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# CLASSIFYING SPACES AND SPECTRAL SEQUENCES

GRAEME SEGAL

The following work makes no great claim to originality. The first three sections are devoted to a very general discussion of the representation of categories by topological spaces, and all the ideas are implicit in the work of Grothendieck. But I think the essential simplicity of the situation has never been made quite explicit, and I hope the present popularization will be of some interest.

Apart from this my purpose is to obtain for a generalized cohomology theory  $k^*$  a spectral sequence connecting  $k^*(X)$  with the ordinary cohomology of  $X$ . This has been done in the past [1], when  $X$  is a CW-complex, by considering the filtration of  $X$  by its skeletons. I give a construction which makes no such assumption on  $X$ : the interest of this is that it works also in the case of an equivariant cohomology theory defined on a category of  $G$ -spaces, where  $G$  is a fixed topological group. But I have not discussed that application here, and I refer the reader to [13]. On the other hand I have explained in detail the context into which the construction fits, and its relation to other spectral sequences obtained in [8] and [12] connected with the bar-construction.

## § 1. SEMI-SIMPLICIAL OBJECTS

A semi-simplicial set is a sequence of sets  $A_0, A_1, A_2, \dots$  together with boundary- and degeneracy-maps which satisfy certain well-known conditions [5]. But it is better regarded as a contravariant functor  $A$  from the category  $Ord$  of finite totally ordered sets to the category of sets. Thus, if  $\mathbf{n}$  denotes the ordered set  $\{0, 1, \dots, n\}$ , we have  $A(\mathbf{n}) = A_n$ . The two boundary-maps  $A_1 \rightarrow A_0$  are induced by the two maps  $\mathbf{0} \rightarrow \mathbf{1}$ , and so on.

More generally, if  $C$  is any category, a semi-simplicial object of  $C$  is a sequence of objects  $A_0, A_1, \dots$  of  $C$  together with various maps; alternatively, it is a contravariant functor  $A : Ord \rightarrow C$ .

A semi-simplicial set  $A$  has a realization  $\Delta(A)$  as a topological space [9]. If, for a finite set  $S$ ,  $\Delta(S)$  denotes the standard simplex with  $S$  as set of vertices, then  $\Delta(A)$  is obtained from the topological sum of all  $\Delta(S) \times A(S)$ , for all finite ordered sets  $S$ , by identifying  $(x, \theta^* a) \in \Delta(S) \times A(S)$  with  $(\theta_* x, a) \in \Delta(T) \times A(T)$  for all  $x \in \Delta(S)$ ,  $a \in A(T)$ , and  $\theta : S \rightarrow T$  in  $Ord$ . ( $S \mapsto \Delta(S)$ ,  $S \mapsto A(S)$  are covariant and contravariant functors, respectively. I have written  $\theta_* = \Delta(\theta)$  and  $\theta^* = A(\theta)$ .)

The product of two semi-simplicial sets  $A$  and  $B$  is defined by

$$(A \times B)(S) = A(S) \times B(S).$$

The natural map  $\Delta(A \times B) \rightarrow \Delta(A) \times \Delta(B)$  is a bijection, and is a homeomorphism if the product on the right is formed in the category of *compactly generated spaces* or *k-spaces* ([7], p. 230; [14], p. 47).

Now the realization process makes sense also when applied to semi-simplicial spaces instead of sets; in particular it takes semi-simplicial *k-spaces* to *k-spaces*, and commutes with products in the latter category, as it is not difficult to verify.

## § 2. CATEGORIES AND CLASSIFYING SPACES

To a category  $C$  one can associate a semi-simplicial set  $NC$ , which one might call the *nerve* of  $C$ , by taking the objects of  $C$  as vertices, the morphisms as 1-simplices, the triangular commutative diagrams as 2-simplexes, and so on. More formally, the definition is as follows. An ordered set  $S$  can be regarded as a category with  $S$  as set of objects and with just one morphism from  $x$  to  $y$  whenever  $x \leq y$ . Then define  $NC(S) = \text{Funct}(S; C)$ , the set of functors  $S \rightarrow C$ .

