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THE REPRESENTATION RING OF A COMPACT LIE GROUP

GRAEME SEGAL

INTRODUCTION

Let G be a compact group.

I shall use the word G-*module* to mean a finite-dimensional complex vector space M together with a continuous linear action of G on M. If M and N are G-modules, one can form their sum $M \oplus N$, and with respect to this operation the isomorphism classes of G-modules form an abelian semigroup. The associated abelian group is called R(G): its elements are formal differences $[M_0]-[M_1]$ of isomorphism classes of G-modules modulo the equivalence relation generated by $[M_0]-[M_1] \sim [M_0 \oplus N]-[M_1 \oplus N]$. Because any G-module is isomorphic to a unique sum of simple G-modules, R(G) is the free abelian group on the set $\hat{G}$ of isomorphism classes of simple G-modules. The tensor product $(M, N) \mapsto M \otimes N$ induces a structure of commutative ring in R(G). If G is abelian the set $\hat{G}$ is an abelian group under $\otimes$, usually called the *dual group* of G, and then R(G) is the group ring $\mathbf{Z}[\hat{G}]$.

A G-module M has a character χ_M, the continuous complex valued function on G defined by $\chi_M(g) = \text{trace}\, g_M$, where g_M is the action of g on M. M is determined by χ_M up to isomorphism. Because $\chi_{M \oplus N} = \chi_M + \chi_N$ and $\chi_{M \otimes N} = \chi_M \cdot \chi_N$ the map $M \mapsto \chi_M$ identifies R(G) with a subring of the ring of complex functions on G.

R(G) is called the *representation ring* or *character ring* of G. The object of this paper is to describe its structure, and in particular to determine its prime ideals. The principal results are Propositions (3.5) and (3.7). I confine myself to the case of a compact *Lie* group G: in general $R(G) = \varinjlim R(G/N)$, where N runs through the compact normal subgroups of G such that G/N is a compact Lie group.

I began the work in order to answer the question: if H is a subgroup of G, and g is an element not conjugate to any element of H, can one find a character χ of G which vanishes on H but not at g? The characters which vanish at g form a prime ideal $\mathfrak{p}$ in R(G), and if there exists such a χ then $R(H)_{\mathfrak{p}}$, i.e. R(H) regarded as an R(G)-module and localized at $\mathfrak{p}$, is zero, and conversely. The ideal $\mathfrak{p}$ depends only on the conjugacy class of the cyclic subgroup generated by g. In Proposition (3.7) I describe the prime ideals $\mathfrak{p}$ of R(G) in terms of cyclic subgroups, and say when $R(H)_{\mathfrak{p}} = 0$. In particular I show that maximal ideals are associated to finite cyclic subgroups.

Because the characters χ_M are class functions on G, i.e. $\chi_M(gg'g^{-1}) = \chi_M(g')$ the problem falls into two parts:

(i) to determine the class functions, i.e. to describe the space of conjugacy classes of G; and

(ii) to decide when a class function is a character.

The first part — which I shall regard as empty when G is finite — is quite easy, and perhaps well-known; but nevertheless I have devoted § 1 to establishing the result in a convenient form. I introduce a class of subgroups which take over the role played by maximal tori in connected groups, and I generalize the three basic properties of maximal tori [13]:

(i) each group element is contained in a maximal torus (cf. (1.3) below).

(ii) any two maximal tori are conjugate (cf. (1.6)).

(iii) two elements of a maximal torus are conjugate in the group if and only if they are conjugate in the normalizer of the torus (cf. (1.8)).

Most of the results of this section have been obtained by Siebenthal [15] by completely different methods.

For the second part of the problem the tool needed is a generalization of the " induced representation " homomorphism $R(H) \to R(G)$ which is familiar when H is a subgroup of finite index in G. The definition is quite simple, but to justify it and to establish its properties is more complicated: it involves elliptic differential operators. The characters of the induced elements are calculated by means of the Lefschetz formula of Atiyah and Bott [2]. § 2 contains the definition and essential properties of the construction; it follows the lines of a paper of Bott [6].

In § 3, I describe the structure of R(G). When G is a finite group this has been done already in [14], [8] and [1]. I have formulated the results to be found there somewhat differently, and extended them to the general case. When G is connected the results are very simple and well-known, for a class function is a character if and only if its restriction to a maximal torus T of G is a character of T ([4], § 4); but that is not trivial.

Finally, in § 4, I point out the crudity of the method I have used, and try to put it in perspective. It appears that the work is not really a contribution to the theory of Lie groups in the ordinary sense, as, if one is content to presuppose all that is known concerning the structure of compact Lie groups, then the fundamental case is that when the identity component of G is a torus. On the other hand none of the discussion would be simplified if one confined oneself to that case, and I think it is illuminating to show how much one can do without using any of the classical theory. (For groups whose identity components are tori one can write down the induced representation more or less explicitly, but not in a convenient form.)

I am grateful to Professor Atiyah for helping me constantly with the following work, much of which is based on published and unpublished papers of his. And I am grateful also for conversations I have had with Professor Bott.

§ 1. CONJUGACY IN COMPACT LIE GROUPS

In this section G is a compact Lie group, G^0 is its identity component, and $\Gamma = G/G^0$ is its group of components; *subgroup* will always mean closed subgroup; and *cyclic group* will mean a compact Lie group containing an element, called the generator, whose powers are dense, i.e. it will mean the product of a finite cyclic group and a torus.

Definition (1.1). — *A subgroup* S *of* G *is a Cartan subgroup if it is cyclic and of finite index in its normalizer* N(S). *The finite group* N(S)/S *is called the Weyl group of* S, *and is denoted by* W_S.

Unfortunately this definition conflicts with that of Chevalley in [10].

Proposition (1.2). — *Each element* g *of* G *is contained in a Cartan subgroup* S.

Proof. — Let T be a maximal torus of the centralizer $Z(g)$ of g. (I shall not presuppose any properties of maximal tori, however.) Let S be the subgroup generated by T and g. The identity component S^0 of S contains T; but if it contained any 1-parameter subgroup L not in T then L and T would generate a torus of $Z(g)$ strictly containing T. So $S^0 = T$. S/S^0 is cyclic, generated by gS^0, so S is also cyclic. Finally $(N(S) : S) = (N(S) : Z(S))(Z(S) : S)$; but $N(S)/Z(S)$ is a compact subgroup of the discrete group of automorphisms of S, so it is finite; and $Z(S)/S$ is finite because $Z(S)^0 = S^0$. So $(N(S) : S)$ is finite, and S is a Cartan subgroup.

