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ON SOME BRUHAT DECOMPOSITION AND THE STRUCTURE OF THE HECKE RINGS OF $\mathfrak{p}$-ADIC CHEVALLEY GROUPS

by N. IWAHORI *and* H. MATSUMOTO

INTRODUCTION

The purpose of this note is to give a sort of Bruhat decomposition for a Chevalley group G over a $\mathfrak{p}$-adic field K and to give some applications of this decomposition. To be more precise, we consider the Chevalley group G (see Chevalley [6]) associated with a pair of a complex semi-simple Lie algebra $\mathfrak{g}_{\mathbf{C}}$ and a field K with non-trivial discrete valuation. (The residue class field $k=\mathfrak{O}/\mathfrak{P}$ of K is not assumed to be finite.) Let $\mathfrak{h}_{\mathbf{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ and Δ the root system of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$. Then for any $\alpha\in\Delta$, there is associated a homomorphism $\Phi_\alpha : \mathrm{SL}(2, \mathrm{K})\to\mathrm{G}$. We denote as usual the image of $\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix}$ under Φ_α by $x_\alpha(t)$, $x_{-\alpha}(t)$ respectively. Now let P_r be the subgroup of $\mathfrak{h}^*_{\mathbf{C}}$ ($=$the dual of $\mathfrak{h}_{\mathbf{C}}$) generated by Δ. Then for any $\chi\in\mathrm{Hom}(\mathrm{P}_r, \mathrm{K}^*)$ there is associated an element $h(\chi)$ of G (see [6]). Now let us define the subgroups U, B of G which will be our main subject in this note. We denote by U the subgroup of G generated by the

$$\mathfrak{X}_{\alpha,\mathfrak{O}}=\{x_\alpha(t)\,;\,t\in\mathfrak{O}\}\ (\alpha\in\Delta) \qquad \text{and} \qquad \mathfrak{H}_{\mathfrak{O}}=\{h(\chi)\,;\,\chi\in\mathrm{Hom}(\mathrm{P}_r, \mathrm{K}^*),\ \chi(\mathrm{P}_r)\subset\mathfrak{O}^*\}$$

where $\mathfrak{O}^*$ is the group of all units in $\mathfrak{O}$ ($=$the ring of integers of K). Let B be the subgroup of U generated by the $\mathfrak{X}_{-\alpha,\mathfrak{O}}$ ($\alpha\in\Delta^+$ ($=$the positive roots)),

$$\mathfrak{X}_{\alpha,\mathfrak{P}}=\{x_\alpha(t)\,;\,t\in\mathfrak{P}\} \quad (\alpha\in\Delta^+)$$

and $\mathfrak{H}_{\mathfrak{O}}$. Then it turns out that U coincides with the subgroup of G consisting of elements which keep invariant the Chevalley lattice $\mathfrak{g}_{\mathfrak{O}}=\mathfrak{O}\otimes_{\mathbf{Z}}\mathfrak{g}_{\mathbf{Z}}$ (in the sense of Bruhat [4]) (see Cor. 2.17), and that B is the full inverse image of a Borel subgroup B_k of the Chevalley group G_k of $\mathfrak{g}_{\mathbf{C}}$ over $k=\mathfrak{O}/\mathfrak{P}$ under the reduction (mod. $\mathfrak{P}$) homomorphism $\mathrm{U}\to\mathrm{G}_k$ (see Prop. 2.4). When K is locally compact, U is a maximal compact subgroup and it is shown that the condition (I) of Satake [12] (i.e. a sort of Iwasawa decomposition) is valid (see Prop. 2.33). Also, in a sense Satake's condition (II) is also verified (see Cor. 2.35). In fact, we can show that the Hecke ring $\mathscr{H}(\mathrm{G}, \mathrm{U})$ (for the definition of Hecke rings, see § 3 or [10, § 1]) is commutative and is isomorphic to the polynomial ring $\mathbf{Z}[X_1, \ldots, X_l]$ where l is the rank of $\mathfrak{g}_{\mathbf{C}}$, not assuming the completeness of K, but assuming that $\mathfrak{O}/\mathfrak{P}$ is finite. However this will be treated in a subsequent paper.

Now let G′ be the commutator subgroup of G; also let $\mathfrak{W}$ be the subgroup of G generated by $\mathfrak{H}=\{h(\chi);\ \chi\in \mathrm{Hom}(P_r, K^*)\}$ and the $\omega_\alpha=\Phi_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right)$ $(\alpha\in\Delta)$. Then we shall show that the triple $(G', B', \mathfrak{W}')$ where $B'=B\cap G'$, $\mathfrak{W}'=\mathfrak{W}\cap G'$, satisfies all the hypotheses of Tits [16] (see Th. 2.44). In this case $B'\cap\mathfrak{W}'=\mathfrak{H}'_{\mathfrak{O}}=\mathfrak{H}_{\mathfrak{O}}\cap G'$ is a distinguished subgroup of $\mathfrak{W}'$ and the quotient group $\widetilde{W}'=\mathfrak{W}'/\mathfrak{H}_{\mathfrak{O}'}$ is isomorphic to the infinite group generated by the reflections with respect to the hyperplanes $P_{\alpha,k}=\{x\in\mathfrak{h}^*_{\mathbf{R}};\ (\alpha, x)=k\}$ $(\alpha\in\Delta,\ k\in\mathbf{Z})$, where $\mathfrak{h}^*_{\mathbf{R}}=\sum_{\gamma\in\Delta}\mathbf{R}\gamma$, and (α, x) means the Killing form; i.e. $\widetilde{W}'$ is the semi-direct product of the Weyl group W and the group D′ consisting of translations $T(d): T(d)x=x+d$ $(d\in P^{\perp}$, where P is the subgroup of $\mathfrak{h}^*_{\mathbf{C}}$ generated by all weights of $\mathfrak{g}_{\mathbf{C}}$ and $P^{\perp}=\{x\in\mathfrak{h}^*_{\mathbf{R}};\ (x,\lambda)\in\mathbf{Z}$ for all $\lambda\in P\})$. Thus after Tits [16], all subgroups H of G′ such that $G'\supset H\supset B'$ are in one-to-one correspondence with the subsets L of the set J of some generators of $\widetilde{W}'$. J is given explicitly in Prop. 2.23 and we can determine in particular the conjugacy classes of maximal subgroups of G′ containing a conjugate of B′ (see Prop. 2.30). When K is locally compact, we can determine the conjugacy classes of maximal compact subgroups of G′ containing a conjugate of B′ (see Prop. 2.32). Also we can prove that some analogous phenomenon as in Tits [16] is true for the triple $(G, B, \mathfrak{W})$ (see Prop. 2.8, Cor. 2.7, Th. 2.22). Here $B\cap\mathfrak{W}=\mathfrak{H}_{\mathfrak{O}}$ and $\widetilde{W}=\mathfrak{W}/\mathfrak{H}_{\mathfrak{O}}$ is the semi-direct product of the translation group $D=\{T(f);\ f\in P_r^{\perp}\}$ and W, where $P_r^{\perp}=\{x\in\mathfrak{h}^*_{\mathbf{R}};\ (x,\alpha)\in\mathbf{Z}$ for all $\alpha\in\Delta\}$. $\widetilde{W}$ is a semi-direct product of $\widetilde{W}'$ and a finite abelian group Ω which is isomorphic to P/P_r ($\cong$ the fundamental group of the adjoint group) (see § 1.7). Namely G is decomposed into a disjoint union of double cosets: $G=\bigcup_{\sigma\in\widetilde{W}}B\omega(\sigma)B$ ($\omega(\sigma)$ is an element of $\mathfrak{W}$ contained in σ) (Prop. 2.16) and some basic conditions of Tits [16] are verified; for example, $\omega(w_i)B\omega(\sigma)\subset B\omega(\sigma)B\cup B\omega(w_i\sigma)B$ (w_i is an involutive element in the system of standard generators; see § 2.3) and $\omega(w_i)B\omega(w_i)^{-1}\neq B$. Then again we can determine the subgroups H of G containing B using a similar discussion as in [16] (see Prop. 2.88). In particular when K is locally compact, we shall determine the conjugacy classes of maximal compact subgroups of G containing a conjugate of B (see Prop. 2.31). On the other hand, when G is of classical type, H. Hijikata has determined recently [9] all the conjugacy classes of maximal compact subgroups of G, which shows that our conjugacy classes given above exhaust all the conjugacy classes. Thus it seems to us that the number given in Prop. 2.31 for exceptional groups will also give the number of all conjugacy classes of maximal compact subgroups. However this is still an open question to us.

In § 3, we assume that $k=\mathfrak{O}/\mathfrak{P}$ is finite, and using the above structure of G, we shall determine the structure of the Hecke rings $\mathscr{H}(\mathrm{G}, \mathrm{B})$ and $\mathscr{H}(\mathrm{G}', \mathrm{B}')$. If $\mathfrak{g}_{\mathbf{C}}$ is simple, $\mathscr{H}(\mathrm{G}', \mathrm{B}')$ is generated by $l+1$ double cosets (l being the rank of $\mathfrak{g}_{\mathbf{C}}$) $\mathrm{S}_i = \mathrm{B}\omega(w_i)\mathrm{B}$ ($i = 0, 1, \ldots, l$) corresponding to the bounding hyperplanes $\mathrm{P}_0, \mathrm{P}_1, \ldots, \mathrm{P}_l$ of the simplex $\mathfrak{D}_0$, the fundamental domain of the discontinuous group $\widetilde{\mathrm{W}}' = \mathrm{D}'\mathrm{W}$, together with the defining relations which are analogous to those given in [10, Th. 4.1] for the case where K is finite (see Th. 3.5). Now Ω acts on $\mathscr{H}(\mathrm{G}', \mathrm{B}')$ as a group of automorphisms and $\mathscr{H}(\mathrm{G}, \mathrm{B})$ is isomorphic to the "twisted" tensor product $\mathbf{Z}[\Omega] \underset{\mathbf{Z}}{\otimes} \mathscr{H}(\mathrm{G}', \mathrm{B}')$ with respect to this action (see Prop. 3.8). Also for $x \in \mathrm{G}$, we shall prove that the index $[\mathrm{B} : \mathrm{B} \cap x^{-1}\mathrm{B}x]$ (which is equal to the number of cosets of the form $\mathrm{B}\xi$ in the double coset $\mathrm{B}x\mathrm{B}$) is always equal to a power of $q = [\mathfrak{O} : \mathfrak{P}]$ and we shall give an explicit formula for the exponent (see Prop. 3.2). We denote by $\lambda(x)$ the exponent: $q^{\lambda(x)} = [\mathrm{B} : \mathrm{B} \cap x^{-1}\mathrm{B}x]$. These theorems in § 3 are also given in Goldman-Iwahori [8], by a different method, for the case where $\mathrm{G} = \mathrm{GL}(n, \mathrm{K})$ and B is the corresponding subgroup. The " Poincaré series " $\sum_x t^{\lambda(x)}$ where the summation is taken over the representatives of the double coset space $\mathrm{B}\backslash\mathrm{G}/\mathrm{B}$ turns out to have some relation with the Poincaré series of the loop space of the compact Lie group associated to $\mathfrak{g}_{\mathbf{C}}$ (see Bott [2]) and is given explicitly in Prop. 1.30. Also using this, a formula for the order of W is given (see Prop. 1.32).

The contents of § 1 are rather classical facts about the structure of the groups $\widetilde{\mathrm{W}}$, $\widetilde{\mathrm{W}}'$ as transformation groups on the euclidean space $\mathfrak{h}^*_{\mathbf{R}}$, which are given in E. Cartan [5], Stiefel [14], Borel-de Siebenthal [1]. But we gave them together with proofs to make the reading easier. We hope that some proofs are new. The main proposition in § 1 is Prop. 1.15 which is the main tool for reaching the defining relations for the generators of the Hecke ring $\mathscr{H}(\mathrm{G}', \mathrm{B}')$. (This proposition 1.15 is the analogue for the group $\widetilde{\mathrm{W}}'$ of the proposition given in [10, Th. 2.6] for the Weyl group W.) As a corollary of Prop. 1.15, we shall give a system of defining relations for the generators w_i ($0 \leqslant i \leqslant l$) of $\widetilde{\mathrm{W}}'$, where w_i is the reflection map with respect to a bounding hyperplane P_i of the fundamental simplex (see Cor. 1.16).

Finally we should like to express our deep thanks to Professor F. Bruhat for the suggesting and helpful conversations during his stay in Tokyo in 1963.

§ 1. On the Weyl group extended by translations.

1.1. Let $\mathfrak{g}_{\mathbf{C}}$ be a semi-simple Lie algebra over the complex number field $\mathbf{C}$ and $\mathfrak{h}_{\mathbf{C}}$ a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Denote by Δ the set of all non-zero roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$. Let $\mathfrak{h}^*_{\mathbf{C}}$ be the dual vector space of $\mathfrak{h}_{\mathbf{C}}$ and $\mathfrak{h}^*_{\mathbf{R}}$ the real subspace of $\mathfrak{h}^*$ spanned by Δ. The restriction of the Killing form of $\mathfrak{g}_{\mathbf{C}}$ on $\mathfrak{h}^*_{\mathbf{R}}$ will be denoted by (x, y) for $x, y \in \mathfrak{h}^*_{\mathbf{R}}$. This restriction (x, y) is a symmetric, positive definite bilinear form on $\mathfrak{h}^*_{\mathbf{R}}$ and thus $\mathfrak{h}^*_{\mathbf{R}}$ is a Euclidean space. The length of $x \in \mathfrak{h}^*_{\mathbf{R}}$ will be denoted by $||x|| : ||x|| = (x, x)^{\frac{1}{2}}$.

Now let Π be a fundamental root system of Δ and fix a lexicographical linear ordering of $\mathfrak{h}_{\mathbf{R}}^*$ such that Π becomes the set of all simple roots in Δ with respect to this ordering. Denote by Δ^+ (resp. by Δ^-) the set of all positive (resp. negative) roots in Δ.

We denote by $P_{\alpha,k}$ ($\alpha\in\Delta$, $k\in\mathbf{Z}$; $\mathbf{Z}$ means the ring of rational integers) the hyperplane of $\mathfrak{h}_{\mathbf{R}}^*$ defined by

$$P_{\alpha,k}=\{x\in\mathfrak{h}_{\mathbf{R}}^*;\ (\alpha, x)=k\}.$$

Also we denote by $\widetilde{\Delta}$ the set of all $P_{\alpha,k}$ ($\alpha\in\Delta$, $k\in\mathbf{Z}$). Now let us denote by $w_{\alpha,k}$ the reflection mapping of $\mathfrak{h}_{\mathbf{R}}^*$ onto itself with respect to $P_{\alpha,k}$. Thus

$$w_{\alpha,k}(x)=x-(x, \alpha)\alpha^*+k\alpha^* \qquad (x\in\mathfrak{h}_{\mathbf{R}}^*)$$

where α^* means the element $2\alpha/(\alpha, \alpha)$ of $\mathfrak{h}_{\mathbf{R}}^*$ for $\alpha\in\Delta$. We denote by $T(d)$ for each $d\in\mathfrak{h}_{\mathbf{R}}^*$ the translation mapping of $\mathfrak{h}_{\mathbf{R}}^*$ onto itself defined by

$$T(d)x=x+d.$$

Also we denote $w_{\alpha,0}$ by w_α. Then we have

$$w_{\alpha,k}=T(k\alpha^*)\circ w_\alpha.$$

Let W be the Weyl group of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$, i.e. W is the group generated by the w_α ($\alpha\in\Delta$). It is known that W is generated by the w_α ($\alpha\in\Pi$) (cf. [13, Exposé 16]).

We denote by P the set of all weights of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$ for all linear representations of $\mathfrak{g}_{\mathbf{C}}$, i.e. $P=\{\lambda\in\mathfrak{h}_{\mathbf{R}}^*;\ (\lambda, \alpha^*)\in\mathbf{Z}$ for any $\alpha\in\Delta\}$. Then P is a $\mathbf{Z}$-submodule of $\mathfrak{h}_{\mathbf{R}}^*$. We denote by P_r the $\mathbf{Z}$-submodule of P generated by Δ. It is known that both P and P_r are stable under the action of W. P and P_r are both free abelian groups with l generators where l is the rank of $\mathfrak{g}_{\mathbf{C}}$: $l=\dim_{\mathbf{C}}\mathfrak{h}_{\mathbf{C}}$. More precisely, let $\Pi=\{\alpha_1, \dots, \alpha_l\}$; then $P_r=\sum_{i=1}^{l}\mathbf{Z}\alpha_i$ (cf. [13, Exp. 10]). Also we have $P=\sum_{i=1}^{l}\mathbf{Z}\Lambda_i$ where $\{\Lambda_1, \Lambda_2, \dots, \Lambda_l\}$ is the fundamental weight system associated with $\Pi=\{\alpha_1, \alpha_2, \dots, \alpha_l\}$, i.e. $\Lambda_1, \dots, \Lambda_l$ are defined by

$$(\Lambda_i, \alpha_j^*)=\delta_{ij} \qquad (1\leq i, j\leq l).$$

The quotient group P/P_r is isomorphic to the center $\mathfrak{z}$ of the simply connected Lie group $G_{\mathbf{C}}$ which has $\mathfrak{g}_{\mathbf{C}}$ as its Lie algebra (cf. [6, § 1]).

1.2. Now let us denote by $P^\perp$, $P_r^\perp$ the $\mathbf{Z}$-submodules of $\mathfrak{h}_{\mathbf{R}}^*$ defined by

$$P^\perp=\{x\in\mathfrak{h}_{\mathbf{R}}^*;\ (x, \lambda)\in\mathbf{Z} \text{ for any } \lambda\in P\},$$
$$P_r^\perp=\{x\in\mathfrak{h}_{\mathbf{R}}^*;\ (x, \alpha)\in\mathbf{Z} \text{ for any } \alpha\in P_r\}.$$

Then $P^\perp$ and $P_r^\perp$ are both free abelian groups of rank l and we have $P^\perp\subset P_r^\perp$ and $P_r^\perp/P^\perp\cong P/P_r\cong\mathfrak{z}$. We have in fact

$$P_r^\perp=\sum_{i=1}^{l}\mathbf{Z}\varepsilon_i,$$
$$P^\perp=\sum_{i=1}^{l}\mathbf{Z}\alpha_i^*,$$

where $\varepsilon_1, \ldots, \varepsilon_l$ are the elements in $\mathfrak{h}_{\mathbf{R}}^*$ defined by

$$(\varepsilon_i, \alpha_j) = \delta_{ij} \qquad (1 \leq i, j \leq l).$$

In other words $(\varepsilon_1, \ldots, \varepsilon_l)$ are given by

$$\varepsilon_i = 2\Lambda_i/(\alpha_i, \alpha_i) \qquad (1 \leq i \leq l).$$

Since P, P_r are stable under W, $\mathrm{P}^\perp$, $\mathrm{P}_r^\perp$ are also stable under W.

We denote by D the group consisting of the translations of the form $\mathrm{T}(d)$, $d \in \mathrm{P}_r^\perp$. Clearly the map $d \to \mathrm{T}(d)$ is an isomorphism from $\mathrm{P}_r^\perp$ onto D and we may identify $\mathrm{P}_r^\perp$ and D by the map $d \to \mathrm{T}(d)$. We also denote by D′ the subgroup of D consisting of the translations of the form $\mathrm{T}(d)$, $d \in \mathrm{P}^\perp$. D′ may be identified with $\mathrm{P}^\perp$ by the above isomorphism and we have $\mathrm{D}/\mathrm{D}' \cong \mathrm{P}/\mathrm{P}_r \cong \mathfrak{z}$. Note that

$$\alpha_i = \sum_{j=1}^{l} a_{ij} \Lambda_j \qquad (1 \leq i \leq l)$$

$$\alpha_j^* = \sum_{i=1}^{l} a_{ij} \varepsilon_i \qquad (1 \leq j \leq l)$$

where $a_{ij} = (\alpha_i, \alpha_j^*)$ $(1 \leq i, j \leq l)$ are the Cartan integers.

Now using the obvious relation $w\mathrm{T}(d)w^{-1} = \mathrm{T}(w(d))$ $(w \in \mathrm{W}, d \in \mathfrak{h}_{\mathbf{R}}^*)$, we see that DW(=WD) is a subgroup of the group of all motions of the Euclidean space $\mathfrak{h}_{\mathbf{R}}^*$ and that D is a distinguished subgroup of DW. Obviously we have $\mathrm{D} \cap \mathrm{W} = \{1\}$. Similarly D′W(=WD′) is a subgroup of DW containing D′ as a distinguished subgroup.

Now the group D′W is generated by the reflections $w_{\alpha,k}$ $(\alpha \in \Delta, k \in \mathbf{Z})$. In fact the equality $w_{\alpha,k} = \mathrm{T}(k\alpha^*)w_\alpha$ shows that every $w_{\alpha,k}$ is in D′W and also that D′ and W are contained in the subgroup generated by the $w_{\alpha,k}$ $(\alpha \in \Delta, k \in \mathbf{Z})$. Thus D′W is the group generated by the $w_{\alpha,k}$ $(\alpha \in \Delta, k \in \mathbf{Z})$.

The set $\widetilde{\Delta}$ of the hyperplanes $\mathrm{P}_{\alpha,k}$ $(\alpha \in \Delta, k \in \mathbf{Z})$ is stable under the group DW. In fact we have

$$\mathrm{T}(d)w(\mathrm{P}_{\alpha,k}) = \mathrm{P}_{w(\alpha), k + (w(\alpha), d)}$$

for any $d \in \mathrm{P}_r$, $w \in \mathrm{W}$, $k \in \mathbf{Z}$, $\alpha \in \Delta$. Also we see that the subgroup D′W is a distinguished subgroup of DW. In fact, $\sigma(\mathrm{P}_{\alpha,k}) = \mathrm{P}_{\beta,m}$ $(\sigma \in \mathrm{DW}; \alpha, \beta \in \Delta; k, m \in \mathbf{Z})$ implies that $\sigma w_{\alpha,k}\sigma^{-1} = w_{\beta,m}$. Then it is easily seen that $\mathrm{DW}/\mathrm{D}'\mathrm{W} \cong \mathrm{D}/\mathrm{D}' \cong \mathrm{P}/\mathrm{P}_r \cong \mathfrak{z}$.

1.3. Now the union $\bigcup_{\alpha,k} \mathrm{P}_{\alpha,k}$ is obviously a closed subset of $\mathfrak{h}_{\mathbf{R}}^*$ and is stable under DW ($\bigcup_{\alpha,k} \mathrm{P}_{\alpha,k}$ is called the diagram of $\mathrm{G}_{\mathbf{C}}$). Hence the complement $\mathfrak{h}_{\mathbf{R}}^* - \bigcup_{\alpha,k} \mathrm{P}_{\alpha,k}$ is an open subset of $\mathfrak{h}_{\mathbf{R}}^*$. Any connected component $\mathfrak{D}$ of $\mathfrak{h}_{\mathbf{R}}^* - \bigcup_{\alpha,k} \mathrm{P}_{\alpha,k}$ is called a *cell.* Since $\mathfrak{h}_{\mathbf{R}}^* - \bigcup_{\alpha,k} \mathrm{P}_{\alpha,k}$ is stable under DW, the group DW acts in an obvious manner on the set $\mathfrak{F}$ of all cells. It is easy to see that the open set

$$\mathfrak{D}_0 = \{x \in \mathfrak{h}_{\mathbf{R}}^*;\ 0 < (\alpha, x) < 1 \text{ for any } \alpha \in \Delta^+\}$$

is a connected component of $\mathfrak{h}_{\mathbf{R}}^* - \bigcup_{\alpha,k} \mathrm{P}_{\alpha,k}$, i.e. $\mathfrak{D}_0$ is a cell. $\mathfrak{D}_0$ is called the *fundamental cell.*

We note that if $P_{\alpha,k}$ and $P_{\beta,m}$ are not parallel, then the angle θ betwen α and β is equal to one of the following 4 values, $(1-\frac{1}{\nu})\pi$ $(\nu=2,3,4,6)$, and the order of $w_{\alpha,k}w_{\beta,m}$ is equal to ν in the respective cases. (We may assume $\pi>\theta\geq\frac{\pi}{2}$ since $P_{\alpha,k}=P_{-\alpha,-k}$.)