The semi-simplicial set  $NC$  obviously determines  $C$ ; Grothendieck has pointed out [6] that a category can be defined as a semi-simplicial set  $A$  with the property that the natural map  $A(S_1 \amalg_{S_0} S_2) \rightarrow A(S_1) \times_{A(S_0)} A(S_2)$  is an isomorphism whenever the amalgamated sum on the left exists.

I shall write  $BC$  for the realization of  $NC$ , and shall call it the *classifying space* of  $C$ .

More generally, let me define a *topological category* as a category in which the set of objects and the set of morphisms have topologies for which the four structural maps are continuous. If  $C$  is a topological category then  $NC$  is a semi-simplicial space, and I define  $BC$  as its realization, just as before.

The functor  $C \mapsto NC$  obviously commutes with products; if one replaces the category of topological spaces by the category of *k-spaces* throughout, as I shall do tacitly from now on, then  $B$  too commutes with products. This has the following interesting consequence.

*Proposition (2.1).* — *If  $C, C'$  are topological categories and  $F_0, F_1 : C \rightarrow C'$  are continuous functors, and  $F : F_0 \rightarrow F_1$  is a morphism of functors, then the induced maps  $BF_0, BF_1 : BC \rightarrow BC'$  are homotopic.*

*Proof.* —  $F$  can be regarded as a functor  $C \times J \rightarrow C'$ , where  $J$  is the ordered set  $\{0, 1\}$  regarded as a category. So  $F$  induces  $BF : B(C \times J) \rightarrow BC'$ . But  $B(C \times J) \cong BC \times BJ$ , and  $BJ$  is the unit interval  $I$ , so  $BF$  is a homotopy between  $BF_0$  and  $BF_1$ .

*Remark.* — Because  $BJ$  is compact this proposition is true either for topological spaces or for *k-spaces*.

### § 3. THE CLASSIFYING SPACE OF A TOPOLOGICAL GROUP

Let  $G$  be a topological group. It can be identified with a topological category with  $\text{ob}(G) = \text{point}$ ,  $\text{mor}(G) = G$ . Its semi-simplicial space  $NG$  is given by  $NG_k = G^k = G \times \dots \times G$  ( $k$  times).

The space  $BG$  is often a classifying space for  $G$  in the usual sense, as one can see as follows. Consider the category  $\overline{G}$  with  $\text{ob}(\overline{G}) = G$  and with a unique isomorphism between each pair of elements of  $G$ , i.e.  $\text{mor}(\overline{G}) = G \times G$ . It is equivalent to the trivial category with one object and one morphism, so  $B\overline{G}$  is contractible by (2.1). There is a functor  $\overline{G} \rightarrow G$  which takes the morphism  $(g_1, g_2)$  to  $g_1^{-1}g_2$ , and it induces a map  $B\overline{G} \rightarrow BG$ . Now  $N\overline{G}$  is  $(G, G \times G, \dots)$ , a semi-simplicial  $G$ -space on which  $G$  acts freely, so  $B\overline{G}$  is a free  $G$ -space. We have  $B\overline{G}/G \xrightarrow{\cong} BG$ , because  $N\overline{G}/G \xrightarrow{\cong} NG$  and quotient formations commute among themselves. (If one allows that  $\Delta$  commutes with fibre products it is immediate that  $G \times B\overline{G} \xrightarrow{\cong} B\overline{G} \times_{BG} B\overline{G}$ , so  $G$  acts freely on  $B\overline{G}$  in the strong sense.)