Proposition (1.3). — *If* S *is a Cartan subgroup generated by* x, *then each element* g *of* G^0x *is conjugate to an element of* S.

Proof. — For any $a \in G$ consider the map f_a of the compact differentiable manifold G/S defined by $f_a(yS) = ayS$. If yS is a fixed point of f_g then $gy \in yS$ and $y^{-1}gy \in S$, as required. But f_g is homotopic to f_x, so to show f_g has a fixed point it suffices to show that f_x has non-zero Lefschetz number. The fixed point set of f_x is N(S)/S, which is finite. Now one can suppose f_x is an isometry, and an isolated fixed point of an isometry of a differentiable manifold must have index $+1$. So f_x has non-zero Lefschetznumber.

Proposition (1.3) can be sharpened slightly, as follows.

Proposition (1.4). — *If* S *is a Cartan subgroup of* G *generated by* x, *then any* $g \in xG_0$ *is conjugate by an element of* G^0 *to an element of* xS^0.

(Notice that there might be several components of S in xG^0.)

Proof. — Proceed as in (1.3) but consider the map $h_g : G^0/S^0 \to G^0/S^0$ defined by $h_g(yS^0) = gyx^{-1}S^0$. A fixed point of h_g is a coset yS^0 with $y \in G^0$ such that $y^{-1}gy \in xS^0$.

Proposition **(1.5)**. — *The projection*

$$\{\textit{Cartan subgroups of } G\} \to \{\textit{cyclic subgroups of } \Gamma\}$$

induces a bijection of conjugacy classes.

Proof. — One half of this follows from (1.2); the other half, that two Cartan subgroups are conjugate if they have the same projection into Γ, follows from (1.3). (For a Cartan subgroup cannot be conjugate to a proper subgroup of itself.)

It remains to decide when two elements of a Cartan subgroup S are conjugate. One sees from the case of finite groups that it is not reasonable to expect that two elements of S will be conjugate only if they are conjugate in N(S). The appropriate result is the following one, where S^* denotes the subset $\{g \in S : gS^0 \text{ generates } S/S^0\}$ of S.

Proposition **(1.6)**. — *If* S *is a Cartan subgroup of* G, *then two elements of* S^* *are conjugate in* G *if and only if they are conjugate in* N(S).

Proof. — If x and gxg^{-1} are both in S^*, then S and $g^{-1}Sg$ are two Cartan subgroups of $Z(x)$, and $x \in S^*$, $x \in (g^{-1}Sg)^*$. So S and $g^{-1}Sg$ are conjugate in $Z(x)$, i.e. there is a $z \in Z(x)$ such that $zg^{-1}Sgz^{-1} = S$. Then $zg^{-1} \in N(S)$, and $(zg^{-1})^{-1}x(zg^{-1}) = gxg^{-1}$, as required.

Corollary **(1.7)**. — *If* [G] *is the space of conjugacy classes of* G, *and* $[\Gamma]$ *is that of* Γ, *then in the projection* $[G] \to [\Gamma]$ *the inverse image of a conjugacy class* $\gamma \in [\Gamma]$ *is isomorphic to* S^*/W_S, *where* S *is a Cartan subgroup of* G *with a generator in* γ.

Definition **(1.8)**. — *An element of* G *is regular if it generates a Cartan subgroup, and singular otherwise.*

Thus I shall call more elements singular than is usual: usually one calls g regular if the dimension of $Z(g)$ is the same as that of a Cartan subgroup containing g. But with my definition it is clearly still true that regular elements are dense in G.

Proposition **(1.9)**. — *If* g *is a regular element of* G, *and* H *is a subgroup of* G, *then* g *acts with only a finite number of fixed points on* G/H.

Proof. — Let S be the Cartan subgroup generated by g. The coset $yH \in G/H$ is fixed by g if and only if $y^{-1}Sy \subset H$. So there are no fixed points at all unless S is conjugate to a subgroup of H, in which case one may as well assume $S \subset H$. Because there are only a finite number of conjugacy classes of Cartan subgroups of H one can choose a finite subset A of G such that $y^{-1}Sy \subset H \Leftrightarrow y^{-1}Sy = h^{-1}a^{-1}Sah$ for some $h \in H$ and $a \in A$. That is: yH fixed $\Leftrightarrow y \in N(S).a.H$ for some $a \in A$. But

$$(N(S).a.H)/H = N(S)/(aHa^{-1} \cap N(S)),$$

which is a quotient of the finite set N(S)/S. So the fixed point set is finite.

Remarks. — One should not assume, when S is a Cartan subgroup of G covering C in Γ, that $(S:S^0)=(C:1)$. A counterexample is given by the Cartan subgroup $\mathbf{Z}/4$ of $(\mathbf{T}^1\tilde{\times}(\mathbf{Z}/4))/(\mathbf{Z}/2)$, where $\mathbf{T}^1\tilde{\times}(\mathbf{Z}/4)$ means the non-trivial semi-direct product of the circle-group $\mathbf{T}^1$ and $\mathbf{Z}/4$, and $\mathbf{Z}/2$ is embedded diagonally. Of course $(S:S^0)$ is a multiple of $(C:1)$. In fact it must divide $(C:1)^2$, as one can prove as follows.

One can choose (see § 4) a subgroup Q of G containing S such that $Q^0=Q\cap G^0$ is a maximal torus T of G. Then the kernel of $S/S^0\to C$ is $(S\cap T)/S^0$, which is a cyclic subgroup of the group of components of the fixed point set of the action of $C=S/(S\cap T)$ on T. This group of components is $H^1(C;\pi_1 T)$, and hence is annihilated by $(C:1)$. (By writing the cohomology sequence for the canonical short exact sequence $0\to\pi_1 T\to V\to T\to 0$ one finds that whenever a finite group F acts on T the group of components of T^F is $H^1(F;\pi_1 T)$.)

§ 2. INDUCED REPRESENTATIONS

If one has a homomorphism $i:H\to G$ of compact groups one can regard any G-module M as an H-module i^*M; and this process induces a homomorphism of rings $i^*:R(G)\to R(H)$. The object of this section is to define a map $i_!:R(H)\to R(G)$ in the opposite direction. It will be a generalization of the classical construction of a G-module i_*M, called the " induced representation ", from an H-module M when G and H are finite groups.