Proposition **1.1** (cf. [5], [14]). — *Let*

$$\Delta=\Delta^{(1)}\cup\ldots\cup\Delta^{(r)}$$

the orthogonal decomposition of Δ *associated with the decomposition* $\mathfrak{g}_{\mathbf{C}}=\mathfrak{g}_{\mathbf{C}}^{(1)}+\ldots+\mathfrak{g}_{\mathbf{C}}^{(r)}$ *of* $\mathfrak{g}_{\mathbf{C}}$ *into simple ideals* $\mathfrak{g}_{\mathbf{C}}^{(1)},\ldots,\mathfrak{g}_{\mathbf{C}}^{(r)}$. *Let* Γ *be the subgroup of* D′W *generated by* w_α $(\alpha\in\Pi)$ *and* $w^{(1)},\ldots,w^{(r)}$, *where* $w^{(i)}=w_{\alpha_0^{(i)},1}$, $\alpha_0^{(i)}$ *being the highest root of* $\Delta^{(i)}$, $i=1,\ldots,r$. *Then* Γ *is transitive on the set* $\mathfrak{F}$ *of all cells.*

Proof. — Obviously we may assume $\mathfrak{g}_{\mathbf{C}}$ to be simple. Then the fundamental cell $\mathfrak{D}_0$ is an open simplex given by

$$\mathfrak{D}_0=\{x\in\mathfrak{h}_{\mathbf{R}}^*;\ 0<(\alpha_i,x),\ 1>(\alpha_0,x),\ i=1,\ldots,l\}$$

where $\Pi=\{\alpha_1,\ldots,\alpha_l\}$ and α_0 is the highest root of Δ. Let $\mathfrak{D}$ be any cell in $\mathfrak{F}$. We have to show the existence of an element $\sigma\in\Gamma$ such that $\sigma(\mathfrak{D})=\mathfrak{D}_0$. Let $a\in\mathfrak{D}_0$, $b\in\mathfrak{D}$ be fixed elements. Since the D-orbit of b is obviously discrete, the D′W-orbit and hence the Γ-orbit $\Gamma(b)$ of b is also a discrete subset of $\mathfrak{h}_{\mathbf{R}}^*$. Thus $\inf||a-x||$ $(x\in\Gamma(b))$ is attained by some $x=\sigma(b)$, $\sigma\in\Gamma$. It is enough to show that $x\in\mathfrak{D}_0$. (Then we get $\sigma(\mathfrak{D})\cap\mathfrak{D}_0\neq\emptyset$ which implies that $\sigma(\mathfrak{D})=\mathfrak{D}_0$.) Assume that $x\notin\mathfrak{D}_0$. Then, with respect to some bounding hyperplane P of $\mathfrak{D}_0$, x and a belong to different half-spaces. Let w be the reflection map with respect to P. Since P is equal to one of $P_{\alpha_1,0},\ldots,P_{\alpha_l,0}$, $P_{\alpha_0,1}$, w is in Γ. Moreover we have easily

$$||w(x)-a||<||x-a||.$$

This contradicts the choice of x, Q.E.D.

Proposition **1.2**. — *We use the same notations as in Prop.* 1.1. *The group* D′W *is generated by the reflections* w_α $(\alpha\in\Pi)$ *and* $w^{(1)},\ldots,w^{(r)}$ *(i.e. by the reflections with respect to the bounding hyperplanes of the fundamental cell* $\mathfrak{D}_0$*)*; D′W *is transitive on* $\mathfrak{F}$.

Proof. — Let $\alpha\in\Delta$, $k\in\mathbf{Z}$, then the hyperplane $P_{\alpha,k}$ bounds some cell $\mathfrak{D}$. Take an element $\sigma\in\Gamma$ such that $\sigma(\mathfrak{D})=\mathfrak{D}_0$ (Prop. 1.1). Then $\sigma(P_{\alpha,k})$ coincides with some bounding hyperplane P of $\mathfrak{D}_0$. Then $\sigma w_{\alpha,k}\sigma^{-1}$ coincides with the reflection w with respect to P : $\sigma w_{\alpha,k}\sigma^{-1}=w\in\Gamma$. Thus $w_{\alpha,k}\in\Gamma$, which implies immediately that $\Gamma=$ D′W and completes the proof.

1.4. — Now before proceeding to the proof that D′W is simply transitive on $\mathfrak{F}$, let us introduce a few notions. Also in order to avoid the inessential troubles about the description, we assume, in the following part of § 1, that $\mathfrak{g}_{\mathbf{C}}$ is simple. We denote by $\alpha_1,\ldots,\alpha_l$ the fundamental roots and by α_0 the highest root. Also we put $P_i=P_{\alpha_i,0}$ $(i=1,\ldots,l)$, $P_0=P_{\alpha_0,1}$, $w_i=w_{\alpha_i}$ $(i=1,\ldots,l)$, $w_0=w_{\alpha_0,1}=T(\alpha_0^*)w_{\alpha_0}$. Thus $P_0,P_1,\ldots,P_l$ are the bounding hyperplanes of the simplex $\mathfrak{D}_0$ and $w_0,w_1,\ldots,w_l$ generate the group D′W.

Now let $\mathfrak{D}$, $\mathfrak{D}'\in\mathfrak{F}$ and $P_{\alpha,k}\in\widetilde{\Delta}$. We shall write $\mathfrak{D}\sim\mathfrak{D}'(P_{\alpha,k})$ (resp. $\mathfrak{D}\nsim\mathfrak{D}'(P_{\alpha,k})$) if $\mathfrak{D}$ and $\mathfrak{D}'$ belong to the same (resp. different) half-spaces with respect to $P_{\alpha,k}$. Since $\mathfrak{D}\cap P_{\alpha,k}=\emptyset$, $\mathfrak{D}'\cap P_{\alpha,k}=\emptyset$, we get easily the following criterion: Let $a\in\mathfrak{D}$, $b\in\mathfrak{D}'$. Then $\mathfrak{D}\nsim\mathfrak{D}'(P_{\alpha,k})$ if and only if the segment $\overline{ab}$ intersects with $P_{\alpha,k}$. Also, $\mathfrak{D}\nsim\mathfrak{D}'(P_{\alpha,k})$ is equivalent to

$$((\alpha, a)-k)((\alpha, b)-k)<0.$$

Now let us denote by $\widetilde{\Delta}(\mathfrak{D}, \mathfrak{D}')$ the subset of $\widetilde{\Delta}$ defined by

$$\widetilde{\Delta}(\mathfrak{D}, \mathfrak{D}')=\{P_{\alpha,k}\in\widetilde{\Delta};\ \mathfrak{D}\nsim\mathfrak{D}'(P_{\alpha,k})\}.$$

$\widetilde{\Delta}(\mathfrak{D}, \mathfrak{D}')$ is always a finite set. In fact, fixing $a\in\mathfrak{D}$ and $b\in\mathfrak{D}'$, it is easy to see that only a finite number of $P_{\alpha,k}$ intersect with the segment $\overline{ab}$. The following equalities are easy consequences of the definition:

$$\widetilde{\Delta}(\mathfrak{D}, \mathfrak{D}')=\widetilde{\Delta}(\mathfrak{D}', \mathfrak{D})$$

$$\sigma.\widetilde{\Delta}(\mathfrak{D}, \mathfrak{D}')=\widetilde{\Delta}(\sigma\mathfrak{D}, \sigma\mathfrak{D}')$$

for any $\mathfrak{D}, \mathfrak{D}'\in\mathfrak{F}$, $\sigma\in DW$.

Now let $\sigma\in DW$. Then we denote by $\widetilde{\Delta}(\sigma)$ the set $\widetilde{\Delta}(\sigma\mathfrak{D}_0, \mathfrak{D}_0)$. We denote by $\lambda(\sigma)$ the cardinality of the finite set $\widetilde{\Delta}(\sigma)$. $\lambda(\sigma)$ is nothing but the function considered by Bott [2]. By the definition of $\widetilde{\Delta}(\sigma)$, we get easily

$$\sigma^{-1}.\widetilde{\Delta}(\sigma)=\widetilde{\Delta}(\sigma^{-1}),$$

$$\lambda(\sigma^{-1})=\lambda(\sigma)$$

for any $\sigma\in DW$.

1.5. Now let us define a function $l(\sigma)$ on D'W. With respect to the involutive generators $w_0, w_1, \ldots, w_l$, any element $\sigma\in D'W\ (\sigma\neq 1)$ can be written as $\sigma=w_{i_1}\ldots w_{i_r}$ $(0\leq i_1, \ldots, i_r\leq l)$. The $\mathrm{Min}(r)$ for all these expressions of σ will be called the length of σ (with respect to the generators $w_0, w_1, \ldots, w_l$) and we denote by $l(\sigma)$ the length of σ. We also put $l(1)=0$. We shall call a word $w_{i_1}\ldots w_{i_r}$ in D'W *reduced* if $l(w_{i_1}\ldots w_{i_r})=r$. Also an expression $\sigma=w_{i_1}\ldots w_{i_r}$ of $\sigma\in D'W$ will be called reduced if $l(\sigma)=r$. Clearly, if $w_{i_1}\ldots w_{i_r}$ is a reduced word, then $w_{i_2}\ldots w_{i_r}$ and $w_{i_1}\ldots w_{i_{r-1}}$ are both reduced. Also for $\sigma\in D'W, l(\sigma)=1$ means that $\sigma\in\{w_0, w_1, \ldots, w_l\}$. The purpose of this section is to show that $\lambda(\sigma)=l(\sigma)$ for $\sigma\in D'W$. We begin with the

Lemma **1.3.** — *For any* $\sigma\in DW$ *and for any* i, $0\leq i\leq l$, *we have*

$$w_i(\widetilde{\Delta}(\sigma^{-1})-\{P_i\})=\widetilde{\Delta}(w_i\sigma^{-1})-\{P_i\}.$$

Proof. — Let $P_{\alpha,k}\in\widetilde{\Delta}(\sigma^{-1})-\{P_i\}$. Then $\sigma(P_{\alpha,k})\in\widetilde{\Delta}(\sigma)$, $P_{\alpha,k}\neq P_i$. We have to show that $w_i(P_{\alpha,k})\in\widetilde{\Delta}(w_i\sigma^{-1})-\{P_i\}$. Firstly, since $w_i(P_i)=P_i$, we have $w_i(P_{\alpha,k})\neq P_i$.

Now assume $w_i(P_{\alpha,k}) \notin \widetilde{\Delta}(w_i\sigma^{-1}\mathfrak{D}_0, \mathfrak{D}_0)$. Then we have $P_{\alpha,k} \notin \widetilde{\Delta}(\sigma^{-1}\mathfrak{D}_0, w_i\mathfrak{D}_0)$, i.e. $\sigma(P_{\alpha,k}) \notin \widetilde{\Delta}(\mathfrak{D}_0, \sigma w_i \mathfrak{D}_0)$, i.e. $\sigma w_i \mathfrak{D}_0 \sim \mathfrak{D}_0(\sigma(P_{\alpha,k}))$. On the other hand we have, by $\sigma(P_{\alpha,k}) \in \widetilde{\Delta}(\sigma)$, $\sigma\mathfrak{D}_0 \nsim \mathfrak{D}_0(\sigma(P_{\alpha,k}))$. Hence we get $\sigma w_i \mathfrak{D}_0 \nsim \sigma\mathfrak{D}_0(\sigma(P_{\alpha,k}))$, i.e. $w_i\mathfrak{D}_0 \nsim \mathfrak{D}_0(P_{\alpha,k})$, i.e. $P_{\alpha,k} \in \widetilde{\Delta}(w_i)$. Now for any $a \in \mathfrak{D}_0$, the only hyperplane in $\widetilde{\Delta}$ which intersects with the segment $\overline{a, w_i(a)}$ is obviously P_i, i.e. $\widetilde{\Delta}(w_i) = \{P_i\}$. Hence we get $P_{\alpha,k} = P_i$, which is a contradiction. Thus we have shown that $w_i(\widetilde{\Delta}(\sigma^{-1}) - \{P_i\}) \subset \widetilde{\Delta}(w_i\sigma^{-1}) - \{P_i\}$. Now replacing σ by σw_i, we get $w_i(\widetilde{\Delta}(w_i\sigma^{-1}) - \{P_i\}) \subset \widetilde{\Delta}(\sigma^{-1}) - \{P_i\}$, which completes the proof since $w_i^2 = 1$.

Corollary **1.4**. — *For any* $\sigma \in DW$ *and for any* i, $0 \leq i \leq l$, *we have*

$$w_i(\widetilde{\Delta}(\sigma) - \{P_i\}) = \widetilde{\Delta}(w_i\sigma) - \{P_i\}.$$

Proof. — Replace σ^{-1} by σ in Lemma 1.3.

Lemma **1.5**. — *For any* $\sigma \in DW$ *and for any* i, $0 \leq i \leq l$, P_i *is exactly in one of* $\widetilde{\Delta}(\sigma^{-1})$, $\widetilde{\Delta}(w_i\sigma^{-1})$. *We have*

$$\lambda(\sigma w_i) = \lambda(\sigma) - 1 \qquad \textit{if} \qquad P_i \in \widetilde{\Delta}(\sigma^{-1})$$

$$\lambda(\sigma w_i) = \lambda(\sigma) + 1 \qquad \textit{if} \qquad P_i \notin \widetilde{\Delta}(\sigma^{-1})$$

Proof. — Assume that $P_i \in \widetilde{\Delta}(\sigma^{-1})$, $P_i \in \widetilde{\Delta}(w_i\sigma^{-1})$. Then we have $\sigma\mathfrak{D}_0 \nsim \mathfrak{D}_0(\sigma(P_i))$, $\sigma w_i \mathfrak{D}_0 \nsim \mathfrak{D}_0(\sigma(P_i))$. Hence $\sigma\mathfrak{D}_0 \sim \sigma w_i \mathfrak{D}_0(\sigma(P_i))$, i.e. $\mathfrak{D}_0 \sim w_i\mathfrak{D}_0(P_i)$ which is a contradiction. Similarly we get a contradiction if we assume $P_i \notin \widetilde{\Delta}(\sigma^{-1})$, $P_i \notin \widetilde{\Delta}(w_i\sigma^{-1})$. Thus P_i is exactly in one of $\widetilde{\Delta}(\sigma^{-1})$, $\widetilde{\Delta}(w_i\sigma^{-1})$. The second half of the lemma is an obvious consequence of Lemma 1.3.

Corollary **1.6**. — *For any* $\sigma \in D'W$, *we have* $l(\sigma) \geq \lambda(\sigma)$.

Proof. — Let $\sigma = w_{i_1} \ldots w_{i_r}$ be any reduced expression of σ. Then since $\lambda(\tau w_i) \leq \lambda(\tau) + 1$ for any $\tau \in DW$, $0 \leq i \leq l$ (Lemma 1.5), we have $\lambda(\sigma) \leq r = l(\sigma)$, Q.E.D.

Lemma **1.7**. — *Let* $\sigma \in D'W$, $\sigma \neq 1$. *Then* $\widetilde{\Delta}(\sigma)$ *is not empty.*

Proof. — Let $\sigma = w_{i_1} \ldots w_{i_r}$ be any reduced expression of σ. Put

$$\sigma = \sigma_1 = w_{i_1} \ldots w_{i_r}, \qquad \sigma_2 = w_{i_2} \ldots w_{i_r}, \qquad \ldots, \qquad \sigma_r = w_{i_r}.$$

Assume that $\widetilde{\Delta}(\sigma)$ is an empty set. Then by Lemma 1.5 (replacing σ^{-1} there by σ) $P_{i_1} \in \widetilde{\Delta}(w_{i_1}\sigma)$. Hence we get by Cor. 1.4, $\widetilde{\Delta}(\sigma_2) = \{P_{i_1}\}$. Let us assume now that we have proved the following assertion (A_k) for some k, $2 \leq k \leq r$:

$$(A_k) : \widetilde{\Delta}(\sigma_k) = \{w_{i_{k-1}} \ldots w_{i_2}(P_{i_1}), w_{i_{k-1}} \ldots w_{i_3}(P_{i_2}), \ldots, w_{i_{k-1}}(P_{i_{k-2}}), P_{i_{k-1}}\}.$$

We shall show that $2 \leq k < r$ and (A_k) imply (A_{k+1}). In fact, it is enough to show that $P_{i_k} \notin \widetilde{\Delta}(\sigma_k)$. (Then, because of Lemma 1.5 and Cor. 1.4, we have

$\widetilde{\Delta}(\sigma_{k+1})=\widetilde{\Delta}(w_{i_k}\sigma_k)=w_{i_k}\widetilde{\Delta}(\sigma_k)\cup\{P_{i_k}\}$ which is nothing but (A_{k+1}).) Assume $P_{i_k}\in\widetilde{\Delta}(\sigma_k)$. Then by (A_k) there exists some m with $2\leq m\leq k-1$ such that

$$P_{i_k}=w_{i_{k-1}}\dots w_{i_m}(P_{i_{m-1}}),$$

i.e.

$$(w_{i_{k-1}}\dots w_{i_m})w_{i_{m-1}}(w_{i_{k-1}}\dots w_{i_m})^{-1}=w_{i_k},$$

i.e.

$$w_{i_{m-1}}w_{i_m}\dots w_{i_{k-1}}=w_{i_m}\dots w_{i_{k-1}}w_{i_k}.$$

Hence we get

$$w_{i_1}\dots w_{i_r}=(w_{i_1}\dots w_{i_{m-1}})(w_{i_m}\dots w_{i_k})(w_{i_{k+1}}\dots w_{i_r})$$

$$=\begin{cases} w_{i_1}\dots w_{i_{m-2}}w_{i_m}\dots w_{i_{k-1}}w_{i_{k+1}}\dots w_{i_r} & (m>2), \\ w_{i_2}\dots w_{i_{k-1}}w_{i_{k+1}}\dots w_{i_r} & (m=2). \end{cases}$$

This contradicts $l(w_{i_1}\dots w_{i_r})=r$. Thus $(A_2), \dots, (A_r)$ are all valid. In particular (A_r) means that

$$\widetilde{\Delta}(\sigma_r)=\widetilde{\Delta}(w_{i_r})=\{w_{i_{r-1}}\dots w_{i_2}(P_{i_1}), \dots, w_{i_{r-1}}(P_{i_{r-2}}), P_{i_{r-1}}\}.$$

On the other hand $\widetilde{\Delta}(w_{i_r})=\{P_{i_r}\}$. Thus P_{i_r} must coincide with an element in $\{w_{i_{r-1}}\dots w_{i_2}(P_{i_1}), \dots, w_{i_{r-1}}(P_{i_{r-2}}), P_{i_{r-1}}\}$. Then we get a contradiction as above, Q.E.D.

Corollary **1.8** (*cf.* [5], [14]). — *The group* D'W *is simply transitive on* $\mathfrak{F}$.

Proof. — We have only to show that $\sigma\in D'W$ and $\sigma\mathfrak{D}_0=\mathfrak{D}_0$ imply $\sigma=1$ (see Prop. 1.2). If $\sigma\mathfrak{D}_0=\mathfrak{D}_0$, then $\widetilde{\Delta}(\sigma)$ is empty. Hence $\sigma=1$ by Lemma 1.7.

Corollary **1.9**. — *Let* $\sigma\in D'W, \sigma\neq 1$. *Then there is some i with* $0\leq i\leq l$ *such that* $P_i\in\widetilde{\Delta}(\sigma)$.

Proof. — Assume $P_i\notin\widetilde{\Delta}(\sigma)$ for all $i=0, 1, \dots, l$. Then for any $a\in\mathfrak{D}_0$ the segment $\overline{a, \sigma(a)}$ does not intersect with any P_i, $0\leq i\leq l$. Hence the point $\sigma(a)$ belongs to $\mathfrak{D}_0$. Thus we have $\sigma\mathfrak{D}_0=\mathfrak{D}_0$ and $\sigma=1$, which is a contradiction.

Proposition **1.10**. — *For any* $\sigma\in D'W$, *we have* $\lambda(\sigma)=l(\sigma)$.

Proof. — Let us prove the proposition by induction on $\lambda(\sigma)$. If $\lambda(\sigma)=0$, then $\widetilde{\Delta}(\sigma)$ is empty and $\sigma=1$. Hence we have $\lambda(\sigma)=l(\sigma)=0$. Now assume that $\lambda(\sigma)=k>0$ and that we have proved $\lambda(\tau)=l(\tau)$ for any $\tau\in D'W$ with $\lambda(\tau)<k$. By Cor. 1.9, there exists some i with $0\leq i\leq l$ such that $P_i\in\widetilde{\Delta}(\sigma^{-1})$. Then we have $\lambda(\tau)=k-1$ for $\tau=\sigma w_i$ by Lemma 1.5. Hence we get $\lambda(\tau)=l(\tau)=k-1$ by our induction assumption. Thus there exist $j_1, \dots, j_{k-1}$ with $0\leq j_1, \dots, j_{k-1}\leq l$ such that $\tau=w_{j_1}\dots w_{j_{k-1}}$. Hence $\sigma=\tau w_i=w_{j_1}\dots w_{j_{k-1}}w_i$. Thus we have $l(\sigma)\leq(k-1)+1=k=\lambda(\sigma)$, which completes the proof by Cor. 1.6.

Corollary **1.11**. — *Let* $\sigma\in D'W$ *and i be an integer with* $0\leq i\leq l$. *Then there exists a reduced expression of* σ *starting with* w_i *(resp. ending at* w_i*) if and only if* $P_i\in\widetilde{\Delta}(\sigma)$ (*resp.* $P_i\in\widetilde{\Delta}(\sigma^{-1})$).

Proof. — Assume $P_i\in\widetilde{\Delta}(\sigma)$. Then, putting $\tau=w_i\sigma$, we have $P_i\notin\widetilde{\Delta}(\tau)$ and $l(\tau)=\lambda(\tau)=\lambda(\sigma)-1=l(\sigma)-1$ by Lemma 1.5 and Cor. 1.4. Thus for any reduced expression $\tau=w_{j_2}\dots w_{j_k}$ of τ, we get a reduced expression $\sigma=w_i w_{j_2}\dots w_{j_k}$ of σ.

Conversely let $\sigma = w_{j_1} w_{j_2} \ldots w_{j_k}$ be a reduced expression of σ with $j_1 = i$. Then $\tau = w_{j_2} \ldots w_{j_k}$ is also a reduced expression. Hence we have $l(\tau) = l(\sigma) - 1$, i.e. $\lambda(\tau) = \lambda(\sigma) - 1$. Then we get $P_i \in \widetilde{\Delta}(\sigma)$ by Lemma 1.5 and Cor. 1.4, Q.E.D.