The only thing wrong with the fibration  $B\overline{G} \rightarrow BG$  is that it may not be locally trivial. If  $G$  itself is locally well-behaved (to be precise, if any map of a closed subset  $F$  of  $\Delta^n \times G \times \dots \times G$  into  $G$  can be extended to a neighbourhood of  $F$  in  $\Delta^n \times G \times \dots \times G$ ; which is true, for example, if  $G$  is an absolute neighbourhood retract) one can construct local sections by induction on the "skeletons" of  $BG$ , so the fibration is locally trivial; and it can be argued that in the converse case local triviality is not an appropriate concept. But to see the point of the matter one should compare  $BG$  with the space  $\mathcal{B}G = (G * G * \dots) / G$  defined by Milnor ([8];  $*$  denotes join). The principal  $G$ -bundle on  $\mathcal{B}G$  is obviously locally trivial. One can obtain  $BG$  from  $\mathcal{B}G$  by collapsing degenerate simplexes, i.e. those joining elements  $g_1, \dots, g_k$  of  $G$  with two  $g_i$  equal; thus it is related to  $\mathcal{B}G$  in precisely the way that reduced suspensions are related to suspensions. But  $\mathcal{B}G$  fits into my framework, too. If  $C$  is a topological category, let  $C_{\mathbf{N}}$  be the associated category unravelled over the ordered set  $\mathbf{N}$  of natural numbers as follows:  $C_{\mathbf{N}}$  is the subcategory of  $\mathbf{N} \times C$  obtained by deleting all morphisms of the form  $(n, c) \rightarrow (n, c')$  except identity-morphisms. Then  $\mathcal{B}G = BG_{\mathbf{N}}$ , and Milnor's universal space is  $B\overline{G}_{\mathbf{N}}$ . The contractibility of  $B\overline{G}_{\mathbf{N}}$  follows from (2.1), and looking at it from that point of view illuminates the contraction given by Dold [3]. One advantage of  $BG$  over  $\mathcal{B}G$  is that  $B(G \times G') \xrightarrow{\cong} BG \times BG'$ , whereas for Milnor's spaces, since  $(G \times G')_{\mathbf{N}} \cong (G_{\mathbf{N}}) \times_{\mathbf{N}} (G'_{\mathbf{N}})$ , one has  $\mathcal{B}(G \times G') \xrightarrow{\cong} \mathcal{B}G \times_{\Delta} \mathcal{B}G'$ , where  $\Delta = B\mathbf{N}$  is the infinite simplex.

As a further illustration of Proposition (2.1) I might mention that a conjugation in  $G$  induces a map of  $BG$  or of  $\mathcal{B}G$  which is homotopic to the identity; for as functor the conjugation is equivalent to the identity.

Finally, if  $C$  is the category of homogeneous  $G$ -spaces, then  $BC$  is the classifying

space for  $G$ -spaces introduced by Palais [11], or, more precisely, it differs from it in the same way that  $BG$  differs from  $\mathcal{B}G$ . (The category  $G$  is equivalent to the category of *principal* homogeneous  $G$ -spaces.)

#### § 4. THE SPACE ASSOCIATED TO A COVERING

Let  $X$  be a space, and  $U = \{U_\alpha\}_{\alpha \in \Sigma}$  be a covering of  $X$  by subsets. If  $\sigma$  is a subset of  $\Sigma$  define  $U_\sigma = \bigcap_{\alpha \in \sigma} U_\alpha$ . If  $R_U$  is the category whose objects are the non-empty  $U_\sigma$  for finite subsets  $\sigma$  of  $\Sigma$ , and whose morphisms are their inclusions, then  $NR_U$  is the barycentric subdivision of what is ordinarily called the *nerve* of  $U$ . (Observe that a "simplicial complex" does not define a semi-simplicial set until one orders the vertices, but its subdivision has a natural ordering.)