To construct $i_!$ it is convenient to consider infinite-dimensional representations of G as well as finite ones, though the latter are the object of study. So in this section G-*module* will mean a complex topological vector space E which is hausdorff, locally convex, and complete, and on which G acts continuously (i.e. $G\times E\to E$ is continuous). The significance of " locally convex and complete " is that one can integrate continuous functions $f:G\to E$ with respect to the Haar measure of G; in particular there is a continuous projection of E on to its invariant subspace E^G.

I recall that all simple G-modules are finite-dimensional.

If E, F are G-modules, $\mathrm{Hom}^G(E;F)$ means the vector space of continuous equivariant linear maps $E\to F$. When E has finite dimension I shall give it the obvious topology.

If E is a G-module there is a map

$$\varepsilon:\coprod_{P\in\hat{G}}(P\otimes\mathrm{Hom}^G(P;E))\to E.$$

($\coprod$ denotes the topological direct sum; I recall that a sum of complete spaces is complete.) It is elementary that ε is injective and that its image is the union of the finite-dimensional G-subspaces of E.

It follows from the Peter-Weyl theorem (cf. [12], p. 31) — in fact it is a convenient formulation of the theorem — that the image of ε is dense in E. One can express this situation by saying that ε is a bijection in the category of G-modules. I shall call it the *canonical decomposition* of E.

Now let $i: H \to G$ be a homomorphism of compact groups, and let M be an H-module. In the classical case the induced G-module has the property

$$\mathrm{Hom}^G(N; i_*M) \cong \mathrm{Hom}^H(i^*N; M)$$

for all H-modules N. This property characterizes a G-module i_*M in any case: it is easy to check that $i_*M = \mathrm{Map}^H(G; M)$, where Map^H means the continuous H-equivariant maps with the topology of uniform convergence. (In $\mathrm{Map}^H(G; M)$ one lets H act on G on the right, and G on itself on the left to make $\mathrm{Map}^H(G; M)$ a G-module.) Of course i_*M is usually infinite-dimensional even when M is finite, so it does not define an element of R(G); but if one introduces [6] the larger group R.(G) of formal infinite linear combinations of simple G-modules (i.e. $R.(G) = \mathrm{Map}(\hat{G}; \mathbf{Z}) = \mathrm{Hom}(R(G); \mathbf{Z})$), and looks at the canonical decomposition $\varepsilon : \coprod_{P \in \hat{G}} (P \otimes \mathrm{Hom}^H(i^*P; M)) \to i_*M$, one finds that i_*M defines an element of R.(G). In fact one obtains a homomorphism of R(G)-modules $i_* : R.(H) \to R.(G)$, which is simply the transpose of $i^* : R(G) \to R(H)$.

It is obvious that, if $i: H \to G$, $j: G \to F$ are two homomorphisms of compact groups, then $j_* i_* = (ji)_*$.

I shall always regard R(G) as embedded in R.(G). When $i: H \to G$ is the inclusion of a closed subgroup of a compact Lie group G, I shall introduce an element $\lambda_{G/H}$ in R(H) with the property that $i_*(\xi . \lambda_{G/H}) \in R(G) \subset R.(G)$ for all ξ in R(H). Then I shall define $i_!(\xi) = i_*(\xi . \lambda_{G/H})$. The homomorphism $i_! : R(H) \to R(G)$ of R(G)-modules will take over the role played by i_* in the analysis of R(G) when G is finite: when $(G : H)$ is finite, $\lambda_{G/H}$ will be 1. There is a simple formula for the character of $i_!(\xi)$.

To define $\lambda_{G/H}$, observe that the coset space G/H is a differentiable manifold on which G acts smoothly. Let $T = T_{G/H}$ be its tangent space at the neutral coset, a real H-module. $T_{\mathbf{C}}$ is its complexification. Let $\lambda^i_{G/H}$ be the class in R(H) of the exterior power $\Lambda^i T^*_{\mathbf{C}}$. Then $\lambda_{G/H} = \sum_i (-1)^i \lambda^i_{G/H}$.

To prove that $i_!(\xi)$ is in R(G), and to calculate its character, one should adopt the following point of view. If M is a finite-dimensional H-module one can form [5] a vector bundle $E_M = (G \times M)/H$ on the space G/H. (H acts on $G \times M$ by $h.(g, m) = (gh^{-1}, hm)$.) For example, the complexified tangent bundle is $E_{T_{\mathbf{C}}}$. One can identify $i_*M = \mathrm{Map}^H(G; M)$ with the space of continuous sections of E_M: its G-action arises when one lets G act compatibly on both the base G/H and the total space E_M. The differentiable sections form a dense subspace $i_d M$ in i_*M, but $i_d M$ should be given the usual topology for smooth functions on a compact manifold, so that it is also a G-module. For example $i_d(\Lambda^p T^*_{\mathbf{C}}) = \Omega^p$ is the space of smooth p-forms on G/H. It is well-known that there is an operation called " exterior differentiation " $d : \Omega^p \to \Omega^{p+1}$ for each p, and that the sequence

$$\ldots \to 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \ldots$$

is a complex (i.e. $d^2=0$) whose cohomology is $H^*(G/H; \mathbf{C})$, the cohomology with complex coefficients of the manifold G/H. The vector spaces $H^p(G/H; \mathbf{C})$ are G-modules, because G acts on G/H. (But if G is connected it acts trivially on them because of the homotopy invariance of cohomology groups.) The next lemma shows that

$$\sum_i (-1)^i i_*(\Lambda^i T^*_{\mathbf{C}}) = \sum_i (-1)^i i_d(\Lambda^i T^*_{\mathbf{C}}) = \sum_i (-1)^i [H^i(G/H; \mathbf{C})]$$

in R.(G). Because the cohomology groups are finite-dimensional this implies that $i_*(\lambda_{G/H})$ is in R(G).

Lemma. — *If* $0 \to E^0 \to E^1 \to \ldots \to E^m \to 0$ *is an exact sequence of* G-*modules in the sense that the image of each map is dense in the kernel of the next, then* $\sum_i (-1)^i [E^i] = 0$ *in* R.(G), *providing that all* $[E^i]$ *are defined.*

Proof. — One must show that for any simple G-module P one has

$$\sum_i (-1)^i \dim \mathrm{Hom}^G(P; E^i) = 0.$$

But $\mathrm{Hom}^G(P; E^i) \cong (P^* \otimes E^i)^G$, so it suffices to show that the two functors $E \mapsto P^* \otimes E$ and $E \mapsto E^G$ are exact in the relevant sense. The first is trivially so, the second because of the continuous projection $E \to E^G$ already remarked.

