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NICHOLAAS H. KUIPER

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ALGEBRAIC EQUATIONS FOR NONSMOOTHABLE 8-MANIFOLDS

by NICOLAAS H. KUIPER (1)

SUMMARY

The singularities of Brieskorn and Hirzebruch are used in order to obtain examples of algebraic varieties of complex dimension four in $P^5(\mathbf{C})$, which are homeomorphic to closed combinatorial 8-manifolds, but not homeomorphic to any differentiable manifold. Analogous nonorientable real algebraic varieties of dimension 8 in $P^{10}(\mathbf{R})$ are also given. The main theorem states that every closed combinatorial 8-manifold is homeomorphic to a Nash-component with at most one singularity of some real algebraic variety. This generalizes the theorem of Nash for differentiable manifolds.

§ 1. Introduction. The theorem of Wall.

From the smoothing theory of Thom [1], Munkres [2] and others and the knowledge of the groups of differential structures on spheres due to Kervaire, Milnor [3], Smale and Cerf [4] follows a.o. that closed combinatorial n-manifolds for $n \leq 7$ are *smoothable*. That is, they admit a combinatorially compatible differential structure. This structure is unique up to equivalence for $n \leq 6$. By a *manifold* we mean a connected closed combinatorial manifold. We will consider manifolds of dimension eight. In § 1, 2, and 3 all manifolds will be oriented. Let X be an oriented 8-manifold and X^k the k-skeleton of some triangulation of X. If the number of vertices is N, then let X^0 be the set of end-points of N orthonormal unitvectors in euclidean vector N-space E^N. The simplices of X^k are then fixed and X lies embedded in E^N. For any $W \subset X \subset E^N$ and $\delta > 0$ we define the neighbourhood $U(W, \delta) = \{x \in X \mid \text{distance}(x, W) < \delta\}$ of W in X.

For small δ, say $\delta < N^{-1}$, $U(X^6, \delta)$ can be given a differential structure $\mathscr{D}$ and this is unique up to equivalence. Next we construct a differential structure on $U(X^7, \delta^2)$ which equals the first structure $\mathscr{D}$ on $U(X^6, \delta^2)$. For that we have to define for every 7-simplex Δ_7 of X^7 some differential structure on $U(\Delta_7, \delta^2)$ which agrees with $\mathscr{D}$ on $U(X^6, \delta^2) \cap U(\Delta_7, \delta^2) = U(\partial\Delta_7, \delta^2)$. This is possible in essentially 28 different ways, because the difference between two such smoothings corresponds with a smoothing

(1) Institut des Hautes Etudes Scientifiques and University of Amsterdam.

of $S^7 \times (0, 1)$ or of S^7, hence with an element of the group $\Gamma_7 \simeq \mathbf{Z}_{28}$ of differential structures on S^7 modulo those that can be extended to the 8-ball.

For each oriented 8-simplex Δ_8 of X^8 we have now smoothed some neighbourhood of the boundary $U(\partial\Delta_8, \delta^2) \cap \Delta_8$. If this smoothing, restricted to $U(X^7, \delta^3) \cap \Delta_8$, can be extended over Δ_8, then we assign to Δ_8 the element $0 \in \Gamma_7$. More generally, following M. Hirsch [5], we observe that any smooth oriented 7-manifold in $U(\partial\Delta_8, \delta^2) \cap \Delta_8$, which is combinatorially isotopic to $\partial\Delta_8$, will have the same structure of an exotic 7-sphere, and so it determines an element $\gamma(\Delta_8) \in \Gamma_7$. Indeed any two such manifolds are h-cobordant with some third that bounds a combinatorial 8-disc containing both, and hence all three are diffeomorphic.

In the top dimensional case the sheaf of coefficients (local coefficients of Munkres) is constant and it can be identified with $\Gamma_{n-1} = \Gamma_7$. This is the case for X orientable as well as X nonorientable.

The function γ on oriented simplices in X^8:

$$\gamma : \Delta_8^{(i)} \to \gamma(\Delta_8^{(i)}) \in \Gamma_7$$

is a cochain, which is a cocycle as there are no 9-simplices in X. So the value of γ on the fundamental cycle of the oriented X is

$$\gamma([X]) = \sum_i \gamma(\Delta_8^{(i)}) \in \Gamma_7.$$

If we change our choice of differential structure at one of the 7-simplices by $\xi \in \Gamma_7$, then the cochain value in the two adjacent 8-simplices alters by ξ and $-\xi$ respectively, and we obtain a cohomologous cochain with unaltered value $\gamma([X]) \in \Gamma_7$. This element represents a cohomology-class $\bar{\gamma}(X) \in H^8(X, \Gamma_7) = \Gamma_7$ which is *an invariant of the combinatorial 8-manifold.*

As X is connected, there is one choice of differential structures in the 7-simplices such that $\gamma(\Delta_8^{(i)}) = 0$ for all except at most one of the 8-simplices. We transport all obstruction to smoothing to *one* 8-simplex. Then on that 8-simplex the value of the cochain is $\bar{\gamma}(X)$. We see that the nonsmoothability of an 8-manifold can be concentrated in an arbitrarily small neighbourhood $N(p)$ of any point p. Any subdivision of the given triangulation, for which $N(p)$ is interior to an 8-simplex therefore gives the same value for $\bar{\gamma}(X)$, which is then an invariant not only of the triangulation but of the combinatorial structure of X. From the above procedure follows :

Lemma 1. — The 8*-manifold* X *has a compatible smoothing if and only if the combinatorial invariant* $\bar{\gamma}(X) \in \Gamma_7 \simeq \mathbf{Z}_{28}$ *vanishes.*

If X and Y are 8-manifolds, then the *connected sum* $X \# Y$ is the oriented 8-manifold obtained by deleting from X and Y each one 8-simplex and identifying the boundaries, say linear on each 7-simplex of this boundary, so that a connected manifold is obtained and the injections of the remaining parts of X and Y are imbedded in $X \# Y$ with preservation of orientation. The negative of X is the same non-oriented manifold with the other orientation.

Lemma 2. — *For any 8-manifold* X, $X\#(-X)$ *is smoothable.*

Proof. — Let Δ and Δ' be two 8-simplices of a triangulation of X, p an interior point of Δ' with neighbourhood $U(p, \delta)\subset\Delta'$. Take a smoothing of $X-U(p,\delta)$, in which $\partial\Delta$ is a smooth usual 7-sphere. Take the smooth connected sum of X and $-X$ along $\partial\Delta\subset X$ and $-\partial\Delta\subset -X$. The combinatorial manifold $X\#(-X)$ has then a natural smoothing, except in $U(p,\delta)\subset X-(\Delta)$ and in the corresponding neighbourhood in $(-X)-(-\Delta)$.

