ROBIN HARTSHORNE

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AMPLE VECTOR BUNDLES

by ROBIN HARTSHORNE (1)

CONTENTS

§ 0. Introduction.

For line bundles (or for divisor classes, to which they correspond in a one-to-one way) on a projective variety, the notion of amplitude has been quite well studied [9, 13]. The purpose of this paper is to extend these ideas to vector bundles of arbitrary rank, and to transport as much of the theory as possible.

In § 2 we give the definition of an ample vector bundle in terms of sheaves generated by global sections. In § 3 we show that this condition is equivalent to three other quite natural conditions which generalize criteria for ample line bundles (Propositions 3.2, 3.3, and 3.5). Therefore this seems to be a " good " notion of ample vector bundle. In § 4 we give various properties of ample bundles under change of base scheme.

The multiplicative properties of ample bundles seem to be more subtle than the additive ones. In particular, we do not know whether the tensor product of two ample bundles is ample. However in characteristic zero we can show this is true (§ 5) using the theory of finite dimensional representations of the general linear group, which we include for convenience in an Appendix. In the case of characteristic $p \neq 0$ we introduce

(1) Junior Fellow, Harvard University.

the notion of a p-ample bundle, and prove that the tensor product of two p-ample bundles is p-ample (§ 6). Every p-ample bundle is ample, but we do not know whether the converse is true, except in the case of curves (§ 7). On a curve we have a fairly good grasp on the ample bundles, and can give necessary and sufficient conditions for a bundle of rank two on a curve in characteristic zero to be ample (Corollary 7.6).

In the last section (§ 8) we give a theorem on finite dimensionality of certain cohomology groups of non-proper schemes, which indeed was one of the principal motivations for developing the theory of ample vector bundles.

There are still many problems left open. One is to study the conditions for a high symmetric product of a bundle to give an embedding into the Grassmannian (which is not equivalent to being ample: see Remark at end of § 3). Another is to study numerical properties of ample bundles. For example, if E is an ample bundle, then is the i^{th} Chern class $c_i(E)$ numerically ample in the sense that it has positive intersection with every effective cycle of complementary dimension? Even for an ample bundle E of rank 2 on a non-singular projective surface one does not know whether $c_2(E)>0$. We hope to study these questions more fully in a later paper.

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§ 1. Review of ample line bundles.

In this paper we will use the work " bundle " as a synonym for " locally free sheaf of finite rank ". Thus " line bundle " will mean " locally free sheaf of rank one " or what is the same thing " invertible sheaf ". When we wish to speak of the scheme bundle associated to the locally free sheaf E, we will use the notation **V**(E) [EGA, **II**, 1.7.8].

We will deal exclusively with schemes of finite type over an algebraically closed field k, and leave to the interested reader the pleasure of relaxing these perhaps excessively strict hypotheses.

In this section we recall without proof various known results on ample line bundles.

Definition (cf. [EGA, **II**, 4.5.5]). — *Let* X *be a scheme (of finite type over k algebraically closed, as always), and let* L *be a line bundle on* X. *We say* L *is ample if for every coherent sheaf* F *on* X, *there exists an integer* n_0, *such that for every* $n \geqslant n_0$, *the sheaf* $F \otimes L^{\otimes n}$ *is generated by its global sections (as an* $\mathcal{O}_X$*-module).*

Proposition (**1.1**). — *With* X *as above, and* L *a line bundle on* X, *the following conditions are equivalent :*

(i) L *is ample.*

(ii) *For some* $n>0$, $L^{\otimes n}$ *is very ample*, i.e. $L^{\otimes n}=j^*(\mathcal{O}_P(1))$ *for some immersion* $j : X \to P = \mathbf{P}^n_k$ *of* X *into a suitable projective space over* k.

[EGA, **II**, 4.4.2 and **II**, 4.5.10.]

(iii) (Grauert's criterion). *Let* $\mathbf{V}(L)=\operatorname{Spec}\sum_{n\geqslant 0} L^{\otimes n}$ *be the vector bundle scheme of* L, *and let* $j: X\to\mathbf{V}(L)$ *be the zero section. Then there exists a scheme* C *(of finite type over k) and a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{j} & \mathbf{V}(L) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \operatorname{Spec} k & \xrightarrow{\varepsilon} & C \end{array}$$

such that g restricted to $\mathbf{V}(L)-j(X)$ *is an open immersion into* $C-\varepsilon(\operatorname{Spec} k)$. *(For short, we say " the zero-section of* $\mathbf{V}(L)$ *can be collapsed to a point ".)*

[EGA, **II**, 8.9.1.]

Proposition (**1.2**). — *If moreover* X *is proper over k, then* (i), (ii) *and* (iii) *above are equivalent to*

(iv) *For every coherent sheaf* F *on* X, *there is an integer* n_0 *such that for all* $n\geqslant n_0$ *and all* $i>0$,

$$H^i(X, F\otimes L^{\otimes n})=0$$

[EGA, **III**, 2.6.1.]

Proposition (**1.3**). — *Let* X *be a scheme, and let* L, M *be line bundles. Then :*

a) *If* $n>0$ *is an integer,* L *is ample* $\Leftrightarrow L^{\otimes n}$ *is ample.*

b) *If* L, M *are ample, then* $L\otimes M$ *is ample.*

c) *If* L *is ample, and* M *is arbitrary, then* $L^{\otimes n}\otimes M$ *is ample for large enough n,*

[EGA, **II**, 4.5.6, **II**, 4.5.7 and **II**, 4.5.8.]

Proposition (**1.4**) (Nakai's criterion). — *Let* X *be projective and non-singular over k. Let x be the class of a divisor* D. *Then* D *is ample* $\Leftrightarrow$ *for every numerical equivalence class y of cycles of dimension* $r>0$ *containing a positive cycle,* $x^r y>0$.

[13, theorem 4.]

§ 2. Definition and elementary properties of ample bundles.

Definition. — *A vector bundle* E *on a scheme* X *is ample if for every coherent sheaf* F, *there is an integer* $n_0>0$, *such that for every* $n\geqslant n_0$, *the sheaf* $F\otimes S^n(E)$ *(where* $S^n(E)$ *is the* n^{th} *symmetric product of* E) *is generated as an* $\mathcal{O}_X$*-module by its global sections.*

Proposition (**2.1**). — *In order for* E *to be ample, it is sufficient that for every coherent sheaf* F, *and for every closed point* $x\in X$, *there be an integer* $n_0>0$ *such that for every* $n\geqslant n_0$ *the global sections of the sheaf* $F\otimes S^n(E)$ *generate its stalk at the point x, as a module over the local ring at that point.*

Proof. — Let E be a bundle which satisfies the condition of the proposition. Fix a point $x\in X$, and a coherent sheaf F. First we apply the statement to $\mathcal{O}_X$, and conclude that for some $n_1>0$, $S^{n_1}(E)_x$ is generated by global sections. Hence there is also a neighborhood U_1 of x in which the sheaf $S^{n_1}(E)$ is generated by global sections.

Now we apply the statement to F at the point x, and conclude that there is an $n_2>0$ such that for all $n\geqslant n_2$, $((F\otimes S^n(E))_x$ is generated by global sections. For each $r=0, 1, \ldots, n_1$ there is therefore a neighborhood V_r of x such that the sheaf $F\otimes S^{n_2+r}(E)$ is generated by global sections in V_r. Let

$$U_x=U_1\cap V_0\cap V_1\cap\ldots\cap V_{n_1}.$$

Then for any $m>0$, the sheaf

$$F\otimes S^{n_2+r}(E)\otimes S^{n_1}(E)^{\otimes m}$$

is generated by global sections in U_x. Since any large enough n can be written in the form n_2+r+n_1m, we conclude that, for all large enough n (say $n\geqslant n(x)$), $F\otimes S^n(E)$ (which is a quotient of the above) is generated by global sections in U_x.

Do this for each $x\in X$. By quasi-compacity we can cover X with a finite number of these neighborhoods, say $U_{x_1}, \ldots, U_{x_s}$. Take $n_0=\max(n(x_i))$. Then for $n\geqslant n_0$, $F\otimes S^n(E)$ is generated by global sections, so E is ample.

Proposition **(2.2)**. — *Any quotient of an ample bundle is ample. If* E_1, E_2 *are two bundles, then* $E_1\oplus E_2$ *is ample if and only if* E_1 *and* E_2 *are both ample.*

Proof. — For the first statement, let E'' be a quotient of an ample bundle E. Then given a coherent sheaf F, $F\otimes S^n(E)$ is generated by global sections for large enough n. But $S^n(E'')$ is a quotient of $S^n(E)$. Hence $F\otimes S^n(E'')$ is generated by global sections. This remark proves the " only if " part of the second statement.

Thus we have to prove that if E_1 and E_2 are ample, then $E_1\oplus E_2$ is ample. Since

$$S^n(E_1\oplus E_2)=\sum_{\substack{p+q=n\\ p,q\geqslant 0}} S^p(E_1)\otimes S^q(E_2),$$

it is sufficient to prove that if F is a coherent sheaf on X, then there is an integer n_0 such that if $p+q\geqslant n_0$, then the sheaf

$$F\otimes S^p(E_1)\otimes S^q(E_2)$$

is generated by global sections.

Given F, we proceed as follows:

a) Choose $n_1>0$ such that for $n\geqslant n_1$, $S^n(E_1)$ is generated by global sections (which is possible since E_1 is ample).

b) Choose $n_2>0$ such that for $n\geqslant n_2$, $F\otimes S^n(E_2)$ is generated by global sections (E_2 ample applied to F).

c) For each $r=0, 1, \ldots, n_1-1$, choose m_r such that for $n\geqslant m_r$,

$$F\otimes S^r(E_1)\otimes S^n(E_2)$$

is generated by global sections (E_2 ample applied to $F\otimes S^r(E_1)$).

d) For each $s=0, 1, \ldots, n_2-1$, choose l_s such that for $n\geqslant l_s$,

$$F\otimes S^n(E_1)\otimes S^s(E_2)$$

is generated by global sections (E_1 ample applied to $F\otimes S^s(E_2)$).

Now take $n_0 = \max_{r,s}(r+m_r; l_s+s)$. I claim that if $p+q \geqslant n_0$, then

$$F \otimes S^p(E_1) \otimes S^q(E_2)$$

is generated by global sections. Indeed, there are three cases:

1) Suppose $p<n_1$. Then $p+q \geqslant n_0 \geqslant p+m_p$, so $q \geqslant m_p$, and we are done, by *c)*.

2) Suppose $q<n_2$. Then similarly $p \geqslant l_q$, and we have the situation of *d)*.

3) If $p \geqslant n_1$, and $q \geqslant n_2$, then $S^p(E_1)$ and $F \otimes S^p(E_2)$ are generated by global sections, by *a)* and *b)*, so their tensor product is also.

q.e.d.

Corollary **(2.3)**. — *If* E_1 *is ample, and* E_2 *is generated by global sections, then* $E_1 \otimes E_2$ *is ample.*

Proof. — Since E_2 is generated by global sections and since it is a coherent sheaf on a quasi-compact scheme X, we can write it as a quotient of a trivial bundle of finite rank [EGA, **0,** 5.2.3]

$$\mathcal{O}_X^r \to E_2 \to 0.$$

Tensoring with E_1 we have

$$E_1^r \to E_1 \otimes E_2 \to 0.$$

Now E_1^r is ample by the proposition, and hence $E_1 \otimes E_2$ is also ample.

Proposition **(2.4)**. — *If* E *is ample, then* $S^n(E)$ *is ample for all large enough n. Conversely, if* $S^n(E)$ *is ample for some n, then* E *is ample.*

Proof. — If E is ample, then $S^{n-1}(E)$ is generated by global sections, for all large enough n. Therefore, by the Corollary, $E \otimes S^{n-1}(E)$ is ample, and hence also its quotient $S^n(E)$.

Conversely, suppose $S^m(E)$ is ample for some $m>0$. By what we have just proved, a suitable symmetric power $S^p(S^m(E))$, and hence also its quotient $S^{pm}(E)$, will be ample and generated by global sections. So we may assume $S^m(E)$ is generated by global sections.