The lemma implies that $[i_*(\lambda^i_{G/H})] = [i_d(\lambda^i_{G/H})]$, because of the exact sequence $0 \to i_d(\lambda^i_{G/H}) \to i_*(\lambda^i_{G/H}) \to 0$; and also, by a standard argument, that a complex can be replaced by its cohomology. (But in fact I shall not use that.)

I want to prove that $i_*(\lambda_{G/H} \cdot [M])$ is in R(G) for any finite G-module M. One can choose a connection in the vector bundle E_M, i.e. a homomorphism $d_M : i_d M \to i_d(M \otimes T^*)$. By averaging over G one can obtain an equivariant connection, i.e. one can suppose d_M is a G-map. It is well-known that d_M induces maps $d_M : i_d(M \otimes \Lambda^p T^*_{\mathbf{C}}) \to i_d(M \otimes \Lambda^{p+1} T^*_{\mathbf{C}})$ for each p. But unfortunately $d^2_M \neq 0$ in general: in fact d^2_M is the curvature of the connection. For that reason one introduces the adjoint of d_M: by choosing a G-invariant Riemannian metric on G/H, and a G-invariant hermitian metric on E_M (which amounts to choosing H-invariant metrics in $T_{G/H}$ and M), one obtains pairings $i_d(M \otimes \Lambda^p T^*_{\mathbf{C}}) \times i_d(M \otimes \Lambda^p T^*_{\mathbf{C}}) \to \mathbf{C}$ with respect to which d_M has an adjoint operator (still equivariant) $d^*_M : i_d(M \otimes \Lambda^p T^*_{\mathbf{C}}) \to i_d(M \otimes \Lambda^{p-1} T^*_{\mathbf{C}})$. So one can consider

$$d_M + d^*_M : \coprod_{p \text{ even}} i_d(M \otimes \Lambda^p T^*_{\mathbf{C}}) \to \coprod_{p \text{ odd}} i_d(M \otimes \Lambda^p T^*_{\mathbf{C}}).$$

This is an elliptic differential operator, so its kernel and cokernel have finite dimension. The lemma above implies $i_*([M] \cdot \lambda_{G/H}) = [\ker(d_M + d^*_M)] - [\mathrm{coker}(d_M + d^*_M)] \in R.(G)$. The last element is in R(G), as desired.

The character $\chi_{i_! M}$ of $i_! M$ can be calculated by the Lefschetz formula [2] of Atiyah and Bott, for it suffices to determine it at regular elements g of G, and they act with isolated fixed points on G/H, by (1.9). One obtains $\chi_{i_! M}(g) = \sum_{x \in F} \chi_M(x^{-1} g x)$, where F is the finite set of fixed points of the action of g on G/H, so that $x \in F \Leftrightarrow x^{-1}gx \in H$, where $x^{-1}gx$ is determined up to conjugation in H.

When (G : H) is finite this is the classical formula for $\chi_{i_* M}$; when G is connected and H is a maximal torus it is related to the Weyl character formula: I shall explain the difference in § 4.

Notice that $i_!$ is always zero if H does not contain a Cartan subgroup of G, for then any regular element of G acts without any fixed points on G/H.

If $j : G \to F$ is another inclusion of compact Lie groups then $T_{F/H} \cong T_{G/H} \oplus i^* T_{F/G}$ as H-module, so $\lambda_{F/H} = \lambda_{G/H} . i^* \lambda_{F/G}$. Therefore

$$\begin{aligned} j_! i_!(\xi) &= j_*(i_*(\xi . \lambda_{G/H}) . \lambda_{F/G}) = j_* i_*(\xi . \lambda_{G/H} . i^* \lambda_{F/G}) \\ &= (ji)_*(\xi . \lambda_{F/G}) = (ji)_!(\xi), \end{aligned}$$

so the induction process is transitive.

So far I have defined $i_!$ only when i is an inclusion, but one can extend it easily to the case where i has a finite kernel N. If $i : H \to H/N$ is the canonical map then $i_* M = M^N$, and one should define $i_! = i_*$. This fits together with the other definition so that induction is transitive, but I shall not go into the details.

Finally, if $i : H \to G$ is split, i.e. if there is a homomorphism $p : G \to H$ such that $pi = \mathrm{id}$, then $i_!(\xi) = \lambda_N . p^*(\xi)$, where $\lambda_N = \sum_k (-1)^k [\Lambda^k T_{N, \mathbf{C}}]$, and $T_{N, \mathbf{C}}$ is the complexified tangent-space to $N = \ker p$ at the neutral element. (G acts on N, hence on T_N.) Alternatively one can say $\lambda_N = \sum_k (-1)^k [H^k(N; \mathbf{C})]$. If H^0 and G^0 are tori then any inclusion $i : H \to G$ is the composite of a split inclusion and a homomorphism with finite kernel and cokernel, so one can give a fairly explicit description of $i_!$ in that case.

§ 3. THE STRUCTURE OF R(G)

I shall begin with some examples:

(i) If $G = \mathbf{Z}/n$, $R(G) \cong \mathbf{Z}[X]/(X^n - 1)$.

(ii) If $G = \mathbf{T}^1$, $R(G) \cong \mathbf{Z}[X, X^{-1}]$, i.e. $\mathbf{Z}[X, Y]/(XY - 1)$.

(iii) If $G = \mathbf{T}^k = \mathbf{T}^1 \times \ldots \times \mathbf{T}^1$ (k factors), $R(G) \cong \mathbf{Z}[X_1, \ldots, X_k, (X_1 X_2 \ldots X_k)^{-1}]$.

(iv) If G is the unitary group $U(n)$, and $T = \mathbf{T}^n$ is a maximal torus of $U(n)$ — say the diagonal matrices — then the restriction $R(G) \to R(T)$ is injective, for each element of G is conjugate to an element of T. So R(G) can be identified with a subring of $R(T) \cong \mathbf{Z}[X_1, \ldots, X_n; (X_1 \ldots X_n)^{-1}]$. But the entries of a diagonal matrix can be permuted by conjugation in $U(n)$, so R(G) consists of symmetric expressions in $X_1, \ldots, X_n$. In fact

Proposition **(3.1)**. — $R(U(n)) = \mathbf{Z}[s_1, \ldots, s_n, s_n^{-1}]$, *where* s_k *is the k-th symmetric function of* $X_1, \ldots, X_n$ *in* R(T). *In particular* R(U(n)) *is a noetherian ring.*

Proof. — s_k is in $R(U(n))$, for it is the character of the exterior power $\Lambda^k M$, where $M = \mathbf{C}^n$ is the standard $U(n)$-module. And s_n^{-1} is the character of $(\Lambda^n M)^*$. On the other hand $\mathbf{Z}[s_1, \ldots, s_n, s_n^{-1}]$ contains all the symmetric functions in $R(T)$, so $R(U(n))$ cannot be any bigger. It is well-known that $s_1, \ldots, s_n$ are algebraically independent.

