of ω, we shall mean a continuous map $\varphi : [0, 1] \to H$, such that $\varphi(1) \in H(1)$, $\varphi(0) = 0$ and, except for denumerable values of t, $d\varphi(u)/du = \omega(\varphi(u))$ for all u sufficiently near t. We shall first prove:

(4.1.1) *For every point* $z \in H - \{o\}$, *there exists at least one complete integral curve of* ω *through* z.

Consider various continuous maps $\psi : (a, b) \to H$, such that $a < |oz| < b$, $\psi(|oz|) = z$ and, except for denumerable values of t, $d\psi(u)/du = \omega(\psi(u))$ for all u sufficiently near t. First of all, $\psi(t) = y$ implies $t = |oy|$ for every t. In fact, this is true for $y = z$ and $d|\psi(t)|/dt = (\omega(\psi(t)), o\psi(t))/|o\psi(t)| = 1$ for almost all t, so that it remains true for all t. Now consider the ordering among those ψ, by defining $\psi_1 < \psi_2$ to mean that ψ_2 is an extension of ψ_1. A maximal element exists and we call it $\bar{\psi} : (p, q) \to H$. For $p < t_1 < t_2 < q$,

$$|\bar{\psi}(t_2) - \bar{\psi}(t_1)| \leq \int_{t_1}^{t_2} |\omega(\bar{\psi}(t))| \, dt \leq \int_{t_1}^{t_2} dt/\sqrt{1 - \beta_H(\bar{\psi}(t))^2} \leq (t_2 - t_1)/\sqrt{1 - \delta^2}.$$

Hence, δ being < 1, we see that $\lim_{t \to p} \bar{\psi}(t)$ and $\lim_{t \to q} \bar{\psi}(t)$ exist. Call these $\bar{\psi}(p)$ and $\bar{\psi}(q)$. In this extended sense, $\bar{\psi}$ is a continuous map: $[p, q] \to H$. If $p > 0$, then $y = \bar{\psi}(p)$ is a point of a stratum Y of H, and Y carries a differentiable vector field $\omega | Y$ within a neighborhood of y. This then enables us to extend $\bar{\psi}$ further. Thus the maximality of $\bar{\psi}$ implies $p = 0$. Similarly we get $q = 1$. The extended $\bar{\psi}$ is hence a complete integral curve of ω. This proves (4.1.1). Next let us pick any positive δ such that $2\delta < 1$ and $1 + e^{-1}\sqrt{r}(\sqrt{2})^{2+e}\delta/(1 - 2\delta) < \sqrt{2}$. We shall then prove:

(4.1.2) *For every complete integral curve* φ *of* ω, *we have*

$$|\varphi(t)/|\varphi(t)| - \varphi(s)/|\varphi(s)|| \leq \{e^{-1}\sqrt{r}(\sqrt{2})^{4+e}\delta/(1 - 2\delta)\} |\varphi(t)|^e$$

for all (s, t) *with* $0 < s < t \leq 1$.

Since any orthogonal transformation of $\mathbf{R}^r$ does affect neither the assumptions nor the conclusion, we may assume that $\varphi(s) = (\varphi_1(s), \ldots, \varphi_r(s)) = (s, 0, \ldots, 0)$. We have

$$(\sum_{i,j} |\varphi_i(u)\varphi_j'(u) - \varphi_j(u)\varphi_i'(u)|^2)^{1/2} = |\varphi(u)| \cdot |\varphi'(u)| \cdot \beta_H(\varphi(u)) \leq \delta |\varphi'(u)| \cdot |\varphi(u)|^{e+1}$$

whenever $\varphi'(u) = d\varphi(u)/du = \omega(\varphi(u))$. This implies

$$(*) \qquad |\varphi_i(u)\varphi_j'(u) - \varphi_j(u)\varphi_i'(u)| \leq \delta\,|\varphi'(u)|\,.\,|\varphi(u)|^{e+1}$$

for every $i \neq j$.