Given F coherent, for each $r=0, 1, \ldots, m-1$ choose l_r such that for $n \geqslant l_r$,

$$F \otimes S^r(E) \otimes S^n(S^m(E))$$

is generated by global sections ($S^m(E)$ ample applied to $F \otimes S^r(E)$). Then the quotient sheaf

$$F \otimes S^{mn+r}(E)$$

is also generated by global sections. Taking $n_0 = m \cdot \max(l_r)$, we see that any $k \geqslant n_0$ can be written in the form $k = mn+r$ with $0 \leqslant r \leqslant m$ and $n \geqslant l_r$, hence $F \otimes S^k(E)$ is generated by global sections, and E is ample.

Corollary **(2.5)**. — *Let* E *be a vector bundle, and let* L *be an ample line bundle. Then* E *is ample if and only if* $L^\vee \otimes S^n(E)$ *is generated by global sections for some* $n>0$. *(Here* $L^\vee = \mathcal{H}om(L, \mathcal{O}_x)$ *is the dual of* L.)

Proof. — If E is ample, then $L^\vee \otimes S^n(E)$ is generated by global sections for all large enough n, by definition. Conversely, suppose $L^\vee \otimes S^n(E)$ is generated by global sections. Then since L is ample, $L \otimes L^{\vee} \otimes S^n(E) = S^n(E)$ is ample by Corollary 2.3. Hence E is ample by the proposition.

Corollary **(2.6)**. — *If* E *is ample on* X, *of rank* r, *then its highest exterior power* $\Lambda^r E$ *is also ample.*

Proof. — For suitable $n > 0$, $S^n(E)$ is ample and generated by global sections. Let its rank be s. Then also $S^n(E)^{\otimes s}$ and its quotient $\Lambda^s(S^n(E))$ are ample. But this latter is a line bundle. Therefore, since the only linear representations of $\mathbf{GL}(r)$ are the powers of the determinant, it must be a power of $\Lambda^r E$, say $(\Lambda^r E)^m$. But then $\Lambda^r E$ is also ample.

Remark. — In § 5 we will complete this result, at least in the case of characteristic zero, by showing that all exterior powers, symmetric powers and tensor powers of an ample bundle are ample.

§ 3. Some criteria for ampleness.

In this section we show that a bundle E on X is ample if and only if the tautological line bundle $L = \mathcal{O}_{\mathbf{P}(E)}(1)$ is ample on $\mathbf{P}(E)$ (see [EGA, **II**, 4.1.1] for definition of $\mathbf{P}(E)$). We also generalize two of the conditions of § 1 for ample line bundles, namely the cohomological one, and Grauert's criterion. Finally, we show by example that the expected generalization to Grassmannians of the condition about embeddings in projective space does not work.

Lemma **(3.1)**. — *Let* E *be a vector bundle on* X, *let* $p : \mathbf{P}(E) \to X$ *be the natural projection map, and let* $L = \mathcal{O}_{\mathbf{P}(E)}(1)$ *be the tautological line bundle on* $\mathbf{P}(E)$. *Then for every coherent sheaf* F *on* X *and for every* $n \geq 0$ *we have a natural isomorphism*

$$\alpha : F \otimes S^n(E) \xrightarrow{\sim} p_*(p^*(F) \otimes L^{\otimes n})$$

and

$$R^i p_*(p^*(F) \otimes L^{\otimes n}) = 0$$

for $i > 0$.

Proof. — There is a natural map α as shown [EGA, **II**, 3.3.2], so the question is local on X, and we may assume E is a trivial bundle of rank r, and $X = \operatorname{Spec} A$ affine with A noetherian.

We prove the second statement by descending induction on i. We know already that $R^i p_*(G) = 0$ for $i > r-1$, for every coherent sheaf G on $\mathbf{P}(E)$ [EGA, **III**, 2.2.2]. Suppose by induction we have shown

$$R^{i+1} p_*(p^*(F) \otimes L^{\otimes n}) = 0$$

for every coherent F on X. Then the functor

$$F \mapsto R^i p_*(p^*(F) \otimes L^{\otimes n})$$

is right exact, since p^* and $\otimes L^{\otimes n}$ are both exact functors. Since X is noetherian and affine, we can find a short resolution

$$\mathcal{O}_X^{n_1} \to \mathcal{O}_X^{n_2} \to F \to 0$$

and thus have only to show $R^i p_*(p^*(\mathcal{O}_X)\otimes L^{\otimes n})=0$ for $i>0$, $n\geqslant 0$. This follows from the explicit calculations of cohomology [EGA, **III**, 2.1.12].

For the first statement, we observe that both sides are right-exact functors (by virtue of what we have just proven) and so as above we reduce to the case $F=\mathcal{O}_X$. That is, we must show that

$$\alpha : S^n(E) \to p_*(L^{\otimes n})$$

is an isomorphism. This also follows from the explicit calculations [*ibid.*].

Proposition **(3.2)**. — *Let* E *be a bundle on* X, *and let* L *be the tautological line bundle on* $\mathbf{P}(E)$, *as above. Then* E *is ample on* X *if and only if* L *is ample on* $\mathbf{P}(E)$.

Proof. — *a)* Suppose E is ample on X. Given a coherent sheaf G on $\mathbf{P}(E)$, we must show that $G\otimes L^{\otimes n}$ is generated by global sections, for all large enough n. Let $p:\mathbf{P}(E)\to X$ be the projection. Since L is relatively ample for p [EGA, **II**, 4.4.2 and 4.6.11], there is an $n_1>0$ such that for all $n\geqslant n_1$,

$$p^*p_*(G\otimes L^{\otimes n}) \to G\otimes L^{\otimes n}$$

is surjective. Hence also for all $n>0$,

$$p^*p_*(G\otimes L^{\otimes n_1})\otimes L^{\otimes n} \to G\otimes L^{\otimes (n+n_1)}$$

is surjective. Hence it will be sufficient to prove that the former is generated by global sections for large n. Let $F=p_*(G\otimes L^{\otimes n_1})$. Applying the same argument to $p^*(F)$, we conclude that it is sufficient to show that $p^*p_*(p^*(F)\otimes L^{\otimes n})$ is generated by global sections for n large. But p^* is exact, so it is enough to show $p_*(p^*(F)\otimes L^{\otimes n})$ is generated by global sections for n large. But by the lemma, this is $F\otimes S^n(E)$, which is generated by global sections for n large, since E is ample.

b) Suppose L is ample on $\mathbf{P}(E)$. To show that E is ample, we use the criterion of Proposition 2.1. So let a coherent sheaf F on X and a closed point $x\in X$ be given. We must show that for all large enough n, the stalk $(F\otimes S^n(E))_x$ is generated by global sections of $F\otimes S^n(E)$.

Using Nakayama's Lemma on the local ring $\mathcal{O}_{X,x}$ of x, it will be sufficient to show $F\otimes S^n(E)\otimes \boldsymbol{k}(x)$ is generated by global sections, i.e. that

$$\Gamma(X, F\otimes S^n(E)) \to \Gamma(X, F\otimes S^n(E)\otimes \boldsymbol{k}(x))$$

is surjective.

Using the Lemma, this is equivalent to the condition on $P=\mathbf{P}(E)$ that

$$\Gamma(P, p^*(F)\otimes L^{\otimes n}) \to \Gamma(P, p^*(F)\otimes L^{\otimes n}\otimes \mathcal{O}_Y)$$

be surjective, where $Y=p^{-1}(x)$ and $\mathcal{O}_Y$ is its structure sheaf with the reduced induced structure. (Note $p^*(F\otimes \boldsymbol{k}(x))=p^*(F)\otimes\mathcal{O}_Y$). Using the exact sequence of cohomology, this is equivalent to showing that

$$\beta : H^1(P, I_Y p^*(F)\otimes L^{\otimes n}) \to H^1(P, p^*(F)\otimes L^{\otimes n})$$

is injective for large enough n, where I_Y is the sheaf of ideals of Y. (Note that if X is proper over k, we are now done. For then P will be proper over k, and both of these H^1's will vanish for large n, by Proposition 1.2.)

Choose $r>0$ such that $M=L^{\otimes r}$ is very ample, say $M=\mathcal{O}_P(1)$ for a suitable projective embedding $\rho : P\to \mathbf{P}_k^N$. For each $i=0, 1, \dots, r-1$, extend $p^*(F)\otimes L^{\otimes i}$ to a coherent sheaf G_i on $\overline{P}$, the projective closure of P in $\mathbf{P}_k^N$. Let $Z=\overline{P}-P$, which is closed in $\overline{P}$.

Now apply the exact sequence of local cohomology to the closed subset Z of $\overline{P}$ [10, Corollary 1.9]. We have exact sequences

$$\begin{array}{ccccc} H^1(\overline{P}, I_Y G_i\otimes M^{\otimes n}) & \longrightarrow & H^1(P, I_Y G_i\otimes M^{\otimes n}) & \longrightarrow & H_Z^2(I_Y G_i\otimes M^{\otimes n}) \\ \downarrow & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \gamma} \\ H^1(\overline{P}, G_i\otimes M^{\otimes n}) & \longrightarrow & H^1(P, G_i\otimes M^{\otimes n}) & \longrightarrow & H_Z^2(G_i\otimes M^{\otimes n}) \end{array}$$

For large enough n, the two terms on the left become zero, since $\overline{P}$ is projective and M is very ample. Since Y, being the fibre of a proper morphism, is proper over k, $\overline{Y}$ does not meet Z. Hence $I_Y\cong\mathcal{O}_P$ in a neighborhood of Z, and by excision, γ is an isomorphism [10, Proposition 1.3]. Hence β' is injective for large n. But on P we have

$$G_i\otimes M^{\otimes n}\cong p^*(F)\otimes L^{\otimes(i+nr)}$$

and the expression $i+nr$, for $i=0, \dots, r-1$, and n large, covers all large enough numbers.

q.e.d.

Remark. — In the course of the proof we have in fact established the following result: let P be a scheme over k, let L be an ample invertible sheaf on P, and let Y be a subscheme of P, proper over k. Then for every coherent sheaf F on P, there is an integer n_0 such that for all $n\geq n_0$, the natural map

$$\Gamma(P, F\otimes L^{\otimes n})\to\Gamma(P, F\otimes\mathcal{O}_Y\otimes L^{\otimes n})$$

is surjective.

Proposition (3.3). — *Let* E *be a vector bundle on* X, *and suppose the following condition satisfied:*

(*) *For every coherent sheaf* F *on* X, *there is an integer* $n_0>0$ *such that for* $n\geq n_0$ *and* $i>0$,

$$H^i(X, F\otimes S^n(E))=0$$

Then E *is ample. Conversely, if* E *is ample, and* X *is proper over* k, *then* (*) *holds.*

Proof. — *a)* Assume (*). To show E is ample, we use the criterion of Proposition 2.1. Let F be a coherent sheaf, and $x\in X$ a closed point. Then we must show that for all large enough n, the stalk

$$(F\otimes S^n(E))_x$$

is generated by global sections of $F\otimes S^n(E)$. Using Nakayama's Lemma, it is sufficient to show that

$$\Gamma(X, F\otimes S^n(E))\to\Gamma(X, F\otimes S^n(E)\otimes \boldsymbol{k}(x))$$

is surjective, or that $H^1(X, I_x F\otimes S^n(E))=0$

where I_x is the sheaf of ideals of the subscheme $\{x\}$. But this follows directly from (*) applied to the sheaf $I_x F$.

b) Conversely, assume E is ample, and X proper over k. Then $P = \mathbf{P}(E)$ is also proper over k, and the tautological line bundle L on P is ample, by the previous proposition. Given a coherent sheaf F on X, we use Proposition 1.2 to conclude that there is an $n_0 > 0$ such that for $n \geqslant n_0$ and $i > 0$,

$$H^i(P, p^*(F) \otimes L^{\otimes n}) = 0.$$

Now there is a spectral sequence, for any coherent sheaf G on P,

$$E_2^{pq} = H^p(X, R^q p_*(G)) \Rightarrow E^n = H^n(P, G).$$

Applying this to $G = p^*(F) \otimes L^{\otimes n}$ and using Lemma 3.1, we find that for every $n \geqslant 0$ and $i > 0$,

$$H^i(X, F \otimes S^n(E)) \cong H^i(P, p^*(F) \otimes L^{\otimes n}).$$

Now the right hand member is zero for large n, as we have shown, so the left hand one is also, and we have condition (*).