The cochain on the triangulation of $X\#(-X)$ has values $\gamma([X])$ and $-\gamma([X])$ on the two exceptional 8-simplices and zero elsewhere. Hence $\overline{\gamma}(X\#(-X))=0$ and lemma 2 follows from lemma 1.

The 8-manifolds X and Y are called *equi-smoothable* or *equal modulo smooth manifolds*, $X\sim Y$, if $X\#(-Y)$ is smoothable.

Lemma 3. — *Equi-smoothability is an equivalence relation.*

Proof. — Applying the above procedure of concentrating the essential contribution of the cochain γ into one 8-simplex, to the 8-manifolds X and Y, it follows immediately from lemma 1 that

$$X\sim Y \Leftrightarrow \overline{\gamma}(X)=\overline{\gamma}(Y).$$

The Theorem of C. T. C. Wall [6]. — *The equi-smoothability classes of oriented 8-manifolds (also called the combinatorial modulo smoothable 8-manifold classes) form a group isomorphic with* $\Gamma_7\simeq(\mathbf{Z}_{28},+)$ *under connected sum* $\#$.

Proof. — Again by the choice of special cochains for X and Y one sees:

$$\overline{\gamma}(X\#Y)=\overline{\gamma}(X)+\overline{\gamma}(Y).$$

Then $\overline{\gamma}$ defines a homomorphism of the associative semi-group of oriented 8-manifolds with connected sum, onto $\Gamma_7\simeq\mathbf{Z}_{28}$. By the proof of lemma 3 the equivalence classes are the 28 fibres of this map.

§ 2. Topological invariance of $\overline{\gamma}$.

D. Sullivan [20] proved that any two combinatorial structures on a simply connected closed topological manifold of dimension ≥ 6 without 2-torsion in $H^3(-, \mathbf{Z})$, are combinatorially equivalent (Hauptvermutung). Hence $\overline{\gamma}$ is a topological invariant for such manifolds.

C. T. C. Wall kindly brought to my attention that the topological invariance of the rational Pontrjagin classes, obtained by Novikov [12], implies that $\overline{\gamma}$ is a topological invariant for an even larger class of 8-manifolds. This can be seen as follows.

Borel and Hirzebruch proved in [7], p. 494, that for a smooth closed oriented manifold X

$$\hat{A}(X, d/2)=e^{d/2}\sum_{j=0}^{\infty}\hat{A}_j(p_1, \ldots, p_j)[X]$$

is an integer. Here $d\in H^2(X,\mathbf{Z})$ is any element which reduces in $H^2(X,\mathbf{Z}_2)$ to the second Whitney-class $w_2(X)$. So we have to *assume the existence of d.* For complex manifolds d exists and can be taken to be the first Chern class c_1.

One finds, with

$$\hat{A}_1 = \frac{1}{24} p_1, \qquad \hat{A}_2 = \frac{2^{-7}}{45}(-4p_2 + 7p_1^2),$$

and with the formula for the signature:

$$\sigma(X) = \frac{1}{45}(7p_2 - p_1^2)[X],$$

that

$$\hat{A}\left(X, \frac{d}{2}\right) = \left(\frac{p_1^2 - 4\sigma}{896} - \frac{d^2 p_1}{192} + \frac{d^4}{384}\right)[X].$$

We now prove the formula

$$(*) \qquad \bar{\gamma}(X) \equiv -28\,\hat{A}\left(X, \frac{d}{2}\right) \bmod 28.$$

Proof. — Let W be Milnor's example of a parallelisable 8-manifold with as boundary the exotic 7-sphere ∂W that represents the generator of Γ_7. M is the closed combinatorial manifold obtained by closing W with an 8-ball. Then $\sigma(M) = 8$. $Y = X \# M \# \ldots \# M$ is the connected sum of X and m copies of M.

One obtains, because X and Y have p_1^2 and d in common,

$$\hat{A}\left(Y, \frac{d}{2}\right) = \hat{A}\left(X, \frac{d}{2}\right) - \frac{4m}{896}\sigma(M) = \hat{A}\left(X, \frac{d}{2}\right) - \frac{m}{28}.$$

By § 1, Y has a smoothing compatible with the given combinatorial structure for exactly one value of m mod 28. This value is given by

$$0 = \bar{\gamma}(Y) = \bar{\gamma}(X) + m \bmod 28.$$

For that value of m we also have

$$0 = \hat{A}\left(Y, \frac{d}{2}\right) = \hat{A}\left(X, \frac{d}{2}\right) - \frac{m}{28} = 0 \bmod 1,$$

and the formula follows.

Consequently the right hand side of $(*)$ is mod 28 independent of the choice of d, as long as d reduces to $w_2(X)$. Then it depends only on the rational Pontrjagin class p_1, on the signature σ and on $w_2(X)$, which are all topological invariants.

Finally $\bar{\gamma}(X)$, the left hand side of $(*)$, is therefore also a topological invariant. We summarize:

Theorem 1. — If the oriented closed 8-manifold X *is simply connected and has no 2-torsion in* $H^3(X, \mathbf{Z})$, *or if* $w_2(X)$ *is the reduction of a* **Z**-*cohomology class d, and* $\bar{\gamma}(X) \neq 0$, *then* X *has no smoothing.* $\bar{\gamma}$ *is a topological invariant for such spaces* X.

§ 3. Complex algebraic varieties as examples.

Brieskorn [8], Milnor [9] and Hirzebruch [10], using Pham [11], have studied isolated singularities of complex algebraic varieties, for which some neighbourhood of the singular point has the natural topological and combinatorial structure of a cone over a smooth possibly exotic 7-sphere which bounds it. In particular this is the case for the singularity at $0 \in \mathbf{C}^5$ of the affine variety ([10])

$$\left.\begin{aligned} f_1(z_1, \ldots, z_5) = z_1^{n-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0 \\ n = 6k > 0 \end{aligned}\right\}. \tag{1}$$

The intersection of (1) with $|z| = \sqrt{\Sigma_{i=1}^5 z_i \bar{z}_i} \leq 9c$, is for small $c > 0$ homeomorphic with an 8-ball, and its boundary, obtained as intersection of (1) and

$$|z| = c \tag{2}$$

is the exotic sphere with value

$$k \, . \, 1 \in \mathbf{Z}_{28} \simeq \Gamma_7. \tag{3}$$

For $k = 1$ the generator with value $1 \in \mathbf{Z}_{28} \simeq \Gamma_7$ is found.