Let us pick any t', $s < t' \leq t$, such that:

$$(**) \qquad \varphi_1(u) \geq |\varphi(u)|/\sqrt{2} \qquad \text{for all } u \in [s, t'].$$

This implies:

$$(***) \qquad \varphi_1'(u) \geq \{(1-2\delta)/\sqrt{2}\}\,|\varphi'(u)| \qquad \text{for all } u \in [s, t'].$$

In fact:

$$\big|\varphi_1(u)/|\varphi(u)| - \varphi_1'(u)/|\varphi'(u)|\big| \leq \big|\varphi(u)/|\varphi(u)| - \varphi'(u)/|\varphi'(u)|\big|$$
$$= \sqrt{2}\,(1-(1-\beta_H(\varphi(u))^2)^{1/2})^{1/2} \leq \sqrt{2}\,\beta_H(\varphi(u)) \leq \sqrt{2}\,\delta.$$

By (*), (**) and (***), we get

$$|\varphi_i'(u)/\varphi_1'(u) - \varphi_i(u)/\varphi_1(u)| \leq \{(\sqrt{2})^{2+e}\delta/(1-2\delta)\}\,.\,\varphi_1(u)^e$$

for every $i > 1$. Let $z_i(u) = \varphi_i(u)/\varphi_1(u)$, and call $k(\delta)$ the number in the last { } divided by e. Then we get $|z_i'(u)/\varphi_1'(u)| \leq k(\delta)e\varphi_1(u)^{e-1}$ so that

$$|z_i(t') - z_i(s)| \leq k(\delta)(\varphi_1(t')^e - \varphi_1(s)^e) \leq k(\delta)\varphi_1(t')^e.$$

Hence $|z(t') - z(s)| \leq \sqrt{r}\,k(\delta)\varphi_1(t')^e$, where $z(u) = (z_1(u), \ldots, z_r(u))$. Since $z(t')/|z(t')| = \varphi(t')/|\varphi(t')|$ and $z(s) = \varphi(s)/|\varphi(s)|$,

$$\big|\varphi(t')/|\varphi(t')| - \varphi(s)/|\varphi(s)|\big| \leq \big|z(t')/|z(t')| - z(t')\big| + |z(t') - z(s)|$$
$$\leq \big|1 - |z(t')|\big| + |z(t') - z(s)| \leq 2\,|z(t') - z(s)| \leq 2\sqrt{r}\,k(\delta)\,|\varphi(t')|^e.$$

Thus (4.1.2) follows if we check that t' can be t. We see this as follows. Since $|z(s)| = 1$, $|z(t')| \leq 1 + \sqrt{r}\,k(\delta)$, i.e., $\varphi_1(t')/|\varphi(t')| \geq (1+\sqrt{r}\,k(\delta))^{-1} > (\sqrt{2})^{-1}$ by the assumption on δ. This being a strict inequality, the continuity of φ (and φ_1) implies that (**) holds for all $u \in [s, t]$. Now, (4.1) follows immediately from (4.1.1) and (4.1.2) if we take δ so small that the number in {} of (4.1.2) is less than ε (in addition to the preceeding inequalities). (Note that every complete integral curve φ of ω has the property: $\varphi(t) = y$ implies $t = |oy|$.)

5. Strict Whitney conditions.

If L and K are two subsets of a euclidian space E, we denote by $\mathrm{dist}_E(L, K)$ the distance between the two sets L and K in E, i.e., the greatest lower bound of $|xy|$ with $x \in L$ and $y \in K$.

Definition (5.1). — *A differentiably stratified space* $A = \bigcup X$ *is said to satisfy the* strict Whitney conditions, *if for every points* y_0 *of every stratum* Y *of* A, *there exists a local* Y*-cartesian imbedding of* A *around* y_0 *into* E *and a positive real number* e *such that*

$$\lim_{x \to y} \alpha_{X,Y}(x)/\mathrm{dist}_E(x, Y)^e = \lim_{x \to y} \beta_{X,Y}(x)/\mathrm{dist}_E(x, Y)^e = 0$$

for every stratum $X \succ Y$ *and every point* $y \in Y$ *(whenever the expressions make sense).*

Lemma **(5.2).** — *For a complex- (or real-)analytic stratification of a complex- (or real-) analytic space, Whitney conditions imply strict Whitney conditions.*