The following theorem of Atiyah is fundamental for the sequel.

Proposition **(3.2)**. — *If* H *is a subgroup of* G *then the restriction* $R(G) \to R(H)$ *makes* $R(H)$ *a finite* $R(G)$*-module.*

Proof. — Embed G in a unitary group $U = U(n)$. It suffices to show that $R(H)$ is finite over $R(U)$, i.e. one can assume $G = U$. Furthermore the restriction $R(H) \to \prod_S R(S)$, where S runs through the finite set of conjugacy classes of Cartan subgroups of H, is injective, so — because $R(U)$ is noetherian — it suffices to show each $R(S)$ is finite over $R(U)$.

For given S one can choose a maximal torus T of U so that $S \subset T \subset U$. $R(T) = \mathbf{Z}[X_1, \ldots, X_n, s_n^{-1}]$ is finite over $R(U)$, for each X_i satisfies the integral equation $\sum_k (-1)^k X_i^k s_{n-k} = 0$ over $R(U)$. Finally $R(S)$ is finite over $R(T)$, indeed $R(T) \to R(S)$ is surjective: for $R(T) = \mathbf{Z}[\hat{T}]$, $R(S) = \mathbf{Z}[\hat{S}]$, and $S \to T$ induces a surjection $\hat{T} \to \hat{S}$.

Corollary **(3.3)**. — $R(G)$ *is a finitely generated ring for any compact Lie group* G. *In particular, it is noetherian.*

For it is finite over some $R(U)$.

Now I can begin to discuss the prime ideals of $R(G)$. I recall that the set of prime ideals of any commutative ring R can be made into a topological space Spec R which depends contravariantly on R [11].

Consider the restriction $R(G) \to \prod_S R(S)$, where S runs through the conjugacy classes of Cartan subgroups of G. It is injective, and $\prod_S R(S)$ is finite over $R(G)$, by the theorem of Cohen-Seidenberg ([7], Chap. 5, § 2, N° 1, Th. 1), $\operatorname{Spec} \prod_S R(S) \to \operatorname{Spec} R(G)$ is surjective. $\operatorname{Spec} \prod_S R(S)$ is the topological sum $\coprod_S \operatorname{Spec} R(S)$, i.e. every prime ideal $\mathfrak{p}$ of $R(G)$ comes from some prime of some $R(S)$.

For a given prime $\mathfrak{p}$ one can consider the set of subgroups H of G such that $\mathfrak{p} \in \operatorname{Im}(\operatorname{Spec} R(H) \to \operatorname{Spec} R(G))$. This set has minimal elements because compact Lie groups obey the descending chain condition. I shall show that the minimal elements are all conjugate, and I shall call any one of them the *support* of $\mathfrak{p}$. The support is a cyclic subgroup, because if $\mathfrak{p}$ comes from H it comes from a Cartan subgroup of H.

The proof has two parts:

a) I shall show that if $\mathfrak{p}$ comes from a cyclic subgroup S but not from a proper subgroup of S, and if $\bar{S}$ is a Cartan subgroup of G covering $S.G^0$ in G/G^0, then $\bar{S}$ is determined up to conjugacy by $\mathfrak{p}$; furthermore, that if $\mathfrak{p}$ comes from the primes $\mathfrak{p}_0$ and $\mathfrak{p}'_0$ of $R(\overline{S})$, then $\mathfrak{p}_0$ and $\mathfrak{p}'_0$ are related by the action of the normalizer of $\overline{S}$ on Spec $R(\overline{S})$.

b) I shall prove the statement for an abelian group such as a Cartan subgroup.

The second part is very easy: the point is simply to show that if $\mathfrak{p}$ comes from subgroups S_1 and S_2 of S then it comes from $S_1 \cap S_2$ (and hence from a cyclic subgroup of $S_1 \cap S_2$). Recall that if $R \to R_1$, $R \to R_2$ are morphisms of rings the induced map

$$\mathrm{Spec}(R_1 \otimes_R R_2) \to (\mathrm{Spec}\, R_1) \times_{(\mathrm{Spec}\, R)} (\mathrm{Spec}\, R_2)$$

is surjective ([11], I (3.4.7)). This means that the result we want follows from

Lemma **(3.4)**. — *If* S *is abelian, and* S_1, S_2 *are subgroups, then*

$$R(S_1) \otimes_{R(S)} R(S_2) \xrightarrow{\cong} R(S_1 \cap S_2)$$

by the natural map.

Proof. — For any subgroup A of S write $J_A = \ker(R(S) \to R(A))$. Then $R(A) \cong R(S)/J_A$. J_A is, as ideal in R(S), generated by the elements $\alpha - 1$, for all homomorphisms $\alpha : S \to \mathbf{C}^*$ such that $\alpha | A = 1$. In fact any element of J_A is a sum of elements of J_A of the form $\sum_i n_i \alpha_i$, where the n_i are integers and the α_i are homomorphisms $S \to \mathbf{C}^*$ which all have the same restriction to A; and one can write $\sum_i n_i \alpha_i = \sum_i n_i \alpha_1 (\alpha_i \alpha_1^{-1} - 1)$, which is of the required form.

The left-hand-side of (3.4) is $R(S)/(J_{S_1} + J_{S_2})$, so one must show that $J_{S_1} + J_{S_2} = J_{S_1 \cap S_2}$. But if $\alpha : S \to \mathbf{C}^*$ is a homomorphism such that $\alpha | (S_1 \cap S_2) = 1$, one can choose $\beta : S \to \mathbf{C}^*$ so that $\beta | S_1 = \alpha | S_1$ and $\beta | S_2 = 1$, for $(S/S_2)^\wedge \to (S_1/S_1 \cap S_2)^\wedge$ is surjective. Then $\alpha - 1 = (\alpha - \beta) + (\beta - 1) \in J_{S_1} + J_{S_2}$.