Remarks. — 1. In condition (*) it is sufficient to take $i = 1$, as one sees from the proof.

2. The converse of this proposition is false without the hypothesis X proper over k, even for line bundles. For example, let X be $\mathbf{P}_k^2$ minus a closed point x. Let $L = \mathcal{O}(1)$ restricted to X, and let $F = \mathcal{O}_X$. Then using the exact sequence of local cohomology

$$H^1(\mathbf{P}^2, F \otimes L^{\otimes n}) \to H^1(X, F \otimes L^{\otimes n}) \to H_x^2(F \otimes L^{\otimes n}) \to H^2(\mathbf{P}^2, F \otimes L^{\otimes n})$$

we see that for $n \geqslant 0$, the two outside terms are zero, so

$$H^1(X, F \otimes L^{\otimes n}) \cong H_x^2(F \otimes L^{\otimes n}).$$

But $F \otimes L^{\otimes n}$ is isomorphic to $\mathcal{O}_X$ in a neighborhood of x, so this latter group is $H_x^2(\mathcal{O}_X) = I_x$ which is the injective hull of $\boldsymbol{k}(x)$ over the local ring $\mathcal{O}_{X,x}$. In particular, it is non-zero. [10, Proposition 4.13.]

Corollary **(3.4)**. — *Let* X *be proper over k and let*

$$0 \to E' \to E \to E'' \to 0$$

be a short exact sequence of vector bundles on X, *with* E′ *and* E″ *ample. Then* E *is ample.*

Proof. — By Proposition 2.2 we know that $E' \oplus E''$ is ample. Hence for each coherent sheaf F there is an $n_0 > 0$ such that for $n \geqslant n_0$ and $i > 0$,

$$H^i(F \otimes S^p(E' \oplus E'')) = 0.$$

In other words, for $p + q \geqslant n_0$, $p, q \geqslant 0$ and $i > 0$,

$$H^i(F \otimes S^p(E') \otimes S^q(E'')) = 0,$$

since $S^n(E' \oplus E'') = \sum_{\substack{p+q=n \\ p,q \geqslant 0}} S^p(E') \otimes S^q(E'')$, and cohomology commutes with direct sums.

Now $S^n(E)$ has a filtration whose successive quotients are $S^p(E')\otimes S^q(E'')$ for $p+q=n$ and $p, q\geqslant 0$. Hence, using the exact sequence of cohomology and proceeding step by step up the filtration, we deduce that for $n\geqslant n_0$ and $i>0$,

$$H^i(F\otimes S^n(E))=0.$$

Therefore E is ample.

Remark. — We do not know whether this Corollary remains true without the hypothesis that X is proper over k.

Proposition (**3.5**) (Grauert's criterion). — E *is ample on* X *if and only if there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{j} & \mathbf{V}(E) \\ \downarrow & & \downarrow{\scriptstyle g} \\ \operatorname{Spec} k & \xrightarrow{\varepsilon} & C \end{array}$$

where $\mathbf{V}(E)=\operatorname{Spec}(\sum_n S^n(E))$ *is the vector bundle scheme of* E, j *is the closed immersion of* X *onto the zero-section of* $\mathbf{V}(E)$, C *is a scheme of finite type over* k, *and the restriction of* g *to* $\mathbf{V}(E)-j(X)$ *is an open immersion into* $C-\varepsilon(\operatorname{Spec} k)$.

Proof (The equivalence of this condition with the condition (*) of proposition 3.3 was proved in the analytic case by Grauert [7]). — Let L be the tautological line bundle on $P=\mathbf{P}(E)$. We use Proposition 3.2 above, and Grauert's criterion for L (Proposition 1.1 (iii)).

a) Suppose the condition of the proposition is satisfied. Considering the natural map $h:\mathbf{V}(L)\to\mathbf{V}(E)$, we get a diagram as shown below.

$$\begin{array}{ccccc} P & & \xrightarrow{j'} & & \mathbf{V}(L) \\ \downarrow{\scriptstyle p} & & & \swarrow{\scriptstyle h} & \\ X & \xrightarrow{j} & \mathbf{V}(E) & & \downarrow{\scriptstyle g'} \\ \downarrow & & & \searrow{\scriptstyle g} & \\ \operatorname{Spec} k & & \xrightarrow{\varepsilon} & & C \end{array}$$

Note that L is relatively ample over X, and p is proper. Therefore by Lemma 3.1 and [EGA, **II**, 8.8.2, 8.8.4], h restricted to $\mathbf{V}(L)-j'(P)$ is an isomorphism onto $\mathbf{V}(E)-j(X)$.

Let $g'=g\circ h$. Then g' restricted to $\mathbf{V}(L)-j'(P)$ is an open immersion into $C-\varepsilon(\operatorname{Spec} k)$. Thus L is ample over Spec k, so E is ample over X.

b) Conversely, suppose E is ample on X.

Let $C=\operatorname{Spec}(\sum_n\Gamma(X, S^n(E))=\operatorname{Spec}(\sum_n\Gamma(P, L^{\otimes n}))$, and build the same diagram as above. Then since L is ample over Spec k, g' restricted to $\mathbf{V}(L)-j'(P)$ is an open immersion into $C-\varepsilon(\mathrm{Spec}\ k)$. But h is an isomorphism of $\mathbf{V}(L)-j'(P)$ to $\mathbf{V}(E)-j(X)$, so we conclude g is an open immersion of $\mathbf{V}(E)-j(X)$ into $C-\varepsilon(\mathrm{Spec}\ k)$.

Remark. — One is tempted to generalize the criterion for line bundles, " L is ample if and only if $L^{\otimes n}=\mathcal{O}_X(1)$ for some n and for a suitable projective embedding " to vector bundles, by considering the mappings into Grassmannians defined by vector bundles. One gets a condition which is too weak: it may happen that a non-ample vector bundle gives an embedding into the Grassmannian. For example, let X be the projective line, and let E be $\mathcal{O}_X\oplus\mathcal{O}_X(1)$. Then E is not ample, but one checks easily that the mapping of X into the Grassmannian given by the sections $1+0$, $0+x$, $0+y$ of E is an embedding.

§ 4. Functorial properties.

In this section we discuss the behavior of ample vector bundles under change of the scheme X. As before, all schemes X are of finite type over an algebraically closed field k.

Proposition **(4.1)**. — *Let* E *be an ample vector bundle on* X, *and let* Y *be a (locally closed) subscheme of* X. *Then* E *restricted to* Y *is ample on* Y.

Proof. — It is sufficient to treat the cases Y open and Y closed. If $Y=U$ is open, any coherent sheaf F on U can be extended to a coherent sheaf $\overline{F}$ on X [EGA, **I**, 9.4.5]. Now if $\overline{F}\otimes S^n(E)$ is generated by global sections, so is $F\otimes S^n(E|U)$, which it its restriction to U. Therefore $E|U$ is ample. On the other hand, if Y is closed, and if F is any coherent sheaf on Y, then $F\otimes_{\mathcal{O}_X}S^n(E)\cong F\otimes_{\mathcal{O}_Y}S^n(E\otimes\mathcal{O}_Y)$ is generated by sections for large n. Therefore $E\otimes\mathcal{O}_Y$ is ample.

Proposition **(4.2)**. — *A vector bundle* E *on* X *is ample if and only if* E_{red} *on* X_{red} *is ample.*

Proof. — Note that L_{red} on $\mathbf{P}(E)_{red}$ is the same as the tautological bundle L′ on $\mathbf{P}(E_{red})$. Therefore by Proposition 3.2 we reduce to the case of a line bundle [EGA, **II**, 4.5.14].

Proposition **(4.3)**. — *Let* $g: X'\to X$ *be a finite, surjective morphism, and let* E *be a bundle on* X. *Assume that* X *contains a subset* Z, *proper over* k, *such that for every point* $x\in X-Z$, *either* X *is normal at* x, *or* g *is flat at every point of* X′ *lying over* x. *Then* E *is ample on* X *if and only if* $g^*(E)$ *is ample on* X′.

Proof. — By passing to $\mathbf{P}(E)$ and $\mathbf{P}(g^*(E))$, we reduce immediately to the case E is a line bundle, which is [EGA, **III**, 2.6.2].

Examples. — The hypothesis of the proposition is satisfied if X is proper or X is a reduced curve, or X is normal, or g is flat.

Proposition **(4.4)**. — *Let* $f: X\to Y$ *be a proper morphism, and let* E *be a bundle on* X. *Suppose for some* $y\in Y$, *the restriction* E_y *of* E *to the fibre* X_y *is ample. Then there is a neighborhood* U *of* y *such that* E *restricted to* $f^{-1}(U)$ *is ample. (In particular, the restriction* $E_{y'}$ *of* E *to any nearby fibre* $X_{y'}$, *with* $y'\in U$, *is ample.)*

Proof. — Again, passing to $\mathbf{P}(E)$, we reduce to the case of a line bundle, which follows from [EGA, **III**, 4.7.1].

Proposition (4.5). — *Let* $f: X \to Y$ *be a finite, faithfully flat morphism, and let* E *be a bundle on* X. *Then the sheaf* $f_* E$ *is locally free of finite rank on* Y, *and if it is ample, so is* E.

Proof. — The fact that $f_* E$ is locally free of finite rank follows from the hypotheses on f. Now suppose that $f_* E$ is ample. Then for any coherent sheaf F on X, $f_* F$ is coherent on Y (since f is finite), so $f_* F \otimes S^n(f_* E)$ is generated by global sections for large enough n. Since f^* is right exact, $f^*(f_* F \otimes S^n(f_* E))$ is also generated by global sections. But since f is affine, this maps onto $F \otimes S^n(E)$ hence the latter is generated by global sections, and E is ample.

Remark. — The converse of this proposition is false. Take for example $X = Y = \mathbf{P}^1_k$, and for f the square mapping $x \to x^2$. Let $E = \mathcal{O}_X(1)$. Then $f_* E$ is a bundle of rank two, since the mapping is of degree two. Hence it is of the form $\mathcal{O}_X(n) \oplus \mathcal{O}_X(m)$ for suitable n, m [8]. But $\Gamma(E) = \Gamma(f_* E)$ is a vector space of dimension two. Therefore either $n = m = 0$ or $n = 1$ and $m < 0$. In either case, $f_* E$ is not ample.

§ 5. Bundles in characteristic zero.

In this section we use the theory of finite-dimensional representations of the general linear group (see Appendix) to study the behavior of amplitude under various tensor operations performed on a bundle. The theory works best in characteristic zero, and in that case we can prove that the tensor product of two ample bundles is ample. We will discuss the case of characteristic p in section 6.

Let k be an algebraically closed field (of arbitrary characteristic) and let $G = \mathbf{GL}(r, k)$ be the general linear group of rank r over k. We refer to the Appendix for the classification of the irreducible representations of G. In particular, if $c = (n_1, \dots, n_r)$ is a set of integers with $n_1 \geq n_2 \geq \dots \geq n_r$, we will denote by V_c the corresponding irreducible representation of G (see Theorem A 7), and we will denote by V the standard representation of G.

Definition. — *If* V_c *is an irreducible representation of* G, *with* $c = (n_1, \dots, n_r)$, *then we call* $\sum_i n_i$ *the weight of* V_c. *We say that* V_c *is positive if* $\sum_i n_i > 0$ *and* $n_r \geq 0$. *If* W *is any representation of* G, *we say* W *is positive if all the irreducible representations in a composition series for* W *are positive. Finally, we say that* W *is homogeneous if all the irreducible representations in a composition series for* W *have the same weight, and in that case we call that weight the weight of* W.