1.6.

Proposition **1.12**. — *Let* $\sigma \in D'W$ *and* $\sigma = w_{i_1} \ldots w_{i_r}$ *be any reduced expression of* σ. *Then we have*

$$\widetilde{\Delta}(\sigma) = \{P_{i_1}, w_{i_1}(P_{i_2}), w_{i_1} w_{i_2}(P_{i_3}), \ldots, w_{i_1} \ldots w_{i_{r-1}}(P_{i_r})\}.$$

Proof. — We prove the proposition by induction on $\lambda(\sigma)$. If $r = \lambda(\sigma) = 1$, then $\sigma = w_i$ and we obviously have $\widetilde{\Delta}(\sigma) = \{P_i\}$. Now assume that $r > 1$ and that our assertion is valid for $\tau \in D'W$ with $\lambda(\tau) < r$. Put $\tau = w_{i_1} \sigma$. Then $\lambda(\tau) = l(\tau) = r - 1$. Thus we get by our induction assumption that $\widetilde{\Delta}(\tau) = \{P_{i_2}, w_{i_2}(P_{i_3}), \ldots, w_{i_2} \ldots w_{i_{r-1}}(P_{i_r})\}$. Now we have $P_{i_1} \in \widetilde{\Delta}(\sigma)$ by Cor. 1.11. Hence $P_{i_1} \notin \widetilde{\Delta}(\tau)$ and we have

$$\widetilde{\Delta}(\sigma) - \{P_{i_1}\} = w_{i_1}(\widetilde{\Delta}(\tau) - \{P_{i_1}\}) = w_{i_1} \widetilde{\Delta}(\tau).$$

Hence $\widetilde{\Delta}(\sigma) = \{P_{i_1}\} \cup w_{i_1} \widetilde{\Delta}(\tau)$ which is what was to be proved.

Corollary **1.13**. — *Let* $\sigma, \tau, \rho \in D'W$ *and* $\sigma = \tau\rho$. *Then we have* $\lambda(\sigma) = \lambda(\tau) + \lambda(\rho)$ *if and only if* $\widetilde{\Delta}(\sigma)$ *is a disjoint union of* $\widetilde{\Delta}(\tau)$ *and* $\tau\widetilde{\Delta}(\rho)$.

Proof. — If $\widetilde{\Delta}(\sigma)$ is a disjoint union of $\widetilde{\Delta}(\tau)$ and $\tau\widetilde{\Delta}(\rho)$, then we obviously get $\lambda(\sigma) = \lambda(\tau) + \lambda(\rho)$. Conversely let $\lambda(\sigma) = \lambda(\tau) + \lambda(\rho)$. Then for any reduced expressions $\tau = w_{i_1} \ldots w_{i_r}$, $\rho = w_{j_1} \ldots w_{j_s}$, $\sigma = w_{i_1} \ldots w_{i_r} w_{j_1} \ldots w_{j_s}$ is a reduced expression of σ. Then by Prop. 1.12, we get $\widetilde{\Delta}(\sigma) = \widetilde{\Delta}(\tau) \cup \tau\widetilde{\Delta}(\rho)$. This is a disjoint union since $\lambda(\sigma) = \lambda(\tau) + \lambda(\rho)$, Q.E.D.

Lemma **1.14**. — *Let* $w_{i_1} \ldots w_{i_r} = w_{j_1} \ldots w_{j_r}$ *be a reduced word in* D'W. *If* $P_{j_1} \notin \widetilde{\Delta}(w_{i_1} \ldots w_{i_s})$, *then there exists an integer* m *such that*

$$s + 1 \leq m \leq r \quad \textit{and} \quad w_{i_1} \ldots w_{i_m} = w_{j_1} w_{i_1} \ldots w_{i_{m-1}}.$$

Proof. — Put $\tau = w_{i_1} \ldots w_{i_s}$, $\rho = w_{i_{s+1}} \ldots w_{i_r}$. Then $\sigma = \tau\rho$ and $l(\sigma) = l(\tau) + l(\rho)$. Hence $\widetilde{\Delta}(\sigma) = \widetilde{\Delta}(\tau) \cup \tau\widetilde{\Delta}(\rho)$ (disjoint) by Cor. 1.13. Now $P_{j_1} \in \widetilde{\Delta}(\sigma)$ by Cor. 1.11. Also $P_{j_1} \notin \widetilde{\Delta}(\tau)$ by the assumption. Hence

$$P_{j_1} \in \tau\widetilde{\Delta}(\rho) = \tau\{P_{i_{s+1}}, w_{i_{s+1}}(P_{i_{s+2}}), \ldots, w_{i_{s+1}} \ldots w_{i_{r-1}}(P_{i_r})\}.$$

Thus there exists some integer m with $s + 1 \leq m \leq r$ such that $P_{j_1} = \tau w_{i_{s+1}} \ldots w_{i_{m-1}}(P_{i_m})$, i.e. $P_{j_1} = w_{i_1} \ldots w_{i_{m-1}}(P_{i_m})$. Hence we get $w_{j_1} = (w_{i_1} \ldots w_{i_{m-1}}) w_{i_m} (w_{i_1} \ldots w_{i_{m-1}})^{-1}$ which completes the proof.

Now let $\theta_{ij} = \theta_{ji}$ be the angle between the fundamental roots α_i and α_j, $1 \leq i \neq j \leq l$. It is known (cf. [13, Exp. 10]) that $\pi/2 \leq \theta_{ij} < \pi$ for $i \neq j$. Also let $\theta_{0i} = \theta_{i0}$ $(i = 1, \ldots, l)$ be the angle between $-\alpha_0, \alpha_i$. Since $\alpha_0 + \alpha_i \notin \Delta$, we have

$2(\alpha_0, \alpha_i)/(\alpha_i, \alpha_i) \geq 0$. Hence we have also $\pi/2 \leq \theta_{0i} \leq \pi$ $(i = 1, \ldots, l)$. It is also well known that, for $0 \leq i+j \leq l$, θ_{ij} is of the form $(1 - \frac{1}{\nu})\pi$, $\nu = 2, 3, 4, 6$ and we have

$$(*) \quad \begin{cases} w_i w_j = w_j w_i & \text{if} \quad \theta_{ij} = \pi/2, \\ w_i w_j w_i = w_j w_i w_j & \text{if} \quad \theta_{ij} = 2\pi/3, \\ (w_i w_j)^2 = (w_j w_i)^2 & \text{if} \quad \theta_{ij} = 3\pi/4, \\ (w_i w_j)^3 = (w_j w_i)^3 & \text{if} \quad \theta_{ij} = 5\pi/6. \end{cases}$$

Proposition **1.15**. — *Let* $\mathfrak{g}_C$ *be a complex simple Lie algebra; we use the notations as above for* $w_0, \ldots, w_l, \theta_{ij}$ $(0 \leq i, j \leq l)$, W, D'W. *Let* $\mathfrak{S}$ *be any associative semi-group and* $\Delta_0, \Delta_1, \ldots, \Delta_l$ *be* $l+1$ *elements in* $\mathfrak{S}$ *satisfying the following relations*:

$$\begin{array}{lll} \Delta_i \Delta_j = \Delta_j \Delta_i & \textit{if} & \theta_{ij} = \pi/2, \\ \Delta_i \Delta_j \Delta_i = \Delta_j \Delta_i \Delta_j & \textit{if} & \theta_{ij} = 2\pi/3, \\ (\Delta_i \Delta_j)^2 = (\Delta_j \Delta_i)^2 & \textit{if} & \theta_{ij} = 3\pi/4, \\ (\Delta_i \Delta_j)^3 = (\Delta_j \Delta_i)^3 & \textit{if} & \theta_{ij} = 5\pi/6. \end{array}$$

Then for any reduced words $w_{i_1} \ldots w_{i_r} = w_{j_1} \ldots w_{j_r}$ *in* D'W, *we have*

$$\Delta_{i_1} \ldots \Delta_{i_r} = \Delta_{j_1} \ldots \Delta_{j_r}.$$

Proof. — Using Lemma 1.14, the proof is given exactly in the same manner as in Iwahori [10, Th. 2.6].

Corollary **1.16**. — *The defining relations for the generators* $w_0, w_1, \ldots, w_l$ *of* D'W *are given by* (*) *above and*

$$w_i^2 = 1 \ (0 \leq i \leq l).$$

Proof. — Using Prop. 1.15, the proof is given exactly in the same manner as in [10, Cor. 2.7].

1.7. Let us define a subgroup Ω of DW by

$$\Omega = \{\sigma \in \mathrm{DW};\ \sigma\mathfrak{D}_0 = \mathfrak{D}_0\}.$$

Clearly Ω is defined also by

$$\Omega = \{\sigma \in \mathrm{DW};\ \lambda(\sigma) = 0\}.$$

Now since D'W is simply transitive on $\mathfrak{F}$, we have easily the following decomposition of DW into a semi-direct product of Ω and D'W:

$$\mathrm{DW} = \Omega . (\mathrm{D'W}), \qquad \Omega \cap \mathrm{D'W} = \{1\}.$$

Hence we have $\Omega \cong \mathrm{DW}/\mathrm{D'W} \cong \mathrm{D}/\mathrm{D'} \cong \mathrm{P}/\mathrm{P}_r = \mathfrak{z}$. Thus Ω is a finite abelian group isomorphic to the center $\mathfrak{z}$ of G_C. It is also easy to see that

$$\lambda(\rho\sigma\rho') = \lambda(\sigma)$$

for any $\sigma\in DW$ and $\rho, \rho'\in\Omega$. In fact,

$$\widetilde{\Delta}(\rho\sigma\rho')=\widetilde{\Delta}(\rho\sigma\rho'\mathfrak{D}_0, \mathfrak{D}_0)=\widetilde{\Delta}(\rho\sigma\mathfrak{D}_0, \mathfrak{D}_0)=\rho\widetilde{\Delta}(\sigma\mathfrak{D}_0, \mathfrak{D}_0)=\rho\widetilde{\Delta}(\sigma)$$

implies that $\lambda(\rho\sigma\rho')=\lambda(\sigma)$.

Proposition **1.17**. — *The intersection of* $P_r^{\perp}$ *with the closure* $\overline{\mathfrak{D}}_0$ *of* $\mathfrak{D}_0$ *consists of* 0 *and the* ε_i *with* $(\alpha_0, \varepsilon_i)=1$.

Proof. — Let $x=\sum_{i=1}^{l}\mu_i\varepsilon_i\in\overline{\mathfrak{D}}_0\cap P_r^{\perp}$, $x\neq 0$, $\mu_i\in\mathbf{Z}\,(1\leq i\leq l)$. Then by $0\leq(\alpha_i, x)$, $(\alpha_0, x)\leq 1$, we get $\mu_i\geq 0\,(1\leq i\leq l)$ and $\sum_{i=1}^{l}\mu_i m_i\leq 1$ where $m_i=(\alpha_0, \varepsilon_i)$. It is known that all m_i are positive integers ([13, Exp. 17]). Since $x\neq 0$, some $\mu_i>0$. Thus $m_i=\mu_i=1$ and all the other μ_j must be 0. Hence $x=\varepsilon_i$ for some i, $1\leq i\leq l$, with $(\alpha_0, \varepsilon_i)=1$. Conversely, if $(\alpha_0, \varepsilon_i)=1$, ε_i is obviously in $\overline{\mathfrak{D}}_0\cap P_r^{\perp}$, Q.E.D.

Now let us give an explicit description of Ω. Let $\sigma=T(d)w\in DW$ be an element of Ω where $d\in P_r^{\perp}$, $w\in W$. Assume $\sigma\neq 1$. Then $d\neq 0$ since $\Omega\cap W\subset\Omega\cap D'W=\{1\}$. Now since $\sigma\mathfrak{D}_0=\mathfrak{D}_0$, we have $\sigma\overline{\mathfrak{D}}_0=\overline{\mathfrak{D}}_0$. Hence $\sigma(0)\in\overline{\mathfrak{D}}_0$, i.e. $w(0)+d=d\in\overline{\mathfrak{D}}_0\cap P_r^{\perp}$. Hence $d=\varepsilon_i$ with some ε_i such that $(\alpha_0, \varepsilon_i)=1$. Note that w is uniquely determined by d. In fact, if we have $w, w'\in W$, $T(d)w\in\Omega$, $T(d)w'\in\Omega$, then we get $w^{-1}w'\in\Omega\cap W=\{1\}$, hence $w=w'$.

Now let us show conversely that if $d=\varepsilon_i$, $(\alpha_0, \varepsilon_i)=1$, then there exists an element $w\in W$ such that $T(d)w\in\Omega$ (w is unique as was remarked above). It is known that there exists in W an element w_Π such that $w_\Pi(\Pi)=-\Pi$ ([13, Exp. 16]). w_Π is unique and satisfies $w_\Pi^2=1$. Similarly, if we denote the subset $\Pi-\{\alpha_i\}$ by Π_i, then the subgroup W_i of W generated by $w_1, \ldots, \hat{w}_i, \ldots, w_l$ ($\hat{w}_i$ means that w_i is omitted) contains an element w_{Π_i} such that $w_{\Pi_i}(\Pi_i)=-\Pi_i$. w_{Π_i} is uniquely determined in W_i and satisfies $w_{\Pi_i}^2=1$. We claim that $T(\varepsilon_i)w_{\Pi_i}w_\Pi\in\Omega$, i.e. $T(\varepsilon_i)w_{\Pi_i}w_\Pi(\mathfrak{D}_0)=\mathfrak{D}_0$. Clearly we have $w_\Pi(\mathfrak{D}_0)=-\mathfrak{D}_0$. Let $a\in\mathfrak{D}_0$. Then $b=w_\Pi(a)\in-\mathfrak{D}_0$. It is enough to show that $w_{\Pi_i}(b)+\varepsilon_i\in\mathfrak{D}_0$. Now since w_{Π_i} is a product of the w_j's with $j\neq i$, we have $w_{\Pi_i}(\alpha_i)=\alpha_i+\sum_{j\neq i}\nu_j\alpha_j$ for some $\nu_j\in\mathbf{Z}$. Hence $w_{\Pi_i}(\alpha_i)>0$. Also we have $w_{\Pi_i}(\alpha_0)>0$. Now if $j\neq i$, $(\alpha_j, w_{\Pi_i}(b)+\varepsilon_i)=(\alpha_j, w_{\Pi_i}(b))=(w_{\Pi_i}(\alpha_j), b)>0$ since $w_{\Pi_i}(\alpha_j)\in-\Pi_i$, $b\in-\mathfrak{D}_0$. Also we have $(\alpha_i, w_{\Pi_i}(b)+\varepsilon_i)=1+(w_{\Pi_i}(\alpha_i), b)>0$ since $w_{\Pi_i}(\alpha_i)\in\Delta^+$ and $b\in-\mathfrak{D}_0$ imply that $(w_{\Pi_i}(\alpha_i), b)>-1$. Finally $(\alpha_0, w_{\Pi_i}(b)+\varepsilon_i)=1+(w_{\Pi_i}(\alpha_0), b)<1$ since $w_{\Pi_i}(\alpha_0)\in\Delta^+$ and $b\in-\mathfrak{D}_0$ imply that $(w_{\Pi_i}(\alpha_0), b)<0$. Thus we get $T(\varepsilon_i)w_{\Pi_i}w_\Pi(\mathfrak{D}_0)=\mathfrak{D}_0$ and we have proved the following

Proposition **1.18**. — *The mapping from the set* $\{0\}\cup\{\varepsilon_i;\ (\alpha_0, \varepsilon_i)=1\}$ *onto* Ω *defined by* $0\to 1$, $\varepsilon_i\to T(\varepsilon_i)w_{\Pi_i}w_\Pi$ *is bijective.*

Corollary **1.19**. — *The order of the group* Ω (*i.e. the index* $[P:P_r]$) *is equal to* $1+N$, *where* N *is the number of i's such that* $(\alpha_0, \varepsilon_i)=1$.

Corollary **1.20** (cf. [5]). — *For any cell* $\mathfrak{D}$, *the intersection* $\overline{\mathfrak{D}}\cap P^{\perp}$ *consists of a single element. In particular* $\overline{\mathfrak{D}}_0\cap P^{\perp}=\{0\}$.

Proof. — Since $P^{\perp}$ is stable under D'W and D'W is transitive on $\mathfrak{F}$, it is enough to show that $\overline{\mathfrak{D}}_0\cap P^{\perp}=\{0\}$. Let $x\neq 0$ be in $\overline{\mathfrak{D}}_0\cap P^{\perp}$. Then since $P^{\perp}\subset P_r^{\perp}$, there is some i with $x=\varepsilon_i$, $(\alpha_0, \varepsilon_i)=1$. Now since $x\in P^\perp$, we have $T(x)=T(\varepsilon_i)\in D'$. Hence $T(\varepsilon_i)w_{\Pi_i}w_\Pi\in D'W\cap\Omega=\{1\}$ which is a contradiction, Q.E.D.

The unique intersection point $\overline{\mathfrak{D}}\cap P^\perp$ is called the *lattice point* associated with the cell $\mathfrak{D}$. Note that for $\sigma, \tau\in D'W$, $\sigma\mathfrak{D}_0$ and $\tau\mathfrak{D}_0$ have the same associated lattice point if and only if $\sigma W=\tau W$. In fact, the lattice point associated with $\sigma\mathfrak{D}_0$ is clearly $\sigma(0)$, hence it is enough to show that

$$\sigma(0)=\tau(0)\Leftrightarrow\sigma W=\tau W.$$

But this is obvious since $\sigma(0)=\tau(0)\Leftrightarrow\sigma^{-1}\tau(0)=0\Leftrightarrow\sigma^{-1}\tau\in W$.

1.8. We shall now consider the automorphism $\sigma\to\rho\sigma\rho^{-1}$ of $D'W$ defined by $\rho\in\Omega$. Since $\lambda(\rho\sigma\rho^{-1})=\lambda(\sigma)$, this automorphism induces a permutation of the set $\{w_0, w_1, \ldots, w_l\}$. Thus we get a homomorphism from Ω onto a permutation group of $l+1$ letters $w_0, w_1, \ldots, w_l$. This homomorphism is injective. In fact, if a non-trivial element $\rho=T(\varepsilon_i)w_{\Pi_i}w_\Pi\in\Omega$, with $(\alpha_0, \varepsilon_i)=1$ induces the identity, we get $\rho w_j\rho^{-1}=w_j\ (0\leq j\leq l)$. In particular we get

$$w_jT(\varepsilon_i)w_{\Pi_i}w_\Pi w_j^{-1}=T(\varepsilon_i)w_{\Pi_i}w_\Pi \quad (j=1, \ldots, l).$$

Hence we have $w_jT(\varepsilon_i)w_j^{-1}=T(\varepsilon_i)$, i.e. $w_j(\varepsilon_i)=\varepsilon_i$, i.e. $(\alpha_j, \varepsilon_i)=0$ for $1\leq j\leq l$. Hence $\varepsilon_i=0$, which is a contradiction.

Proposition **1.21.** — (i) *Let* $\rho=T(\varepsilon_i)w_{\Pi_i}w_\Pi\in\Omega$, $(\alpha_0, \varepsilon_i)=1$. *Then* $\rho w_0\rho^{-1}=w_i$.

(ii) *Let* $\varphi: DW\to W$ *be the natural homomorphism. Then* φ *is injective on* Ω *and the set* $\{\alpha_1, \ldots, \alpha_l, -\alpha_0\}$ *is stable under the subgroup* $W_\Omega=\varphi(\Omega)$ *of* W.

Proof. — (i) Let us show first that $\rho w_0\rho^{-1}\in W$, i.e. $\rho w_0\rho^{-1}(0)=0$, i.e. $\rho^{-1}(0)\in P_{\alpha_0,1}$. Now $\rho^{-1}(0)=w_{\Pi_i}w_\Pi(-\varepsilon_i)$. Since $w_j(\varepsilon_i)=\varepsilon_i\ (j\neq 1)$ we have $w_{\Pi_i}(\varepsilon_i)=\varepsilon_i$, hence $\rho^{-1}(0)=-w_\Pi(\varepsilon_i)$. Thus we have to show that $(\alpha_0, -w_\Pi(\varepsilon_i))=1$, i.e. $(w_\Pi(\alpha_0), -\varepsilon_i)=1$. Now $w_\Pi(\Pi)=-\Pi$ implies that $w_\Pi(\alpha_0)=-\alpha_0$ and we have $(w_\Pi(\alpha_0), -\varepsilon_i)=(\alpha_0, \varepsilon_i)=1$. Hence we get $\rho w_0\rho^{-1}\in W$. Thus $\rho w_0\rho^{-1}\in\{w_1, \ldots, w_l\}$. Now the natural homomorphism $\varphi: DW\to W$ is injective on Ω, since $\Omega\cap D=\{1\}$ by Prop. 1.18. Hence to determine the element $\rho w_0\rho^{-1}\in W$, it is enough to determine the image of $\rho w_0\rho^{-1}$ under this homomorphism $DW\to W$. Now this image is clearly given by $w_{\Pi_i}w_\Pi w_{\alpha_0}w_\Pi w_{\Pi_i}=w_{\Pi_i}w_{\alpha_0}w_{\Pi_i}$ since $w_\Pi(\alpha_0)=-\alpha_0$ and $w_\Pi w_{\alpha_0}w_\Pi=w_{-\alpha_0}=w_{\alpha_0}$. Thus the image is equal to w_β where $\beta=w_{\Pi_i}(\alpha_0)$. On the other hand $\beta\in\pm\Pi$ since $\rho w_0\rho^{-1}\in\{w_1, \ldots, w_l\}$. As was remarked in the proof of Prop. 1.18, $w_{\Pi_i}(\alpha_0)>0$. Hence $\beta\in\Pi$. Also, since α_0 is of the form $\alpha_i+\sum_{j\neq i}m_j\alpha_j$ and w_{Π_i} is a product of the w_j's $(j\neq 1)$, $\beta=w_{\Pi_i}(\alpha_0)$ is also of the form $\alpha_i+\sum_{j\neq i}\mu_j\alpha_j$, $\mu_j\in\mathbf{Z}$. Thus β must coincide with α_i and we get $\rho w_0\rho^{-1}=w_i$.

(ii) Let $\rho=T(\varepsilon_i)w_{\Pi_i}w_\Pi$ be a non-trivial element in Ω. We have seen above that

$$\varphi(\rho)(-\alpha_0)=w_{\Pi_i}w_\Pi(-\alpha_0)=\alpha_i.$$

Put $\rho^{-1}=T(\varepsilon_j)w_{\Pi_j}w_\Pi$. Then $w_{\Pi_j}w_\Pi=(w_{\Pi_i}w_\Pi)^{-1}=w_\Pi w_{\Pi_i}$. Hence

$$\varphi(\rho)(\alpha_j)=w_{\Pi_i}w_\Pi(\alpha_j)=w_\Pi w_{\Pi_j}(\alpha_j)=-\alpha_0.$$

(Since $w_{\Pi_j} w_{\Pi}(-\alpha_0) = \alpha_j$.) Also for $\alpha_k \in \Pi - \{\alpha_j\}$, we get

$$\varphi(\rho)(\alpha_k) = w_{\Pi} w_{\Pi_j}(\alpha_k) \in w_{\Pi}(-\Pi_j) \subset \Pi.$$

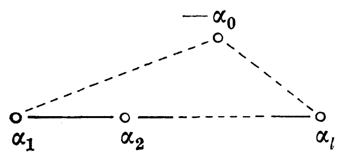
Thus $\varphi(\rho)$ keeps the set $\{\alpha_1, \ldots, \alpha_l, -\alpha_0\}$ stable, Q.E.D.