Proof. — It is enough to take the real-analytic case. Since the question is local, let us assume that we have a real-analytic Y-cartesian imbedding of the entire A into $E = E_1 \times E_2$ with $E_i = \mathbf{R}^{r_i}$. Let us pick any X to check the conditions of (5.1). We may then assume that X is open in A, A is smooth in X and Y is closed in A. We consider the case of $\beta_{X,Y}(x)$. (The proof for $\alpha_{X,Y}(x)$ is quite similar.) Let $\pi_i : E \to E_i$ denote the projection for $i = 1, 2$. We have the canonical $\mathbf{R}^*$-bundle $\lambda : E_2 - \{0_2\} \to P_2$ with the real projective space P_2 of dimension $r_2 - 1$. Let $n = \dim X$ and, for each $x \in X$, $\tau(x)$ denotes the point of $\mathbf{Grass}(n, r)$ which corresponds to the tangent space of X at x (identified as a vector subspace of E). We then get a map $g = \lambda\pi_2 \times \tau : X \to P_2 \times \mathbf{Grass}(n, r)$. We can show that there exists a closed real-analytic subspace W of $A \times P_2 \times \mathbf{Grass}(n, r)$ which induces the graph of g in $X \times P_2 \times \mathbf{Grass}(n, r)$. The projection induces a proper morphism $h : W \to A$, which induces an isomorphism $X' \to X$ where $X' = h^{-1}(X)$. Let $Y' = h^{-1}(Y)$. We have a real-analytic function β on $P_2 \times \mathbf{Grass}(n, r)$ such that $\beta_{X,Y} = \beta g$. Namely, $\beta(L, T) = \max_{u,v}\{|(u, v)|/|u| . |v|\}$, where L (resp. T) is a vector subspace of dimension 1 (resp. n) and u (resp. v) runs through the vectors $\neq 0$ in L (resp. the orthogonal complement of T). Let β' be the function on W induced by β. Whitney condition on $\beta_{X,Y}$ is equivalent to saying that β' vanishes on $\overline{X'} \cap Y'$. This then implies the strong vanishing of β' of the type (5.1) due to the real-analyticity of the space and the strata. A quick way to see this is to apply the resolution of singularities to the pair (W, Y'), i.e., a proper real-analytic map $f : V \to W$, such that V is smooth, that f induces an isomorphism $X'' \to X'$ with $X'' = f^{-1}(X')$ and that $Y'' = f^{-1}(Y')$ has only normal crossings. Let Z be the connected component of V containing X'', and $Y^* = Y'' \cap Z$. Let $\mathscr{I}$ be the ideal sheaf of Y in A, which is generated by the coordinate functions of E_2. Now the function β^* on Z induced by β' vanishes on Y^*, for the complement of X'' in Z is real-analytic and hence X'' is dense in Z. Locally at every point of Y^*, we can find a coordinate system $(z_1, \ldots, z_n)$ of Z such that $\mathscr{I}\mathcal{O}_Z$ is generated by a monomial in the z_i. Thanks to this, it is easy to prove that the radical of $\mathscr{I}\mathcal{O}_Z$ contains the function β^* which vanishes on $Y^* = \mathrm{Supp}(\mathcal{O}_Z/\mathscr{I}\mathcal{O}_Z)$. Then there exists an integer $q > 1$ such that $\mathscr{I}\mathcal{O}_Z$ contains β^{*q-1}. It then follows that the limit condition of (5.1) is satisfied with the number $e = 1/q$.

6. Normal pseudo-flatness theorem.

In this section, $A = \bigcup X$ will denote either a complex-, or real-, analytic stratification of a complex-, or real-, analytic space A.

Theorem **(6.1).** — *If* $A = \bigcup X$ *satisfies Whitney conditions, then* A *is normally pseudo-flat along each stratum.*

Proof. — It is enough to consider the real-analytic case. Let us pick a stratum Y of A and a point y_0 of Y. The question being local, we may assume that there is given a global Y-cartesian imbedding of A into $E = E_1 \times E_2$ and that A is contained in $\pi^{-1}(Y)$ where $\pi : E \to E_1 \times 0_2$ denotes the projection. Whitney limit condition (1.1) on $\alpha_{X,Y}$ for each stratum $X \succ Y$ implies that π induces a smooth morphism $X \cap U \to Y$ with an open neighborhood U of Y in E. We may assume that $U = E$ for all $X \succ Y$, and furthermore that Y is a boundary of every other stratum X of A. By Lemma (5.2), we may assume that there exists $e > 0$ such that