Remark. — In fact $R(S_1) \otimes_{R(S)} R(S_2) \xrightarrow{\cong} R(S_1 \times_S S_2)$ for any fibre product of abelian groups.

Turning now to the statement *a)*, the inclusion $\mathbf{Z} \to R(G)$ induces a projection $\pi : \mathrm{Spec}\, R(G) \to \mathrm{Spec}\, \mathbf{Z}$ which assigns to $\mathfrak{p}$ the prime $\mathfrak{p} \cap \mathbf{Z}$ of $\mathbf{Z}$, perhaps zero. Alternatively, $\pi(\mathfrak{p})$ is the " residual characteristic " of $\mathfrak{p}$, the characteristic of the quotient-field of $R(G)/\mathfrak{p}$. It is expedient to consider the fibres of π separately.

Let $\mathbf{F}_p$ be the prime field of characteristic p, so that $\mathbf{F}_0$ is the rationals. Then $\pi^{-1}(p)$ can be identified with Spec $R_p(G)$, where $R_p(G) = R(G) \otimes_{\mathbf{Z}} \mathbf{F}_p$ ([11], I (3.6.1)).

If $\mathfrak{p}$ is a prime of R(G) above p, and S is a minimal cyclic subgroup from which it arises, then $(S : S^0)$ is not divisible by p. For otherwise one can write $S = T \times (\mathbf{Z}/p^k)$, so that $R(S) \cong R(T)[X]/(X^{p^k} - 1)$, and $R_p(S) = R_p(T)[X]/(X-1)^{p^k}$. Then $R_p(S) \to R_p(T)$, given by $X \mapsto 1$, has a nilpotent kernel, and induces a homeomorphism of spectra ([11], I (1.2.7)), so $\mathfrak{p}$ comes from T. I shall call a cyclic group S *p-regular* if p does not divide $(S : S^0)$.

If S is a cyclic group generated by g, I shall write $\mathfrak{p}_S$ for the prime ideal of characters of S which vanish at g. This ideal does not depend on the generator chosen: it is zero if S is a torus, and if $(S:S^0)=n$ it is the principal ideal $(\varphi_n(X))$, where φ_n is the n-th cyclotomic polynomial, and $X:S\to\mathbf{C}^*$ is a homomorphism with kernel S^0. I define $\widetilde{R}(S)=R(S)/\mathfrak{p}_S$, and $\widetilde{R}_p(S)=\widetilde{R}(S)\otimes_{\mathbf{Z}}\mathbf{F}_p$. If S is p-regular then $R_p(S)\xrightarrow{\cong}\prod_T\widetilde{R}_p(T)$, where T runs through the subgroups of finite index in S; thus the primes of $\widetilde{R}_p(S)$ are the primes of $R_p(S)$ which do not come from any proper subgroup T of finite index in S.

If S is a Cartan subgroup of G, the Weyl group W_S acts naturally on $\widetilde{R}_p(S)$. Furthermore, $R_p(S)^{W_S}\cong R(S)^{W_S}\otimes_{\mathbf{Z}}\mathbf{F}_p$, as both are $\mathbf{F}_p[\hat{S}/W_S]$; and if S is p-regular, $\widetilde{R}_p(S)^{W_S}\cong\widetilde{R}(S)^{W_S}\otimes_{\mathbf{Z}}\mathbf{F}_p$.

The relation between primes and Cartan subgroups, and in particular the statement *a)*, is established by the following proposition.

Proposition **(3.5)**. — *Consider the restriction* $R_p(G)\to\prod_S\widetilde{R}_p(S)^{W_S}$, *where* S *runs through the conjugacy classes of p-regular Cartan subgroups.*

Its kernel is the nilradical (set of nilpotent elements) of $R_p(G)$.

If G *is finite, or if* $p=0$, *it is surjective. In general, given* $y\in\prod_S\widetilde{R}_p(S)^{W_S}$, *one can find* $k\in\mathbf{N}$ *such that* y^k *comes from* $R_p(G)$.

In any case the map $\coprod_S(\operatorname{Spec}\widetilde{R}_p(S))/W_S\to\operatorname{Spec}R_p(G)$ *is a homeomorphism. (*$\coprod$ *means " topological sum ".)*

Remark. — It is perhaps worth pointing out that if G is finite then $R_p(G)/(\text{nilradical})\cong R(G;p)\otimes_{\mathbf{Z}}\mathbf{F}_p$, where $R(G;p)$ is the ring of representations of G in an algebraically closed field of characteristic p.

The proof will be preceded by a lemma which is a particular case of the general result. I should point out that the case $p=0$, which is much easier than other cases, is included, rather trivially, in the following discussion, provided that one sometimes interprets p as 1, in a familiar way.

Lemma **(3.6)**. — *Let* $S\to H\to P$ *be an extension of a p-group* P *of order* $q=p^k$ *by a p-regular cyclic group* S. *Then the kernel of the restriction* $r:R_p(H)\to R_p(S)^P$ *is the nilradical of* $R_p(H)$; *if the extension is split r is surjective; in general, if* $y\in R_p(S)^P$, *then* y^q *comes from* $R_p(H)$.

Proof. — We know that any prime of $R_p(H)$ comes from a p-regular cyclic subgroup T. But any such T lies in S, so $\operatorname{Spec}R_p(S)\to\operatorname{Spec}R_p(H)$ is surjective. This implies that the kernel of r consists of nilpotent elements. There are no nilpotent elements in $R_p(S)$, so the kernel must be precisely the nilradical.

Now I shall show that $R(H)\to R(S)^P$ is surjective if the extension is split. We can identify $R(S)^P$ with the free abelian group generated by the orbits of the set S under P.

Given $\alpha \in \hat{S}$, let $K \subset H$ be its stabilizer, i.e. $K = \{h \in H : \alpha(hsh^{-1}) = \alpha(s) \text{ for all } s \in S\}$. Then $\alpha : S \to \mathbf{C}^*$ extends to $\bar{\alpha} : K \to \mathbf{C}^*$ in $R(K)$ because the extension $S \to K \to K/S$ is split. The character of H induced from $\bar{\alpha}$ restricts to $\sum_{h \in H/K} h.\alpha$, a typical basis-element of $R(S)^P$.