Corollary **1.22.** — *If* $(\alpha_0, \varepsilon_i) = 1$, *then* $w_{\Pi_i}(\alpha_0) = \alpha_i$.

We note here that the order of $\rho = \mathrm{T}(\varepsilon_i) w_{\Pi_i} w_{\Pi} \in \Omega$ is equal to the order of $w_{\Pi_i} w_{\Pi}$ since the homomorphism $\varphi : \mathrm{DW} \to \mathrm{W}$ is injective on Ω. Thus, if the Weyl group W has a non-trivial center, then $w_{\Pi} = -1$ and the order of ρ is equal to 2. Hence for types B_l, C_l, D_l ($l =$ even), G_2, F_4, E_7, E_8, every element ρ of Ω ($\rho \neq 1$) is of order 2.

We shall give in the following the table of the action of $\rho \in \Omega$ on the set $\{w_0, w_1, \ldots, w_l\}$ defined by $w_i \to \rho w_i \rho^{-1}$. We refer to Borel-de Siebenthal [1] for the coefficients m_i in the expression of $\alpha_0 = \sum_{i=1}^{l} m_i \alpha_i$. It is also noted that the permutation $w_i \to \rho w_i \rho^{-1}$ $(0 \leq i \leq l)$ of the set $\{w_0, w_1, \ldots, w_l\}$ induced by $\rho \in \Omega$ coincides with the permutation of the Dynkin diagram of $\{-\alpha_0, \alpha_1, \ldots, \alpha_l\}$ induced by $\varphi(\rho) \in \mathrm{W}_\Omega \subset \mathrm{W}$. Since $\varphi(\rho)$ preserves the angle between $-\alpha_0, \alpha_1, \ldots, \alpha_l$, $\varphi(\rho)$ is an automorphism of the Dynkin diagram of $\{-\alpha_0, \alpha_1, \ldots, \alpha_l\}$.

(A_l) :



$\alpha_0 = \alpha_1 + \ldots + \alpha_l$

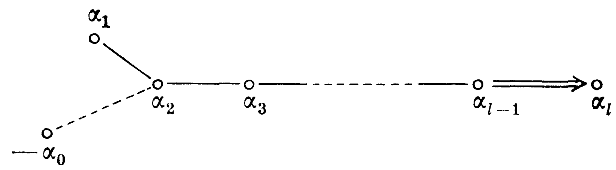
$\Omega \cong \mathbf{Z}_{l+1}$ (cyclic group of order $l+1$),

$\rho = \mathrm{T}(\varepsilon_1) w_{\Pi_1} w_{\Pi}$ generates Ω and

$\rho w_1 \rho^{-1} = w_2$, $\rho w_2 \rho^{-1} = w_3$, $\ldots$, $\rho w_l \rho^{-1} = w_0$.

$\rho w_0 \rho^{-1} = w_1$.

(B_l) :

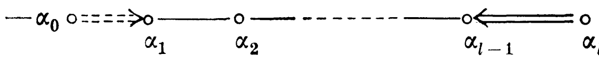


$\alpha_0 = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_l)$

$\Omega \cong \mathbf{Z}_2$, $\Omega = \{1, \rho\}$, $\rho = \mathrm{T}(\varepsilon_1) w_{\Pi_1} w_{\Pi}$.

$\rho w_0 \rho^{-1} = w_1$, $\rho w_1 \rho^{-1} = w_0$, $\rho w_i \rho^{-1} = w_i$ $(2 \leq i \leq l)$.

(C_l) :

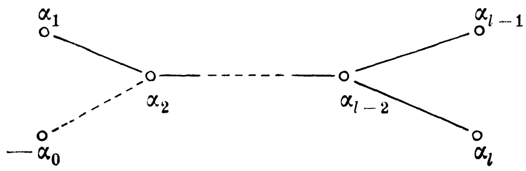


$\alpha_0 = 2(\alpha_1 + \ldots + \alpha_{l-1}) + \alpha_l$

$\Omega \cong \mathbf{Z}_2$, $\Omega = \{1, \rho\}$, $\rho = \mathrm{T}(\varepsilon_l) w_{\Pi_l} w_{\Pi}$.

$\rho w_0 \rho^{-1} = w_l$, $\rho w_1 \rho^{-1} = w_{l-1}$, $\ldots$, $\rho w_l \rho^{-1} = w_0$.

(D_l) :



$$\alpha_0 = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l$$

$\Omega = \{1, \rho_1, \rho_{l-1}, \rho_l\}$ where

$$\rho_i = T(\varepsilon_i) w_{\Pi_i} w_{\Pi} \qquad (i = 1, l-1, l).$$

$l = \text{even}$: $\Omega \cong \mathbf{Z}_2 \times \mathbf{Z}_2$,

$$\rho_1 w_0 \rho_1^{-1} = w_1, \quad \rho_1 w_1 \rho_1^{-1} = w_0, \quad \rho_1 w_i \rho_1^{-1} = w_i \qquad (2 \leq i \leq l-2),$$

$$\rho_1 w_{l-1} \rho_1^{-1} = w_l, \quad \rho_1 w_l \rho_1^{-1} = w_{l-1}.$$

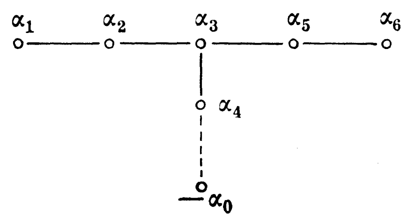
$$\rho_l w_i \rho_l^{-1} = w_{l-i} \ (0 \leq i \leq l), \quad \rho_{l-1} = \rho_l \rho_1 = \rho_l \rho_1.$$

$l = \text{odd}$: $\Omega \cong \mathbf{Z}_4$, ρ_l generates Ω and $\rho_1 = \rho_l^2$, $\rho_{l-1} = \rho_l^3$.

$$\rho_l w_0 \rho_l^{-1} = w_l, \quad \rho_l w_1 \rho_l^{-1} = w_{l-1}, \quad \rho_l w_i \rho_l^{-1} = w_{l-i} \qquad (2 \leq i \leq l-2),$$

$$\rho_l w_{l-1} \rho_l^{-1} = w_0, \quad \rho_l w_l \rho_l^{-1} = w_1.$$

(E_6) :



$$\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_2 + 2\alpha_4 + 2\alpha_5 + \alpha_6$$

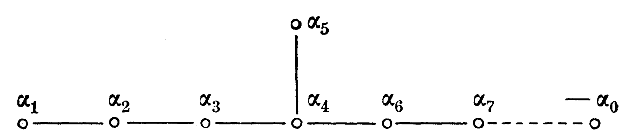
$\Omega \cong \mathbf{Z}_3$, $\quad \rho = T(\varepsilon_1) w_{\Pi_1} w_{\Pi}$ generates Ω,

$$\rho w_0 \rho^{-1} = w_1, \quad \rho w_1 \rho^{-1} = w_6, \quad \rho w_6 \rho^{-1} = w_0,$$

$$\rho w_4 \rho^{-1} = w_2, \quad \rho w_2 \rho^{-1} = w_5, \quad \rho w_5 \rho^{-1} = w_4$$

$$\rho w_3 \rho^{-1} = w_3.$$

(E_7) :



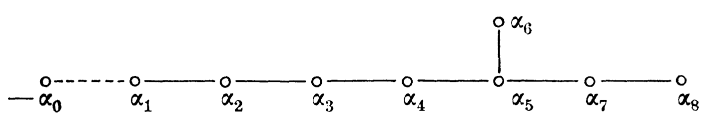
$$\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7$$

$\Omega \cong \mathbf{Z}_2$, $\quad \Omega = \{1, \rho\}$, $\quad \rho = T(\varepsilon_1) w_{\Pi_1} w_{\Pi}$.

$$\rho w_0 \rho^{-1} = w_1, \quad \rho w_7 \rho^{-1} = w_2, \quad \rho w_6 \rho^{-1} = w_3,$$

$$\rho w_4 \rho^{-1} = w_4, \quad \rho w_5 \rho^{-1} = w_5.$$

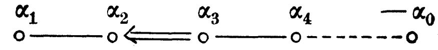
(E_8) :



$$\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 3\alpha_6 + 4\alpha_7 + 2\alpha_8$$

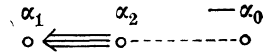
$$\Omega = \{1\}.$$

(F_4) :



$$\alpha_0 = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$$
$$\Omega = \{1\}.$$

(G_2) :



$$\alpha_0 = 3\alpha_1 + 2\alpha_2$$
$$\Omega = \{1\}.$$

1.9. We shall give in this section a formula for $\lambda(\sigma)$ and applications of this formula.

Let $\sigma = \mathrm{T}(d)w \in \mathrm{DW}$, $d \in \mathrm{P}_r^{\perp}$, $w \in \mathrm{W}$. Then for a hyperplane $\mathrm{P}_{\alpha,k} \in \widetilde{\Delta}$, the relation $\mathrm{P}_{\alpha,k} \in \widetilde{\Delta}(\sigma)$ is equivalent to

$$((\alpha, a) - k)((\alpha, \sigma(a)) - k) < 0$$

where a is any point in $\mathfrak{D}_0$ (see § 1.4). Now since $\mathrm{P}_{\alpha,k} = \mathrm{P}_{-\alpha,-k}$, we may assume always that $\alpha \in \Delta^+$. Let us denote by ν_α the number of $k \in \mathbf{Z}$ satisfying the above inequality for fixed $\alpha \in \Delta^+$, $a \in \mathfrak{D}_0$. Then we have

$$\lambda(\sigma) = \sum_{\alpha \in \Delta^+} \nu_\alpha.$$

Now let us compute ν_α. Since $\sigma(a) = w(a) + d$ and

$$(\alpha, \sigma(a)) = (\alpha, w(a) + d) = (w^{-1}(\alpha), a) + (\alpha, d),$$

ν_α is equal to the number of $k \in \mathbf{Z}$ satisfying the following inequality:

$$(\alpha, a) \gtrless k \gtrless (w^{-1}(\alpha), a) + (d, \alpha).$$

Now ν_α is independent of the choice of $a \in \mathfrak{D}_0$. Taking a sufficiently close to the origin of $\mathfrak{h}_{\mathbf{R}}^*$, we see easily that

$$\nu_\alpha = \begin{cases} |(\alpha, d)| & \text{if} \quad w^{-1}(\alpha) > 0, \\ |(\alpha, d) - 1| & \text{if} \quad w^{-1}(\alpha) < 0. \end{cases}$$

Thus we get the following

Proposition **1.23**. — *Let* $d \in \mathrm{P}_r^{\perp}$, $w \in \mathrm{W}$. *Then*

$$\lambda(\mathrm{T}(d)w) = \sum_{\substack{\alpha > 0 \\ w^{-1}(\alpha) > 0}} |(\alpha, d)| + \sum_{\substack{\alpha > 0 \\ w^{-1}(\alpha) < 0}} |(\alpha, d) - 1|.$$

Let $w \in \mathrm{W}$. Then we denote by Δ_w^+ the subset of Δ^+ defined by $\Delta_w^+ = w^{-1}\Delta^- \cap \Delta^+$. We also denote by $n(w)$ the cardinality of the set Δ_w^+. Then by Prop. 1.23 we get easily the

Corollary **1.24**. — $\lambda(w) = n(w)$ *for any* $w \in \mathrm{W}$.

As applications of Prop. 1.23, we shall compute $\underset{\sigma \in \mathrm{T}(d)\mathrm{W}}{\mathrm{Min}} \lambda(\sigma)$, $\underset{\sigma \in \mathrm{T}(d)\mathrm{W}}{\mathrm{Max}} \lambda(\sigma)$ for a given $d \in \mathrm{P}_r^{\perp}$. Put

$$\Delta_1 = \{\alpha \in \Delta^+;\ (\alpha, d) \leq 0\}, \qquad \Delta_2 = \{\alpha \in \Delta^+;\ (\alpha, d) > 0\}.$$

Then $\Theta=(-\Delta_1)\cup\Delta_2$ obviously satisfies

$$\Delta=\Theta\cup(-\Theta),\qquad \Theta\cap(-\Theta)=\emptyset.$$

Moreover, Θ is additively closed in Δ, i.e. $\alpha\in\Theta$, $\beta\in\Theta$, $\alpha+\beta\in\Delta$ imply that $\alpha+\beta\in\Theta$. Hence there exists a unique element $w^*\in W$ such that $w^*\Delta^+=\Theta$. (See Borel-Hirzebruch, *Amer. J. Math.*, 80 (1958), Chap. I, § 4, or R. Steinberg, *Trans. Amer. Math. Soc.*, 105 (1962), 118-125.) Then we have $\Delta^+_{(w^*)^{-1}}=\Delta_1$ and $\Delta^+-\Delta^+_{(w^*)^{-1}}=\Delta_2$. Thus we get by Prop. 1.23,

$$\lambda(T(d)w^*)=\sum_{\alpha\in\Delta_1}(|(\alpha,d)|+1)+\sum_{\alpha\in\Delta_2}|(\alpha,d)|.$$

Then it is obvious that we have $\lambda(T(d)w^*)=\operatorname*{Max}_{\sigma\in T(d)W}\lambda(\sigma)$. Similarly, there exists a unique element $w^{**}\in W$ such that $w^{**}\Delta^+=\Delta_1\cup(-\Delta_2)=-\Theta$. Hence $\Delta^+_{(w^{**})^{-1}}=\Delta_2$, $\Delta^+-\Delta^+_{(w^{**})^{-1}}=\Delta_1$ and we have

$$\lambda(T(d)w^{**})=\sum_{\alpha\in\Delta_2}(|(\alpha,d)|-1)+\sum_{\alpha\in\Delta_1}|(\alpha,d)|.$$

Then we obviously have $\lambda(T(d)w^{**})=\operatorname*{Min}_{\sigma\in T(d)W}\lambda(\sigma)$, $\lambda(T(d)w^*)-\lambda(T(d)w^{**})=|\Delta^+|$, where $|\Delta^+|$ means the cardinality of the set Δ^+. Moreover we get $w^*=w^{**}w_\Pi$ since $w^{**}\Delta^+=-w^*\Delta^+=w^*w_\Pi\Delta^+$. Now let us show that the element $w\in W$ which attains the $\operatorname*{Max}_{w\in W}\lambda(T(d)w)$ is unique. More precisely we shall show

$$\lambda(T(d)w^*w)=\lambda(T(d)w^*)-n(w)$$

for any $w\in W$. In fact, we have $l(w)=\lambda(w)=n(w)$, hence

$$\lambda(T(d)w^*w)\geq\lambda(T(d)w^*)-\lambda(w)=\lambda(T(d)w^*)-n(w).$$

by Lemma 1.5. Put $w'=w^{-1}w_\Pi$; then we easily get $n(w')=n(w_\Pi)-n(w)=|\Delta^+|-n(w)$ (observe that $\Delta^+=(-w_\Pi\Delta^+_{w'})\cup\Delta^+_{w^{-1}}$ is a disjoint union and $\Delta^+_{w^{-1}}=-w\Delta^+_w$) and we have

$$\begin{aligned}&\lambda(T(d)w^*)-|\Delta^+|=\lambda(T(d)w^{**})=\\&\quad\lambda(T(d)w^*ww')\geq\lambda(T(d)w^*w)-n(w')\geq\lambda(T(d)w^*)-n(w)-n(w')=\lambda(T(d)w^*)-|\Delta^+|.\end{aligned}$$

Thus we get the equalities everywhere and hence we have $\lambda(T(d)w^*w)=\lambda(T(d)w^*)-n(w)$. Similarly we get

$$\lambda(T(d)w^{**}w)=\lambda(T(d)w^{**})+n(w)$$

for any $w\in W$. Hence the element $w\in W$ which attains the $\operatorname*{Min}_{w\in W}\lambda(T(d)w)$ is also unique. Thus we have proved the

Proposition **1.25**. — *Let* $d\in P_r^\perp$. *Then* $\operatorname*{Max}_{w\in W}\lambda(T(d)w)$ *and* $\operatorname*{Min}_{w\in W}\lambda(T(d)w)$ *are attained by unique elements* $w^*, w^{**}\in W$ *respectively. Moreover we have*

$$\lambda(T(d)w^*)=\frac{1}{2}N_d+|S_d|,$$

$$\lambda(T(d)w^{**})=\frac{1}{2}N_d-|S'_d|;$$

where $|S_d|$ *(resp.* $|S'_d|$*) means the cardinality of the subset of* Δ^+ *defined by*

$$S_d=\{\alpha\in\Delta^+;\ (d,\alpha)\leq 0\}\quad (\textit{resp.}\ \ S'_d=\{\alpha\in\Delta^+;\ (d,\alpha)>0\})$$

and $N_d=\sum_{\alpha\in\Delta}|(d,\alpha)|$.

Furthermore, we have, for any $w\in W$,

$$\lambda(T(d)w^*w)=\lambda(T(d)w^*)-n(w),$$
$$\lambda(T(d)w^{**}w)=\lambda(T(d)w^{**})+n(w);$$

We also have $w^*=w^{**}w_\Pi$, $\lambda(T(d)w^*)-\lambda(T(d)w^{**})=|\Delta^+|$.

Corollary **1.26.** — *Let* $d\in P_r^{\perp}$. *Then* $\operatorname*{Max}_{w\in W}\lambda(w.T(d))$ *and* $\operatorname*{Min}_{w\in W}\lambda(w.T(d))$ *are attained by unique elements* $w^{(1)}$, $w^{(2)}$ *respectively. We have moreover*

$$\lambda(w^{(1)}.T(d))=\frac{1}{2}N_d+|R_d|,$$

$$\lambda(w^{(2)}.T(d))=\frac{1}{2}N_d-|R'_d|,$$

where $R_d=\{\alpha\in\Delta^+;\ (d,\alpha)\geq 0\}$, $R'_d=\{\alpha\in\Delta^+;\ (d,\alpha)<0\}$. *We also have for any* $w\in W$

$$\lambda(ww^{(1)}T(d))=\lambda(w^{(1)}T(d))-n(w),$$
$$\lambda(ww^{(2)}T(d))=\lambda(w^{(2)}T(d))+n(w),$$

and $w^{(1)}=w_\Pi w^{(2)}$, $\lambda(w^{(1)}T(d))-\lambda(w^{(2)}T(d))=|\Delta^+|$.

Corollary **1.27.** — *Let* $\sigma\in DW$. *Then* $\operatorname*{Min}_{w\in W}\lambda(w\sigma)$ *is attained by* $w=1$ *if and only if* $\sigma\mathfrak{D}_0$ *is contained in the positive Weyl chamber* $\{x\in\mathfrak{h}_R^*;\ (\alpha_i,x)>0$ *for all* $i=1,\ldots,l\}$. *Also* $\operatorname*{Max}_{w\in W}\lambda(w\sigma)$ *is attained by* $w=1$ *if and only if* $\sigma\mathfrak{D}_0$ *is contained in the negative Weyl chamber* $\{x\in\mathfrak{h}_R^*;\ (\alpha_i,x)<0$ *for all* $i=1,\ldots,l\}$.

Proof. — By Cor. 1.26, $\lambda(\sigma)=\operatorname*{Min}_{w\in W}\lambda(w\sigma)$ is equivalent to $\lambda(w_i\sigma)>\lambda(\sigma)$ $(i=1,\ldots,l)$, i.e. to $P_i\notin\widetilde{\Delta}(\sigma)$ $(i=1,\ldots,l)$; which is in turn equivalent to $\sigma\mathfrak{D}_0\sim\mathfrak{D}_0(P_i)$ $(i=1,\ldots,l)$, i.e. to the fact that $\sigma\mathfrak{D}_0$ is contained in the positive Weyl chamber. The second half is also proved similarly.

Remark. — Let J be any proper subset of $\{0,1,\ldots,l\}$. Then the subgroup $\widetilde{W}_J$ of D′W generated by $\{w_j;j\in J\}$ is finite. More precisely, the natural homomorphism $D'W\to W$ is injective on $\widetilde{W}_J$, i.e. $D'\cap\widetilde{W}_J=\{1\}$. In fact, since J is a proper subset of $\{0,1,\ldots,l\}$, $\bigcap_{j\in J}P_j$ is not empty. Let $a\in\bigcap_{j\in J}P_j$ and $\sigma\in D'\cap\widetilde{W}_J$. Then $\sigma(a)=a$. However, the only element $\sigma\in D'$ which has a fixed point is 1. Thus we get $D'\cap\widetilde{W}_J=\{1\}$, whence $\widetilde{W}_J$ is isomorphic to a subgroup of W. Now, using $\widetilde{W}_J$ instead of W, Prop. 1.26 and Cor. 1.27 are still valid under a suitable modification. However we shall not use this fact in this paper and shall return to a detailed treatment of this question in a subsequent paper.

For later use, we give a criterion for P_i to belong to $\widetilde{\Delta}(\sigma^{-1})$, i.e. a criterion for $\lambda(\sigma w_i)<\lambda(\sigma)$ $(\sigma\in DW)$.

Proposition **1.28**. — *Let* $\sigma=T(d)w$, $d\in P_r^\perp$, $w\in W$ *and i an integer with* $1\leq i\leq l$. *Then we have*

$$\text{(i)}\qquad \lambda(\sigma w_i)<\lambda(\sigma)\quad \textit{if}\quad w(\alpha_i)>0,\ (w(\alpha_i), d)>0,\quad \textit{or if}\quad w(\alpha_i)<0,\ (w(\alpha_i), d)\geq 0.$$

$$\text{(ii)}\qquad \lambda(\sigma w_i)>\lambda(\sigma)\quad \textit{if}\quad w(\alpha_i)>0,\ (w(\alpha_i), d)\leq 0,\quad \textit{or if}\quad w(\alpha_i)<0,\ (w(\alpha_i), d)<0.$$

Proof. — Let $a\in\mathfrak{D}_0$. Then $\lambda(\sigma w_i)<\lambda(\sigma)$ is equivalent to $P_i\in\widetilde{\Delta}(\sigma^{-1})$, i.e. to $(\alpha_i, a)(\alpha_i, \sigma^{-1}(a))<0$. This is equivalent to $(\alpha_i, \sigma^{-1}(a))<0$ since $(\alpha_i, a)>0$. Now $\sigma^{-1}(a)=w^{-1}(a-d)$. Hence $(\alpha_i, \sigma^{-1}(a))=(w(\alpha_i), a)-(w(\alpha_i), d)$. Since a can be taken arbitrarily close to the origin, $(\alpha_i, \sigma^{-1}(a))<0$ is equivalent to $(w(\alpha_i), d)>0$ (resp. $(w(\alpha_i), d)\geq 0$) if $w(\alpha_i)>0$ (resp. if $w(\alpha_i)<0$). Thus we have proved (i). (ii) is shown similarly.

The following proposition is also proved similarly.