Proposition (5.1). — *For each set of integers* $c = (n_1, \dots, n_r)$ *with* $n_1 \geq n_2 \geq \dots \geq n_r \geq 0$ *one can find integers* $q_1, \dots, q_t$, *with* $t = r!$ *such that*

a) *The character* λ *given by*

$$\lambda(h) = h_1^{n_1} \dots h_r^{n_r}$$

for every diagonal matrix $h = (h_1, \dots, h_r)$, *is an upper weight (see definition in Appendix) of*

$$S^{q_1}(V) \otimes \dots \otimes S^{q_t}(V),$$

where V *is the standard representation of* G, *and*

b) *the minimum of the* q_i *tends to infinity with* $\sum_i n_i$.

Proof. — To prove *a)*, we must show that there is a non-zero vector

$$v \in S(q_1, \ldots, q_t) = S^{q_1}(V) \otimes \ldots \otimes S^{q_t}(V)$$

which is an eigenvector for the subgroup P of G consisting of the upper triangular matrices, and which gives rise to the character λ of $H = P/X$ (using the notation of the Appendix). We proceed in several steps.

1. If $v \in S(a_1, \ldots, a_j)$ and $w \in S(b_1, \ldots, b_k)$ are two vectors, we define a vector

$$v \otimes w \in S(a_1, \ldots, a_j, b_1, \ldots, b_k)$$

in the obvious way. If v and w are non-zero eigenvectors for P with characters λ, μ, then clearly $v \otimes w$ is also non-zero, and is an eigenvector for P with character $\lambda\mu$.

If $v \in S(a)$ and $w \in S(b)$ are two vectors, then there is a natural product $vw \in S(a+b)$. Since $\sum_{a \geqslant 0} S(a)$ with this multiplication is a polynomial algebra in r indeterminates (where $r = \dim V$), we see that if $v, w \neq 0$, then $vw \neq 0$. This product extends in a natural way to give, for $v \in S(a_1, \ldots, a_j)$ and $w \in S(b_1, \ldots, b_k)$ a product

$$vw \in S(a_1 + b_1, \ldots, a_j + b_j, \ldots, b_k)$$

(say $j \leqslant k$). If v, w are non-zero eigenvectors for P with characters λ, μ then vw is also non-zero and is an eigenvector for P with character $\lambda\mu$.

We will use these notations and results below.

2. We now define some specific eigenvectors for P. Let $e_1, \ldots, e_r$ be the basis of V relative to which G acts. Define, for each $i = 1, \ldots, r$ a vector

$$f_i \in V^{\otimes i} = S(1, 1, \ldots, 1)$$

by

$$f_i = \sum_{\sigma \in \mathfrak{S}_i} \operatorname{sgn} \sigma . e_{\sigma 1} \otimes \ldots \otimes e_{\sigma i}$$

where $\mathfrak{S}_i$ is the symmetric group on i letters. Then clearly f_i is an eigenvector for P, with character λ_i given by

$$\lambda_i(h) = h_1 \ldots h_i.$$

3. Now let $c = (n_1, \ldots, n_r)$ be given. Let $s_i = n_i - n_{i+1}$ for $i = 1, \ldots, r-1$; $s_r = n_r$, and let $t = r!$ For each $i = 1, \ldots, r$ we perform a division:

$$s_i = \frac{t}{i} Q_i + R_i \quad \text{with} \quad 0 \leqslant R_i < \frac{t}{i}.$$

Let

$$Q = Q_1 + \ldots + Q_r,$$

and let

$$q_i = \begin{cases} Q + R_i + R_{i+1} + \ldots + R_r & \text{for } i = 1, \ldots, r \\ Q & \text{for } i = r+1, \ldots, t. \end{cases}$$

Define $v \in S(q_1, \ldots, q_t)$ by

$$v = \prod_{i=1}^{r} (f_i^{\otimes \frac{t}{i}})^{Q_i} (f_i)^{R_i}$$

Then by construction, v is a non-zero eigenvector for P with character λ given by

$$\lambda(h) = h_1^{n_1} \ldots h_r^{n_r}$$

as required.

4. To check statement *b)* note that

$$is_i = tQ_i + iR_i \qquad \text{with } 0 \leqslant iR_i < t.$$

Hence

$$is_i \leqslant (t+1)Q_i$$

and

$$\sum_i n_i = \sum_i is_i \leqslant (t+1)Q.$$

But $Q = \min(q_i)$, so we are done.

Now let X be a scheme over k, let E be a vector bundle (i.e. a locally free sheaf) of rank r on X, and let T be a representation of $\mathbf{GL}(r, k)$ on a vector space of dimension n over k. That is, T can be thought of as a homomorphism of group varieties from $\mathbf{GL}(r, k)$ to $\mathbf{GL}(n, k)$. With this data we define a vector bundle T(E) of rank n on X as follows.

Since E is a locally free sheaf, we can describe it by taking an affine open covering (X_i) of X, and giving transition functions $g_{ij} \in \mathbf{GL}(r, A_{ij})$ where $A_{ij} = \Gamma(X_i \cap X_j, \mathcal{O}_X)$, satisfying the usual cocycle condition. Now all the A_{ij} are k-algebras, so by base extension we obtain homomorphisms of groups, for each i, j,

$$T_{ij} : \mathbf{GL}(r, A_{ij}) \to \mathbf{GL}(n, A_{ij}).$$

Now the collection of functions $T_{ij}(g_{ij})$ allows us to define the new bundle T(E) of rank n, which of course does not depend (up to isomorphism) on any of the choices made.

We will call T(E) a *tensor bundle* of E, and will carry over to T(E) terminology which applies to T, such as *irreducible*, *positive*, *homogeneous*, *weight*. Examples of tensor bundles are tensor products, symmetric products, and exterior products.

Theorem (5.2). — *Assume* char $k = 0$. *Let* E *be an ample vector bundle on* X. *Then any positive tensor bundle of* E *is also ample.*

Proof. — Since any direct sum of ample bundles is ample (Proposition 2.2) and since in characteristic zero, every representation of $\mathbf{GL}(r)$ (where $r = \text{rank } E$) is a direct sum of irreducible representations (see Appendix), we have only to consider positive irreducible tensor bundles of E.

Pick $n_0 > 0$ such that for all $n \geqslant n_0$, $S^n(E)$ is ample and is generated by global sections (Proposition 2.4). Pick an $m_0 > 0$ such that whenever $c = (n_1, \ldots, n_r)$ is a set of integers with $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 0$ and $\sum_i n_i \geqslant m_0$, then the minimum of the q_i of Proposition 5.1 is $\geqslant n_0$. Then if T_c is the irreducible representation of $\mathbf{GL}(r)$ corresponding to c, $T_c(E)$ will be a direct summand of

$$S^{q_1}(E) \otimes \ldots \otimes S^{q_t}(E)$$

which is ample (Corollary 2.3), and so $T_c(E)$ will be ample. Thus any positive irreducible tensor bundle of E of large enough weight is ample.

Finally let T(E) be any positive irreducible tensor bundle of E. Then for n large,

$S^n(T(E))$ will be a direct sum of positive irreducible tensor bundles of E of large weight, hence ample. Therefore T(E) will be ample by Proposition 2.4.

Corollary **(5.3)** (char $k=0$). — *Let* E *be an ample bundle on* X. *Then* $S^n(E)$ *and* $E^{\otimes n}$ *are ample for every* $n>0$, *and* $\Lambda^n E$ *is ample for* $n=1, 2, \ldots, r$, *where* $r=\operatorname{rank} E$. *If* E_1 *and* E_2 *are ample bundles, then* $E_1 \otimes E_2$ *is ample.*

Proof. — These statements are all special cases of the theorem, except for the last. If E_1 and E_2 are ample, then $E_1 \oplus E_2$ is ample. By the Theorem,

$$S^2(E_1 \oplus E_2) = S^2(E_1) \oplus (E_1 \otimes E_2) \oplus S^2(E_2)$$

is ample, and so $E_1 \otimes E_2$ is ample.

Proposition **(5.4)** (char $k=0$). — *Let* E *be an ample vector bundle on* X. *Then for every coherent sheaf* F *on* X, *there is an integer* $m_0>0$ *such that for every positive homogeneous tensor bundle* T(E) *of* E *of weight* $\geqslant m_0$, $F \otimes T(E)$ *is generated by global sections.*

Proof. — Choose $n_0>0$ such that for $n \geqslant n_0$, $F \otimes S^n(E)$ and $S^n(E)$ are generated by global sections. Then also

$$(*) \qquad F \otimes S^{q_1}(E) \otimes \ldots \otimes S^{q_t}(E)$$

will be generated by global sections whenever $\min(q_i) \geqslant n_0$. As before it is sufficient to consider irreducible positive tensor bundles of E. If T(E) is such a bundle of large enough weight, then by Proposition 5.1, $F \otimes T(E)$ will be a direct summand of a sheaf of the form (*) above, with $\min(q_i) \geqslant n_0$, and hence will be generated by global sections.

Exercise. — In case X is projective over k, find out whether the analogous statement about the cohomology groups $H^i(F \otimes T(E))$ being zero for $i>0$ is true.

Corollary **(5.5)** (char $k=0$). — *Let* E *be a bundle on* X. *Then* E *is ample if and only if for every coherent sheaf* F *on* X *there is an integer* $n_0>0$ *such that for every* $n \geqslant n_0$, $F \otimes E^{\otimes n}$ *is generated by global sections.*

Proof. — One implication follows from the Proposition, and the other follows from the fact that $S^n(E)$ is a quotient of $E^{\otimes n}$.

§ 6. Bundles in characteristic $p \neq 0$.

In this section we try to extend the results of the previous section to the case of characteristic $p \neq 0$. We introduce the notion of a p-ample vector bundle which is *a priori* stronger than the notion of an ample bundle. We prove that tensor products, symmetric products and exterior products of p-ample bundles are p-ample, and hence ample. However we do not know whether every ample bundle is p-ample, except over a complete non-singular curve in which case it is (see section 7).

Proposition **(6.1)**. — *Let* X *be a quasi-projective scheme over a field k of characteristic* $p \neq 0$. *Then there is a unique functor* $\pi : F \mapsto F^{(p)}$ *from coherent sheaves on* X *to coherent sheaves on* X *with the following properties:*

a) π *is additive and right exact;*

b) *if* L *is a line bundle,* $\pi(L) = L^{\otimes p}$;

c) *if* $u : L_1 \to L_2$ *is a morphism of line bundles, then* $\pi(u) : L_1^{\otimes p} \to L_2^{\otimes p}$ *is* $u^{\otimes p}$.

Proof. — To show π is unique if it exists, let F be any coherent sheaf on X. Then since X is quasi-projective, we can find a short resolution of F

$$M_1 \xrightarrow{u} M_0 \rightarrow F \rightarrow 0$$

where M_0 and M_1 are finite direct sums of line bundles. Now by *a)* and *b)* we know what π does to M_1 and M_0, and by *c)* we know what it does to u. Finally, since π is right exact, we know $\pi(F)$ by the five-lemma.

For the existence we proceed as follows. Define a new scheme X_p over k by taking the same scheme $(X, \mathcal{O}_X)$, but where the k-algebra structure of $\mathcal{O}_X$ is given by multiplying by the p^{th} roots of elements of k. We define a k-morphism $f: X \rightarrow X_p$ (the *Frobenius morphism*) by giving the identity map on the spaces, and the p^{th} power map on the structure sheaves. (Note by the way that since X is of finite type over k, f is a finite morphism: it is sufficient to check that a polynomial ring $k[t_1, \ldots, t_r]$ is a finite module over its subring $k[t_1^p, \ldots, t_r^p]$.) Now for any coherent sheaf F on X, denote by F_p the same sheaf considered on X_p, and take $\pi(F) = f^*(F_p)$. On checks easily that π has the required properties.

Remarks. — 1. One can describe $F^{(p)}$ a bit more simply as a tensor product with explanation: $F^{(p)} = F \otimes_{\mathcal{O}_X} \mathcal{O}_X$, where F is an $\mathcal{O}_X$-module in the usual way; $\mathcal{O}_X$ is an $\mathcal{O}_X$-module via the p^{th} power map, and the result is an $\mathcal{O}_X$-module by multiplying in the factor on the right in the usual way.