In general the extension $S \to H \to P$ corresponds to an element $c \in H^2(P; S)$. Let $S_0 = \ker(S \xrightarrow{\times q} S)$. The extension $S/S^0 \to H/S^0 \to P$ corresponds to the image of c in $H^2(P; S/S^0)$, which is zero as $S \to S/S^0$ factorizes through $S \xrightarrow{\times q} S$ and $q.c = 0$. So $S/S^0 \to H/S^0 \to P$ is split. Now consider the diagram

$$\begin{array}{ccc} R_p(H/S_0) & \longrightarrow & R_p(S/S_0)^P \\ \downarrow & & \downarrow \\ R_p(H) & \longrightarrow & R_p(S)^P \end{array}$$

The top arrow is surjective. And if $\alpha \in \hat{S}$ then $\alpha^q \in (S/S_0)\hat{}$. If $y = \sum_i \alpha_i$ is a general element of $R_p(S)^P$ then $y^q = \sum_i \alpha_i^q$, which comes from $R_p(S/S_0)^P$, hence from $R_p(H/S_0)$, and *a fortiori* from $R_p(H)$.

Proof of Proposition **(3.5)**. — We know that the map induces a surjection of spectra, so its kernel is nilpotent. And again there are no nilpotent elements on the right.

Now suppose given $y \in R(S)^{W_S}$. One must find $x \in R(G)$ which has the same image as y^{p^k} in $\tilde{R}_p(S)$ and has image 0 in the other $\tilde{R}_p(S')$. Let P be a Sylow p-subgroup of $W_S = N(S)/S$, and let H be its inverse image in $N(S)$. I shall show that a suitable x can be induced from H.

Choose $\theta \in R(S)$ such that χ_θ vanishes on all subgroups T of finite index in S and takes the constant value $m = (S : S^0)$ on S^*. (I.e. $\theta = \prod_\alpha (\alpha - 1)$, where α runs through the non-zero elements of $(S/S^0)\hat{}$.) Then $\theta.y$ is in $R(S)^P$, and one can find $\bar{y} \in R(H)$ which restricts to $(\theta.y)^q$, where $q = (H : S)$. Let $i : H \to G$ be the inclusion, and consider $x = i_!(\bar{y}) \in R(G)$. If g is a regular element of G then $\chi_x(g) = \sum_{\gamma H} \chi_{\bar{y}}(\gamma^{-1} g \gamma)$, where the sum is over cosets γH of H such that $\gamma^{-1} g \gamma \in H$. If g generates a p-regular Cartan subgroup of H not conjugate to S then $\chi_x(g) = 0$, for any p-regular subgroup of H is contained in S. If g generates S then $\chi_x(g) = (N(S) : H) m^q \chi_y(g)^q$, which is good enough for our purpose, as $(N(S) : H) m^q$ is invertible in $\mathbf{F}_p$.

When G is finite the groups H are always split, so the morphism of (3.5) is surjective.

The statement about spectra follows from the others, for whenever a finite group W acts on a ring R one has $(\text{Spec } R)/W \xrightarrow{\cong} \text{Spec } R^W$ ([7], Chap. 5, § 2, N° 2, Th. 2).

In view of Proposition (3.5) and the preceding discussion we can describe the primes of R(G) as follows.

Proposition **(3.7)**. — (i) *Each prime* $\mathfrak{p}$ *of* R(G) *has a support* S *which is a cyclic subgroup of* G *determined up to conjugation in* G. *The primes of* R(S) *with image* $\mathfrak{p}$ *are permuted transitively by the finite group* N(S)/Z(S).

(ii) $\mathfrak{p}$ *is maximal if and only if its support is finite, and its residual characteristic non-zero.*

(iii) *The minimal primes correspond one-to-one to the conjugacy classes of Cartan subgroups of* G, *or alternatively to the conjugacy classes of cyclic subgroups of* G/G^0.

(iv) *If* H *is a subgroup of* G *the following are equivalent*:

a) $\mathfrak{p}$ *comes from* R(H);

b) $\mathfrak{p}$ *contains* $\ker(R(G) \to R(H))$;

c) *the localized module* $R(H)_{\mathfrak{p}}$ *is non-zero*;

d) *the support of* $\mathfrak{p}$ *is conjugate to a subgroup of* H.

(v) *If* $\mathfrak{p} \subset \mathfrak{p}'$ *then the support of* $\mathfrak{p}'$ *is conjugate to a subgroup of the support of* $\mathfrak{p}$.

Proof. — (i) is already established.

(ii) If $\mathfrak{p}$ is maximal then it is the kernel of a homomorphism of R(G) onto a field K, necessarily finite. The homomorphism factorizes through $R_p(S) \cong \mathbf{F}_p[\hat{S}] \to K$, where S is the support of $\mathfrak{p}$. Such a morphism is induced by a homomorphism of $\hat{S}$ onto a finite cyclic subgroup $\hat{S}_0$ of K^*. So $R_p(S) \to K$ factorizes through $R_p(S_0)$, where S_0 is a finite subgroup of S. Because S is the support of $\mathfrak{p}$, $S = S_0$.

Conversely, if S is finite cyclic, then $R_p(S)$ is a sum of fields, so any prime coming from S is maximal (if $p \neq 0$).

(iii) is clear from (3.5).

(iv) is a trivial consequence of (i), stated for future reference.

(v) follows from (iv), because, if S is the support of $\mathfrak{p}$, then $\mathfrak{p}$ contains the kernel of $R(G) \to R(S)$, so $\mathfrak{p}'$ contains it also, so $\mathfrak{p}'$ comes from S.

Example. — There is a homomorphism of rings $R(G) \to \mathbf{Z}$ defined by $[M] \mapsto \dim M$. Its kernel is a prime called the *augmentation ideal* and denoted by I_G. Its support is the subgroup $\{1\}$. For future reference I shall give the generalizations of two propositions of [1].

Proposition **(3.8)** ([1], (6.7)). — *If* H *is a subgroup of* G *then the primes of* R(H) *containing* I_H *are the same as those containing the restriction of* I_G.

Proof. — If $\mathfrak{p}_0$ in R(H) contains the restriction of I_G then its image in R(G) contains I_G, so its support is $\{1\}$. But then the support of $\mathfrak{p}_0$ is also $\{1\}$, so $\mathfrak{p}_0$ contains I_H.

Corollary **(3.9)**. — *The* I_G*-adic topology on* R(H), *when it is regarded as an* R(G)*-module, coincides with its* I_H*-adic topology.*

Corollary (3.9) can also be proved by a dévissage precisely parallel to that of (3.2).