There is also another category  $X_U$  associated to  $U$ . It is a topological category whose objects are the pairs  $(x, U_\sigma)$  with  $x \in U_\sigma$ , and whose morphisms  $(x, U_\sigma) \rightarrow (y, U_\tau)$  are inclusions  $i: U_\sigma \rightarrow U_\tau$  such that  $i(x) = y$ . I.e.  $\text{ob}(X_U) = \coprod_{\sigma} U_\sigma$ , the sum being over all finite subsets of  $\Sigma$ , and  $\text{mor}(X_U) = \coprod_{\sigma \subset \tau} U_\tau$ , with the sum over all pairs of finite subsets  $\sigma \subset \tau$  of  $\Sigma$ . This category  $X_U$  occurs in some places in nature. For example, if  $G$  is a topological group, to define a principal  $G$ -bundle  $P$  on  $X$  which is locally trivial with respect to the covering  $U$  one can prescribe transition functions which amount in fact to a functor  $X_U \rightarrow G$ , where  $G$  is regarded as a category as in § 3. Two functors define the same  $P$  if and only if they are equivalent <sup>(1)</sup>. So to  $P$  is associated a map  $BX_U \rightarrow BG$  determined up to homotopy. Now we shall see in a moment that if the covering  $U$  is numerable [3] there is a natural homotopy-equivalence between  $X$  and  $BX_U$ , so one has a map  $X \rightarrow BG$  determined up to homotopy. This is a classifying map for  $P$  in the usual sense, for if  $\bar{U}$  is the covering of  $P$  induced by  $U$  then a trivialization of  $P$  with respect to  $U$  is a functor  $P_{\bar{U}} \rightarrow \bar{G}$  which induces a map  $P \rightarrow B\bar{G}$  covering  $X \rightarrow BG$ . Furthermore the functors  $X_U \rightarrow G$ ,  $P_{\bar{U}} \rightarrow \bar{G}$  factorize through  $G_{\mathbf{N}}$ ,  $\bar{G}_{\mathbf{N}}$ ; so they induce maps into Milnor's spaces too.

Let us look at the space  $BX_U$  more closely. The obvious functor  $X_U \rightarrow R_U$  induces a map  $BX_U \rightarrow BR_U$ , and the inverse image of a point in the interior of the simplex  $[\sigma_0 \subset \dots \subset \sigma_p]$  of  $BR_U$  is  $U_{\sigma_p}$ . In fact  $BX_U$  can be identified with  $U[\sigma_0 \subset \dots \subset \sigma_p] \times U_{\sigma_p} \subset BR_U \times X$ , the last space being the classifying space of the category formed like  $X_U$  but with all the  $U_\alpha$  replaced by  $X$ . (But the topology of  $BX_U$  may be finer than that induced from  $BR_U \times X$ .)

*Proposition (4.1).* — *The projection  $\text{pr}: BR_U \rightarrow X$  is a homotopy-equivalence if  $U$  is numerable.*

<sup>(1)</sup> Thus the set called  $H^1(U; G)$  is the set of equivalence classes of functors  $X_U \rightarrow G$ , just as, if  $\Gamma$  is a group,  $H^1(\Gamma; G)$  is the set of conjugacy classes of homomorphisms  $\Gamma \rightarrow G$ .

*Proof.* — A locally finite partition of unity  $\{\varphi_\alpha\}$  subordinate to  $U$  defines a map  $\varphi : X \rightarrow BR_U$ . The product  $\varphi \times id : X \rightarrow BR_U \times X$  factorizes through a map  $\psi : X \rightarrow BX_U$ . For, if  $x \in X$ , let  $\sigma = \{\alpha \in \Sigma : \varphi_\alpha(x) \neq 0\}$ . Then  $x \in U_\sigma$ , and  $\varphi(x)$  is in some open simplex  $[\sigma_0 \subset \dots \subset \sigma_p]$  of  $BR_U$  with  $\sigma_p = \sigma$ . To see that  $\psi : X \rightarrow BX_U$  is continuous it suffices, by localization and functoriality, to consider the case of a finite closed covering, in which case  $BX_U$  has the product topology. The map  $\psi$  is a homotopy-inverse to  $pr : BX_U \rightarrow X$ , for  $pr \circ \psi = id$ , while  $\psi \circ pr$  can be joined to the identity by a linear homotopy.