If we embed $\mathbf{C}^5$ as the complement of a hyperplane, the so called " hyperplane at infinity " in $\mathbf{P}^5(\mathbf{C})$, then (1) can be considered as the affine equation of an algebraic variety in $\mathbf{P}^5(\mathbf{C})$.

In homogeneous coordinates $z_1, \ldots, z_5, w$, it has the equation

$$z_1^{n-1} + z_2^3 w^{n-4} + (z_0^2 + z_4^2 + z_5^2) w^{n-3} = 0.$$

This algebraic variety has, apart from the old singularity, many more singularities namely at infinity $(w = 0)$. In order to avoid new extra singularities we modify our function f_1 and choose the new function f as follows:

$$f = z_1^{n-1} + z_2^3 + \sum_{i=3}^{5} z_i^2 + \sum_{j=1}^{5} \lambda^{j-1} z_j^n \tag{4}$$

$$\lambda \in \mathbf{R} \subset \mathbf{C}, \; n = 6k$$

This function is locally near $0 \in \mathbf{C}^5$ equivalent to f_1 by a holomorphic change of coordinates of the kind

$$z_j' = \Phi_j(z_j) \qquad (j = 1, \ldots, 5)$$

with

$$\begin{aligned} (\Phi_1(u))^{n-1} &= u^{n-1} + u^n \\ (\Phi_2(u))^3 &= u^3 + \lambda u^n \\ (\Phi_i(u))^2 &= u^2 + \lambda^{i-1} u^n \qquad (i = 3, 4, 5). \end{aligned}$$

Therefore the affine variety $f = 0$ has near $0 \in \mathbf{C}^5$ a singularity with the same local properties as mentioned for $f_1 = 0$ (take c small). We now search for the singularities on the variety $f = 0$. They obey the equations:

$$\partial_1 f = \partial_2 f = \partial_3 f = \partial_4 f = \partial_5 f = 0 \tag{5}$$

and

(6) $$f=0.$$

Solution of $z_1, \ldots, z_5$ from (5) and substitution in (6) yields for

$$(z_1, \ldots, z_5) \neq (0, \ldots, 0) = 0 \in \mathbf{C}^5$$

and for different choices of the solutions, rational algebraic equations, which can be combined into one rational algebraic equation. It expresses a necessary condition on λ, for having at least one more singular point on (6). So for only a finite number of values of λ there are other singularities. In particular for $\lambda = e$, an arbitrary transcendental number, the only singularity on the affine variety is $0 \in \mathbf{C}^5$.

We imbed $\mathbf{C}^5$ as the complement of the hyperplane $w=0$ in the complex projective 5-space $P^5(\mathbf{C})$ and close the image of the affine variety. Then we obtain the algebraic variety $V_k \subset P^5(\mathbf{C})$ with equation in homogeneous coordinates $(z_1, \ldots, z_5, w)$:

(7) $$V_k : z_1^{n-1}w + z_2^3 w^{n-3} + \sum_{i=3}^{5} z_i^2 w^{n-2} + \sum_{j=1}^{5} e^{j-1} z_j^n = 0$$

V_k has clearly no singularities at infinity $(w=0)$. It has exactly one singular point $p=(0, 0, 0, 0, 0, 1)$, and $V_k - \{p\}$ is a smooth 8-manifold. Then by the result of Brieskorn V_k is a topological manifold at p, as well as all over. However, it also has its natural triangulation as an algebraic real 8-dimensional variety, where near p the triangulation is obtained by triangulating the cone on the (possibly) exotic 7-sphere described above. We compute the invariant $\overline{\gamma}(V_k)$ as follows. Take a triangulation such that p is interior point of some 8-simplex. Take in $V_k - \{p\}$ the differential structure from the (there!) differential manifold V_k. Then the cochain γ so obtained has value zero on all simplices outside p. At p the value is therefore $\overline{\gamma}(V_k) = k.1 \in \mathbf{Z}_{28}$.

Theorem 2. — Every class of combinatorial modulo smoothable 8-manifolds can be represented by a complex algebraic hypersurface $V_k \subset P^5(\mathbf{C})$, $k=1, 2, \ldots, 28$. *Among these, only* V_{28} *is homeomorphic to a smooth manifold. In particular the algebraic variety* $V_1 \in P^5(\mathbf{C})$ *with affine equation*

(8) $$z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 + \sum_{j=1}^{5} e^{j-1} z_j^6 = 0$$

is a topological 8-manifold without any smoothing.

Proof. — By theorem 1 it is sufficient to prove the existence of $d \in H^2(V_k, \mathbf{Z})$ reducing to $w_2(V_k) \in H^2(V_k, \mathbf{Z}_2)$. For a sufficiently close approximation of equation (8) we can obtain a complex manifold without singularity Y with the following properties.

There exists a manifold with boundary V_k^0 obtained from V_k by deleting a small disc containing the singularity. $i : V_k^0 \to V_k$ is the inclusion. There is an embedding $j : V_k^0 \to Y$, because outside some neighbourhood of the singularity, V_k and Y are near to each other with first derivatives included and hence diffeomorphic. Now the first Chern class $c_1(Y) \in H^2(Y, \mathbf{Z})$ reduces to $w_1(Y) \in H^2(Y, \mathbf{Z}_2)$. Hence $d = (i^*)^{-1} j^* c_1(Y) \in H^2(V_k, \mathbf{Z})$ reduces to $w_1(V_k) = (i^*)^{-1} j^* w_1(Y) \in H^2(V_k, \mathbf{Z}_2)$.

§ 4. Nonorientable 8-manifolds

We first recall the nonorientable version of Wall's theorem.

Theorem. — The connected nonorientable closed combinatorial 8-manifolds modulo smooth manifolds form a group of two elements.