Proposition **1.29**. — *Let* $\sigma=T(d)w$, $d\in P_r^\perp$, $w\in W$. *Then we have*

$$\text{(i)}\qquad \lambda(\sigma w_0)<\lambda(\sigma)\quad \textit{if}\quad w(\alpha_0)>0,\ 0\geq(w(\alpha_0), d)+1,\quad \textit{or if}\quad w(\alpha_0)<0,\ 0>(w(\alpha_0), d)+1.$$

$$\text{(ii)}\qquad \lambda(\sigma w_0)>\lambda(\sigma)\quad \textit{if}\quad w(\alpha_0)>0,\ 0<(w(\alpha_0), d)+1,\quad \textit{or if}\quad w(\alpha_0)<0,\ 0\leq(w(\alpha_0), d)+1.$$

1.10. In this section a few comments about the Poincaré series $P(DW, t)$, $P(D'W, t)$ will be given, where

$$P(DW, t)=\sum_{\sigma\in DW} t^{\lambda(\sigma)},\qquad P(D'W, t)=\sum_{\sigma\in D'W} t^{\lambda(\sigma)}.$$

(cf. Bott [2, §§ 9, 13]). Since DW is a semi-direct product of Ω and D'W and since $\lambda(\rho\tau)=\lambda(\tau)$ for $\rho\in\Omega$, $\tau\in D'W$, we have $P(DW, t)=|\Omega|.P(D'W, t)$ where $|\Omega|$ is the order of Ω.

Now let $d\in P_r^\perp$, $w\in W$. We shall say that d is *related to* w if Min $\lambda(\sigma)$ for $\sigma\in T(d)W$ is attained by $T(d)w$. By Prop. 1.25, if d is related to w, then we have

$$w\Delta^+=\{\alpha\in\Delta^+;\ (\alpha, d)\leq 0\}\cup\{\alpha\in\Delta^-;\ (\alpha, d)<0\},$$

i.e.

$$w\Delta^-=\{\alpha\in\Delta^-;\ (\alpha, d)\geq 0\}\cup\{\beta\in\Delta^+;\ (\beta, d)>0\},$$

and also we have

$$\begin{aligned}\lambda(T(d)w)&=\sum_{\substack{\alpha>0\\(\alpha,d)\leq 0}}(\alpha, d)+\sum_{\substack{\alpha>0\\(\alpha,d)>0}}((\alpha, d)-1)\\&=\sum_{\beta\in w\Delta^-\cap\Delta^-}(d, \beta)+\sum_{\beta\in w\Delta^-\cap\Delta^+}(d, \beta)-|w\Delta^-\cap\Delta^+|\\&=\sum_{\beta\in\Delta^-}(d, w\beta)-n(w).\end{aligned}$$

Let $\sum_{\alpha>0}\alpha=a_1\alpha_1+\ldots+a_l\alpha_l$ where $a_1, \ldots, a_l$ are positive integers. Then we get from the above equality

$$\lambda(\mathrm{T}(d)w)=\sum_{i=1}^{l}a_i(d, -w\alpha_i)-n(w).$$

Now fix $w\in\mathrm{W}$. Then $d\in\mathrm{P}_r^{\perp}$ is related to w if and only if

$$(d, w\beta)\geq 0 \quad \text{for } \beta\in\Delta^-\cap w^{-1}\Delta^- \text{ and}$$
$$(d, w\beta)>0 \quad \text{for } \beta\in\Delta^-\cap w^{-1}\Delta^+.$$

These conditions are equivalent to

$$(d, w\alpha)\geq 0 \quad \text{for } \alpha\in(-\Pi)\cap w^{-1}\Delta^- \text{ and}$$
$$(d, w\alpha)>0 \quad \text{for } \alpha\in(-\Pi)\cap w^{-1}\Delta^+.$$

In fact, let $-\Pi_1=(-\Pi)\cap w^{-1}\Delta^-$, $-\Pi_2=(-\Pi)\cap w^{-1}\Delta^+$. Then Π_1, Π_2 form a partition of Π. Let $d\in\mathrm{P}_r^{\perp}$ satisfy $(d, w\alpha)\geq 0$ (for any $\alpha\in-\Pi_1$) and $(d, w\alpha)>0$ (for any $\alpha\in-\Pi_2$). Let $\beta\in\Delta^-$ and $\beta=\sum_{\alpha\in-\Pi_1}\nu_\alpha.\alpha+\sum_{\gamma\in-\Pi_2}\nu_\gamma.\gamma$ where ν_α, ν_γ are non-negative integers. Now if $\beta\in\Delta^-\cap w^{-1}\Delta^-$, then $(d, w\beta)=\sum_{\alpha\in-\Pi_1}\nu_\alpha(d, w\alpha)+\sum_{\gamma\in-\Pi_2}\nu_\gamma(d, w\gamma)\geq 0$. Also if $\beta\in\Delta^-\cap w^{-1}\Delta^+$, then $w\beta=\sum_{\alpha\in-\Pi_1}\nu_\alpha.w\alpha+\sum_{\gamma\in-\Pi_2}\nu_\gamma.w\gamma>0$. Hence we have $\nu_\gamma>0$ for some $\gamma\in-\Pi_2$. Thus we get $(d, w\beta)=\sum_{\alpha\in-\Pi_1}\nu_\alpha(d, w\alpha)+\sum_{\gamma\in-\Pi_2}\nu_\gamma(d, w\gamma)>0$.

Let $\Theta(w)$ be the set of all $d\in\mathrm{P}_r^{\perp}$ which are related to $w\in\mathrm{W}$. Let

$$-\Pi_1=(-\Pi)\cap w^{-1}\Delta^-=\{-\alpha_1, \ldots, -\alpha_r\},$$
$$-\Pi_2=(-\Pi)\cap w^{-1}\Delta^+=\{-\alpha_{r+1}, \ldots, -\alpha_l\}.$$

Then by what we have seen above, $d\in\mathrm{P}_r^{\perp}$ is in $\Theta(w)$ if and only if $\xi_1\leq 0, \ldots, \xi_r\leq 0$, $\xi_{r+1}<0, \ldots, \xi_l<0$ where $w^{-1}(d)=\sum_{i=1}^{l}\xi_i\varepsilon_i$, $\xi_i\in\mathbf{Z}$ $(1\leq i\leq l)$. Moreover if $d\in\Theta(w)$, we have

$$\lambda(\mathrm{T}(d)w)=-\sum_{i=1}^{l}a_i\xi_i-n(w).$$

Thus we have obtained for a fixed element $w\in\mathrm{W}$

$$\sum_{d\in\Theta(w)}t^{\lambda(\mathrm{T}(d)w)}=t^{-n(w)}\sum_{\eta_1=0}^{\infty}\ldots\sum_{\eta_r=0}^{\infty}\sum_{\eta_{r+1}=1}^{\infty}\ldots\sum_{\eta_l=1}^{\infty}t^{a_1\eta_1+\ldots+a_l\eta_l}$$
$$=t^{-n(w)}\frac{1}{1-t^{a_1}}\cdots\frac{1}{1-t^{a_r}}\frac{t^{a_{r+1}}}{1-t^{a_{r+1}}}\cdots\frac{t^{a_l}}{1-t^{a_l}}$$

Let us denote by $a(w)$ the integer defined by

$$a(w)=\sum_{\alpha_i\in w^{-1}\Delta^-}a_i \qquad (w\in\mathrm{W}).$$

Then we have $\sum_{d\in\Theta(w)} t^{\lambda(\mathrm{T}(d)w)} = \dfrac{t^{a(w)-n(w)}}{\prod_{i=1}^{l}(1-t^{a_i})}$. Since DW is a disjoint union of the subsets $\Theta'(w)\mathrm{W}$, where $\Theta'(w)=\{\mathrm{T}(d)w;\, d\in\Theta(w)\}$ $(w\in\mathrm{W})$ (see Prop. 1.25), we get

$$\mathrm{P}(\mathrm{DW}, t) = \sum_{w\in\mathrm{W}} \sum_{\sigma\in\Theta'(w)\mathrm{W}} t^{\lambda(\sigma)}.$$

Now $\sum_{\sigma\in\Theta'(w)\mathrm{W}} t^{\lambda(\sigma)} = \sum_{\tau\in\Theta'(w)} \sum_{w'\in\mathrm{W}} t^{\lambda(\tau)+n(w')} = \mathrm{P}(\mathrm{W}, t)\sum_{\tau\in\Theta'(w)} t^{\lambda(\tau)}$ (see Prop. 1.25), where

$$\mathrm{P}(\mathrm{W}, t) = \sum_{w'\in\mathrm{W}} t^{n(w')},$$

hence we get
$$\mathrm{P}(\mathrm{DW}, t) = \frac{\mathrm{P}(\mathrm{W}, t)}{\prod_{i=1}^{l}(1-t^{a_i})} \sum_{w\in\mathrm{W}} t^{a(w)-n(w)}.$$

Thus we have proved

Proposition **1.30.**
$$\mathrm{P}(\mathrm{DW}, t) = \frac{\mathrm{P}(\mathrm{W}, t)}{\prod_{i=1}^{l}(1-t^{a_i})} \sum_{w\in\mathrm{W}} t^{a(w)-n(w)},$$

$$\mathrm{P}(\mathrm{D'W}, t) = \frac{\mathrm{P}(\mathrm{W}, t)}{|\Omega|\prod_{i=1}^{l}(1-t^{a_i})} \sum_{w\in\mathrm{W}} t^{a(w)-n(w)}.$$

where $\sum_{\alpha\in\Delta^+} \alpha = a_1\alpha_1 + \ldots + a_l\alpha_l$, $a(w) = \sum_{\alpha_i\in\Pi\cap w^{-1}\Delta^-} a_i$.

Similarly, using $\operatorname{Max}_{w\in\mathrm{W}} \lambda(\mathrm{T}(d)w)$ we get

$$\mathrm{P}(\mathrm{DW}, t) = \frac{\mathrm{P}(\mathrm{W}, t)}{\prod_{i=1}^{l}(1-t^{a_i})} t^{-|\Delta^+|} \sum_{w\in\mathrm{W}} t^{b(w)+n(w)},$$

where $b(w) = \sum_{\alpha_i\in\Pi\cap w^{-1}\Delta^+} a_i$. Hence $a(w)+b(w) = a_1+\ldots+a_l$. We note that

$$a(w_\Pi w) = b(w), \quad n(w_\Pi w) = |\Delta^+| - n(w).$$

Hence $\sum_{w\in\mathrm{W}} t^{a(w)-n(w)}$ is self-reciprocal: $(a(w)-n(w)) + (a(w_\Pi w) - n(w_\Pi w)) = \sum_{i=1}^{l} a_i - |\Delta^+|$.

Now using Cor. 1.26, 1.27, similarly as in Prop. 1.30, we obtain

$$\mathrm{P}(\mathrm{DW}, t) = \mathrm{P}(\mathrm{W}, t)\sum_{\sigma\in\Gamma} t^{\lambda(\sigma)}$$

where Γ is the set of elements σ in DW such that $\sigma\mathfrak{D}_0$ is contained in the positive Weyl chamber. Let $\Gamma' = \Gamma\cap\mathrm{D'W}$. Then we have

$$\mathrm{P}(\mathrm{D'W}, t) = \mathrm{P}(\mathrm{W}, t)\sum_{\sigma\in\Gamma'} t^{\lambda(\sigma)}.$$

Let $m_1, \ldots, m_l$ be the exponents of W, i.e. let the Poincaré polynomial of the compact form of G_C be $\prod_{i=1}^{l}(1+t^{2m_i+1})$. Then by Bott [2, § 13],

$$\sum_{\sigma \in \Gamma'} t^{\lambda(\sigma)} = \frac{1}{\prod_{i=1}^{l}(1-t^{m_i})}.$$

Also it is known that ([6, p. 44])

$$P(W, t) = \prod_{i=1}^{l}(1+t+\ldots+t^{m_i}).$$

Thus we have

Proposition **1.31**. — $P(DW, t) = |\Omega| \prod_{i=1}^{l} \frac{1+t+\ldots+t^{m_i}}{1-t^{m_i}}$.

Also we get an explicit form of the polynomial denoted by $Q(t)$ in Bott [2, p. 277], i.e.

$$Q(t) = \sum_{\sigma \in \Gamma_1} t^{\lambda(\sigma)}$$

where Γ_1 is the set of elements σ in D'W such that $\sigma\mathfrak{D}_0$ is contained in the parallelotope $\{x \in \mathfrak{h}_R^*; 0 < (\alpha_i, x) < 1 \text{ for } i = 1, \ldots, l\}$. By [2, § 13]

$$\sum_{\sigma \in \Gamma'} t^{\lambda(\sigma)} = \frac{Q(t)}{\prod_{i=1}^{l}(1-t^{a_i})},$$

hence we have

$$P(DW, t) = |\Omega| P(W, t) \frac{Q(t)}{\prod_{i=1}^{l}(1-t^{a_i})}.$$

Comparing this with Prop. 1.30, we get

$$|\Omega| . Q(t) = \sum_{w \in W} t^{a(w)-n(w)}.$$

Putting $t=1$, we get a formula for the order $|W|$ of W:

$$|W| = |\Omega| . Q(1).$$

The value $Q(1)$ is given by [2]: $Q(1) = l! \prod_{i=1}^{l} d_i$,

where $d_i = (\alpha_0, \varepsilon_i)$ $(1 \leq i \leq l)$, i.e. $\alpha_0 = \sum_{i=1}^{l} d_i \alpha_i$. Thus we have a formula for the order $|W|$ of the Weyl group W:

Proposition **1.32**. — $|W| = |\Omega| l! \prod_{i=1}^{l} d_i$.

Since $|W| = \prod_{i=1}^{l}(1+m_i)$, we also have

$$|\Omega| = \frac{\prod_{i=1}^{l}(1+m_i)}{l! \prod_{i=1}^{l} d_i}.$$

§ 2. On a generalized Bruhat decomposition of a Chevalley group over a $\mathfrak{p}$-adic field.

2.1. Let K be a field with a non-trivial non-Archimedean discrete valuation $|\ |$, i.e. $\xi \to |\xi|$ is a map from K into the real number field $\mathbf{R}$ such that

(i) $|\xi| \geq 0$ for any $\xi \in K$ and $|\xi| = 0$ if and only if $\xi = 0$.

(ii) $|\xi\eta| = |\xi| . |\eta|$ for any $\xi, \eta \in K$.

(iii) $|\xi + \eta| \leq \sup(|\xi|, |\eta|)$ for any $\xi, \eta \in K$.

(iv) $\{|\xi|; \xi \in K^* = K - \{0\}\}$ is an infinite cyclic subgroup of $\mathbf{R}_+ = \{a \in \mathbf{R}; a > 0\}$.

Then $\mathfrak{O} = \{\xi \in K; |\xi| \leq 1\}$ is a subring of K called the ring of integers of K and $\mathfrak{P} = \{\xi \in K; |\xi| < 1\}$ is the unique maximal ideal of $\mathfrak{O}$. The complement $\mathfrak{O}^*$ of $\mathfrak{P}$ in $\mathfrak{O}$ is the group of units of $\mathfrak{O}$. We denote by k the residue class field $\mathfrak{O}/\mathfrak{P}$. There exists an element π in $\mathfrak{P}$ which attains $\mathrm{Max}\{|\xi|; \xi \in \mathfrak{P}\}$. An element π in $\mathfrak{P}$ attains $\mathrm{Max}\{|\xi|; \xi \in \mathfrak{P}\}$ if and only if $\mathfrak{P} = \pi\mathfrak{O}$. Such an element π is called a prime element. We fix once for all a prime element π.

Now let $\mathfrak{g}_{\mathbf{C}}$ be a complex semi-simple Lie algebra and $\mathfrak{h}_{\mathbf{C}}$ a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. We keep the notations of § 1, i.e. Π is a fundamental root system of the root system Δ of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$ and so on. Let $\mathfrak{g}_{\mathbf{Z}}$ denote the Lie subring (over $\mathbf{Z}$) of $\mathfrak{g}_{\mathbf{C}}$ introduced by Chevalley [6, p. 32]:

$$\mathfrak{g}_{\mathbf{Z}} = \mathfrak{h}_{\mathbf{Z}} + \sum_{\alpha \in \Delta} \mathbf{Z} X_\alpha.$$

Let us denote by Φ_α the homomorphism from $SL(2, K)$ into the automorphism group of the Lie algebra $\mathfrak{g}_K = K \otimes \mathfrak{g}_{\mathbf{Z}}$ over K which was defined in [6, p. 33]. (We keep the notational conventions in [6, p. 36].) Let us consider the Chevalley group G associated with the pair $\mathfrak{g}_{\mathbf{C}}$, K ([6, p. 37]); G is generated by the subgroups $\{\mathfrak{X}_\alpha; \alpha \in \Delta\}$ and $\mathfrak{H}$ where

$$\mathfrak{X}_\alpha = \{x_\alpha(t); t \in K\}, \qquad x_\alpha(t) = \Phi_\alpha\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right),$$
$$\mathfrak{H} = \{h(\chi); \chi \in \mathrm{Hom}(P_r, K^*)\}.$$

As in [6] $\mathfrak{U}$ (resp. $\mathfrak{V}$) denotes the subgroup of G generated by the $\{\mathfrak{X}_\alpha; \alpha \in \Delta^+\}$ (resp. by the $\{\mathfrak{X}_\alpha; \alpha \in \Delta^-\}$).

We now introduce some subgroups of G: let U be the subgroup of G generated by the subgroups $\{\mathfrak{X}_{\alpha, \mathfrak{O}}; \alpha \in \Delta\}$ and $\mathfrak{H}_{\mathfrak{O}}$, where

$$\mathfrak{X}_{\alpha, \mathfrak{O}} = \{x_\alpha(\xi); \xi \in \mathfrak{O}\},$$
$$\mathfrak{H}_{\mathfrak{O}} = \{h(\chi); \chi \in \mathrm{Hom}(P_r, \mathfrak{O}^*)\}.$$

We denote by B the subgroup of U generated by the subgroups $\{\mathfrak{X}_{\alpha, \mathfrak{O}}; \alpha \in \Delta^-\}$, $\{\mathfrak{X}_{\alpha, \mathfrak{P}}; \alpha \in \Delta^+\}$ and $\mathfrak{H}_{\mathfrak{O}}$, where

$$\mathfrak{X}_{\alpha, \mathfrak{P}} = \{x_\alpha(\xi); \xi \in \mathfrak{P}\}.$$

We denote by $\mathfrak{W}_{\mathfrak{O}}$ the subgroup of U generated by the elements $\Phi_{\alpha}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ $(\alpha\in\Delta)$ and $\mathfrak{H}_{\mathfrak{O}}$. Let ζ be the homomorphism from $\mathfrak{W}$ onto the Weyl group W defined in [6, p. 37], where $\mathfrak{W}$ is the subgroup of G generated by the elements $\Phi_{\alpha}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ $(\alpha\in\Delta)$ and $\mathfrak{H}$. Then it is seen easily that the restriction of ζ to $\mathfrak{W}_{\mathfrak{O}}$ is a surjective homomorphism from $\mathfrak{W}_{\mathfrak{O}}$ onto W with the kernel $\mathfrak{H}_{\mathfrak{O}}$, since $\mathfrak{W}=\mathfrak{W}_{\mathfrak{O}}\mathfrak{H}$, $\mathfrak{H}_{\mathfrak{O}}=\mathfrak{W}_{\mathfrak{O}}\cap\mathfrak{H}$.

We denote by D the subgroup of $\mathfrak{H}$ defined by

$$\mathrm{D}=\{h(\chi)\,;\,\chi\in\mathrm{Hom}(\mathrm{P}_r,\{\pi^i;\, i\in\mathbf{Z}\}).$$

Since the map $\chi\to h(\chi)$ from $\mathrm{Hom}(\mathrm{P}_r,\mathrm{K}^*)$ onto $\mathfrak{H}$ is an isomorphism, the group D is isomorphic to the group $\mathrm{Hom}(\mathrm{P}_r,\{\pi^i;\, i\in\mathbf{Z}\})$ via the map h, i.e. $\mathrm{D}\cong\mathrm{Hom}(\mathrm{P}_r,\mathbf{Z})$. On the other hand $\mathrm{Hom}(\mathrm{P}_r,\mathbf{Z})$ may be identified naturally with the module $\mathrm{P}_r^{\perp}$ (§ 1.2) via the map $d\to\chi_d$, where $\chi_d(\alpha)=(d,\alpha)$ for $\alpha\in\mathrm{P}_r$, from $\mathrm{P}_r^{\perp}$ onto $\mathrm{Hom}(\mathrm{P}_r,\mathbf{Z})$. Thus the group D defined above may be identified with the group D defined in § 1.2 via the map $h(\chi_d)\to\mathrm{T}(d)$ $(d\in\mathrm{P}_r^{\perp})$. Since K^* is the direct product of the subgroups $\mathfrak{O}^*$ and $\{\pi^i;\, i\in\mathbf{Z}\}$, $\mathfrak{H}$ is the direct product of the subgroups $\mathfrak{H}_{\mathfrak{O}}$ and D. Hence $\mathfrak{W}$ is the semi-direct product of D and $\mathfrak{W}_{\mathfrak{O}}$ with D as a distinguished subgroup. Thus the quotient group $\widetilde{\mathrm{W}}=\mathfrak{W}/\mathfrak{H}_{\mathfrak{O}}$ is the semi-direct product DW of D and $\mathrm{W}=\mathfrak{W}_{\mathfrak{O}}/\mathfrak{H}_{\mathfrak{O}}$. We denote by $\tilde{\zeta}$ the canonical homomorphism from $\mathfrak{W}$ onto $\widetilde{\mathrm{W}}$. It is easily seen that there exists a unique isomorphism from $\widetilde{\mathrm{W}}$ onto the semi-direct product DW in § 1.2 preserving the elements in D, W. We shall identify these two groups in what follows.

2.2. In this section we shall investigate the fundamental case where $\mathrm{G}=\mathrm{SL}(2,\mathrm{K})$ and

$$\mathrm{SL}(2,\mathfrak{O})=\mathrm{U}=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{SL}(2,\mathrm{K})\,;\, a,b,c,d\in\mathfrak{O}\right\},$$

$$\mathrm{B}=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{U}\,;\, a,d\in\mathfrak{O}^*,\, c\in\mathfrak{O},\, b\in\mathfrak{P}\right\}.$$

Let

$$\mathrm{D}=\left\{\begin{pmatrix} \pi^i & 0 \\ 0 & \pi^{-i} \end{pmatrix};\, i\in\mathbf{Z}\right\},\quad \mathfrak{H}=\left\{\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix};\, \xi\in\mathrm{K}^*\right\}$$

and

$$\mathfrak{W}_{\mathfrak{O}}=\mathfrak{H}_{\mathfrak{O}}\cap\mathfrak{H}_{\mathfrak{O}}w_1,\quad \mathfrak{W}=\mathfrak{H}\cup\mathfrak{H}w_1$$

where

$$\mathfrak{H}_{\mathfrak{O}}=\left\{\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix};\, u\in\mathfrak{O}^*\right\},\quad w_1=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, as is well known, G (resp. U) is generated by the elements $\left\{\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix};\, \xi\in\mathrm{K}\right\}$ $\left(\text{resp. } \left\{\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix};\, \xi\in\mathfrak{O}\right\}\right)$.