### § 5. SPECTRAL SEQUENCES

In this section  $k^* = \{k^q\}_{q \in \mathbb{Z}}$  will be a generalized cohomology theory defined on a category of pairs  $(X, A)$ , where  $X$  is a topological space and  $A$  is a closed subspace. I shall assume  $k^*$  has the following properties :

- (i) It is a contravariant  $\partial$ -functor in the sense of [4].
- (ii) If  $f_0 \simeq f_1 : (X, A) \rightarrow (Y, B)$ , then  $k^*(f_0) = k^*(f_1)$ .
- (iii) If  $f : (X, A) \rightarrow (Y, B)$  is a relative homeomorphism, in the sense that it induces a homeomorphism  $X/A \rightarrow Y/B$ , then  $k^*(f)$  is an isomorphism.
- (iv)  $k^*(\coprod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha k^*(X_\alpha)$  for any family of spaces  $\{X_\alpha\}$ , where  $\coprod$  denotes the topological sum [10].

If  $A$  is a semi-simplicial space its realization  $\Delta A$  has a natural filtration  $\Delta^0 A \subset \Delta^1 A \subset \dots \subset \Delta A$ , where  $\Delta^p A$  is the image of  $\Delta^p \times A_p$  in  $A$ . (In fact  $\Delta^p A$  is a quotient space of  $\Delta^p \times A_p$ .) This filtration leads to a spectral sequence.

*Proposition (5.1).* — *To a semi-simplicial space  $A$  is associated naturally a spectral sequence whose termination is  $k^*(\Delta A)$ , with  $E_2^{pq} = H^p(k^q(A))$ , the  $p$ -th cohomology group of the semi-simplicial cochain complex  $k^q(A)$ .*

*Proof.* — The filtration leads by the method of [2], p. 333 to a spectral sequence with  $E_1^{pq} = k^{p+q}(\Delta^p A, \Delta^{p-1} A)$ . There is a relative homeomorphism

$$(\Delta^p \times A_p, (\Delta^p \times A_p^d) \cup (\dot{\Delta}^p \times A_p)) \rightarrow (\Delta^p A, \Delta^{p-1} A),$$

where  $\dot{\Delta}^p$  is the  $(p-1)$ -skeleton of the simplex  $\Delta^p$ , and  $A_p^d$  is the degenerate part of  $A_p$  (the union of the images of all the maps  $A_r \rightarrow A_p$  with  $r < p$ ). Thus the pair  $(\Delta^p A, \Delta^{p-1} A)$  can be identified with the  $p$ -fold suspension of  $(A_p, A_p^d)$ , and accordingly  $E_1^{pq} \cong k^q(A_p, A_p^d)$ . I shall show that the natural map  $E_1^{pq} \rightarrow k^q(A_p)$  is compatible with the differential of the cochain complex  $k^q(A)$ . When that is done it follows that  $E_2^{pq}$  can be calculated from  $k^q(A)$ , for  $k^q(A_p, A_p^d)$  is a direct summand in  $k^q(A_p)$  complementary to the subgroup of degenerate cochains : indeed  $k^q(A(S)) \cong \bigoplus_T k^q(A(T), A^d(T))$ , where  $T$  runs through the quotients of  $S$ .

The compatibility of differentials follows from the commutativity of the diagram

$$\begin{array}{ccccc}
 & & k^*(\Delta^p A, \Delta^{p-1} A) & \xrightarrow{d} & k^*(\Delta^{p+1} A, \Delta^p A) & & \\
 & \swarrow & & & \searrow & & \\
 k^*(\Delta^p \times A_p, \dot{\Delta}^p \times A_p) & \xrightarrow{\theta} & \prod_p k^*(\Delta^p \times A_{p+1}, \dot{\Delta}^p \times A_{p+1}) & \xleftarrow{\cong} & k^*(\dot{\Delta}^{p+1} \times A_{p+1}, \ddot{\Delta}^{p+1} \times A_{p+1}) & \xrightarrow{d} & k^*(\Delta^{p+1} \times A_{p+1}, \dot{\Delta}^{p+1} \times A_{p+1}) \\
 \uparrow E^p \cong & & \uparrow E^p \cong & & \uparrow E^{p+1} \cong & & \\
 k^*(A_p) & \xrightarrow{\theta} & \prod_p k^*(A_{p+1}) & \xrightarrow{\Sigma} & k^*(A_{p+1}) & & 
 \end{array}$$