Proposition **(3.10)** ([1], (6.9)). — *If* $R(G)^\wedge$ *is the* I_G*-adic completion of* R(G), *then the kernel of the natural map* $R(G) \to R(G)^\wedge$ *consists of the elements whose characters vanish on the components of* G *which have prime power order in* G/G^0.

Proof. — By considering the diagram

$$\begin{array}{ccc} R(G) & \longrightarrow & R(G)^{\wedge} \\ \downarrow & & \downarrow \\ \prod_S R(S) & \longrightarrow & \prod_S R(S)^{\wedge} \end{array}$$

where S runs through the Cartan subgroups of G, so that the vertical arrows are injective, one reduces oneself to the case of a cyclic group, which is easy by the method of [1].

Finally I shall prove a partial generalization of Brauer's theorem concerning finite groups. I shall call a compact Lie group H *hyperelementary* if it is an extension of a finite p-group P by a p-regular cyclic group S. (I.e. $S \to H \to P$; p is a prime number.)

Proposition **(3.11)**. — (i) R(G) *is generated as an abelian group by modules induced from hyperelementary subgroups of* G.

(ii) R(G) *is generated as an abelian group by modules induced from one-dimensional modules.*

(iii) *A class function on* G *is a character if its restriction to each hyperelementary subgroup is a character.*

Proof. — (i) The modules induced from hyperelementary subgroups generate an ideal in R(G). The substance of the proof of (3.5) was to show that this ideal is not contained in any prime ideal of R(G).

(iii) Let R be the ring of class functions on G which restrict to characters of all hyperelementary subgroups. The ring R(G) is a subring of R containing 1: it suffices to show that it is an ideal in R. But the characters induced from hyperelementary subgroups generate an ideal in R, as one sees by looking at the formula for the induced character; and by (i) this ideal is R(G).

(ii) follows from (i) in view of the following proposition, which was pointed out to me by Professor J. Tits.

Proposition **(3.12)**. — *Let* $S \to H \to P$ *be an extension of a nilpotent group* P *by a cyclic group* S. *Then any simple* H-*module is monomial*, i.e. *induced from a one-dimensional module for a subgroup of finite index.*

Proof. — If H is nilpotent this is well-known (cf. [13], Exposé 24). In general, let M be a simple H-module, and let $M = M_1 \oplus \ldots \oplus M_k$ be its decomposition into isotypical S-modules. Let $K = \{h \in H : hM_1 \subset M_1\}$. Then $M = j_* M_1$, where $j : K \to H$, so it suffices to show M_1 is monomial. Let $\overline{K}$ be the image of K in $\mathrm{Aut}(M_1)$. It is a *central* extension of a cyclic group by a nilpotent group, so it is nilpotent, and M_1 is monomial as $\overline{K}$-module. But therefore it is monomial also as K-module.

§ 4. FINAL REMARKS

The theory I have developed in this paper is quite crude. For example I have not proved that $R(G) \xrightarrow{\cong} R(T)^{W_T}$ when G is connected and T is a maximal torus. The reason is that I have used only the differentiable structure of G/T when it is actually a complex manifold, indeed a rational algebraic variety. To put it another way: the element $\lambda_{G/H} \in R(H)$ with the property $i_*(\lambda_{G/H}.R(H)) \subset R(G)$ is by no means optimal. When G/H admits an almost-complex structure, then $T_{G/H} \otimes_{\mathbf{R}} \mathbf{C} \cong T_{G/H} \oplus \overline{T}_{G/H}$ as G-module, and one can factorize $\lambda_{G/H}$ as $\lambda^c_{G/H}.\overline{\lambda}^c_{G/H}$ in R(H), and $i_*(\lambda^c_{G/H}.R(H)) \subset R(G)$. In fact it suffices to assume G/H has a Spin^c-structure [3]. This corresponds to the use of the $\bar{\partial}$-complex, or Dirac operator, instead of the de Rham complex. So one can define in suitable circumstances a more subtle induction process

$$i_c : R(H) \to R(G) \qquad \text{by} \qquad i_c(\xi) = i_*(\xi.\lambda^c_{G/H}).$$

The formula for the character $\chi_{i_c\xi}(g)$ is $\sum_{x \in F} \chi_\xi(x^{-1}gx)/\chi_{\overline{\lambda}^c_{G/H}}(x^{-1}gx)$, the sum being over the same fixed-point set as before.

When G is connected and T is a maximal torus, the last formula is the character formula of Hermann Weyl. Then there is an important identity $i_c(1) = 1$ which derives from the fact that G/T is a rational algebraic variety; it implies that all characters of G can be induced by i_c from T. An analogous statement can be made when G is not connected, as follows.

Proposition **(4.1)**. — *A compact Lie group* G *has a subgroup* Q *with the following properties*:

(i) $Q^0 = G^0 \cap Q = T$, *a maximal torus of* G.

(ii) $Q/Q^0 \cong G/G^0$.

(iii) $G/Q \cong G^0/T$ *is a rational complex algebraic variety on which* G *acts algebraically.*

(iv) *if* $i : Q \to G$ *is the inclusion, then* $i_c(1) = 1$, *and hence every character of* G *can be induced by* i_c *from* Q.

Sketch of Proof. — One considers the complex algebraic group $G_{\mathbf{C}}$ of which G is the real part ([9], Chap. 7). Let T be a maximal torus of G; one can choose a Borel subgroup B [10] of $G_{\mathbf{C}}$ such that $B \cap G = T$. Define $Q = N(B) \cap G$. Then $G^0 \cap Q = G^0 \cap N(B) = G^0 \cap B = T$, proving (i). To prove (ii) it suffices to show that $G^0Q = G$. But if $g \in G$, then gBg^{-1} is a Borel subgroup of G, and any such is of the form $\gamma B \gamma^{-1}$ for some $\gamma \in G^0$. Then $g\gamma^{-1} \in N(B) \cap G = Q$, so $g \in G^0Q$.

(i) and (ii) imply that $G/Q \cong G^0/T$; the rest of (iii) holds because

$$G/Q \cong G_{\mathbf{C}}/N(B) \cong G^0_{\mathbf{C}}/B,$$

which is a rational variety.

(iv) follows from (iii) because, by Dolbeault's theorem, $i_c(1) = \sum_k (-1)^k [H^k(G/Q; \mathcal{O})]$, where $\mathcal{O}$ is the sheaf of holomorphic functions on G/Q; and $H^k(G/Q; \mathcal{O}) = 0$ when $k > 0$ because G/Q is a rational variety [5].

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