2. If E is a vector bundle on X, then one can describe $E^{(p)}$ as the tensor bundle T(E) where T is the representation of $\mathbf{GL}(r)$ given by $T((g_{ij})) = (g_{ij}^p)$.

3. We denote by $F^{(p^n)}$ the result of applying the functor π to F n times.

Proposition **(6.2)** (char $k = p$). — *Let* E *be a vector bundle on* X, *and assume either* a) X *is normal, or* b) X *is proper over* k. *Then* E *is ample if and only if* $E^{(p)}$ *is ample.*

Proof. — With the notation of the proof of Proposition 6.1, the functor $F \mapsto F_p$ is an isomorphism of the category of sheaves on X to the category of sheaves on X_p. Since ampleness of a vector bundle depends only on the category of sheaves on the underlying scheme, we see immediately that E is ample if and only if E_p is ample. Now since f is a finite morphism, and $E^{(p)} = f^*(E_p)$, our result follows from Proposition 4.3.

Definition. — *Let* E *be a vector bundle on a scheme* X *over a field* k *of characteristic* $p \neq 0$. *We say that* E *is p-ample if for every coherent sheaf* F *on* X, *there is an integer* $n_0 > 0$ *such that for every* $n \geq n_0$, *the sheaf* $F \otimes E^{(p^n)}$ *is generated by global sections.*

Proposition **(6.3)**. — *If* X *is projective over* k, *and if* E *is a p-ample bundle on* X, *then* E *is ample.*

Proof. — Let L be an ample invertible sheaf on X. Then for some $n > 0$, $L^{\vee} \otimes E^{(p^n)}$ will be generated by global sections. This implies that $E^{(p^n)}$ is a quotient of a direct sum of copies of L, and hence is ample (Proposition 2.2). Therefore by the previous proposition, E is ample.

Remark. — We do not know whether the converse of this proposition is true, except in the case of line bundles (trivial) and in the case of curves (Proposition 7.3).

We leave to the reader the verification of the following elementary properties of p-ample bundles, whose proofs are analogous to those of Propositions 2.2, 2.3, and 2.4.

Proposition **(6.4)**. — a) *Any quotient of a p-ample bundle is p-ample. The direct sum of two bundles is p-ample if and only if each of them is.*

b) *If* E_1 *is a p-ample bundle, and if* E_2 *is a bundle generated by global sections, then* $E_1 \otimes E_2$ *is p-ample.*

c) *A bundle* E *is p-ample if and only if* $E^{(p)}$ *is.*

Remark. — We do not know whether the analogues of Propositions 3.3 and 3.4 are true for p-ample bundles.

Lemma **(6.5)**. — *Let* E *be a direct sum of ample line bundles on* X. *Then every positive tensor bundle of* E *(see definition in* § 5) *is also a direct sum of ample line bundles, and hence is p-ample.*

Proof. — Indeed, let T be a positive representation of $\mathbf{GL}(r)$, where $r = \operatorname{rank} E$. Since E is a direct sum of line bundles $L_1 \otimes \ldots \otimes L_r$, it is sufficient to consider the representation induced by T on the group $H \cong \mathbf{G}_m^r$ of diagonal matrices. Now the representations of $\mathbf{G}_m$ are completely decomposable, and the irreducible ones are one-dimensional and give bundles of the form $L_1^{n_1} \otimes \ldots \otimes L_r^{n_r}$ [4, exposé 4, théorème 2]. Thus T(E) will be a direct sum of these, and they will all be ample since T is positive and hence in each one, all $n_i \geqslant 0$, and $\Sigma n_i > 0$.

Theorem **(6.6)** (char $k = p \neq 0$). — *Let* X *be a projective scheme over k and let* E *be a p-ample bundle on* X. *Then every positive irreducible tensor bundle of* E *is p-ample; also* $\otimes^n E$, $S^n(E)$ *and* $\Gamma^n(E) = S^n(E^\vee)^\vee$ *are p-ample for* $n > 0$, *and* $\Lambda^n E$ *is p-ample for* $n = 1, \ldots, r = \operatorname{rank} E$.

Proof. — Let L be an ample line bundle on X. Then for some n, $L^\vee \otimes E^{(p^n)}$ will be generated by global sections, and hence $E^{(p^n)}$ will be a quotient of a direct sum of copies of L. If T is any representation of $\mathbf{GL}(r)$ defined over the prime field $\mathbf{F}_p \subseteq k$, as for example any irreducible representation of $\mathbf{GL}(r)$, or any of the representations $\otimes^n$, S^n, Γ^n, Λ^n, then T commutes with the functor π of Proposition 6.1: $T(E^{(p)}) = T(E^{(p)})$. Thus using Proposition 6.4 *c)*, we may replace E by $E^{(p^n)}$ and assume that E is a quotient of a direct sum of ample line bundles.

So let E be a quotient of E_1, which is a direct sum of ample line bundles and let E_1 have rank t. Let T be a positive irreducible representation of $\mathbf{GL}(r)$ corresponding to the set of integers $c = (n_1, \ldots, n_r)$ with $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 0$ in the classification of Theorem A 7. Then the vector space V_c on which it acts is a quotient of the subrepresentation of

$$S^{s_1}(V) \otimes \ldots \otimes S^{s_r}(\Lambda^r V)$$

generated by the vector $e_1^{s_1} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_r)^{s_r}$ (using the notation of Theorem A 7). Now let W be the standard representation of $\mathbf{GL}(t)$, and let T_1 be the action of $\mathbf{GL}(t)$ on the subspace W_c of

$$S^{s_1}(W) \otimes \ldots \otimes S^{s_r}(\Lambda^r W)$$

generated by the vector $f_1^{s_1}\otimes\ldots\otimes(f_1\wedge\ldots\wedge f_r)^{s_r}$ where $f_1, \ldots, f_t$ is the standard basis of W. Then T_1 is a positive representation of $\mathbf{GL}(t)$, and if one maps W to V by sending f_i to e_i for $i=1, \ldots, r$, and f_i to 0 for $i>r$, then V_c is a quotient of W_c, compatible with the two group actions. Therefore T(E) is a quotient of $T_1(E_1)$, which is p-ample by the Lemma, and so T(E) is p-ample.

The last statement of the theorem follows similarly: one has only to note that $\otimes^n E$, $S^n(E)$, $\Gamma^n(E)$ and $\Lambda^n E$ are quotients of $\otimes^n E_1$, $S^n(E_1)$, $\Gamma^n(E_1)$ and $\Lambda^n E_1$, respectively.

Corollary **(6.7)**. — *If* X *is projective over* k, *and if* E_1 *and* E_2 *are two p-ample bundles, then* $E_1\otimes E_2$ *is p-ample.*

Proof. — This follows from the theorem applied to $E_1\oplus E_2$, as in the proof of Corollary 5.3.

Corollary **(6.8)**. — *If* X *is projective over* k *and* E *is p-ample on* X, *then every positive tensor bundle of* E *is ample.*

Proof. — We know by the theorem that the irreducible ones are p-ample and hence ample. But any positive representation of $\mathbf{GL}(r)$ has a composition series whose quotients are positive irreducible representations. So the result follows from Corollary 3.4.

§ 7. Vector bundles on curves.

In this section X will be a non-singular projective curve over k. The fact that any vector bundle has a filtration whose quotients are line bundles [1, Part I, § 4] enables us to give quite precise statements. In particular for bundles of rank two in the characteristic zero case we give necessary and sufficient conditions for amplitude.

We recall for convenience the following well-known proposition:

Proposition **(7.1)**. — *Let* L *be a line bundle on* X. *Then*

a) L *is ample if and only if* deg $L>0$.

b) *Given any coherent sheaf* F *on* X, *there is an integer* d_0 *such that if* $\deg L\geqslant d_0$, *then* $F\otimes L$ *is generated by global sections, and* $H^1(F\otimes L)=0$.

Proof. — *a)* The condition is necessary, because a line bundle of negative degree has no sections. The sufficiency follows from *b)*.

b) Any coherent F is a quotient of a direct sum of line bundles. Hence we may assume F is a line bundle, and we are reduced to showing that there is an integer d_0 such that if $\deg L\geqslant d_0$, then L is generated by global sections, and $H^1(L)=0$. In fact if g is the genus of X, $d_0=2g$ will do [1, Lemma 8].

Proposition **(7.2)**. — *Let* $r>0$ *be fixed. Then*

a) *There is an integer* d_0 *such that every indecomposable bundle* E *on* X *of rank* r *and degree* $d\geqslant d_0$ *is ample.*

b) *Given a coherent sheaf* F, *there is a* d_0 *such that for every indecomposable bundle* E *of rank* r *and degree* $d\geqslant d_0$, $F\otimes E$ *is generated by global sections.*

Proof. — If E is indecomposable of rank r, then E has a filtration whose quotients are line bundles $L_1, \dots, L_r$, with $\deg L_i \geqslant \varphi(\deg E)$ where φ is a function which goes to infinity with deg E [1, Lemma 7].

a) By choosing deg E large enough, we can make sure that $\deg L_i > 0$ for each i. Then each L_i is ample by the previous proposition, and so E is ample by Corollary 3.4.

b) By choosing deg E large enough, we can make sure that $F \otimes L_i$ is generated by global sections and $H^1(F \otimes L_i) = 0$ for each i. Then $F \otimes E$ is generated by global sections and $H^1(F \otimes E) = 0$.

Proposition **(7.3)**. — *Let* char $k = p$ *be different from zero. If* E *is an ample bundle on* X, *then* E *is p-ample.*

Proof. — By induction on $r = \operatorname{rank} E$, the case $r = 1$ being trivial. Let E be ample of rank r. Using Propositions 2.2, 6.2, 6.4 and the induction hypothesis, we may assume that $E^{(p^n)}$ is indecomposable for every $n > 0$. Now $\deg E > 0$, since the degree of E is the degree of $\Lambda^n E$, which is ample by Corollary 2.6, and an ample line bundle has positive degree (Proposition 7.1). Hence

$$\deg E^{(p^n)} = p^n . \deg E$$

grows indefinitely with n, and so E is p-ample by part *b)* of the proposition.

Corollary **(7.4)**. — *If* E_1 *and* E_2 *are ample bundles on* X, *then* $E_1 \otimes E_2$ *is ample.*

Proof. — In characteristic zero this is Corollary 5.3, and in characteristic p it follows from the proposition and Corollary 6.7.

Now we restrict our attention to bundles of rank two, and try to characterize those which are ample.

Proposition **(7.5)**. — *Let* E *be a bundle of rank two on* X, *and write*

$$0 \to L_1 \to E \to L_2 \to 0$$

where L_1 *is a sub-line bundle of maximum degree. Assume that* $\deg L_1 \leqslant 0$ *and either* a) char $k = 0$ *and* $\deg E > 0$, *or* b) char $k = p$ *and* $\deg E > \frac{2}{p}(g-1)$, *where g is the genus of* X. *Then* E *is ample.*

Proof (due essentially to Mumford). — Let $P = \mathbf{P}(E)$, let $\pi : P \to X$ be the projection, and let L be the tautological line bundle on P. By Proposition 3.2, we need only show that L is ample, and to do that, we apply Nakai's criterion (Proposition 1.4) to the divisor class d of L on the ruled surface P. Recall that the divisor class group of P is $\mathbf{Z} \oplus \mathbf{Z}$, and is generated by d and the class f of a fibre of P over X. To show that L is ample, we must show $d^2 > 0$, and $d.e > 0$ for every effective divisor class e, and it will be sufficient to consider those e which are the class of an irreducible curve C on P.

1) I claim there is a section Y of P over X. Indeed, let M be any quotient line bundle of E, as for example L_2. Then $\mathbf{P}(L_2) = X$, and there is an open set $U \subseteq X$ and a map $s : U \to P$ such that $\pi \circ s = \mathrm{id}$ [EGA, **II**, 3.5.1]. Now since X is a non-singular curve, and P is complete, s extends to a section $s : X \to P$. We put $Y = s(X)$.