Proof. — Let X be a nonorientable connected 8-manifold with k-skeleton X^k. We smooth some neighbourhood of X^7 as before, such that the non-smoothability of X is concentrated in one 8-simplex Δ. On this oriented 8-simplex let it be given by $x \in \Gamma_7$. As X is nonorientable, there exists (assuming the triangulation of X fine enough) a sequence of 8-simplices $\Delta^{(i)}$, $i=1, \ldots, N+1$ with $\Delta^{(1)}=\Delta^{(N+1)}=\Delta$, $\Delta_7^{(i)}=\Delta^{(i)} \cap \Delta^{(i+1)}$ is a common face, such that the union $\bigcup_{i=1}^{N} \Delta^{(i)}$ is a nonorientable neighbourhood of a closed curve in X. Any element $y \in \Gamma_7$ can be represented by a change of smoothing in the oriented face $\Delta_7^{(1)}$ of Δ, which can be neutralized with respect to smoothability of $\Delta^{(2)}$ by a suitable change of smoothing in $\Delta_7^{(2)}$. Etc. After coming back to $\Delta^{(N+1)}=\Delta$ the non-smoothability is again completely concentrated in the oriented 8-simplex Δ, but represented with value $x-y+(-y)=x-2y \in \Gamma_7$.

In the nonorientable case the 8-simplices of $X^{(8)}$ have no preferred orientation. Then reducing the constant local coefficient sheaf $\Gamma_7 \simeq \mathbf{Z}_{28}$, modulo 2, there remains from the theory in § 1, a $\mathbf{Z}_2$-cocycle $\gamma(X, \mathbf{Z}_2)$ in $H^8(X, \mathbf{Z}_2)$ which is an invariant of the nonorientable manifold X. In order to be able to smooth X, it is necessary that $\gamma(X, \mathbf{Z}_2)$ vanishes. But above we have seen that it is also sufficient: Take y such that $2y=x \in \Gamma_7$. From the construction as in § 1 it is seen that $\gamma(X \# Y, \mathbf{Z}_2)=\gamma(X, \mathbf{Z}_2)+\gamma(Y, \mathbf{Z}_2) \in \mathbf{Z}_2$, for X and Y orientable or not. Then the theorem follows. Formally the obstruction to smoothing lies in $H^8(X, \Gamma_7)=H_0(X, \text{orientation} \otimes \Gamma_7)=H_0(X, \mathbf{Z}_2)=\mathbf{Z}_2$.

Theorem 2. — The real algebraic 8-variety $W_1 \in P^{10}(\mathbf{R})$ *in real projective 10-space with affine equations in* $x_1, y_1, \ldots, x_5, y_5$:

$$z_1^5+z_2^3+z_3^2+z_4^2+z_5^2+\sum_{j=1}^{5} e^{j-1} z_j^6=0 \tag{9}$$

$$z_j=x_j+iy_j$$

is a closed nonorientable combinatorial 8-manifold without compatible smoothing. It represents the nonsmoothable class.

N.B. — In this nonorientable case we cannot decide that the manifold is not homeomorphic to any smooth manifold (with a different combinatorial structure).

Proof. — W_1 is an algebraic real variety with exactly one singularity of type $\pm 1 \in \Gamma_7$. At this singularity W_1 is a combinatorial manifold and not smooth. There remains to prove that W_1 is nonorientable.

Take a suitable diffeomorphism of $\mathbf{C}^5=\mathbf{R}^{10}$ onto the open ball $|z|<1$, which commutes with rotations, leaves each real half ray from o invariant, and is the identity near $o \in \mathbf{C}^5$. Let $\mathring{W}$ be the image of $W \cap \mathbf{R}^{10}$. $\mathring{W}$ can be closed by the 7-manifold $\partial\mathring{W}$:

$$(\sum_j e^{j-1} z_j^6 = 0) \cap (\sum_j z_j \bar{z}_j) = 1.$$

The diametrical map $\delta : (z_1, \ldots, z_5) \to (-z_1, \ldots, -z_5)$ leaves $\mathring{W}$ invariant and preserves orientation in $\mathring{W}$ as well as in $\mathbf{R}^{10}$. Now W is essentially obtained from $\mathring{W} \cup \partial\mathring{W}$ by identifying diametrical points in $\partial\mathring{W}$. [This is analogous to obtaining $P^{10}(\mathbf{R})$ from $\sum_j z_j \bar{z}_j \leq 1$ by identifying diametrical points on $\sum_j z_j \bar{z}_j = 1$.] Hence W is nonorientable.

Remark. — If a manifold with one singularity is " exotic " at that singularity, then it still may globally admit some smoothing. For example this is the case with the variety $W_2 \subset P^{10}(\mathbf{R})$ with real affine equations

$$\left.\begin{array}{l} z_1^{11} + z_2^3 + z_3^2 + z_4^2 + z_5^2 + \sum_{j=1}^{5} e^{j-1} z_j^{12} = 0 \\ z_j = x_j + i y_j \end{array}\right\}.$$

It has the same exotic singularity at o as V_2. The same holds for any nonorientable 8-manifold with one singularity, in case that singularity is like that of V_k for some *even* $k \neq 0 \bmod 28$.

Exercise. — If the oriented 8-manifold X admits an orientation reversing combinatorial involution without fixed point, then it has a smoothing.

§ 5. Formulation of the main theorem. A lemma on polynomial approximation.

A closed connected, C^∞-manifold X, C^∞-embedded in $\mathbf{R}^n$, is called a *Nash manifold*, if there exists a polynomial map $g : \mathbf{R}^n \to \mathbf{R}^q$ for some q, and $X \subset g^{-1}(0) \subset \mathbf{R}^n$ with $\dim X = \dim g^{-1}(0)$. A C^∞-map $f : X \to Y$ between Nash manifolds $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ is called a *morphism*, if its graph $\{(x, f(x)) : x \in X\} \subset \mathbf{R}^m \times \mathbf{R}^n = \mathbf{R}^{m+n}$ is a Nash manifold. We now recall the classical

Theorem of Nash [13, 14]. — *Every closed* C^∞*-manifold* X *admits the structure of a Nash manifold and this structure is unique up to isomorphism.*

As every closed combinatorial manifold of dimension $k \leq 7$ has a compatible C^∞-manifold structure (unique for $k \leq 6$), it also has a Nash-manifold structure (unique for $k \leq 6$). On the 7-sphere S^7 there are 28 Nash-manifold structures as there are 28 differential structures.