Let $\mathfrak{U}_{\mathfrak{P}}=\left\{\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix};\, \xi\in\mathfrak{P}\right\}$, $\mathfrak{B}_{\mathfrak{O}}=\left\{\begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix};\, \xi\in\mathfrak{O}\right\}$. Then the following proposition for $\mathrm{SL}(2,\mathrm{K})$ is easily verified by a direct computation.

Proposition **2.1.**

(i) $\mathrm{B} = \mathfrak{B}_{\mathfrak{O}}\mathfrak{H}_{\mathfrak{O}}\mathfrak{U}_{\mathfrak{P}} = \mathfrak{U}_{\mathfrak{P}}\mathfrak{H}_{\mathfrak{O}}\mathfrak{B}_{\mathfrak{O}}$.

(ii) $\mathrm{U} = \mathrm{B} \cup \mathrm{B}w_1\mathrm{B}$ *(disjoint union) and* $\mathrm{B}w_1\mathrm{B} = \mathrm{B}w_1\mathfrak{B}_{\mathfrak{O}}$.

(iii) $\mathrm{G} = \mathrm{B}\mathfrak{W}\mathrm{B} = \bigcup_{\sigma\in\widetilde{\mathrm{W}}} \mathrm{B}\omega(\sigma)\mathrm{B}$ *(disjoint union), where* ω *is a map from* $\widetilde{\mathrm{W}} = \mathfrak{W}/\mathfrak{H}_{\mathfrak{O}}$ *into* $\mathfrak{W}$ *such that* $\widetilde{\zeta}(\omega(\sigma)) = \sigma$ *for any* $\sigma\in\widetilde{\mathrm{W}}$. ($\widetilde{\zeta}$ *is the natural homomorphism* $\mathfrak{W}\to\widetilde{\mathrm{W}} = \mathfrak{W}/\mathfrak{H}_{\mathfrak{O}}$.)

The involutive elements $w_0 = \widetilde{\zeta}\left(\begin{pmatrix} 0 & \pi \\ -\pi^{-1} & 0\end{pmatrix}\right)$ and $w_1 = \widetilde{\zeta}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right)$ form a system of generators of $\widetilde{\mathrm{W}}$. Noting this fact, it is easy to prove the

Proposition **2.2.** — *For the system* (G, B, $\mathfrak{W}$) *and the involutive generators* w_0, w_1 *of* $\widetilde{\mathrm{W}}$, *the hypotheses of Tits* [16] *are all satisfied.*

To be more precise, we note that B and $\mathfrak{W}$ generate G, that $\mathrm{B}\cap\mathfrak{W} = \mathfrak{H}_{\mathfrak{O}}$ is a distinguished subgroup of $\mathfrak{W}$ and that $\mathfrak{W}/\mathfrak{H}_{\mathfrak{O}} = \widetilde{\mathrm{W}}$ is generated by w_0, w_1. Moreover, the conditions (iii), (vii) of Tits [16] are easily verified: $\omega(w_i)\mathrm{B}\omega(\sigma) \subset \mathrm{B}\omega(\sigma)\mathrm{B} \cup \mathrm{B}\omega(w_i\sigma)\mathrm{B}$ for any $\sigma\in\widetilde{\mathrm{W}}$ and $i = 0, 1$; $\omega(w_i)\mathrm{B}\omega(w_i) \neq \mathrm{B}$ for $i = 0, 1$.

Thus, by Tits [16], U and $\mathrm{V} = \mathrm{B}\cup\mathrm{B}\omega(w_0)\mathrm{B}$ are the only subgroups H of G such that $\mathrm{G} \supsetneq \mathrm{H} \supsetneq \mathrm{B}$. They are not conjugate in G (see [16]), but they are conjugate in GL(2, K) by the element $\begin{pmatrix} 0 & \pi \\ 1 & 0\end{pmatrix}$ which normalizes B.

The following proposition is also easy to check and gives an "Iwasawa decomposition" of SL(2, K).

Proposition **2.3.** — $\mathrm{G} = \mathrm{U}\mathfrak{H}\mathfrak{U} = \mathrm{UD}\mathfrak{U}$, *where*

$$\mathfrak{U} = \left\{\begin{pmatrix} 1 & \xi \\ 0 & 1\end{pmatrix}; \xi\in\mathrm{K}\right\}.$$

2.3. Now let us return to the notations of § 2.1.

Proposition **2.4.**

$$\begin{aligned}\mathrm{U} &= \mathfrak{U}_{\mathfrak{P}}\mathfrak{B}_{\mathfrak{O}}\mathfrak{W}_{\mathfrak{O}}\mathfrak{B}_{\mathfrak{O}} \\ &= \bigcup_{w\in\mathrm{W}} \mathfrak{U}_{\mathfrak{P}}\mathfrak{B}_{\mathfrak{O}}\mathfrak{H}_{\mathfrak{O}}\,\omega(w)\mathfrak{B}_{\mathfrak{O}} \qquad \textit{(disjoint union)};\end{aligned}$$

where $\mathfrak{U}_{\mathfrak{P}}$ *(resp.* $\mathfrak{B}_{\mathfrak{O}}$*) is the subgroup of* $\mathfrak{U}$ *(resp. of* $\mathfrak{B}$*) generated by* $\{\mathfrak{X}_{\alpha,\mathfrak{P}};\alpha\in\Delta^+\}$ *(resp.* $\{\mathfrak{X}_{\alpha,\mathfrak{O}};\alpha\in\Delta^-\}$*), and* ω *is a map from* W *into* $\mathfrak{W}_{\mathfrak{O}}$ *such that* $\zeta(\omega(w)) = w$ *for any* $w\in\mathrm{W}$.

Proof. — As in the proof of [6, Lemme 4, p. 38], we see that $\mathfrak{X}_{\alpha,\mathfrak{O}}$, $\mathfrak{X}_{-\alpha,\mathfrak{O}}$ $(\alpha\in\Pi)$ and $\mathfrak{H}_{\mathfrak{O}}$ generate the group U. Therefore, to prove $\mathrm{U} = \mathfrak{U}_{\mathfrak{P}}\mathfrak{B}_{\mathfrak{O}}\mathfrak{W}_{\mathfrak{O}}\mathfrak{B}_{\mathfrak{O}}$, it is enough to show that $z\mathfrak{U}_{\mathfrak{P}}\mathfrak{B}_{\mathfrak{O}}\mathfrak{W}_{\mathfrak{O}}\mathfrak{B}_{\mathfrak{O}} \subset \mathfrak{U}_{\mathfrak{P}}\mathfrak{B}_{\mathfrak{O}}\mathfrak{W}_{\mathfrak{O}}\mathfrak{U}_{\mathfrak{O}}$ for any element z in the system of generators $\{\mathfrak{H}_{\mathfrak{O}}, \mathfrak{X}_{\alpha,\mathfrak{O}}, \mathfrak{X}_{-\alpha,\mathfrak{O}}\,(\alpha\in\Pi)\}$. To begin with, we note the following facts (cf. Chevalley [6, § III]):

(i) $\mathfrak{U}_{\mathfrak{P}}$ (resp. $\mathfrak{B}_{\mathfrak{O}}$) is a distinguished subgroup of the group $\mathfrak{U}_{\mathfrak{P}}\mathfrak{H}_{\mathfrak{O}}$ (resp. $\mathfrak{B}_{\mathfrak{O}}\mathfrak{H}_{\mathfrak{O}}$).

(ii) $\mathfrak{U}_{\mathfrak{P}} = \prod_{\alpha\in\Delta^+}\mathfrak{X}_{\alpha,\mathfrak{P}}$ (resp. $\mathfrak{B}_{\mathfrak{O}} = \prod_{\beta\in\Delta^-}\mathfrak{X}_{\beta,\mathfrak{O}}$), where the product is taken in the ascending (resp. descending) order of the roots. (We assume here that the linear ordering of the roots is regular in the sense of [6, p. 20] i.e. the height $h(\alpha)$ of $\alpha\in\Delta$ with respect to Π is an increasing function in $\alpha : h(\alpha)\geq h(\beta)$ if $\alpha>\beta$.)

(iii) $\mathfrak{U}_{\mathfrak{P}}=\mathfrak{X}_{\alpha_i,\mathfrak{P}}\mathfrak{U}_{\mathfrak{P}}^{(i)}$ where $\mathfrak{U}_{\mathfrak{P}}^{(i)}=\prod_{\substack{\alpha\in\Delta^+\\ \alpha\neq\alpha_i}}\mathfrak{X}_{\alpha,\mathfrak{P}}$, $\alpha_i\in\Pi$.

$\mathfrak{V}_{\mathfrak{O}}=\mathfrak{X}_{-\alpha_i,\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}^{(i)}$ where $\mathfrak{V}_{\mathfrak{O}}^{(i)}=\prod_{\substack{\alpha\in\Delta^+\\ \alpha\neq\alpha_i}}\mathfrak{X}_{-\alpha,\mathfrak{O}}$, $\alpha_i\in\Pi$.

$z\mathfrak{U}_{\mathfrak{P}}^{(i)}z^{-1}\subset\mathfrak{U}_{\mathfrak{P}}^{(i)}$ for any z in $\mathfrak{X}_{\alpha_i,\mathfrak{O}}$.

$z\mathfrak{V}_{\mathfrak{O}}^{(i)}z^{-1}\subset\mathfrak{V}_{\mathfrak{O}}^{(i)}$ for any z in $\mathfrak{X}_{-\alpha_i,\mathfrak{O}}$.

Now the statement (i) implies immediately that $z\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\mathfrak{W}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\subset\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\mathfrak{W}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}$ for any $z\in\mathfrak{H}_{\mathfrak{O}}$. Let α_i be a fundamental root. By the statements (ii) and (iii), we have $\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\subset\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\mathfrak{X}_{\alpha_i,\mathfrak{P}}\mathfrak{X}_{-\alpha_i,\mathfrak{O}}$, and more generally, for any z in $\mathfrak{X}_{\alpha_i,\mathfrak{O}}$ or in $\mathfrak{X}_{-\alpha_i,\mathfrak{O}}$, we get

$$z\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\subset\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}z\mathfrak{X}_{\alpha_i,\mathfrak{P}}\mathfrak{X}_{-\alpha_i,\mathfrak{O}}\subset\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\Phi_{\alpha_i}(\mathrm{SL}(2,\mathfrak{O}));$$

therefore

$$z\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\mathfrak{W}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\subset\bigcup_{w\in W}\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\Phi_{\alpha_i}(\mathrm{SL}(2,\mathfrak{O}))\omega(w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}.$$

Now by Prop. 2.1, we have

$$\Phi_{\alpha_i}(\mathrm{SL}(2,\mathfrak{O}))\subset\mathfrak{X}_{\alpha_i,\mathfrak{P}}\mathfrak{X}_{-\alpha_i,\mathfrak{O}}\mathfrak{H}_{\mathfrak{O}}\cup\mathfrak{X}_{\alpha_i,\mathfrak{P}}\mathfrak{X}_{-\alpha_i,\mathfrak{O}}\omega(w_{\alpha_i})\mathfrak{H}_{\mathfrak{O}}\mathfrak{X}_{-\alpha_i,\mathfrak{O}};$$

hence, if $w^{-1}(-\alpha_i)<0$, we have

$$\begin{aligned}&\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\Phi_{\alpha_i}(\mathrm{SL}(2,\mathfrak{O}))\omega(w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\subset\\ &\subset\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\mathfrak{X}_{\alpha_i,\mathfrak{P}}\mathfrak{X}_{-\alpha_i,\mathfrak{O}}\omega(w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\cup\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\mathfrak{X}_{\alpha_i,\mathfrak{P}}\mathfrak{X}_{-\alpha_i,\mathfrak{O}}\omega(w_{\alpha_i})\mathfrak{X}_{-\alpha_i,\mathfrak{O}}\omega(w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\subset\\ &\subset\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\omega(w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\cup\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\omega(w_{\alpha_i}w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\end{aligned}$$

(by the statement (iii)); if $w^{-1}(-\alpha_i)>0$, we have $w^{-1}w_{\alpha_i}^{-1}(-\alpha_i)<0$,

$$\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\Phi_{\alpha_i}(\mathrm{SL}(2,\mathfrak{O}))\omega(w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\subset\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\Phi_{\alpha_i}(\mathrm{SL}(2,\mathfrak{O}))\omega(w_{\alpha_i}w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}};$$

hence, as in the preceding case,

$$\mathfrak{U}_{\mathfrak{P}}^{(i)}\mathfrak{V}_{\mathfrak{O}}^{(i)}\Phi_{\alpha}(\mathrm{SL}(2,\mathfrak{O}))\omega(w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\subset\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\omega(w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}\cup\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\omega(w_{\alpha_i}w)\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}.$$

Thus we have proved $\mathrm{U}=\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\mathfrak{W}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}$.

Now let us consider the homomorphism ρ defined by the reduction mod. $\mathfrak{P}$ from U onto the Chevalley group G_k of $\mathfrak{g}_{\mathbf{C}}$ over the residue class field $k=\mathfrak{O}/\mathfrak{P}$. ρ satisfies $\rho(x_\alpha(\xi))=x_\alpha(\bar{\xi})$ for any $\alpha\in\Delta$, $\xi\in\mathfrak{O}$, where $\bar{\xi}$ is the residue class of ξ, and $\rho(h(\chi))=h(\bar{\chi})$ where $\chi\in\mathrm{Hom}(\mathrm{P}_r,\mathfrak{O}^*)$ and $\bar{\chi}\in\mathrm{Hom}(\mathrm{P}_r,k^*)$ is such that $\bar{\chi}(\alpha)$ is the residue class of $\chi(\alpha)$ for $\alpha\in\mathrm{P}_r$. Let B_k be the Borel subgroup of G_k generated by $\rho(\mathfrak{V}_{\mathfrak{O}})$ and $\rho(\mathfrak{H}_{\mathfrak{O}})$; we have $\rho(\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\mathfrak{H}_{\mathfrak{O}})\subset\mathrm{B}_k$. Therefore, from the decomposition of U which we have just shown it follows that

$$\mathrm{G}_k=\mathrm{B}_k\rho(\mathfrak{W}_{\mathfrak{O}})\mathrm{B}_k=\bigcup_{w\in W}\mathrm{B}_k\rho(\omega(w))\mathrm{B}_k.$$

This is nothing but the Bruhat decomposition of G_k with respect to B_k and G_k is the disjoint union of the double cosets $B_k\rho(\omega(w))B_k$, $w\in W$ (see [6, Th. 2]). It follows immediately from this that U is the disjoint union of the subsets $\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\mathfrak{H}_{\mathfrak{O}}\omega(w)\mathfrak{V}_{\mathfrak{O}}$, $w\in W$, and that the inverse image $\rho^{-1}(B_k)$ of B_k by ρ is equal to $\mathfrak{U}_{\mathfrak{P}}\mathfrak{V}_{\mathfrak{O}}\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}=\mathfrak{U}_{\mathfrak{P}}\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}$. The proof is now complete.

Theorem **2.5.** — $B=\mathfrak{U}_{\mathfrak{P}}\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}$.

In fact, as we have just seen, $\mathfrak{U}_{\mathfrak{P}}\mathfrak{H}_{\mathfrak{O}}\mathfrak{V}_{\mathfrak{O}}$ is a subgroup of G, and has the same system of generators as B.

This theorem is our fundamental tool, which will play an important part in our later discussions.

We remark that, since Th. 2.5 is established, Prop. 2.4 gives the double coset decomposition of U with respect to B.

Let $d=h(\chi)$ be an element in D. As we have remarked in § 2.1, d is identified with an element in $P_r^{\perp}$ which is also denoted by d and we have $\chi(\alpha)=\pi^{(d,\alpha)}$ for any $\alpha\in P_r$.

Assume now that $\mathfrak{g}_{\mathbf{C}}$ is simple and let α_0 be the highest root in Δ. Put $w_0=\widetilde{\zeta}\left(\Phi_{\alpha_0}\left(\begin{pmatrix} 0 & \pi \\ -\pi^{-1} & 0\end{pmatrix}\right)\right)$; we then have $w_0=w_{\alpha_0}d_0$, where $w_{\alpha_0}\in W$ is the reflection with respect to the the hyperplane $\{x\in\mathfrak{h}_{\mathbf{R}}^*;\ \alpha_0(x)=0\}$ and $d_0\in D$ is given by $(d_0,\alpha)=-\alpha(H_{\alpha_0})=-2(\alpha,\alpha_0)/(\alpha_0,\alpha_0)=-(\alpha,\alpha_0^*)$ for any α in P_r. (Hence it is easily checked that this element w_0 is identified with the element w_0 defined in § 1.4, via the identification in § 2.1.) For each α_i in $\Pi=\{\alpha_1,\ldots,\alpha_l\}$, put $w_i=w_{\alpha_i}=\widetilde{\zeta}\left(\Phi_{\alpha_i}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right)$.

Let ω be a map from $\widetilde{W}=DW$ into $\mathfrak{W}$ such that $\widetilde{\zeta}(\omega(\sigma))=\sigma$ for any $\sigma\in\widetilde{W}$. We then observe that the cosets $B\omega(\sigma)$, $\omega(\sigma)B$ and the double coset $B\omega(\sigma)B$ are independent of the choice of the map ω and depend only on $\sigma\in\widetilde{W}$, since B contains the kernel $\mathfrak{H}_{\mathfrak{O}}$ of the homomorphism $\widetilde{\zeta}:\mathfrak{W}\to\widetilde{W}$. Also the subgroup $\omega(\sigma)B\omega(\sigma)^{-1}$ depends only on $\sigma\in\widetilde{W}$ but not on ω. Thus we have $B\omega(\sigma)\omega(\tau)=B\omega(\sigma\tau)$, $(B\omega(\sigma))^{-1}=\omega(\sigma^{-1})B$ for any $\sigma,\tau\in\widetilde{W}$. Under these notations, we have the

Proposition **2.6.** — *Assume that* $\mathfrak{g}_{\mathbf{C}}$ *is simple. Let*

$$\Gamma_i=B\cap\omega(w_i)^{-1}B\omega(w_i)\quad(0\le i\le l),$$

and let $\{t_\nu\}$ *be a representative system in* $\mathfrak{O}$ *of* $k=\mathfrak{O}/\mathfrak{P}$. *Then we have*

(i) $B=\bigcup_\nu\Gamma_i x_{-\alpha_i}(t_\nu)$ *is a disjoint union for any* $i=1,\ldots,l$.

(ii) $B=\bigcup_\nu\Gamma_0 x_{\alpha_0}(\pi t_\nu)$ *is a disjoint union.*

Proof. — (i) Let b be an element in B. Then b can be written as $b=uhv$, $u\in\mathfrak{U}_{\mathfrak{P}}$, $h\in\mathfrak{H}_{\mathfrak{O}}$, $v\in\mathfrak{V}_{\mathfrak{O}}$ by Th. 2.5. Since $\mathfrak{V}_{\mathfrak{O}}=\mathfrak{V}_{\mathfrak{O}}^{(i)}\mathfrak{X}_{-\alpha_i,\mathfrak{O}}$ we may write $v=v'x_{-\alpha_i}(t)$, $v'\in\mathfrak{V}_{\mathfrak{O}}^{(i)}$, $t\in\mathfrak{O}$. Now by $\omega(w_i)\mathfrak{X}_{\alpha,\mathfrak{O}}\omega(w_i)^{-1}=\mathfrak{X}_{w_i(\alpha),\mathfrak{O}}$ and $\omega(w_i)\mathfrak{X}_{\alpha,\mathfrak{P}}\omega(w_i)^{-1}=\mathfrak{X}_{w_i(\alpha),\mathfrak{P}}$ (for any $\alpha\in\Delta$, $1\le i\le l$), we have

$$\omega(w_i)\mathfrak{U}_{\mathfrak{P}}^{(i)}\omega(w_i)^{-1}\subset\mathfrak{U}_{\mathfrak{P}}^{(i)},\quad \omega(w_i)\mathfrak{V}_{\mathfrak{O}}^{(i)}\omega(w_i)^{-1}\subset\mathfrak{V}_{\mathfrak{O}}^{(i)}$$

because $\alpha\in\Delta^-$, $\alpha\neq-\alpha_i$ implies that $w_i(\alpha)\in\Delta^-$, $w_i(\alpha)\neq-\alpha_i$. These relations together with $\omega(w_i)\mathfrak{H}_{\mathfrak{D}}\omega(w_i)^{-1}=\mathfrak{H}_{\mathfrak{D}}$ show that $\omega(w_i)b\omega(w_i)^{-1}$ is in B if and only if $\omega(w_i)x_{-\alpha_i}(t)\omega(w_i)^{-1}$ is in B. In other words, $b\in\Gamma_i$ is equivalent to $x_{-\alpha_i}(t)\in\mathrm{B}$. Now from the fact that $\mathfrak{V}\mathfrak{H}\cap\mathfrak{U}=\{1\}$ ([6, p. 42]) and Th. 2.5, it is seen easily that $x_{-\alpha_i}(t)\in\mathrm{B}$ is equivalent to $t\in\mathfrak{P}$. Thus we have shown that

$$\Gamma_i=\mathfrak{U}_{\mathfrak{P}}\mathfrak{H}_{\mathfrak{D}}\mathfrak{V}^{(i)}_{\mathfrak{D}}\mathfrak{X}_{-\alpha_i,\mathfrak{P}}$$

and $\mathrm{B}=\Gamma_i\mathfrak{X}_{-\alpha_i,\mathfrak{D}}$, $\Gamma_i\cap\mathfrak{X}_{-\alpha_i,\mathfrak{D}}=\mathfrak{X}_{-\alpha_i,\mathfrak{P}}$. Then we easily get the disjoint union $\mathrm{B}=\bigcup\limits_{\nu}\Gamma_i x_{-\alpha_i}(t_\nu)$.