where:  $\dot{\Delta}^p$  (resp.  $\ddot{\Delta}^p$ ) means the  $(p-1)$ -skeleton (resp.  $(p-2)$ -skeleton) of  $\Delta^p$ ,  
 the maps  $\theta$  are induced by the  $p+2$  injections  $\mathbf{p} \rightarrow \mathbf{p}+\mathbf{1}$ ,  
 $E^p$  denotes  $p$ -fold suspension, and  
 $\Sigma$  denotes summation with alternating signs, so that the bottom line of the diagram is the differential of the semi-simplicial cochain complex.

Proposition (5.1) applies in particular to the classifying space for a topological category  $\mathbf{C}$ . (I shall write  $B^p \mathbf{C}$  for  $\Delta^p \mathbf{NC}$ .) For instance when  $\mathbf{G}$  is a topological group we have a spectral sequence beginning with the semi-simplicial cochain complex

$$k^*(\text{point}) \rightarrow k^*(\mathbf{G}) \rightarrow k^*(\mathbf{G} \times \mathbf{G}) \rightarrow \dots,$$

and ending with  $k^*(B\mathbf{G})$ . This has been used by Milnor, Moore, Steenrod, Rothenberg, etc. [8], [12].

If the category  $\mathbf{C}$  is discrete, and  $k^*$  is ordinary cohomology, the spectral sequence collapses (for  $E_1^{pq} = 0$  unless  $q=0$ ), and the cohomology of  $B\mathbf{C}$  — which one might well call the cohomology of  $\mathbf{C}$  — can be calculated from the complex  $H^*(\mathbf{NC})$ . In the case of a group this is the bar-construction.

The case of the category associated in § 4 to a covering  $U = \{U_\alpha\}$  of a space  $X$  is interesting and less well-known. Then the  $E_2$ -term is the cohomology of the nerve of the covering with coefficients in the system  $\sigma \mapsto k^*(U_\sigma)$ . The termination is  $k^*(B\mathbf{X}_U)$ . But if  $U$  is numerable we have seen that the natural map  $k^*(X) \rightarrow k^*(B\mathbf{X}_U)$  is an isomorphism. The resulting spectral sequence  $H^p(U; k^q) \Rightarrow k^*(X)$  is the Leray spectral sequence of the covering  $U$  in the theory  $k^*$ .

One can prove that  $k^*(X) \xrightarrow{\cong} k^*(B\mathbf{X}_U)$  in some other cases too, for example when the covering  $U$  is finite-dimensional and closed. Let  $X_k$  be the part of  $X$  contained

in at least  $k + 1$  sets of  $U$ , and let  $B_k = \text{pr}^{-1}X_k$ . Then  $B_k - B_{k+1} = (X_k - X_{k+1}) \times \Delta^k$ , whence

$$k^*(X_k, X_{k+1}) \xrightarrow{\cong} k^*(B_k, B_{k+1}),$$

both being  $\prod_{\dim \sigma = k} k^*(U_\sigma, U_\sigma \cap X_{k+1})$ ; so  $k^*(X, X_k) \xrightarrow{\cong} k^*(B, B_k)$  for all  $k$ .

In the case of a covering by two sets  $X = X_1 \cup X_2$  the Leray spectral sequence reduces to the Mayer-Vietoris sequence. Then  $BX_U = (X_1 \times 0) \cup (X_1 \times I) \cup (X_2 \times I) \subset X \times I$ , and the Mayer-Vietoris sequence is the exact sequence for the pair

$$(BX_U, B^0X_U) = (X_1 \times 0) \cup (X_2 \times I).$$

This way of obtaining the sequence, unlike the hexagonal argument of Eilenberg and Steenrod, depends on the homotopy axiom for  $k^*$ . It would be interesting to generalize the hexagonal argument to obtain the Leray spectral sequence for a finite covering without using the homotopy axiom.