2) We use Y to conclude that $d^2 = \deg E > 0$. Indeed, let the class of Y be $d+mf$ where $m \in \mathbf{Z}$. Then we have an exact sequence of sheaves on P,

$$0 \to \mathcal{O}_P(-d-mf) \to \mathcal{O}_P \to \mathcal{O}_Y \to 0.$$

Or, tensoring by $L = \mathcal{O}_P(d)$ we have

$$0 \to \mathcal{O}_P(-mf) \to L \to L \otimes \mathcal{O}_Y \to 0.$$

Now $\deg(L \otimes \mathcal{O}_Y) = d.(d+mf) = d^2 + m$ (since $d.f = 1$).

To calculate this we apply π_* (noting that $\pi_*(\mathcal{O}_P(-mf)) = \mathcal{O}_X(-m)$ is a line bundle of degree $-m$, and $R^1\pi_*(\mathcal{O}_P(-mf)) = 0$):

$$0 \to \mathcal{O}_X(-m) \to E \to \pi_*(L \otimes \mathcal{O}_Y) \to 0.$$

But $\pi_*(L \otimes \mathcal{O}_Y)$ is isomorphic to $L \otimes \mathcal{O}_Y$ via the section $s : X \to Y$. Hence, taking degrees, we have

$$\deg E = -m + (d^2 + m) = d^2.$$

Hence $d^2 = \deg E > 0$.

Note also that in this construction E was found to have a sub-line bundle of degree $-m$. Hence from our hypotheses we conclude $m \geqslant 0$, and so $d.Y = d^2 + m > 0$ also.

3) Now let C be any irreducible curve on P. We wish to show that $C.d > 0$. Let the class of C be $nd + mf$. If $n < 0$, we would have $C.f < 0$ which is impossible for an effective curve. If $n = 0$, then $m = 1$ since C is irreducible. Hence $C = f$ and $C.d = 1 > 0$. If $n = 1$, then C is a section of P over X, and we use the argument of 2) above to show $C.d > 0$. We are left with the case $n > 1$, which we treat below.

4) We calculate the canonical divisor K on P, and show that the class of K is

$$K = -2d + (2g - 2 + d^2)f.$$

Indeed, for any curve C on P, we have the formula

$$2p_a(C) - 2 = C.(C + K).$$

Applying this to a fibre f we have

$$-2 = f.(f + K),$$

so $K.f = -2$. Applying it to the section Y above, we have

$$2g - 2 = Y.(Y + K) = (d + mf).(d + mf + K),$$

or

$$2g - 2 = d^2 + 2m + d.K + mf.K,$$

so

$$d.K = 2g - 2 - d^2.$$

The formula for K above follows from this.

5) Let $C = nd + mf$ with $n > 1$, and suppose either $\operatorname{char} k = 0$ or $n < p$. Then we can apply the Hurwitz formula to the projection of C onto X, and find

$$2p_a(C) - 2 = n(2g - 2) + (\#\text{ ramification pts}) + (\text{contribution of singularities}).$$

In particular, $$2p_a(C)-2 \geqslant n(2g-2).$$

On the other hand, we can calculate $p_a(C)$ using the canonical divisor:

$$\begin{aligned}2p_a(C)-2 &= C.(C+K)\\ &= (nd+mf)(nd+mf-2d+(2g-2+d^2)f)\\ &= n(n-2)d^2+n(m+2g-2+d^2)+m(n-2)\\ &= n(n-1)d^2+n(2g-2)+2m(n-1).\end{aligned}$$

Combining, we have $$(n-1)(nd^2+2m)\geqslant 0.$$

Since $n>1$, this implies $$nd^2+2m\geqslant 0$$

or $$nd^2+m\geqslant -m.$$

We wish to show $C.d=nd^2+m$ is positive. We know $n>0$ and $d^2>0$. Hence $nd^2>0$, so if $m\geqslant 0$, $nd^2+m>0$. On the other hand, if $m<0$, we have $nd^2+m\geqslant -m>0$. Thus $C.d>0$ in any case.

6) Finally, suppose char $k=p\neq 0$, and $n\geqslant p$. Then we cannot apply the Hurmitz formula to $C=nd+mf$, but we have the weaker inequality

$$2p_a(C)-2\geqslant 2g-2.$$

As above, we deduce $$(n-1)(nd^2+2g-2+2m)\geqslant 0$$

and therefore $$nd^2+2g-2+2m\geqslant 0$$

or $$nd^2+m\geqslant -m-2g+2$$

We wish to show that $C.d=nd^2+m>0$. We have the additional hypothesis that $d^2=\deg E>\frac{2}{p}(g-1)$ and $n\geqslant p$. Hence $nd^2>2g-2$. So if $m\geqslant -2g+2$, then $nd^2+m>0$. On the other hand if $m<-2g+2$, then

$$nd^2+m\geqslant -m-2g+2>0.$$

7) In conclusion, we have shown that $d^2>0$, and $C.d>0$ for every irreducible curve C on P. Hence L is ample, and therefore E is ample.

Corollary **(7.6)**. — *Let* char $k=0$. *Then a bundle* E *of rank two on* X *is ample if and only if* $\deg E>0$ *and* $\deg L>0$ *for every quotient line bundle* L *of* E.

Proof. — The condition is clearly necessary. To show that it is sufficient, choose a sub-line bundle L_1 of maximum degree of E, and write

$$0\to L_1\to E\to L_2\to 0.$$

If $\deg L_1>0$, we have also $\deg L_2>0$ by hypothesis, so E is an extension of two ample line bundles, and hence ample. If $\deg L_1\leqslant 0$ we are under the conditions of the Proposition, so E is ample.

Corollary **(7.7).** — *Let* char $k=p$. *If* E *is a bundle of rank two on* X *and* $\deg E > \frac{2}{p}(g-1)$ *and* $\deg L > 0$ *for every quotient line bundle of* E, *then* E *is ample.*

Proof. — The same as for the previous corollary.

Corollary **(7.8).** — *An indecomposable bundle* E *of rank two on an elliptic curve* X *(in arbitrary characteristic) is ample if and only if* $\deg E > 0$.

Proof. — Using the two previous corollaries, we need only show that $\deg L > 0$ for every quotient line bundle of E. Suppose to the contrary there is an exact sequence

$$0 \to L_1 \to E \to L_2 \to 0$$

with $\deg L_2 \leqslant 0$. Then $\deg L_1 > 0$ since $\deg E > 0$. The extension will be classified by an element of $H^1(L_1 \otimes L_2^{-1})$. But this group is zero since $\deg L_1 \otimes L_2^{-1} > 0$, and X is elliptic $(2g-2=0)$. Therefore the extension splits, and E decomposes, which is a contradiction.

Examples. — 1. On the projective line, every vector bundle is a direct sum of line bundles, so the ample ones are just the sums of ample line bundles.

2. For indecomposable bundles E of rank 2 on a curve X of genus $g > 1$ there is no numerical criterion on deg E for E to be ample. We show this by constructing two indecomposable bundles of degree $2g-2$, one of which is ample, and the other of which is not.

a) Let K be the canonical divisor on X, and choose a non-trivial extension

$$0 \to K \to E \to \mathcal{O} \to 0,$$

which is possible since $H^1(K) \cong k \neq 0$. If E should decompose into the direct sum of two line bundles L_1, L_2, then at least one of them, say L_1, has degree>0, since $\deg E = 2g-2 > 0$. Thus there are no non-zero homomorphisms of L_1 into $\mathcal{O}$. Therefore there must be a surjection of L_2 into $\mathcal{O}$, so $L_2 \cong \mathcal{O}$. Hence $L_1 \cong K$, and the extension above splits, which is a contradition. Thus E is indecomposable. And E is not ample, since it has as a quotient $\mathcal{O}$, which is not ample.

b) Let F be a non trivial extension

$$0 \to \mathcal{O} \to F \to \mathcal{O} \to 0$$

which exists since $H^1(\mathcal{O}) \neq 0$. Then reasoning as above, we see F is indecomposable. Let L be a line bundle of degree $g-1$, and let $E = F \otimes L$. Then E is also indecomposable, and being an extension of L by L (both ample line bundles) it is ample. Its degree is $2g-2$.

§ 8. Two applications.

In this section we give two elementary applications of the theory of ample vector bundles. One says that certain cohomology group of coherent sheaves on the complement of a closed subscheme of projective space are finite dimensional. The other says that there are no non-constant holomorphic functions in the neighborhood of a non-singular subvariety of positive dimension of projective space.

Theorem (**8.1**). — *Let* X *be the projective space* $\mathbf{P}^r_k$, *and let* Y *be a non-singular closed subscheme of* X *(not necessarily connected) of codimension* p. *Assume either that* $\operatorname{char} k = 0$ *or that* Y *is a curve. Then*

a) *For every coherent sheaf* F *on* X—Y, *and for every* $i \geq p$, *the k-vector space* $H^i(X-Y, F)$ *is finite-dimensional.*

b) *For every locally free sheaf* F *on* X, *and for every* $i < r-p = \dim Y$, *the k-vector space* $H^i(\hat{X}, \hat{F})$ *is finite-dimensional, where* $\hat{X}$ *is the formal completion of* X *along* Y, *and* $\hat{F}$ *is the completion of* F (see [EGA, ch. **I**, § 10] for definitions).

Lemma (**8.2**). — *Under the hypotheses of the theorem, let* F *be a locally free sheaf on* X, *and let* I *be the sheaf of ideals of* Y. *Then for each* $i > p$, *there is an* n_0 *such that for* $n \geq n_0$,

$$\operatorname{Ext}^i_{\mathcal{O}_X}(I^n/I^{n+1}, F) = 0.$$

Proof. — This Ext group is the abutment of a spectral sequence whose initial term is

$$E_2^{pq} = \operatorname{Ext}^p_{\mathcal{O}_Y}(I^n/I^{n+1}, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, F)).$$

Now since Y being non singular is locally a complete intersection of codimension p, we can calculate locally with a Koszul resolution [FGA, exp. 149, § 3] and find

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_Y, F) = \begin{cases} 0 & \text{for } i \neq p. \\ G, \text{ a locally free sheaf on Y, for} & i = p. \end{cases}$$

Therefore the spectral sequence degenerates, and

$$\operatorname{Ext}^i_{\mathcal{O}_X}(I^n/I^{n+1}, F) \cong \operatorname{Ext}^{i-p}_{\mathcal{O}_Y}(I^n/I^{n+1}, G)$$

for each i.

Let E be the normal bundle to Y in X, namely $(I/I^2)'$. It is a bundle of rank p on Y. From the exact sequences

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{r+1} \to T_X \to 0$$

and

$$0 \to T_Y \to T_X \otimes \mathcal{O}_Y \to E \to 0,$$

where T_X and T_Y are the tangent bundles to X and Y, respectively, we deduce that E is a quotient of $\mathcal{O}_Y(1)^{r+1}$, and hence is ample.

Since I^n/I^{n+1} is isomorphic to $S^n(I/I^2)$, it is locally free on Y, and we have

$$\operatorname{Ext}^i_{\mathcal{O}_Y}(I^n/I^{n+1}, G) \cong \operatorname{Ext}^i_{\mathcal{O}_Y}(\mathcal{O}_Y, G \otimes (I^n/I^{n+1})') \cong H^i(Y, G \otimes (I^n/I^{n+1})').$$

But $(I^n/I^{n+1})' = (S^n(E'))' = \Gamma^n(E)$. So our problem is to show that for large enough n, and $i > 0$,

$$H^i(Y, G \otimes \Gamma^n(E)) = 0.$$

In case char $k=0$, $\Gamma^n(E)$ is isomorphic to $S^n(E)$, so we are done, since E is ample. In case Y is a curve, there is only the group H^1 to consider, and H^1 is a right exact functor. Hence it is sufficient to observe that $\Gamma^n(E)$ is a quotient of $\Gamma^n(\mathcal{O}_Y(1)^{r+1})=S^n(\mathcal{O}_Y(1)^{r+1})$ and again we are done, since $\mathcal{O}_Y(1)^{r+1}$ is ample.