If $g : \mathbf{R}^n \to \mathbf{R}^q$ is a polynomial map and X is a real analytic closed subset of $g^{-1}(0)$ of the same dimension as $g^{-1}(0)$, then X is called a Nash space and also a Nash component of $g^{-1}(0)$. A Nash space X which is a topological manifold, and except at one point x_0 a C^∞-manifold, will be called a *Nash manifold with one singularity at* x_0. Examples are described in theorems 2 and 3 above. (In order to meet the definition strictly we have to embed $P^5(\mathbf{C})$ and $P^{10}(\mathbf{R})$ as real algebraic varieties in $\mathbf{R}^N$ for some N.)

In the remaining part of this paper we prove an analogue of Nash's theorem:

Main theorem 4. — *Every closed combinatorial 8-manifold* X *has the structure of a Nash manifold with one singularity, embedded in* $\mathbf{R}^{16}$. *It is a Nash component of the algebraic set* $g^{-1}(0)$ *for some polynomial map* $g: \mathbf{R}^{16} \to \mathbf{R}^q$.

We first prove an important lemma which we need later. For any C^∞-function $f: W \to \mathbf{R}^q$, defined on a neighbourhood W of 0 in $\mathbf{R}^n$, and for any natural number s, we denote by f_s the polynomial function of degree s, which at $0 \in \mathbf{R}^n$ has all derivatives of orders $\leq s$ in common with f. f_s is therefore the *Taylor series of f at* 0, *up to and included terms of degree s.*

Lemma 4. — Let W, with closure $\overline{W}$, and W′ be bounded open sets in $\mathbf{R}^n$ and $0 \in W \subset \overline{W} \subset W'$; $s \geq 0$; $\varepsilon > 0$; $|x| = \sqrt{\sum_{i=1}^{n} (x_i)^2}$ for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$.

For any C^∞*-function* $f: W' \to \mathbf{R}^q$, *there exists a polynomial function* $\psi: W \to \mathbf{R}^q$ *with the same Taylor-s-part at* 0:

$$\psi_s = f_s$$

and ε*-near to f on* W *in the* C^s*-metric*:

$$|(\partial_\alpha \psi)(x) - (\partial_\alpha f)(x)| < \varepsilon \qquad \text{for} \qquad |\alpha| \leq s,\ x \in W.$$

Here, if α is the multiindex $\alpha = i_1, \ldots, i_r$, then

$$\partial_\alpha = \partial_{i_1} \ldots \partial_{i_r} \qquad \text{and} \qquad |\alpha| = r \geq 0.$$

Proof. — Because we can C^∞-extend the restriction of f to $\overline{W}$ over $\mathbf{R}^n$, we may just as well assume that W and W′ are bounded open balls with centre in $0 \in \mathbf{R}^n$. It is well known that given f and $\delta > 0$, there exists a polynomial function Φ, for which

$$(10) \qquad |\partial_\alpha (f - \Phi)(x)| < \delta \qquad \text{for} \qquad |\alpha| \leq s,\ x \in W.$$

We refer to Graves [16] and only recall that Φ can be obtained for example for a sufficiently large integer m, and

$$c_m^{-1} = \int_{|u| \leqslant m} (1 - u^2/m^2)^{m^4} du, \qquad u^2 = \langle u, u \rangle,$$

as the convolution (an averaging process):

$$\Phi(x) = \int_{\mathbf{R}^m} f(u) \cdot c_m \cdot [1 - (x-u)^2/m^2]^{m^4} du$$

with $f(x) = 0$ by definition for $x \notin W'$.

Φ is then a polynomial function of highest degree $\leq 2m^4$.

From (10) we obtain in particular at $0 \in \mathbf{R}^n$:

$$|\partial_\alpha (f - \Phi)(0)| = |\partial_\alpha (f_s - \Phi_s)(0)| < \delta \qquad \text{for} \qquad |\alpha| \leq s.$$

By integrating along half rays starting at $0 \in \mathbf{R}^m$ we see that a constant $C > 0$ exists, such that

$$|\partial_\alpha (f_s - \Phi_s)(x)| < C\delta \qquad \text{for} \qquad |\alpha| < s,\ x \in \overline{W}.$$

C depends only on $\overline{W}$, and not on f or Φ.

Let δ be so small that $(1+C)\delta<\varepsilon$. The required function is then

$$\psi=\Phi+(f_s-\Phi_s). \tag{11}$$

It has the properties:

$$(\partial_\alpha\psi)(0)=(\partial_\alpha f_s)(0)=(\partial_\alpha f)(0) \qquad \text{for} \qquad |\alpha|\leq s,$$

and

$$|(\partial_\alpha\psi)(x)-(\partial_\alpha f)(x)|\leq|(\partial_\alpha(\Phi-f)(x)|+|\partial_\alpha(f_s-\Phi_s)(x)|\leq\delta+C\delta\leq\varepsilon \quad \text{for} \quad |\alpha|\leq s,\ x\in W.$$

§ 6. Construction of an embedding of the closed combinatorial 8-manifold X in $\mathbf{R}^{16}$ as a C^∞-manifold with one specific singularity at $0\in\mathbf{R}^{16}$.

This construction follows completely the proof of Whitney's embedding theorem for C^∞-manifolds. We smooth (see § 1) the complement $X-U_0$ of an open 8-ball U_0 in X. The boundary ∂U_0 is an exotic 7-sphere representing $\overline{\gamma}(X)\in\Gamma_7=\mathbf{Z}_{28}$. Let $k=\overline{\gamma}(X)\bmod 28$ and $0<k<28$. (In the case $k=0$, X is smoothable and we are done by Nash's theorem.) The closed 8-ball $\overline{U}_0$ is embedded by a map i_0 onto the standard model with $n=6k$:

$$i_0(\overline{U}_0)=\{z|f_1(z)=z_1^{n-1}+z_2^3+z_3^2+z_4^2+z_5^2=0 \text{ and } |z|\leq 9c\}\subset\mathbf{C}^5=\mathbf{R}^{10}. \tag{12}$$

For later use we define

$$U_0(t)=i_0(\{z|\ |z|<t\}), \qquad 0<t\leq 9c; \tag{13}$$

∂U_0 has by virtue of i_0 an induced differential structure, which represents $\overline{\gamma}(X)$ by the choice of $n=6k$.

We can assume the two smoothings of ∂U_0 to be equal, and $\overline{U}_0$ and $X-U_0$ can be glued along their common boundary to obtain a C^∞-manifold with one singularity (12) at x_0. From now on we assume this structure in the symbol X. Next we construct an embedding of X in some euclidean space.