From the spectral sequence for a covering one can obtain the spectral sequence for a map. This reduces, when the map is the identity, to the spectral sequence mentioned in the introduction linking  $k^*$  to ordinary cohomology.

*Proposition (5.2).* — *If  $X$  is a paracompact space, and  $f : Y \rightarrow X$  is a continuous map, there is a spectral sequence with termination  $k^*(Y)$  and with  $E_2^{pq} = H^p(X; k^q f)$ , where  $k^q f$  is the sheaf associated to the presheaf  $U \mapsto k^q(f^{-1}U)$  on  $X$ .*

*Proof.* — If  $U$  is an open covering of  $X$ , form the spectral sequence  $E(U)$  for the numerable covering  $f^{-1}U$  of  $Y$ . This terminates with  $k^*(Y)$  and begins with the Čech cohomology of the covering  $U$  with coefficients in the presheaf  $V \mapsto k^q(f^{-1}V)$ . The desired spectral sequence is obtained by taking the direct limit of the family  $\{E(U)\}$  indexed by all the open coverings of  $X$ . Notice that, if  $V, W$  are two coverings of  $Y$ , and  $V$  refines  $W$ , there are evident continuous functors  $Y_V \rightarrow Y_W$ ; and if  $F_0, F_1$  are two such functors one can find a third,  $F$ , with morphisms  $F \rightarrow F_0, F \rightarrow F_1$ . So  $BF_0 \simeq BF_1 : BY_V \rightarrow BY_W$ . The homotopy  $BF_t$  between them does not preserve the filtration, but  $BF_t(B^p Y_V) \subset B^{p+1} Y_W$  for all  $t$ , so the two morphisms of spectral sequences coincide from the  $E_2$ -term onwards ([2], p. 336).

As a final application of the method I shall mention the filtration of  $k^*(X)$ . I recall [1] that when  $X$  is a CW-complex it is customary to define

$$k_p^*(X) = \ker(k^*(X) \rightarrow k^*(X^{p-1})),$$

where  $X^{p-1}$  is the  $(p-1)$ -skeleton of  $X$ . For a general space  $X$  I propose to define  $\xi \in k_p^*(X)$  if  $\xi \in \ker(k^*(X) \rightarrow k^*(B^{p-1}X_U))$  for some numerable covering  $U$  of  $X$ . This coincides with the former definition when  $X$  is a finite simplicial complex (for if  $U$  is the star-covering of  $X$  then  $B^p X_U \simeq X^p$ ; but any covering can be refined by the star-covering of a barycentric subdivision). If  $k^*$  is a multiplicative theory, i.e. if there is a functorial product  $k^*(X, A) \otimes k^*(X, B) \rightarrow k^*(X, A \cup B)$ , then



**Proposition (5.3).** —  $k^*(X)$  is a filtered ring, i.e.  $k_p^*(X) \cdot k_q^*(X) \subset k_{p+q}^*(X)$ .

*Proof.* — Suppose  $\xi \in \ker(k^*(X) \rightarrow k^*(B^{p-1}X_U))$  and  $\eta \in \ker(k^*(X) \rightarrow k^*(B^{q-1}X_V))$ . Then I assert  $\xi \cdot \eta \in \ker(k^*(X) \rightarrow k^*(B^{p+q-1}X_{U \cap V}))$ . One can assume  $U = V$ , and I shall write  $B = BX_U$ ,  $B^p = B^p X_U$ . Let  $\bar{\xi}, \bar{\eta}$  be the images of  $\xi, \eta$  in  $k^*(B)$ . Then  $\bar{\xi}$  comes from  $k^*(B, B^{p-1})$ , and  $\bar{\eta}$  from  $k^*(B, B^{q-1})$ . Hence  $\bar{\xi} \cdot \bar{\eta}$  comes by the diagonal map from  $k^*(B \times B, (B \times B^{q-1}) \cup (B^{p-1} \times B))$ , and it suffices to show that its image in  $k^*(B^{p+q-1})$  is zero. That is a consequence of the following lemma.