Proof of theorem. — *a)* We may assume that F is the restriction of a coherent sheaf F on all of X. First we reduce to the case F locally free. Indeed, suppose it is true for locally free sheaves, and suppose also by induction it has been proven for $j>i$ (the case $i=r+1$ being trivial, since all cohomology groups vanish then). Write F as a quotient of a locally free sheaf L :

$$0\to R\to L\to F\to 0.$$

Then we have an exact sequence

$$\dots \to H^i(X-Y, L) \to H^i(X-Y, F) \to H^{i+1}(X-Y, R) \to \dots$$

Since the two outside ones are finite-dimensional by hypothesis, so is the middle one.

Next we use the exact sequence of local cohomology [10, § 1]:

$$\dots \to H^i(X, F) \to H^i(X-Y, F) \to H^{i+1}_Y(F) \to H^{i+1}(X, F) \to \dots$$

Since the two outside ones are known to be finite dimensional by Serre's theorems, it will be sufficient to prove that $H^{i+1}_Y(F)$ is finite-dimensional. Now

$$H^i_Y(F)=\varinjlim_n \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X/I^n, F)$$

[10, Theorem 2.8]. Since these Ext groups are all finite-dimensional, it will be sufficient to show that the maps in the direct system are eventually all surjective. The maps in the direct system are derived from the exact sequences

$$0 \to I^n/I^{n+1} \to \mathcal{O}/I^{n+1} \to \mathcal{O}/I^n \to 0,$$

which give

$$\dots \to \operatorname{Ext}^i(\mathcal{O}/I^n, F) \to \operatorname{Ext}^i(\mathcal{O}/I^{n+1}, F) \to \operatorname{Ext}^i(I^n/I^{n+1}, F) \to \dots$$

Thus using induction on n, we must show that

$$\operatorname{Ext}^i_{\mathcal{O}_X}(I^n/I^{n+1}, F)$$

is zero for large n, and for $i>p$. This is proved by the Lemma.

b) Let F be locally free on X. Then

$$H^i(\hat{X}, \hat{F})=\varprojlim_n H^i(X, F\otimes\mathcal{O}_X/I^n)$$

by [EGA, $\mathbf{0}_{\mathrm{III}}$, 13.3.1]. To show it is finite-dimensional for $i<r-p$, it will be sufficient to show that the groups on the right are finite-dimensional, and that the maps of the inverse system are all eventually injective. By duality on projective space,

$$H^i(X, F\otimes\mathcal{O}_X/I^n)$$

is dual to $$\mathrm{Ext}^{r-i}_{\mathcal{O}_X}(\mathrm{F}\otimes\mathcal{O}_X/\mathrm{I}^n, \omega) = \mathrm{Ext}^{r-i}_{\mathcal{O}_X}(\mathcal{O}_X/\mathrm{I}^n, \mathrm{F}'\otimes\omega)$$

where ω is the sheaf of n-differential forms on X. The inverse system is transformed into a direct system, and we are reduced to the proof of *a)* above, since $i<r-p$ is the same as $r-i>p$.

Proposition **(8.3)**. — *Let* $X = \mathbf{P}^r_k$, *and let* Y *be a connected, non-singular closed subscheme of* X *of positive dimension. Then there are no non-constant holomorphic functions in a neighborhood of* Y, *i.e.* $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) = k$.

Proof. — Let I be the ideal of Y. Then we write

$$\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) = \varprojlim_n \Gamma(X, \mathcal{O}_X/\mathrm{I}^n),$$

and we prove by induction on n that $\Gamma(X, \mathcal{O}_X/\mathrm{I}^n) = k$. For $n = 1$ we have $\Gamma(X, \mathcal{O}_Y) = k$ since Y is reduced and connected. For $n \geqslant 1$ we use the exact sequence

$$0 \to \mathrm{I}^n/\mathrm{I}^{n+1} \to \mathcal{O}_X/\mathrm{I}^{n+1} \to \mathcal{O}_X/\mathrm{I}^n \to 0$$

whence $$0 \to \Gamma(\mathrm{I}^n/\mathrm{I}^{n+1}) \to \Gamma(\mathcal{O}_X/\mathrm{I}^{n+1}) \to \Gamma(\mathcal{O}_X/\mathrm{I}^n) \to \dots$$

We will show that $\Gamma(\mathrm{I}^n/\mathrm{I}^{n+1}) = 0$ for $n \geqslant 1$. Then since the middle group contains k, and the right-hand one is k by induction, we will have that the middle one is k.

Now since Y is non-singular, I/I^2 is locally free on Y. Since its dual is a quotient of $\mathcal{O}_Y(1)^{r+1}$ as in the proof of Lemma 8.2 above, it itself is a subsheaf of $\mathcal{O}_Y(-1)^{r+1}$. Hence

$$\mathrm{I}^n/\mathrm{I}^{n+1} = \mathrm{S}^n(\mathrm{I}/\mathrm{I}^2) \subseteq \mathrm{S}^n(\mathcal{O}_Y(-1)^{r+1}) = \mathcal{O}_Y(-n)^s.$$

Thus to show $\Gamma(\mathrm{I}^n/\mathrm{I}^{n+1}) = 0$, we have only to show $\Gamma(\mathcal{O}_Y(-n)) = 0$. Let H be a hypersurface of degree n which contains no component of Y. Then we have

$$0 \to \mathcal{O}_Y(-n) \to \mathcal{O}_Y \to \mathcal{O}_{Y\cap H} \to 0$$

where $Y \cap H$ is the scheme-theoretic intersection. Since Y is of positive dimension, $Y \cap H$ is non-empty. Now

$$0 \to \Gamma(\mathcal{O}_Y(-n)) \to \Gamma(\mathcal{O}_Y) \to \Gamma(\mathcal{O}_{Y\cap H}) \to \dots$$

But $\Gamma(\mathcal{O}_Y) = k$, and the map $\Gamma(\mathcal{O}_Y) \to \Gamma(\mathcal{O}_{Y\cap H})$ is non-zero, so $\Gamma(\mathcal{O}_Y(-n)) = 0$.

APPENDIX

REPRESENTATIONS OF $\mathbf{GL}(r)$

In this appendix we give a classification of the irreducible finite-dimensional algebraic representations of the general linear group $\mathbf{GL}(r)$ over a field k, which for convenience we will assume to be algebraically closed. In the case of characteristic zero, the representations of $\mathbf{GL}(r)$ are well known [4; 5; 12]. In the first place, every finite-dimensional representation is a direct sum of irreducible representations [12, exposé 7, théorème 2], and in the second place the irreducible representations are classified by the character associated with an eigenvector for a Borel subgroup of $\mathbf{GL}(r)$, [12, exposés 17, 18, 19]. In the case of characteristic $p \neq 0$ it is no longer true that every finite-dimensional representation is completely decomposable. However, we can still give a classification of the irreducible representations analogous to that in characteristic zero. These results are contained in [4, exposés 15, 16], yet we feel that it is useful to have a self-contained elementary treatment for the general linear group. Our exposition follows closely that of Godement [5, Appendix], except that the proofs are algebraic, and slight changes have been made for the case of characteristic p.

We recall for convenience the basic definitions. A *representation* of $G = \mathbf{GL}(r)$ is an algebraic action of G on a (finite-dimensional) k-vector space V, i.e. a homomorphism $T : \mathbf{GL}(r) \to \mathbf{GL}(V)$ of group varieties. We will denote the action of a matrix $g \in G$ on a vector $v \in V$ by $T(g)v$. A *morphism* of representations is a linear map of the corresponding vector spaces, compatible with the action of the group. A representation is *irreducible* if the vector space has no proper stable subspaces.

Examples. — 1. Let V be an r-dimensional k-vector space with basis $e_1, \ldots, e_r$. Then $G = \mathbf{GL}(r)$ acts on V in the usual way. This is the *standard representation* of G.

2. If V is the standard representation of G, then the symmetric powers $S^n(V)$, the tensor powers $\otimes^n V$ and the exterior powers $\Lambda^n V$ are also representations of G.

3. If T is a representation of G on a vector space V, then we have the *contragredient representation* T′ of G on the dual vector space V′ defined by $T'(g) = {}^tT(g^{-1})$.

4. Let $A(G) = \Gamma(G, \mathcal{O}_G)$ be the (infinite-dimensional) k-vector space of regular functions on G. Then G acts on A(G) by right translation: if $\theta \in A(G)$ and $g \in G$, then

$$(T(g)\theta)(g') = \theta(g'g).$$

For any $\theta \in A(G)$, the translates of θ generate a finite-dimensional subspace of A(G), since they are all polynomials in the g_{ij} and $1/\det(g_{ij})$ of some bounded degree depending on θ (see also [4, exposé 4, lemme 2]). Furthermore, this vector space is stable under the action of G, and hence is a representation of G. We will see that every irreducible representation of G arises in this way.

From now on we will use the following notations:

k is an algebraically closed field
$G = \mathbf{GL}(r, k)$
$A(G) = \Gamma(G, \mathcal{O}_G)$ is the vector space of regular functions on G
$H =$ diagonal matrices $h = (h_1, \ldots, h_r) \in G$
$P =$ upper triangular matrices in G
$X =$ upper triangular matrices with 1's on the diagonal
$Q =$ lower triangular matrices
$Y =$ lower triangular matrices with 1's on the diagonal

Thus P and Q are solvable algebraic groups, with commutator subgroups X and Y respectively, and P/X, Q/Y are isomorphic to H.

Definition. — Let T *be a representation of* G *on a vector space* V. *A coefficient of* T *is any function* $\theta \in A(G)$ *of the form*

$$\theta(g) = (T(g)v, v')$$

where v *is a fixed vector in* V, v' *is a fixed vector in the dual space* V', *and where* (,) *is the scalar product.*

(Note: Since G is a reduced scheme over an algebraically closed field, we allow ourselves to use the old-fashioned terminology of " function on G " for an element of A(G), and we define such functions by giving their values at the rational points of G.)

Lemma A 1. — Let $g = (g_{ij})$ *be an element of* G. *Let* $g_0 = 1$, *and for each* $i = 1, \ldots, r$, *let*

$$g_i = \det \begin{vmatrix} g_{11} & \cdots & g_{1i} \\ \vdots & & \vdots \\ g_{i1} & \cdots & g_{ii} \end{vmatrix}.$$

If all the g_i *are different from zero, then* g *can be written uniquely as* $g = yhx$ *with* $y \in Y$, $h \in H$, *and* $x \in X$. *Furthermore, in that case* $h_i = g_i / g_{i-1}$ *for* $i = 1, \ldots, r$.

Proof. — Left to the reader.

We denote by G_0 the subset of $g \in G$ for which all the g_i are different from zero. It is an open Zariski-dense subset.

Proposition A 2. — Let T *be a representation of a connected solvable linear algebraic group* B *on a non-zero vector space* V *over* k. *Then there is a non-zero eigenvector* $v \in V$ *for* B, i.e. *a vector such that for every* $b \in B$,

$$T(b)v = \chi(b)v,$$

where $\chi : B \to k^*$ *is a suitable character of* B.

Proof. — This is a consequence of the Borel fixed-point theorem [2, Proposition 15.5; 6, theorem 2] as follows. Let $\overline{V}$ be the projective space of lines through the origin in V. Then B acts on $\overline{V}$, which is a complete variety, hence there is a fixed point. Any non-zero $v \in V$ lying over this fixed point will be an eigenvector for B.

Definition. — Let T *be a representation of* G *on a vector space* V. *Since* P *is a connected solvable linear algebraic group, which acts on* V *through* T, *there must be an eigenvector for* P, i. e. *a non-zero* $v \in V$ *and a character* $\lambda : P \to k^*$ *such that for every* $p \in P$,

$$T(p)v = \lambda(p)v.$$

Such a character λ *is called an upper weight of* T.