It is easy to see that a C^∞-map

$$\kappa_0' : \mathbf{C}^5 \to \mathbf{C}^5\times\mathbf{R}=\mathbf{R}^{11}$$

exists with
$$\begin{cases} \kappa_0'(z)=(z, 0) & \text{for } |z|\leq 8c,\\ \kappa_0'(z)=(0, 1) & \text{for } |z|\geq 9c,\\ \kappa_0' \text{ is a } C^\infty\text{-embedding} & \text{for } |z|<9c.\end{cases}$$

The composition $\kappa_0'\circ i_0$, extended by the constant map $\kappa_0(x)=(0, 1)$ for $x\notin U_0(9c)$, determines a map

$$\kappa_0 : X\to\mathbf{R}^{11},$$

which is C^∞ on $X-\{x_0\}$, C^∞-embedding on $U_0(9c)-\{x_0\}$, and " standard " (see (12)) on $U_0(8c)$.

For any point $x\in X-U_0(9c)$ there is an 8-ball neighbourhood $U_x\subset X-U_0(8c)$ and a C^∞-map

$$\kappa_x : X\to\mathbf{R}^9$$

onto the 8-sphere

$$S^8 = \{(u_1, \ldots, u_9) \in \mathbf{R}^9 : \sum_{j=1}^{9} u_j^2 = 2u_9\},$$

such that the restriction $\kappa_x | U_x$ is a C^∞-diffeomorphism onto $S^8 - \{0\}$, and $\kappa_x(y) = 0 \in \mathbf{R}^9$ for $y \notin U_x$.

A finite number of the neighbourhoods U_0 and U_x, say $U_0, U_1, \ldots, U_L$ cover X. Then we obtain a map

$$\kappa : X \to \mathbf{R}^{11+9L}$$

defined by $\kappa(x) = (\kappa_0(x), \kappa_1(x), \ldots, \kappa_L(x))$. κ is an embedding of X onto a C^∞-manifold with one standard singularity $\kappa(U_0(8c)) \subset \mathbf{R}^{10} \times 0 \subset \mathbf{R}^{11+9L}$ near $\kappa(x_0) = 0$.

Finally we decrease the dimension of the target space in the usual manner as follows. The set of chords and tangents of $\kappa(X - \{x_0\})$ is the C^∞-image of a 17-manifold. It is nowhere dense for $11 + 9L > 17$ by Sard's theorem. We then can project $\kappa(X)$ from some point into that linear subspace of $\mathbf{R}^{11+9L}$ on which the last coordinate vanishes, and we obtain an analogous embedding. This process can be repeated until we get an embedding in $\mathbf{R}^{17}$. One more projection yields an immersion with isolated transversal self-intersections in $\mathbf{R}^{16}$. The self intersections can be removed by Whitney's method [15], to obtain the required embedding. Observe that during this process the embedding of $U_0(8c)$ remains unchanged.

From now on we identify X with $\kappa(X) \subset \mathbf{R}^{16}$, the embedded manifold with standard part $U_0(8c) = \kappa(U_0(8c)) \subset \mathbf{R}^{10} \times 0$. So we have a diagram of inclusions:

$$\begin{array}{ccc} U_0(8c) & \longrightarrow & \mathbf{R}^{10} \times 0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{R}^{16} \end{array} \tag{14}$$

§ 7. C^∞-equations for $X \subset \mathbf{R}^{16}$ (1).

In this paragraph we define a diagram of C^∞-maps

$$\begin{array}{ccccc} W & \xrightarrow{\bar{\alpha}} & A_8 \subset G_8 \times \mathbf{R}^{16} & \xrightarrow{p_2} & \mathbf{R}^{16} \\ & & \downarrow p & & \\ X^8 - U_0(c) & \xrightarrow{\alpha} & G_8 & & \end{array} \tag{15}$$

W an open neighbourhood of $X \subset \mathbf{R}^{16}$, such that

$$X^8 = (p_2 \bar{\alpha})^{-1}(0) \cap W. \tag{16}$$

(1) The constructions in § 7 and § 8 are analogous to those of Thom [14] concerning smooth manifolds.

The map $p_2\bar{\alpha}$ therefore determines a set of 16 equations for X^8. $\bar{\alpha}$ has a *singularity* at o, and it is *transversal* to $G_8 \subset A_8$ at all other points of W.

We first define $\bar{\alpha}$ on certain parts $W(t)$ of W.

Consider the normal bundle of $X-U_0(c)$ in $\mathbf{R}^{16}$. The normal exponential map nexp : $(x, \nu) \mapsto x+\nu \in \mathbf{R}^{16}$ is a C^∞-map of its total space into $\mathbf{R}^{16}$. Here ν is a normal vector at $x \in X-U_0(c)$.

There is a constant ε_1 such that the restriction of nexp to the space of normal vectors of length smaller than ε_1, is a diffeomorphism, with image the tubular neighbourhood (= total ε_1-ball bundle space):

$$W(t) = \text{nexp}\{(x, \nu) \mid x \in X-U_0(t), |\nu| < \varepsilon_1\} \qquad c \leq t \leq 9c.$$

The projection of the ε_1-ball bundle is called

$$\mu : W(t) \to X^8 - U_0(t).$$

Let G_8 be the Grassmann manifold of all 8-dimensional vector subspaces of $\mathbf{R}^{16}$, and A_8 the total space of the corresponding open ε_1-ball bundle p:

$$\begin{array}{l} A_8 = \{(g, \nu) \in G_8 \times \mathbf{R}^{16} \mid \nu \in g, |\nu| < \varepsilon_1\} \subset G_8 \times \mathbf{R}^{16} \\ \downarrow p \\ G_8 = G_8 \times o. \end{array}$$

The tangent vector spaces at different points of $\mathbf{R}^{16}$ are all identified with $\mathbf{R}^{16}$. μ is induced from p by the natural bundle map

$$\begin{array}{ccc} W(t) & \xrightarrow{\bar{\beta}} & A_8 \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle p} \\ X^8 - U_0(t) & \xrightarrow{\beta} & G_8 \end{array}$$

where $\beta(x)$ is the normal vector space at x in $\mathbf{R}^{16}$ with respect to X^8, and

$$\bar{\beta}(\text{nexp}(x, \nu)) = (\beta(x), \nu).$$

G_8 is identified with the o-section of p.

We will modify the map β and obtain a map α which is constant near $\partial U_0(5c)$.