**Lemma (5.4).** — If  $A$  is a semi-simplicial space let us give  $\Delta A \times \Delta A$  the product filtration  $(\Delta A \times \Delta A)^n = \bigcup_{p+q=n} (\Delta^p A \times \Delta^q A)$ . Then the diagonal map  $\Delta A \rightarrow \Delta A \times \Delta A$  is homotopic to a filtration-preserving map.

*Proof.* — I shall produce two deformations of the identity-map of  $\Delta A$ . Let us regard an  $n$ -simplex  $\Delta(S)$  as the subspace  $\{t : 0 = t_0 \leq t_1 \leq \dots \leq t_n \leq 1\}$  of  $\mathbf{R}^S$ . Then define  $h_S : \Delta(S) \rightarrow \Delta(S)$  by  $h_S(t)_i = \inf(2t_i, 1)$ .  $h_S$  depends functorially on  $S$ , so it induces a map  $h_A : \Delta A \rightarrow \Delta A$  for any semi-simplicial space  $A$ . The map  $h_S$ , and therefore also  $h_A$ , is linearly homotopic to the identity. Similarly, define  $h'_S : \Delta(S) \rightarrow \Delta(S)$  by  $h'_S(t)_i = \sup(0, 2t_i - 1)$ . This leads to  $h'_A : \Delta A \rightarrow \Delta A$ . The product

$$h_n \times h'_n : \Delta(\mathbf{n}) \rightarrow \Delta(\mathbf{n}) \times \Delta(\mathbf{n})$$

is filtration-preserving, in fact  $(h_n(t), h'_n(t)) \in \Delta(\{0, 1, \dots, p\}) \times \Delta(\{p, p+1, \dots, n\})$ , where  $p = \inf\left\{i : t_i \geq \frac{1}{2}\right\}$ . Hence  $h_A \times h'_A : \Delta A \rightarrow \Delta A \times \Delta A$  is filtration-preserving for any  $A$ , and is the required deformation of the diagonal.

#### REFERENCES

- [1] M. F. ATIYAH and F. HIRZEBRUCH, Vector bundles and homogeneous spaces, *Differential Geometry, Proc. of Symp. in Pure Math.*, **3**, Amer. Math. Soc., 7-38.
- [2] H. CARTAN and S. EILENBERG, *Homological algebra*, Princeton University Press, 1956.
- [3] A. DOLD, Partitions of unity in the theory of fibrations, *Ann. of Math.*, **78** (1963), 223-255.
- [4] S. EILENBERG and N. E. STEENROD, *Foundations of algebraic topology*, Princeton University Press, 1952.
- [5] R. GODEMENT, *Topologie algébrique et théorie des faisceaux*, Paris, Hermann, 1958.
- [6] A. GROTHENDIECK, Théorie de la descente, etc., *Séminaire Bourbaki*, **195** (1959-1960).
- [7] J. L. KELLEY, *General topology*, Princeton, Van Nostrand, 1955.
- [8] J. MILNOR, Construction of universal bundles, I, *Ann. of Math.*, **63** (1956), 272-284.
- [9] J. MILNOR, The realization of a semi-simplicial complex, *Ann. of Math.*, **65** (1957), 357-362.
- [10] J. MILNOR, Axiomatic homology theory, *Pacific J. Math.*, **12** (1962), 337-341.
- [11] R. S. PALAIS, The classification of G-spaces, *Mem. Amer. Math. Soc.*, **36**, 1960.
- [12] M. ROTHENBERG and N. E. STEENROD, The cohomology of classifying spaces of H-spaces, *Bull. Amer. Math. Soc.*, **71** (1965), 872-875.
- [13] G. B. SEGAL, Equivariant K-theory, *Publ. math. I.H.E.S.*, **34** (1968).
- [14] P. GABRIEL and M. ZISMAN, Calculus of fractions and homotopy theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 35, Berlin, Springer, 1967.

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