(Note that since k^* is commutative, and X is the commutator subgroup of P, λ is identically 1 on X, and so is determined by its restriction to $H = P/X$.) Define also a *lower weight of* T by taking an eigenvector for Q.

Proposition A 3. — Let T *be an irreducible representation of* G *on a vector space* V. *Then* T *has a unique upper weight, and the eigenvectors for* P *form a one-dimensional subspace of* V.

Proof. — Let λ be an upper weight of T, corresponding to an eigenvector $v \in V$ for P. Let T′ be the contragredient representation of T, and let μ be any lower weight for T′, corresponding to an eigenvector $v' \in V'$ for Q. Then we will show that λ is determined by μ. But since λ was arbitrary to begin with, this will show that any two are the same, and so T has a unique upper weight.

Consider the coefficient θ of T defined by the two vectors v, v':

$$\theta(g) = (T(g)v, v').$$

For any $g \in G_0$, we can write $g = yhx$ as in Lemma A 1, so that

$$\theta(g) = (T(yhx)v, v').$$

Now transposing the $T(y)$ and using the fact that v, v' are eigenvectors, we have

$$\begin{aligned}\theta(g) &= (T(hx)v, {}^tT(y)v') \\ &= (\lambda(hx)v, \mu(y^{-1})v') \\ &= (\lambda(h)v, v') \\ &= \lambda(h) . \theta(e),\end{aligned}$$

where $e \in G$ is the unit element. But also

$$\begin{aligned}\theta(g) &= (T(x)v, {}^tT(yh)v') \\ &= \mu(h^{-1}) . \theta(e).\end{aligned}$$

Therefore we have $$\lambda(h) . \theta(e) = \mu(h^{-1}) . \theta(e)$$

for all $h \in H$, since $H \subseteq G_0$.

Now I claim that $\theta(e)$ is not zero. Indeed, if it were, then by the above calculation, $\theta(g)$ would be zero for all $g \in G_0$, and hence θ would be identically zero since G_0 is dense. But this is impossible, since $v' \neq 0$, and the vectors $T(g)v$ for $g \in G$ generate all of V since T is irreducible by hypothesis. So we deduce that $\lambda(h) = \mu(h^{-1})$, which shows that λ is determined by μ and so is unique.

To show that the eigenvectors for P form a one-dimensional subspace of V, we will define a map $f : V \to A(G)$, as follows. Let $v' \in V'$ be the eigenvector for Q chosen above, and for each $v \in V$, let $f(v)$ be the coefficient of T defined by v and v':

$$f(v)(g) = (T(g)v, v').$$

As above, the irreducibility of T implies that whenever $v \neq 0$, then also $f(v) \neq 0$, i.e. f is injective. Therefore it is sufficient to show that the coefficients $\theta = f(v)$ coming from an eigenvector $v \in V$ for P form a one-dimensional subspace. But by the calculation above, any such coefficient satisfies

$$\theta(g) = \lambda(h) . \theta(e)$$

for all $g \in G_0$, and thus is determined by the choice of $\theta(e)$ which is a scalar. q.e.d.

Proposition A 4. — *If* λ *is the upper weight of an irreducible representation* T *of* G *on a vector space* V, *then* $\lambda(x)=1$ *for all* $x \in X$, *and there are integers* $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$ *such that*

$$\lambda(h)=h_1^{n_1} \cdots h_r^{n_r}$$

for all $h=(h_1, \ldots, h_r) \in H$.

Proof. — We have already seen that $\lambda(x)=1$ for every $x \in X$. On the other hand, it is well known [4, exposé 4, théorème 2] that every character of $H \simeq (k^*)^r$ is of the form $\lambda(h)=h_1^{n_1} \cdots h_r^{n_r}$ for suitable $n_i \in \mathbf{Z}$. Hence we have only to show that $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$.

As in the previous proof, we can choose vectors $v \in V$ and $v' \in V'$ so that the corresponding coefficient θ of T satisfies

$$\theta(g)=\lambda(h) . \theta(e)$$

for every $g=yhx$ in G_0. Furthermore, multiplying v' by a scalar if necessary, we may assume that $\theta(e)=(v, v')=1$. Defining g_i for $i=0, \ldots, r$ as in Lemma A 1, we have

$$\begin{aligned}\theta(g)&=h_1^{n_1} \cdots h_r^{n_r}\\ &=g_1^{n_1-n_2} \cdots g_{r-1}^{n_{r-1}-n_r} g_r^{n_r}\end{aligned}$$

for each $g \in G_0$. Now our result is a consequence of the following Lemma.

Lemma A 5. — *Let integers* $s_1, \ldots, s_r$ *be given. Then there is a function* θ *in* A(G) *which for each* $g \in G_0$ *is of the form*

$$\theta(g)=g_1^{s_1} \cdots g_r^{s_r}$$

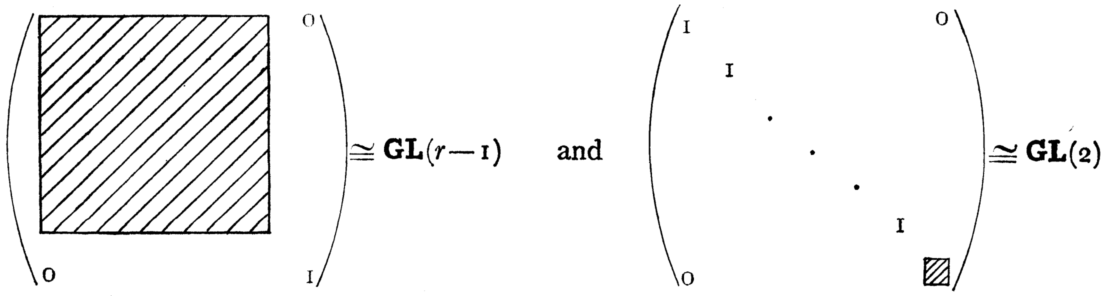
if and only if $s_1, \ldots, s_{r-1} \geqslant 0$ *(with no restriction on* s_r*).*

Proof. — Suppose first that there is such a function θ. Then we proceed by induction on r. If $r=1$ there is nothing to prove. If $r=2$ we consider the matrices

$$g(x)=\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$$

for each $x \in k$, and define a function φ from k to k by $\varphi(x)=\theta(gx))$. For $x \neq 0$ we have $g(x) \in G_0$, and so $\varphi(x)=x^{s_1}$. Since the algebraic functions from k to k are the polynomials in x, we deduce that $s_1 \geqslant 0$.

For $r \geqslant 2$ we consider the following two subgroups of G:



Restricting θ to these subgroups, we get

a) a function θ' on $\mathbf{GL}(r-1)$ of the form

$$\theta'(g)=g_1^{s_1}\cdots g_{r-2}^{s_{r-2}}g_{r-1}^{s_{r-1}+s_r}$$

for $g\in\mathbf{GL}(r-1)_0$, and

b) a function θ'' on $\mathbf{GL}(2)$ of the form

$$\theta''(g)=g_1^{s_{r-1}}g_2^{s_r}$$

for $g\in\mathbf{GL}(2)_0$.

We deduce from *a)* that $s_1, \ldots, s_{r-2}\geqslant 0$, and form *b)* that $s_{r-1}\geqslant 0$.

Conversely, suppose that the integers $s_1, \ldots, s_r$ satisfy $s_1, \ldots, s_{r-1}\geqslant 0$. Then the formula given for θ defines a function for all $g\in G$, because $g_r=\det(g_{ij})$ is invertible.

Now we show how to construct an irreducible representation of G with given upper weight.

Proposition A 6. — Let a set of integers $c=(n_1, \ldots, n_r)$ *with* $n_1\geqslant n_2\geqslant\ldots\geqslant n_r$ *be given. Let* $\theta_c\in A(G)$ *be the function given by*

$$\theta_c(g)=g_1^{s_1}\cdots g_r^{s_r}$$

where $s_i=n_i-n_{i+1}$ *for* $i=1, \ldots, r-1$, *and* $s_r=n_r$. *Let* V_c *be the subspace of* $A(G)$ *generated by the right translates of* θ_c *(as in example 4 above). Then* V_c *is an irreducible representation of* G *with upper weight* λ *given by*

$$\lambda(h)=h_1^{n_1}\cdots h_r^{n_r}.$$

Furthermore, any other irreducible representation of G *with the same upper weight is isomorphic to this one.*

Proof. — First we show that θ_c is an eigenvector for P with upper weight λ. Indeed, note by calculating with matrices that if $p\in P$ and $g\in G$, then $\theta_c(p)=\lambda(p)$, and for each i, $(gp)_i=g_ip_i$. Thus

$$\begin{aligned}(T(p)\theta_c)(g)&=\theta_c(gp)=(gp)_1^{s_1}\cdots(gp)_r^{s_r}\\&=(g_1^{s_1}\cdots g_r^{s_r})(p_1^{s_1}\cdots p_r^{s_r})\\&=\theta_c(g)\,.\,\theta_c(p)\\&=\lambda(p)\,.\,\theta_c(g)\end{aligned}$$

as required.

Thus V_c is a representation of G containing an eigenvector for P with the correct upper weight. If it is not irreducible, we can divide it by a maximal stable subspace not containing θ_c, and thus deduce the existence of some irreducible representation of G with the correct upper weight.

Now let T_1 be any irreducible representation of G with upper weight λ, on a vector space W. Let T_1' be the contragredient representation on the dual vector space W', and let $w'\in W'$ be an eigenvector for Q. Define a mapping $f: W\rightarrow A(G)$ as in the proof of Proposition A 3 by sending each $w\in W$ into the coefficient of T_1 determined by w and w'. Then, as above, f is injective and is compatible with the action of G. Furthermore, if $w \in W$ is an eigenvector for P such that $(w, w') = 1$, then the calculation of the proof of Proposition A 4 shows that $f(w) = \theta_c$. Therefore $f(w) \subseteq A(G)$ is an irreducible representation of G, containing θ_c. It follows that $f(W)$ equals V_c, which proves that V_c is indeed irreducible, and that f is an isomorphism of W onto V_c. q.e.d.

Example. — In characteristic $p \neq 0$ it may happen that a representation of G is generated by an eigenvector for P without being irreducible. For example, let $p = r = 2$. Let V be the standard representation of $\mathbf{GL}(2)$, with basis x, y. Consider $\Gamma^2(V) = S^2(V')'$. Then $\Gamma^2(V)$ has for a basis the symbols $x^2/2$, xy, $y^2/2$ with G acting in the obvious way. $\Gamma^2(V)$ is reducible, since xy generates a one-dimensional subrepresentation isomorphic to $\Lambda^2 V$. On the other hand, $x^2/2$ is an eigenvector for P, and generates the whole vector space under the action of G.

In conclusion we can state the following theorem.

Theorem A 7. — Let k be an algebraically closed field, and let $G = \mathbf{GL}(r, k)$. *Then the isomorphism classes of irreducible algebraic representations of* G *are in one-to-one correspondence with sequence of integers* $c = (n_1, \ldots, n_r)$ *with* $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$. *This correspondence is such that*

a) *The upper weight* λ *of the irreducible representation* V_c *corresponding to* c *is given by*

$$\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$$

for all $h \in H$, *and*

b) *if* V *is the standard representation of* G, *then* V_c *is a quotient of a subrepresentation of*

$$S^{s_1}(V) \otimes S^{s_2}(\Lambda^2 V) \otimes \ldots \otimes S^{s_r}(\Lambda^r V)$$

where $s_i = n_i - n_{i+1}$ *for* $i = 1, \ldots, r-1$ *and* $s_r = n_r$. (Note that $\Lambda^r V$ is a one-dimensional representation, so that expression above makes sense also for negative s_r.)

Proof. — Everything is already proved except for *b)*. Let $e_1, \ldots, e_r$ be the basis of V with respect to which G acts. Then the vector

$$e_1^{s_1} \otimes (e_1 \wedge e_2)^{s_2} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_r)^{s_r}$$

in the representation above, is an eigenvector for P with upper weight λ. Hence V_c is a quotient of the subrepresentation it generates. (In characteristic zero, V_c is isomorphic to the subrepresentation it generates.)

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