The space $\overline{U_0(8c)}-U_0(5c)$ is diffeomorphic with $\Sigma^7 \times I$, the product space of the exotic 7-sphere Σ^7 and a segment. Because $U_0(8c)$ is contained in $\mathbf{R}^{10} \times o \subset \mathbf{R}^{16}$, the normal bundle of this product-space part of X is the direct sum of a trivialized 6-plane bundle and a orientable trivial 2-plane bundle. Then by fibre-bundle theory [$\pi_7(G_{2,8}) = 0$ for $G_{2,8}$ the Grassmann space of 2-planes in $\mathbf{R}^{10}$] there is a C^∞-map α, whose restriction to $\overline{U_0(8c)}-U_0(5c)$ is a homotopy:

$$X^8 - U_0(5c) \xrightarrow{\alpha} G_8,$$

such that: $$\alpha(x)=\begin{cases}\beta(x) & \text{for} \quad x\in X-U_0(7c)\\ g_0 & \text{for} \quad x\in U_0(6c)-U_0(5c).\end{cases}$$

g_0 is the 8-plane $0\times\mathbf{R}^2\times\mathbf{R}^6\subset\mathbf{R}^{16}$;

$\alpha(x)$ is an 8-plane containing the 6-plane, $0\times 0\times\mathbf{R}^6\subset\mathbf{R}^{16}$ for all $x\in U_0(8c)-U_0(5c)$.

The bundle induced from p by α is equivalent to that induced from p by β. Hence we may identify both induced bundles and we have an orthogonal ε_1-ball bundle map, which is a C^∞-map of pairs,

(17) $$\begin{array}{ccc} W(5c) & \xrightarrow{\bar\alpha} & A_8 \\ \Big\downarrow u & & \Big\downarrow p \\ X^8-U_0(5c) & \xrightarrow{\alpha} & G_8 \end{array}$$

and

(17') $$\alpha(U_0(6c)-U_0(5c))=g_0, \qquad \bar\alpha(w)=\bar\beta(w) \quad \text{for} \quad w\in W(7c).$$

A_8 contains the fibre $p^{-1}(g_0)$ which is an ε-ball in $\mathbf{R}^8$. We will in the sequel extend the fibre-bundle map over the base space $U_0(5c)-U_0(3c)$ by a map $(\bar\alpha, \alpha)$ for which α takes the constant value g_0. We will further extend $\bar\alpha$ over some neighbourhood of $U_0(3c)$ in $\mathbf{R}^{16}$, by a map with all values in the ε-ball $p^{-1}(g_0)\mathbf{R}^8$.

In order to define $\bar\alpha$ *near the singularity, we start from the map*

$$f_1 : \mathbf{R}^{10}\to\mathbf{R}^2,$$

which was defined in terms of complex variables by (1) in § 3. Observe that for $t\leq 8c$:

$$f_1^{-1}(0)\cap\{z \mid |z|<t\}=U_0(t)\subset\mathbf{R}^{10}=\mathbf{R}^{10}\times 0\subset\mathbf{R}^{16}.$$

Near to the singularity, that is for some small enough neighbourhood of $U_0(c)$ in $\mathbf{R}^{16}$, we define

(18) $$\bar\alpha=f_1\times \mathrm{id} : \mathbf{R}^{10}\times\mathbf{R}^6\to\mathbf{R}^2\times\mathbf{R}^6.$$

Here id is the identity map of $\mathbf{R}^6$.

For small $\varepsilon>0$ and $B(\varepsilon)=\{y\in\mathbf{R}^2 \mid |y|<\varepsilon\}$, f_1 determines a framing and a trivial fibre bundle with fibres diffeomorphic to $B(\varepsilon)$, group the group of diffeomorphisms of $B(\varepsilon)$, base space $\overline{U_0(4c)}-U_0(c)$, and fibre over x:

$$F_x=f_1^{-1}(B(\varepsilon))\cap\{\mathrm{nexp}(x,\nu) \mid \nu \text{ normal vector at } x\}.$$

F_x is contained in a unique linear two-dimensional variety $L_x\subset\mathbf{R}^{10}$. Let the framing map $\pi_x : B(\varepsilon)\to L_x$ be defined as the inverse of

$$(f_1|F_x) : F_x\to B(\varepsilon).$$

We want to modify π_x (hence f_1) to obtain isometries for $x\in U_0(4c)-U_0(3c)$. Because the oriented differentiable embeddings of $B(\varepsilon)$ in $\mathbf{R}^2$ with fixed origin, retract by deformation into the orthogonal group SO(2), there is for each x a homotopy of C^∞-embeddings

$$\pi_{x,t}: B(\varepsilon) \to L_x, \qquad \pi_{x,t}(0) = x,$$

starting with $\pi_{x,0} = \pi_x$ and ending with an isometry $\pi_{x,1}$. We can choose it such that the mapping $\pi_{x,t}$ depends C^∞ on x and t, and $\pi_{x,t}$ is constant with respect to t for $0 < t < \frac{1}{3}$ and for $\frac{2}{3} < t \leq 1$.

Now we are ready to replace f_1 by a new map α_0. For $x \in \overline{U_0(4c)} - U_0(c)$ let t implicitly be given by

$$x \in \partial U_0(c + 3tc), \qquad 0 \leq t \leq 1.$$

Let $y \in \pi_{x,t}(B(\varepsilon)) \subset L_x \subset \mathbf{R}^{10}$.

Now put $$\alpha_0(y) = (\pi_{x,t})^{-1}(y) \in \mathbf{R}^2.$$

We continue the definition of $\overline{\alpha}$. In some neighbourhood of $\overline{U_0(4c)} - U_0(0)$ in its tubular normal bundle space in $\mathbf{R}^{16}$, we put

$$\overline{\alpha} = \alpha_0 \times \text{id}. \tag{19}$$

Here again id is the identity map of $\mathbf{R}^6$. Observe that (19) agrees with (18).

Over the part $\overline{U_0(4c)} - U_0(3c)$ and over the part $\overline{U_0(6c)} - U_0(5c)$ the mapping $\overline{\alpha}$ into $\mathbf{R}^8 = \mathbf{R}^2 \times \mathbf{R}^6 = p^{-1}(g_0)$ determines orthogonal trivializations of the normal tangent bundle, each splitting of the same trivial trivialization in the vector spaces *parallel* to $0 \times \mathbf{R}^6 \subset \mathbf{R}^{16}$. Recall for this that $\overline{U_0(8c)} \subset \mathbf{R}^{10} \times 0 \subset \mathbf{R}^{16}$. These trivializations therefore reduce to trivializations of 2-plane bundles essentially over seven-spheres. They are homotopic.