(ii) Let $b=vhu\in\mathrm{B}$, $v\in\mathfrak{V}_{\mathfrak{D}}$, $h\in\mathfrak{H}_{\mathfrak{D}}$, $u\in\mathfrak{U}_{\mathfrak{P}}$. Then u can be written as

$$u=u'x_{\alpha_0}(t),\quad u'\in\prod_{\substack{\alpha\in\Delta^+\\ \alpha\neq\alpha_0}}\mathfrak{X}_{\alpha,\mathfrak{P}},\quad t\in\mathfrak{P}.$$

Now we have for any $\alpha\in\Delta$, $t'\in\mathrm{K}$,

$$\omega(w_0)x_\alpha(t')\omega(w_0)^{-1}=x_\beta(\pm\pi^{(\alpha_0^*,\beta)}t')$$

where $\beta=w_{\alpha_0}(\alpha)=\alpha-(\alpha_0^*,\alpha)\alpha_0$. Since $(\alpha,\alpha_0)\geq 0$ for any $\alpha\in\Delta^+$, we see that $(\alpha_0^*,\beta)=(\alpha_0^*,w_{\alpha_0}(\alpha))=(w_{\alpha_0}(\alpha_0^*),\alpha)=-(\alpha_0^*,\alpha)$ is given by

$$(\alpha_0^*,\beta)=\begin{cases}-2 & \text{if}\quad \alpha=\alpha_0,\\ -1 & \text{if}\quad \alpha\in\Delta^+,\quad \alpha\neq\alpha_0,\quad \beta\in\Delta^-,\\ 0 & \text{if}\quad \alpha\in\Delta^+,\quad \beta\in\Delta^+,\end{cases}$$

using the fact that $(\alpha,\alpha)\leq(\alpha_0,\alpha_0)$ (for any $\alpha\in\Delta$). Thus we have $\omega(w_0)u'\omega(w_0)^{-1}\in\mathrm{B}$. Similarly we get $\omega(w_0)v\omega(w_0)^{-1}\in\mathrm{B}$. Obviously we have $\omega(w_0)h\omega(w_0)^{-1}\in\mathrm{B}$. Thus $\omega(w_0)b\omega(w_0)^{-1}\in\mathrm{B}$ is equivalent to $\omega(w_0)x_{\alpha_0}(t)\omega(w_0)^{-1}\in\mathrm{B}$, i.e. $b\in\Gamma_0$ is equivalent to $x_{-\alpha_0}(\pm\pi^{-2}t)\in\mathrm{B}$. From $\mathfrak{U}\mathfrak{H}\cap\mathfrak{V}=\{1\}$ and Th. 2.5, it is easily seen that $x_{-\alpha_0}(\pm\pi^{-2}t)\in\mathrm{B}$ is equivalent to $t\in\mathfrak{P}^2$. Thus we have obtained

$$\Gamma_0=\mathfrak{V}_{\mathfrak{D}}\mathfrak{H}_{\mathfrak{D}}\mathfrak{U}^{(0)}_{\mathfrak{P}}\mathfrak{X}_{\alpha_0,\mathfrak{P}^2}$$

where $\mathfrak{U}^{(0)}_{\mathfrak{P}}=\prod\limits_{\substack{\alpha\in\Delta^+\\ \alpha\neq\alpha^0}}\mathfrak{X}_{\alpha,\mathfrak{P}}$, $\mathfrak{X}_{\alpha_0,\mathfrak{P}^2}=\{x_{\alpha_0}(t)\,;\,t\in\mathfrak{P}^2\}$. Also we see that

$$\mathrm{B}=\Gamma_0\mathfrak{X}_{\alpha_0,\mathfrak{P}},\qquad \Gamma_0\cap\mathfrak{X}_{\alpha_0,\mathfrak{P}}=\mathfrak{X}_{\alpha_0,\mathfrak{P}^2}.$$

Hence we get the disjoint union $\mathrm{B}=\bigcup\limits_{\nu}\Gamma_0 x_{\alpha_0}(\pi t_\nu)$.

Corollary 2.7. — (i) $\omega(w_i)^{-1}\mathrm{B}\omega(w_i)\neq\mathrm{B}$ *for* $i=0,1,\ldots,l$.

(ii) $\mathrm{B}\omega(w_i)\mathrm{B}=\mathrm{B}\omega(w_i)\mathfrak{X}_{-\alpha_i,\mathfrak{D}}$ $(1\leq i\leq l)$ *and* $\mathrm{B}\omega(w_i)\mathrm{B}=\bigcup\limits_{\nu}\mathrm{B}\omega(w_i)x_{-\alpha_i}(t_\nu)$ *is a disjoint union for* $i=1,\ldots,l$.

(iii) $\mathrm{B}\omega(w_0)\mathrm{B}=\mathrm{B}\omega(w_0)\mathfrak{X}_{-\alpha_0,\mathfrak{P}}$ *and* $\mathrm{B}\omega(w_0)\mathrm{B}=\bigcup\limits_{\nu}\mathrm{B}\omega(w_0)x_{\alpha_0}(\pi t_\nu)$ *is a disjoint union.*

Proof. — (i) is clear by Prop. 2.6. (ii), (iii) are seen from the fact that the natural map $\Gamma_i\backslash\mathrm{B}\to\mathrm{B}\backslash\mathrm{B}\omega(w_i)\mathrm{B}$ from the coset space $\Gamma_i\backslash\mathrm{B}=\{\Gamma_i b\,;\,b\in\mathrm{B}\}$ onto the coset space $\mathrm{B}\backslash\mathrm{B}\omega(w_i)\mathrm{B}=\{\mathrm{B}\omega(w_i)b\,;\,b\in\mathrm{B}\}$ defined by $\Gamma_i b\to\mathrm{B}\omega(w_i)b$ is a bijection.

Now using the function λ in § 1.4, we get the

Proposition **2.8.** — *Assume that* $\mathfrak{g}_{\mathbf{C}}$ *is simple. Let* i *be an integer with* $0 \leq i \leq l$ *and* σ *an element in* $\widetilde{W} = DW$. *Then*

(i) *if* $\lambda(w_i\sigma) > \lambda(\sigma)$, *we have* $B\omega(w_i)B\omega(\sigma)B = B\omega(w_i\sigma)B$;

(ii) *if* $\lambda(w_i\sigma) < \lambda(\sigma)$, *we have* $B\omega(w_i)B\omega(\sigma)B = B\omega(\sigma)B \cup B\omega(w_i\sigma)B$.

Proof. — (i) First let $i > 0$. Then

$$B\omega(w_i)B\omega(\sigma)B = B\omega(w_i)\mathfrak{X}_{-\alpha_i, \mathfrak{O}}\omega(\sigma)B =$$
$$= B\omega(w_i)\omega(\sigma)\omega(\sigma)^{-1}\mathfrak{X}_{-\alpha_i, \mathfrak{O}}\omega(\sigma)B = B\omega(w_i\sigma) . \omega(\sigma)^{-1}\mathfrak{X}_{-\alpha_i, \mathfrak{O}}\omega(\sigma)B$$

by Cor. 2.7, (ii). Thus it is enough to show that $\omega(\sigma)^{-1}\mathfrak{X}_{-\alpha_i, \mathfrak{O}}\omega(\sigma) \subset B$ under the assumption $\lambda(w_i\sigma) > \lambda(\sigma)$. Let $\sigma^{-1} = dw$, $d \in D$, $w \in W$. Then

$$\omega(\sigma)^{-1}x_{-\alpha_i}(t)\omega(\sigma) = x_{-w(\alpha_i)}(\pm\pi^{(d, -w(\alpha_i))}t).$$

Now $\lambda(\sigma^{-1}w_i) = \lambda(w_i\sigma) > \lambda(\sigma) = \lambda(\sigma^{-1})$ implies by Prop. 1.28 that $(d, -w(\alpha_i)) \geq 0$ (when $w(\alpha_i) > 0$) and $(d, -w(\alpha_i)) > 0$ (when $w(\alpha_i) < 0$). Therefore we get $\omega(\sigma)^{-1}\mathfrak{X}_{-\alpha_i, \mathfrak{O}}\omega(\sigma) \subset B$. The case where $i = 0$ is also proved similarly using Prop. 1.29.

(ii) First let $i > 0$ and $\{t_\nu\}$ be a representative system in $\mathfrak{O}$ of $k = \mathfrak{O}/\mathfrak{P}$. Then by Prop. 2.6, (i) we have

$$B\omega(w_i)B\omega(\sigma)B = \bigcup_\nu \Gamma_i\,\omega(w_i)x_{-\alpha_i}(t_\nu)\omega(\sigma)B.$$

Now put $w_i\sigma = \tau \in \widetilde{W}$. Then

$$\omega(w_i)x_{-\alpha_i}(t_\nu)\omega(\sigma)B = \omega(w_i)x_{-\alpha_i}(t_\nu)\omega(w_i)^{-1}\omega(w_i)\omega(\sigma)B$$
$$= x_{\alpha_i}(\pm t_\nu)\omega(\tau)B.$$

On the other hand, using the homomorphism $\Phi_{\alpha_i} : SL(2, K) \to G$, it is seen that $t_\nu \notin \mathfrak{P}$ implies $x_{\alpha_i}(\pm t_\nu) \in B\omega(w_i)B$. Thus we have

$$\omega(w_i)x_{-\alpha_i}(t_\nu)\omega(\sigma) \in B\omega(w_i)B\omega(\tau)B = B\omega(w_i\tau)B = B\omega(\sigma)B$$

for $t_\nu \in \mathfrak{O}^*$ since $\lambda(w_i\tau) > \lambda(\tau)$. In other words we have

$$B\omega(w_i)x_{-\alpha_i}(t_\nu)\omega(\sigma)B = B\omega(\sigma)B \qquad \text{for} \quad t_\nu \in \mathfrak{O}^*.$$

If $t_\nu \in \mathfrak{P}$, the preceding computations also show that

$$B\omega(w_i)x_{-\alpha_i}(t_\nu)\omega(\sigma)B = B\omega(\tau)B = B\omega(w_i\sigma)B.$$

Thus we have proved $B\omega(w_i)B\omega(\sigma)B = B\omega(\sigma)B \cup B\omega(w_i\sigma)B$. The case where $i = 0$ is also proved similarly by using Prop. 1.29, Prop. 2.6 and the following fact: $\omega(w_0)x_{\alpha_0}(\pi t_\nu)\omega(w_0)^{-1} \in B\omega(w_0)B$ (for $t_\nu \in \mathfrak{O}^*$), which is seen using the homomorphism $\Phi_{\alpha_0} : SL(2, K) \to G$.

Corollary **2.9.** — *Assume that* $\mathfrak{g}_{\mathbf{C}}$ *is simple. Let* i *be an integer with* $0 \leq i \leq l$ *and* σ *an element in* $\widetilde{W}$. *Then:*

(i) $B \cup B\omega(w_i)B$ *forms a subgroup of* G.

(ii) *If* $\lambda(w_i\sigma) > \lambda(\sigma)$, *then* $B\omega(w_i)B\omega(w_i\sigma) \subset B\omega(\sigma) \cup B\omega(w_i\sigma)B$.

Proof. — (i) Since $(B\omega(w_i)B)^{-1}=B\omega(w_i)B$, we have only to show that $B\omega(w_i)B\omega(w_i)B \subset B\cup B\omega(w_i)B$, but this is an immediate corollary of Prop. 2.8, (ii).

(ii) Since

$B\omega(w_i)B\omega(w_i\sigma)=B\omega(w_i)B\omega(w_i)\omega(\sigma)$ and $B\omega(w_i)B\omega(w_i)\subset B\omega(w_i)B\omega(w_i)B=B\cup B\omega(w_i)B$,

we get $B\omega(w_i)B\omega(w_i\sigma)\subset(B\cup B\omega(w_i)B)\omega(\sigma)\subset B\omega(\sigma)\cup B\omega(w_i)B\omega(\sigma)$.

Now by the assumption $\lambda(w_i\sigma)>\lambda(\sigma)$, $B\omega(w_i)B\omega(\sigma)\subset B\omega(w_i)B\omega(\sigma)B=B\omega(w_i\sigma)B$ (see Prop. 2.8). Hence the proof is complete.

2.4. Let us now consider the subgroup G' of G which is generated by the subgroups $\mathfrak{X}_\alpha$, $\alpha\in\Delta$. Since our ground field K is an infinite field, G' is the commutator group of G) (See [6, Cor. of Th. 3] when $\mathfrak{g}_C$ is simple. This immediately extends to the case where $\mathfrak{g}_C$ is semi-simple, since the Chevalley group of $\mathfrak{g}_C$ is the direct product of the Chevalley groups of the simple factors of $\mathfrak{g}_C$).

Let $\mathfrak{H}'$ be the subgroup of $\mathfrak{H}$ defined in [6, p. 47], i.e. $h(\chi)$, for $\chi\in\mathrm{Hom}(P_r, K^*)$, is in $\mathfrak{H}'$ if and only if there exists an element $\chi'\in\mathrm{Hom}(P, K^*)$ such that $\chi'|P_r=\chi$. We denote by D' the subgroup of D defined by $D'=D\cap\mathfrak{H}'$. Then it is easily seen that this subgroup D' coincides with the group denoted by D' in § 1.2 under the identification in § 2.1.

Now let us consider the subgroup Ω defined in § 1.7. Let us investigate the relationship between Ω and the normalizer $N(B)$ of B in G. Let $\sigma=dw\in\widetilde{W}$, $d\in D$, $w\in W$. Then, $\omega(\sigma)x_\alpha(t)\omega(\sigma)^{-1}=x_{w(\alpha)}(\pm\pi^{(d,w(\alpha))}t)$. Therefore $\omega(\sigma)B\omega(\sigma)^{-1}\subset B$ is equivalent to the following conditions:

$$\begin{array}{lll} (d, w(\alpha))\geq 0 & \text{for} & \alpha\in\Delta^+\cap w^{-1}\Delta^+,\\ (d, w(\alpha))\geq -1 & \text{for} & \alpha\in\Delta^+\cap w^{-1}\Delta^-,\\ (d, w(\alpha))\geq 1 & \text{for} & \alpha\in\Delta^-\cap w^{-1}\Delta^+,\\ (d, w(\alpha))\geq 0 & \text{for} & \alpha\in\Delta^-\cap w^{-1}\Delta^-. \end{array}$$

Thus we see that $\omega(\sigma)B\omega(\sigma)^{-1}\subset B$ is equivalent to the following conditions:

$$\begin{array}{lll} (d, w(\alpha))=1 & \text{for} & \alpha\in\Delta^-\cap w^{-1}\Delta^+,\\ (d, w(\alpha))=0 & \text{for} & \alpha\in\Delta^-\cap w^{-1}\Delta^-. \end{array}$$

In other words, $\omega(\sigma)B\omega(\sigma)^{-1}\subset B$ is equivalent to the following conditions:

$$\begin{array}{lll} (d, \beta)=1 & \text{for} & \beta\in\Delta^+\cap w\Delta^-,\\ (d, \beta)=0 & \text{for} & \beta\in\Delta^+\cap w\Delta^+. \end{array}$$

By Prop. 1.23, these conditions are equivalent to $\lambda(\sigma)=0$, i.e. to $\sigma\in\Omega$. Thus, since $\lambda(\sigma^{-1})=\lambda(\sigma)$, $\omega(\sigma)B\omega(\sigma)^{-1}\subset B$ implies that $\omega(\sigma)^{-1}B\omega(\sigma)\subset B$, hence we have then $\omega(\sigma)\in N(B)$. Thus:

Proposition **2.10.** — *Assume that* $\mathfrak{g}_C$ *is simple. Let* ω *be a map* $\widetilde{W}\to\mathfrak{W}$ *such that* $\widetilde{\zeta}(\omega(\sigma))=\sigma$ *for any* $\sigma\in\widetilde{W}$. *Let* $\sigma\in\widetilde{W}$. *Then we have* $\omega(\sigma)\in N(B)$ *if and only if* $\sigma\in\Omega$.

Now let us prove that the double cosets $B\omega(\sigma)B$, for $\sigma\in\widetilde{W}$, are mutually disjoint. We begin with the

Lemma **2.11.** — $\mathfrak{W}\cap B=\mathfrak{H}_{\mathfrak{D}}$.

Proof. — By [6, Cor. 1 of Th. 2], G is a disjoint union of the subsets $\mathfrak{V}\mathfrak{H}\omega(w)\mathfrak{U}'_w\,(w\in W)$, where $\mathfrak{U}'_w=\prod_{\alpha\in\Delta^+\cap w^{-1}\Delta^-}\mathfrak{X}_\alpha\subset\mathfrak{U}$. Since $\mathfrak{U}'_w=\mathfrak{U}$ for $w=1$, B is contained in $\mathfrak{V}\mathfrak{H}\omega(1)\mathfrak{U}=\mathfrak{V}\mathfrak{H}\mathfrak{U}$ by Th. 2.5. (Note that $B=\mathfrak{V}_{\mathfrak{D}}\mathfrak{H}_{\mathfrak{D}}\mathfrak{U}_{\mathfrak{P}}$.) Thus if $x\in B$ is in $\mathfrak{W}=\bigcup_{w\in W}\mathfrak{H}\omega(w)$, we must have $x\in\mathfrak{H}\omega(1)=\mathfrak{H}$. Now any element in B can be written as vhu with $v\in\mathfrak{V}_{\mathfrak{D}}$, $h\in\mathfrak{H}_{\mathfrak{D}}$, $u\in\mathfrak{U}_{\mathfrak{P}}$. Furthermore, in this expression v, h and u are determined uniquely by $\mathfrak{V}\mathfrak{H}\cap\mathfrak{U}=\{1\}$, $\mathfrak{V}\cap\mathfrak{U}=\{1\}$. Thus we have $\mathfrak{H}\cap B\subset\mathfrak{H}_{\mathfrak{D}}$. Hence we have shown that $\mathfrak{W}\cap B\subset\mathfrak{H}_{\mathfrak{D}}$. Obviously $\mathfrak{W}\cap B\supset\mathfrak{H}_{\mathfrak{D}}$ and this completes the proof.

Corollary **2.12.** — $\widetilde{\zeta}^{-1}(\Omega)\cap B=\mathfrak{H}_{\mathfrak{D}}$.

The proof of the following proposition is essentially the same as the one given in Tits [16]. However, for the covenience of the reader, we shall reproduce his proof here.

Proposition **2.13.** — *Assume that* $\mathfrak{g}_{\mathbf{C}}$ *is simple. Let* $\sigma,\tau\in\widetilde{W}$ *and* $B\omega(\sigma)B=B\omega(\tau)B$, *then* $\sigma=\tau$.

Proof. — Let $\lambda(\sigma)\leq\lambda(\tau)$. We shall prove our assertion by induction on $\lambda(\sigma)$. If $\lambda(\sigma)=0$, then $\omega(\sigma)\in N(B)$. Hence $\omega(\tau)$ is also in $N(B)$. Then we get $B\omega(\sigma)=B\omega(\tau)$, i.e. $\omega(\rho)\in B$ where $\rho=\sigma\tau^{-1}\in\Omega$. Hence $\omega(\rho)\in B\cap\widetilde{\zeta}^{-1}(\Omega)=\mathfrak{H}_{\mathfrak{D}}$ by Cor. 2.12, i.e, $\rho=\widetilde{\zeta}(\omega(\rho))=1$. Thus we get $\sigma=\tau$.

Now let $\lambda(\sigma)=k>0$ and assume that our assertion is true for $B\omega(\sigma')B=B\omega(\tau')B$ with $\lambda(\sigma')\leq\lambda(\tau')$, $\lambda(\sigma')<k$. For some i with $0\leq i\leq l$, we have $\lambda(w_i\sigma)<\lambda(\sigma)$ by Lemma 1.5 and Cor. 1.9. Now $\omega(w_i)\omega(w_i\sigma)B=\omega(\sigma)B\subset B\omega(\tau)B$, hence

$$\omega(w_i\sigma)\,B\subset\omega(w_i)B\omega(\tau)B\subset B\omega(\tau)B\cup B\omega(w_i\tau)B$$

by Prop. 2.8. Therefore $B\omega(w_i\sigma)B$ must coincide with $B\omega(\tau)B$ or with $B\omega(w_i\tau)B$. Hence, by the inductive assumption, we get $w_i\sigma=\tau$ or $w_i\sigma=w_i\tau$. However, $w_i\sigma=\tau$ is impossible since $\lambda(w_i\sigma)<\lambda(\sigma)\leq\lambda(\tau)$. Thus $w_i\sigma=w_i\tau$, i.e. $\sigma=\tau$, Q.E.D.

Remark. — When K is locally compact, Prop. 2.13 can be also proved using a result in Goldman-Iwahori [7, Th. 3.15].

Lemma **2.14.** — $B\mathfrak{W}B$ *is a subgroup of* G.

Proof. — We may assume that $\mathfrak{g}_{\mathbf{C}}$ is simple. Since $(B\mathfrak{W}B)^{-1}=B\mathfrak{W}B$ and $B\mathfrak{W}B=\bigcup_{\sigma\in\widetilde{W}}B\omega(\sigma)B$, we have only to show that $B\omega(\sigma)B.\omega(\tau)B\subset B\mathfrak{W}B$ for any $\sigma,\tau\in\widetilde{W}$. Let $\sigma=\rho\sigma'$, $\rho\in\Omega$, $\sigma'\in D'W$ (note that $\widetilde{W}$ is a semi-direct product of Ω and D'W; cf. § 1). Let $\sigma'=w_{i_1}\ldots w_{i_r}$ be a reduced expression of σ' with respect to the generators $w_0,\ldots,w_l$ of D'W. Then $\lambda(\sigma'_1)<\lambda(\sigma'_2)<\ldots<\lambda(\sigma'_r)$ where $\sigma'_s=w_{i_1}\ldots w_{i_s}\,(1\leq s\leq r)$. Hence we have by Prop. 2.8

$$B\omega(\sigma')B=B\omega(w_{i_1})B\omega(w_{i_2})B\ldots B\omega(w_{i_r})B.$$

Hence $B\omega(\sigma')B\omega(\tau)B \subset B\mathfrak{W}B$ by Prop. 2.8. Now since $\omega(\rho) \in N(B)$ we have $B\omega(\rho) = \omega(\rho)B$ and $B\omega(\sigma)B = B\omega(\rho)B\omega(\sigma')B$. Therefore

$$B\omega(\sigma)B\omega(\tau)B = B\omega(\rho)B\omega(\sigma')B\omega(\tau)B \subset B\omega(\rho)B\mathfrak{W}B = B\omega(\rho)\mathfrak{W}B = B\mathfrak{W}B,$$

which completes the proof.

Lemma **2.15**. — *Assume that* $\mathfrak{g}_{\mathbf{C}}$ *is simple. Let* H *be a subgroup of* G *such that* $G \supset H \supset B$, $H \cap D \neq \{1\}$. *Then* H *contains the subgroup* G′B *of* G.

Proof. — Let $d \in H \cap D$, $d \neq 1$. Then $(d, \alpha) \neq 0$ for some $\alpha \in \Delta$. Hence

$$H \supset \bigcup_{i \in \mathbf{Z}} d^i \mathfrak{X}_{\alpha, \mathfrak{P}} d^{-i} = \mathfrak{X}_\alpha.$$

Also $(d, -\alpha) \neq 0$ implies that $H \supset \mathfrak{X}_{-\alpha}$. Since $\Phi_\alpha(SL(2, K))$ is generated by $\mathfrak{X}_\alpha$ and $\mathfrak{X}_{-\alpha}$, we have then $H \supset \Phi_\alpha(SL(2, K))$. Then $d_1 = \Phi_\alpha\left(\begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix}\right) \in H \cap D$ and $(d_1, \beta) = (\beta, \alpha^*)$ for any $\beta \in \Delta$. Hence, as above, $H \supset \Phi_\beta(SL(2, K))$ for any $\beta \in \Delta$ such that $(\beta, \alpha) \neq 0$. Now, since $\mathfrak{g}_{\mathbf{C}}$ is simple, for any $\beta \in \Delta$ there exists a chain $\gamma_1, \ldots, \gamma_r$ of roots such that $\beta = \gamma_1$, $\alpha = \gamma_r$, $(\gamma_i, \gamma_{i+1}) \neq 0$ for $1 \leq i \leq r-1$. Thus $H \supset \Phi_\beta(SL(2, K)) \supset \mathfrak{X}_\beta$ for any $\beta \in \Delta$. Hence $H \supset G'$, which completes the proof.

Theorem **2.16**. — $G = B\mathfrak{W}B = \bigcup_{\sigma \in \tilde{W}} B\omega(\sigma)B$ *(disjoint union)*.

Proof. — We may assume that $\mathfrak{g}_{\mathbf{C}}$ is simple. $B\mathfrak{W}B$ is a subgroup of G containing B, $\mathfrak{W}$ (Lemma 2.14). Hence $B\mathfrak{W}B \supset D$. Thus $B\mathfrak{W}B \supset G'$ by Lemma 2.15. Also we have $\mathfrak{H} \subset \mathfrak{W} \subset B\mathfrak{W}B$. Hence $\mathfrak{H}G' \subset B\mathfrak{W}B$, i.e. $G = B\mathfrak{W}B = \bigcup_{\sigma \in \tilde{W}} B\omega(\sigma)B$ and this is a disjoint union by Prop. 2.13, Q.E.D.