$$\lim_{x \to y} \beta_{X,Y}(x) / |\pi(x)x|^e = 0$$

for all X and all $y \in Y$. Now let us pick $b > 0$ such that $G = \{x_1 \times 0_2 \in E : |x_1| \leq b\} \subset Y$. Then, by taking a suitable expansion (i.e., a scalar multiplication by a number > 1) in E_2, we may assume that if $V_d = \{x_1 \times x_2 \in \pi^{-1}(G) : |x_2| \leq d\}$, then A is closed and analytic in a neighborhood of V_1. Here note that for any given $\delta > 0$, we can choose the above expansion so large that $\beta_{X,Y}(x) \leq \delta |\pi(x)x|^e$ for all $X \succ Y$ and all $x \in X \cap V_1$. Let $H_y = \pi^{-1}(y) \cap A \cap V_1$ which inherits a stratification $\bigcup X_y$ with $X_y = H_y \cap X$ and an imbedding into the euclidian space $y \times E_2$, where $y \in G$. As π induces a smooth morphism $X \to Y$ for all x, we can apply Lemma (4.1), say with $\varepsilon = 1$, to each $H_y = \bigcup X_y$ in $y \times E_2$. We may thus assume

(!) $\qquad \mathrm{dif}(H_y(0), H_y(d)) \leq d^e$ for all $y \in G$ and all d, $0 \leq d \leq 1$.

Now, if K is any real-analytic space, such as the one in the condition (2.4), and if z is any point of K which is contained in the closure of $K - z$, then there exists a real-analytic map $g : D = (-1, 1) \to K$ such that $g(0) = z$ and $g(t) \in K - z$ for all $t \neq 0$. Thus, to verify the normal pseudo-flatness condition (2.4) of A along Y, it is enough to show that, for every real-analytic map $h : D \to Y$ with its image in G, the extension $q : Q \to D$ with $Q = D \times_Y C^0_{A,Y}$ is an open map. Since y_0 is arbitrary, this then amounts to proving that $q^{-1}(0)$ is contained in the closure of $q^{-1}(D - \{0\})$ for every such q as above. Let $S_d = \{x_1 \times x_2 \in E : |x_2| = d\}$, and $T_1 = \{t \times x_2 \in D \times E_2 : |x_2| = 1\}$. We have $\bigcup_{t \in D - \{0\}} t \times_Y H_{h(t)}(0) \subset \lim_{d \to 0} (\bigcup_{t \in D - \{0\}} t \times_Y H_{h(t)}(d)) \cap ((D - \{0\}) \times E_2)$ and, by (3.1), this is equal to $(D - \{0\}) \times_Y (C^0_{A,Y} \cap S_1) = q^{-1}(D - \{0\}) \cap T_1$. Therefore, (!) implies that for every d, $1 \geq d > 0$, the $2d^e$-neighborhood of $q^{-1}(D - \{0\}) \cap T_1$ (with respect to the euclidian distance in E) contains the closure of the set $\bigcup_{t \in D - \{0\}} t \times_Y H_{h(t)}(d)$. This closure contains $\bigcup_{t \in D} t \times_Y H_{h(t)}(d)$ for all sufficiently small $d > 0$, because $X \to Y$ is smooth surjective and X is transversal with S_d for all strata X by Whitney condition on β. It follows that the closure of $q^{-1}(D - \{0\}) \cap T_1$ contains $\lim_{d \to 0} \bigcup_{t \in D} t \times_Y H_{h(t)}(d)$, which is by (3.1) equal to $(D \times_Y C^0_{A,Y}) \cap T_1$. In particular, $q^{-1}(0) \cap T_1$ is contained in the closure of $q^{-1}(D - \{0\}) \cap T_1$. $C^0_{A,Y}$ being a family of cones, it follows that $q^{-1}(0)$ is in the closure of $q^{-1}(D - \{0\})$.

Corollary **(6.2)**. — *If* $A = \bigcup X$ *is a complex-analytic stratification of a reduced complex space* A *and if it satisfies Whitney conditions, then* A *is equimultiple along every stratum.*

Proof. — Immediate from (6.1) and (3.2).

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Manuscrit reçu le 9 janvier 1969.