The trivializations of the tubular neighbourhoods over $\overline{U_0(4c)} - U_0(3c)$ and over $\overline{U_0(6c)} - U_0(5c)$ as orthogonal ε_1-ball bundles (for ε_1 small enough), correspond one-to-one to the orthogonal trivializations of the normal tangent bundles. Therefore $\overline{\alpha}$ can be extended over the normal tubular bundle over $\overline{U_0(5c)} - U_0(4c)$ in $\mathbf{R}^{16}$ by a map into $\mathbf{R}^8$, which is also isometric on each fibre.

Taking the map $\overline{\alpha}$ of differentiability class C^∞ we have obtained, with (17), (18) and (19), for some neighbourhood W of X in $\mathbf{R}^{16}$, the map

$$w \xrightarrow{\overline{\alpha}} A_8. \tag{15}$$

The restriction to W(3c) is an orthogonal bundle map:

$$\begin{array}{ccc} W(3c) & \xrightarrow{\overline{\alpha}} & A_8 \\ \downarrow & & \downarrow \\ X - U_0(3c) & \xrightarrow{\alpha} & G_8 \end{array} \tag{15 a}$$

The restriction to $W - W(6c)$ is into $p^{-1}(g_0) \subset \mathbf{R}^8$:

(15 *b*) $$W - W(6c) \xrightarrow{\bar{\alpha}} p^{-1}(g_0) \subset \mathbf{R}^8.$$

The restriction to $W - W(c)$ is:

(15 *c*) $$f_1 \times \mathrm{id}.$$

Now the diagram (15) and the properties mentioned after (15) follow immediately.

§ 8. Algebraic equations for X.

We consider again diagram (15). Let G_8 be embedded as an algebraic submanifold in some euclidean space $\mathbf{R}^M$. The normal exponential map defines a tubular neighbourhood Y with radius ε (sufficiently small) of G_8, and with an *algebraic* orthogonal projection (a retraction)

(20) $$\rho : Y \to G_8 \subset Y^M$$

$\rho(y)$ is the point in G_8 that is nearest to $y \in Y$. We now extend diagram (15) by natural inclusions

(21) $$\begin{array}{ccccccc} W & \xrightarrow{\bar{\alpha}} & A_8 \subset G_8 \times \mathbf{R}^{16} \subset & Y \times \mathbf{R}^{16} & \subset \mathbf{R}^{M+16} & & \\ & & \downarrow & \swarrow^{p_1} \quad \searrow^{p_2} & & & \\ & & G_8 \xrightarrow{\subset} Y & & \mathbf{R}^{16} & & \end{array}$$

The retraction ρ in (20) can be covered by a *retraction* $\bar{\rho}$, which is also algebraic:

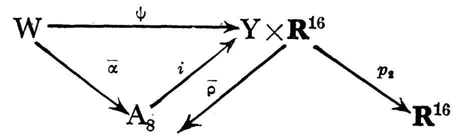
(22) $$\begin{array}{ccc} Y \times \mathbf{R}^{16} & \xrightarrow{\bar{\rho}} & A_8 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\rho} & G_8 \end{array}$$

It is defined by the condition that $\bar{\rho}(y, z)$ is the orthogonal projection of the point $(\rho(y), z) \in Y \times \mathbf{R}^{16}$ which lies in the euclidean 16-space $p_1^{-1}(\rho(y))$, into the euclidean sub-8-space $p^{-1}(\rho(y)) \subset A_8$.

We now call $W : W'$, and let $W \subset \overline{W} \subset W'$ be a smaller analogous neighbourhood of $X \subset \mathbf{R}^{16}$. Then we apply lemma 4 to the map

$$W' \xrightarrow{i\bar{\alpha}} Y \times \mathbf{R}^{16} \subset \mathbf{R}^M \times \mathbf{R}^{16}.$$

We obtain a polynomial map ψ, arbitrary C^S-near to $i\bar{\alpha}$ on W and with the same s-jet at $o \in W \in \mathbf{R}^{16}$. The image of this s-jet therefore lies in A_8! We now have the noncommutative diagram



Let g be the *algebraic* map $\quad g = p_2 i \bar{\rho} \psi : W \to \mathbf{R}^{16}$.

Then $X_1 = (\bar{\sigma}\psi)^{-1}(G_8) = g^{-1}(o) \subset W$ is the required Nash-manifold with one singularity. It is a real analytic manifold with one singularity, locally defined by algebraic equations. (It is an open problem also for smoothable n-manifolds, whether X_1 can be a topological component, or even the whole, of the set of zeros of a set of polynomials. See [13]. If X_1 has a trivial tangent bundle then it can be a topological component.)

The maps $\bar{\alpha}$ and $\bar{\rho}\psi : W \to A_8$ have the same s-jet at o. From Malgrange's preparation theorem [17], as applied to ideals of C^∞-functions by Tougeron [18] and Mather [19], it follows that there exists for s large enough a C^∞-diffeomorphism $\zeta : U \to \zeta(U)$, defined on some neighborhood U of o in $\mathbf{R}^{16}$, as C^2-near as we please to the identity map, such that $\zeta[(p_2\bar{\alpha})^{-1}(o) \cap U] = g^{-1}(o) \cap \zeta(U)$. The singularities of X and X_1 at o are therefore of the same exotic kind. The restriction $\bar{\alpha} | (W - \{o\})$ is transversal to $G_8 \subset A_8$. Hence for any choice of neighborhood U′ of o in $\mathbf{R}^{16}$ also the restriction $\bar{\rho}\psi | (W - U')$ is transversal to G_8 in case ψ is C^1-near enough to $i\bar{\alpha}$.

The map which assigns to any point of $X - U_0(c')$ (c' small) the unique nearest point of X_1, defines a diffeomorphism, C^2-near to the identity map restricted to $X - U_0(c')$. This diffeomorphism can be extended over $X - \{o\}$ such that it equals ζ near o.

Consequently X and X_1 are combinatorially equivalent, and g is the polynomial map required in theorem 4.

We conclude with the formulation of two problems:

Problem. — Which combinatorial 8-manifolds admit a complex manifold structure with one Hirzebruch singularity?

Problem. — Which combinatorial 8-manifolds can be embedded as Nash manifold with one Hirzebruch singularity in a low dimensional euclidean space?

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