Corollary **2.17**. — (i) $G = U\mathfrak{H}U = UDU$.

(ii) U *coincides with the subgroup of* G *consisting of elements* x *such that* $x\mathfrak{g}_{\mathfrak{O}} = \mathfrak{g}_{\mathfrak{O}}$, *where* $\mathfrak{g}_{\mathfrak{O}} = \mathfrak{g}_{\mathfrak{O}} \otimes_{\mathbf{Z}} \mathfrak{O}$ *is the Chevalley lattice in the sense of Bruhat* [4].

Proof. — (i) is seen from $\omega(w) \in U$ for $w \in W$ and $\mathfrak{W} = D\mathfrak{W}_{\mathfrak{O}}$.

(ii) is seen by (i) and the following facts: $x \in U$ implies that $x\mathfrak{g}_{\mathfrak{O}} = \mathfrak{g}_{\mathfrak{O}}$; $d \in D$, $d \neq 1$ implies that $d\mathfrak{g}_{\mathfrak{O}} \neq \mathfrak{g}_{\mathfrak{O}}$.

Corollary **2.18**. (cf. Bruhat [4]). — *If* K *is a locally compact field, then* U *is a maximal compact subgroup of* G *with respect to the natural topology of* G.

Proof. — Obvious by Cor. 2.17.

Corollary **2.19**. — $N(B) = B\tilde{\zeta}^{-1}(\Omega)$, $N(B) = \bigcup_{\rho \in \Omega} B\omega(\rho)$ *(disjoint union) and*

$$N(B)/B \cong \Omega \cong P/P_r.$$

Proof. — Let $x \in N(B)$. We may write $x = b_1 \omega(\sigma) b_2$ where $b_1 \in B$, $b_2 \in B$, $\sigma \in \tilde{W}$. Then $\omega(\sigma) \in N(B)$. Hence $\sigma \in \Omega$ by Prop. 2.10. Thus $N(B) \subset \bigcup_{\rho \in \Omega} B\omega(\rho)$. $N(B) \supset \bigcup_{\rho \in \Omega} B\omega(\rho)$ is obvious and we have $N(B) = \bigcup_{\rho \in \Omega} B\omega(\rho)$. Now this is a disjoint union by Cor. 2.12. Hence we get $N(B)/B \cong \Omega \cong P/P_r$, Q.E.D.

Now assume that $\mathfrak{g}_{\mathbf{C}}$ is simple and let us consider the union $H = \bigcup_{\sigma \in D'W} B\omega(\sigma)B$. This is a subgroup since D′W is generated by $w_0, w_1, \ldots, w_l$ (see the proof of Lemma 2.14).

H contains B and a non-trivial element of D since H contains $\omega(W)$ and $\omega(w_0)$. Hence $H \supset G'B$. Now since we may assume that $\omega(w_i) \in G'$ $(0 \leq i \leq l)$, we have $H \subset G'B$. Thus we get the

Proposition **2.20.** — $G'B = \bigcup_{\sigma \in D'W} B\omega(\sigma)B$ *(disjoint union)*.

Corollary **2.21.** — $G'N(B) = N(B)G' = G$, $N(B) \cap G'B = B$.

Proof. — By

$$N(B) = \bigcup_{\rho \in \Omega} B\omega(\rho), \quad G'B = \bigcup_{\sigma \in D'W} B\omega(\sigma)B \quad \text{and} \quad \widetilde{W} = DW = \Omega(D'W) = (D'W)\Omega,$$

we get

$$\begin{aligned} G'N(B) = (G'B).N(B) &= \bigcup_{\substack{\sigma \in D'W \\ \rho \in \Omega}} B\omega(\sigma)B\omega(\rho)B \\ &= \bigcup_{\substack{\sigma \in D'W \\ \rho \in \Omega}} B\omega(\sigma\rho)B = \bigcup_{\sigma \in \widetilde{W}} B\omega(\sigma)B = G. \end{aligned}$$

Also $\Omega \cap D'W = \{1\}$ implies that $N(B) \cap G'B = B$ using the preceding double coset decompositions.

Now let $\mathfrak{W}^* = \widetilde{\zeta}^{-1}(D'W)$. Then, for $\sigma \in \widetilde{W}$, $\omega(\sigma) \in \mathfrak{W}^*$ is equivalent to $\omega(\sigma) \in G'B \cap \mathfrak{W}$ by Prop. 2.20 and Th. 2.16. Hence $\mathfrak{W}^* = G'B \cap \mathfrak{W}$. By Lemma 2.11, we have $\mathfrak{W}^* \cap B = \mathfrak{H}_{\mathfrak{D}}$. The quotient group $\mathfrak{W}^*/\mathfrak{H}_{\mathfrak{D}}$ is isomorphic to D'W.

Theorem **2.22.** — *The hypotheses of Tits* [16] *are all satisfied for the triple of groups* $(G'B, B, \mathfrak{W}^*)$ *and the involutive generators* w_α $(\alpha \in \Pi)$, $w^{(1)}, \ldots, w^{(r)}$ *of* D'W (cf. Prop. 1.2 *for the notations* $w^{(i)}$).

Proof. — We may assume that $\mathfrak{g}_{\mathbf{C}}$ is simple. We have to show with respect to the involutive generators $w_0, w_1, \ldots, w_l$ of D'W the following facts:

a) $\omega(w_i)B\omega(\sigma) \subset B\omega(w_i\sigma)B \cup B\omega(\sigma)B$ for any w_i and $\sigma \in D'W$, where ω is a map from D'W into $\mathfrak{W}^*$ such that $\widetilde{\zeta}(\omega(\sigma)) = \sigma$ for any $\sigma \in D'W$.

b) $\omega(w_i)B\omega(w_i) \neq B$ for any w_i.

c) B and $\mathfrak{W}^*$ generate the group G'B.

However we have already verified these properties *a)*, *b)* and *c)* in Prop. 2.8, Cor. 2.7 and Prop. 2.20.

Thus we now can apply the theorems of Tits [16] to the group G'B. In particular, when $\mathfrak{g}_{\mathbf{C}}$ is simple, $w_0, w_1, \ldots, w_l$ are the only elements of D'W such that $B \cup B\omega(\sigma)B$ is a subgroup of G'B. Hence, returning to the case where $\mathfrak{g}_{\mathbf{C}}$ is semi-simple, let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}_{\mathbf{C}}^{(1)} + \ldots + \mathfrak{g}_{\mathbf{C}}^{(r)}$ be the decomposition of $\mathfrak{g}_{\mathbf{C}}$ into simple ideals $\mathfrak{g}_{\mathbf{C}}^{(1)}, \ldots, \mathfrak{g}_{\mathbf{C}}^{(r)}$. Let $\Delta = \Delta^{(1)} \cup \ldots \cup \Delta^{(r)}$ be the corresponding orthogonal decomposition of the root system. Let $\alpha_0^{(i)}$ be the highest root of $\Delta^{(i)}$ $(1 \leq i \leq r)$ and $w^{(i)}$ the element of D'W defined by $w^{(i)} = d^{(i)} w_{\alpha_0^{(i)}}$ where $d^{(i)} \in D'$ is given by $(d^{(i)}, \alpha) = 2(\alpha, \alpha_0^{(i)})/(\alpha_0^{(i)}, \alpha_0^{(i)})$ for any $\alpha \in \Delta$. Now let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ and $w_i = w_{\alpha_i}$ $(1 \leq i \leq l)$. Then we get by the above remark the following

Proposition **2.23.** — $w_1, \ldots, w_l, w^{(1)}, \ldots, w^{(r)}$ *are the only elements of* D'W *such that* $B \cup B\omega(\sigma)B$ *is a subgroup of* G'B.

Also, by Tits [16], the subgroups H such that $G'B \supset H \supset B$ (the parabolic subgroups containing B in the sense of [16]) are determined. Namely, for any subset J' of $J = \{w_1, \ldots, w_l, w^{(1)}, \ldots, w^{(r)}\}$, let $\widetilde{W}_{J'}$ be the subgroup of D'W generated by J'. Then $B\omega(\widetilde{W}_{J'})B$ is a subgroup of G'B containing B. The map $J' \rightarrow B\omega(\widetilde{W}_{J'})B$ is a bijection from the set consisting of all subsets J' of J onto the set of all parabolic subgroups containing B. If $B\omega(\widetilde{W}_{J'_1})B$ and $B\omega(\widetilde{W}_{J'_2})B$ are conjugate in G'B, then $J'_1 = J'_2$.

Now let us modify Th. 2.22 to obtain the

Theorem **2.24**. — *Let* $B' = B \cap G'$, $\mathfrak{W}' = \mathfrak{W} \cap G'$, $\mathfrak{H}'_{\mathfrak{D}} = \mathfrak{H}_{\mathfrak{D}} \cap G'$. *Then*

(i) $B' = \mathfrak{U}_{\mathfrak{P}} \mathfrak{H}'_{\mathfrak{D}} \mathfrak{V}_{\mathfrak{D}}$, $B' \cap \mathfrak{W}' = \mathfrak{H}'_{\mathfrak{D}}$.

(ii) $\mathfrak{W}' = \bigcup_{\sigma \in D'W} \mathfrak{H}'_{\mathfrak{D}} \omega(\sigma)$ *is a disjoint union, where* ω *is a map from* D'W *into* $\mathfrak{W}'$ *such that* $\widetilde{\zeta}(\omega(\sigma)) = \sigma$ *for any* $\sigma \in D'W$. *Hence the quotient group* $\mathfrak{W}'/\mathfrak{H}'_{\mathfrak{D}}$ *is isomorphic to* D'W.

(iii) $G' = \bigcup_{\sigma \in D'W} B'\omega(\sigma)B'$ *is a disjoint union.*

(iv) *The triple of groups* $(G', B', \mathfrak{W}')$ *and the involutive generators* w_α $(\alpha \in \Pi)$, $w^{(1)}, \ldots, w^{(r)}$ *of* D'W *satisfy all the hypotheses of Tits* [16].

Proof. — Since $\mathfrak{U}_{\mathfrak{P}} \subset G'$, $\mathfrak{V}_{\mathfrak{D}} \subset G'$, an element $b = uhv$ of B, where $u \in \mathfrak{U}_{\mathfrak{P}}$, $h \in \mathfrak{H}_{\mathfrak{D}}$, $v \in \mathfrak{V}_{\mathfrak{D}}$, is in G' if and only if $h \in \mathfrak{H}'_{\mathfrak{D}}$. Hence $B' = \mathfrak{U}_{\mathfrak{P}} \mathfrak{H}'_{\mathfrak{D}} \mathfrak{V}_{\mathfrak{D}}$. Now

$$B' \cap \mathfrak{W}' = B \cap \mathfrak{W} \cap G' = \mathfrak{H}_{\mathfrak{D}} \cap G' = \mathfrak{H}'_{\mathfrak{D}}.$$

Thus we get (i). Now let $\widetilde{\zeta}'$ be the restriction of the homomorphism $\widetilde{\zeta} : \mathfrak{W} \rightarrow DW$ to $\mathfrak{W}'$. Then, since $\widetilde{\zeta}^{-1}(\sigma) \cap G'$ is not empty for any $\sigma \in D'W$, we have $\widetilde{\zeta}'(\mathfrak{W}') = D'W$ and the kernel of $\widetilde{\zeta}'$ coincides with $\mathfrak{W}' \cap \mathfrak{H}_{\mathfrak{D}} = \mathfrak{H}'_{\mathfrak{D}}$. Thus we have proved (ii). To prove (iii), (iv), we may assume that $\mathfrak{g}_C$ is simple. Then it is not difficult to verify all the analogues of Propositions 2.6 to 2.11, 2.13 to 2.15 replacing B, $\mathfrak{W}$, $\mathfrak{H}_{\mathfrak{D}}$, D, DW by B', $\mathfrak{W}'$, $\mathfrak{H}'_{\mathfrak{D}}$, D', D'W respectively. Hence we get (iii), (iv) quite analogously as above.

Thus the results of Tits [16] are also valid for $(G', B', \mathfrak{W}')$. In particular, there is a bijection of the set of all subgroups H' such that $G' \supset H' \supset B'$ on the set of subsets J' of $J = \{w_1, \ldots, w, w^{(1)}, \ldots, w^{(r)}\}$. Hence, there is a bijection of the set of all parabolic subgroups of $(G', B', \mathfrak{W}')$ containing B' on the set of all parabolic subgroups of $(G'B, B, \mathfrak{W}^*)$ containing B, whose inverse is given by $H \rightarrow H'$, where

$$G'B \supset H \supset B,\ G' \supset H' \supset B',\ H' = H \cap G'.$$

If $H = \bigcup_{\sigma \in \widetilde{W}_{J'}} B\omega(\sigma)B$, we may assume that $\omega(\sigma) \in G'$ and we have $H' = \bigcup_{\sigma \in \widetilde{W}_{J'}} B'\omega(\sigma)B'$. Hence we also have $H = BH'B$. In particular we have

Corollary **2.25**. — *Let* $\sigma \in D'W$, $\omega(\sigma) \in \mathfrak{W}'$, $\widetilde{\zeta}'(\omega(\sigma)) = \sigma$. *Then*

$$(B\omega(\sigma)B) \cap G' = B'\omega(\sigma)B'.$$

2.5. We shall now determine the subgroups H of G containing B. Let H be such a subgroup. Then $H \cap \mathfrak{W} \supset B \cap \mathfrak{W} = \mathfrak{H}_{\mathfrak{D}}$ and $\widetilde{W}_H = \widetilde{\zeta}(\mathfrak{W} \cap H)$ is a subgroup of $\widetilde{W} = DW$. Since $H \supset B$, H has an expression $H = \bigcup_{\sigma \in \Theta} B\omega(\sigma)B$ for some subset Θ

of $\widetilde{W}$. Then we obviously have $\widetilde{\zeta}^{-1}(\Theta)=H\cap\mathfrak{W}$ and $\Theta=\widetilde{W}_H$. Thus $H\to\widetilde{W}_H$ is an injective map from the set $\mathfrak{S}$ of all subgroups H of G containing B into the set $\mathfrak{S}^*$ of all subgroups of $\widetilde{W}$. Let $\mathfrak{S}_0$ be the image of $\mathfrak{S}$ under this injection. We have to determine the set $\mathfrak{S}_0$. Let $H\in\mathfrak{S}$. Then $\Omega_H=\Omega\cap\widetilde{W}_H$ and $\widetilde{W}'_H=D'W\cap\widetilde{W}_H$ are subgroups of Ω and D'W respectively. Let J be the set of all $\sigma\in D'W$ such that $B\cup B\omega(\sigma)B$ forms a subgroup of G. Then in our former notation, $J=\{w_1, \ldots, w_l, w^{(1)}, \ldots, w^{(r)}\}$. Now let us prove that $\widetilde{W}'_H$ is generated by the subset $J_H=\widetilde{W}'_H\cap J$ and that $\widetilde{W}_H=\Omega_H.\widetilde{W}'_H$. To begin with:

Lemma **2.26**. — *Assume that* $\mathfrak{g}_C$ *is simple. Let* $0\leq i\leq l$ *and* $\sigma\in\widetilde{W}$. *If* $\lambda(w_i\sigma)<\lambda(\sigma)$, *then* $\omega(w_i)\in B\omega(\sigma)B\omega(\sigma)^{-1}B$ (cf. Tits [16, Cor. 2 to Th. 1]).

Proof. — By Prop. 2.8, the intersection $B\omega(w_i)B\omega(\sigma)B\cap B\omega(\sigma)B$ is not empty. Hence there exist $b, b_1, b_2\in B$ such that $\omega(w_i)b\omega(\sigma)=b_1\omega(\sigma)b$, i.e. $\omega(w_i)\in B\omega(\sigma)B\omega(\sigma)^{-1}B$, Q.E.D.

Now let $H\in\mathfrak{S}$ and $\sigma\in\widetilde{W}_H$. We can write $\sigma=\tau\rho$ with $\tau\in D'W$, $\rho\in\Omega$. Let $\tau=w_{i_1}\ldots w_{i_r}$ be a reduced expression of τ. Then $\lambda(w_{i_1}\sigma)=\lambda(w_{i_1}\tau)<\lambda(\tau)=\lambda(\sigma)$. Hence we have by Lemma 2.26 $\omega(w_{i_1})\in B\omega(\sigma)B\omega(\sigma)^{-1}B\subset H$, i.e. $w_{i_1}\in\widetilde{W}_H$. Therefore $w_{i_1}\sigma=w_{i_2}\ldots w_{i_r}\rho\in\widetilde{W}_H$. Continuing in the same manner, we get $w_{i_1}, \ldots, w_{i_r}\in\widetilde{W}_H$ and $\rho\in\widetilde{W}_H$. Thus we see that $\widetilde{W}'_H=D'W\cap\widetilde{W}_H$ is generated by $J_H=\widetilde{W}'_H\cap J$ and that $\widetilde{W}_H=\Omega_H.\widetilde{W}'_H$.

Furthermore, J_H is normalized by any element $\rho\in\Omega_H$: $\rho J_H\rho^{-1}=J_H$. In fact, J_H is the set of all $\sigma\in D'W$ such that $\lambda(\sigma)=1$ and $\sigma\in\widetilde{W}_H$ (cf. Prop. 1.10). Hence $\rho J_H\rho^{-1}\subset J_H$ for any $\rho\in\Omega_H$ by using the fact $\lambda(\rho\sigma\rho^{-1})=\lambda(\sigma)$. Therefore we get $\rho J_H\rho^{-1}=J_H$ for any $\rho\in\Omega_H$.

Let now $\mathfrak{S}_1$ be the set of all pairs (Ω', J') consisting of a subgroup Ω' of Ω and a subset J' of J such that $\rho J'_H\rho^{-1}=J'_H$ for any $\rho\in\Omega'$. Then we get as above a map $\mathfrak{S}\to\mathfrak{S}_1$ defined by $H\to(\Omega_H, J_H)$. This is injective since $H=\bigcup_{\sigma\in\widetilde{W}_H} B\omega(\sigma)B$, $\widetilde{W}_H=\Omega_H\widetilde{W}'_H=\widetilde{W}'_H\Omega_H$ and $\widetilde{W}'_H$ is generated by J_H. Now let us show that this map is surjective. Let $(\Omega', J')\in\mathfrak{S}_1$. Let $\widetilde{W}'_{J'}$ be the subgroup of $\widetilde{W}'$ generated by J'. Then obviously $\Omega'\widetilde{W}'_{J'}=\widetilde{W}'_{J'}\Omega'$ is a subgroup of $\widetilde{W}$ containing $\widetilde{W}'_{J'}$ as a distinguished subgroup. Then $H=B\omega(\Omega'\widetilde{W}'_{J'})B$ is a subgroup of G by the same argument as in the proof of Lemma 2.14. It is easy to see that $H\supset B$ and $\Omega'=\Omega_H$, $\widetilde{W}'_{J'}=\widetilde{W}'_H$. Then we get $J'=J_H$ by Tits [16, Cor. 3] since $(G', B', \mathfrak{W}')$ satisfies the hypotheses of Tits. Thus we have proved the

Theorem **2.27**. — *The map* $H\to(\Omega_H, J_H)$ *defined above from the set* $\mathfrak{S}$ *of all subgroups* H *of* G *containing* B *into the set* $\mathfrak{S}_1$ *of all pairs* (Ω', J') *of a subgroup* Ω' *of* Ω *and a subset* J' *of the standard generators* J *of* D'W *is bijective.*

Now we shall consider the conjugacy problem of $H_1, H_2\in\mathfrak{S}$. If $H_1, H_2\in\mathfrak{S}$ are conjugate in G, there is an element $x\in G$ such that $xH_1x^{-1}=H_2$. Now by Th. 2.16

we may write $x=b_1\omega(\sigma)b_2$ with $b_1, b_2\in \mathrm{B}$, $\sigma\in\widetilde{\mathrm{W}}$. Then $\omega(\sigma)\mathrm{H}_1\omega(\sigma)^{-1}=\mathrm{H}_2$. Therefore $\omega(\sigma)\mathrm{B}\omega(\sigma)^{-1}\subset\mathrm{H}_2$. Put $\sigma=\tau\rho$, $\tau\in\mathrm{D}'\mathrm{W}$, $\rho\in\Omega$ and let $\tau=w_{i_1}\dots w_{i_r}$ be a reduced expression of τ. Then by Lemma 2.26 we get as above $\omega(w_{i_1})\in\mathrm{B}\omega(\sigma)\mathrm{B}\omega(\sigma)^{-1}\mathrm{B}\subset\mathrm{H}_2$. Hence $\omega(w_{i_1}\sigma)\mathrm{H}_1\omega(w_{i_1}\sigma)^{-1}=\mathrm{H}_2$ and so on. Therefore finally we get $\omega(\rho)\mathrm{H}_1\omega(\rho)^{-1}=\mathrm{H}_2$. Then we get immediately

$$\Omega_{\mathrm{H}_1}=\Omega_{\mathrm{H}_2},\qquad \rho\mathrm{J}_{\mathrm{H}_1}\rho^{-1}=\mathrm{J}_{\mathrm{H}_2}.$$

Conversely, if these conditions for Ω_{H_1}, Ω_{H_2}, $\mathrm{J}_{\mathrm{H}_1}$, $\mathrm{J}_{\mathrm{H}_2}$ are satisfied for some $\rho\in\Omega$, we have easily $\omega(\rho)\mathrm{H}_1\omega(\rho)^{-1}=\mathrm{H}_2$. Thus we have proved the

Proposition **2.28**. — *Let* H_1, H_2 *be subgroups of* G *containing* B. *If* H_1 *and* H_2 *are conjugate by an element of* G, *then they are conjugate by an element of* N(B). *Moreover,* H_1 *and* H_2 *are conjugate in* G *if and only if* $\Omega_{\mathrm{H}_1}=\Omega_{\mathrm{H}_2}$ *and* $\rho\mathrm{J}_{\mathrm{H}_1}\rho^{-1}=\mathrm{J}_{\mathrm{H}_2}$ *for some* $\rho\in\Omega$.

By a similar argument as above, we have the

Proposition **2.29**. — *Let* $\mathrm{N(H)}=\mathrm{L}$ *be the normalizer of a subgroup* H *with* $\mathrm{G}\supset\mathrm{H}\supset\mathrm{B}$. *Then*

$$\Omega_{\mathrm{L}}=\{\rho\in\Omega;\ \rho\mathrm{J}_{\mathrm{H}}\rho^{-1}=\mathrm{J}_{\mathrm{H}}\},\quad \mathrm{J}_{\mathrm{L}}=\mathrm{J}_{\mathrm{H}}.$$

Now using Prop. 2.28 and the table of the action of Ω on J given in § 1.8, we can determine easily the number of conjugate classes of maximal subgroups of G containing a conjugate of B for each type of simple Lie algebra over **C**. (We note that for $\mathrm{H}_1, \mathrm{H}_2\in\mathfrak{S}$, $\mathrm{H}_1\subset\mathrm{H}_2$ is equivalent to $\Omega_{\mathrm{H}_1}\subset\Omega_{\mathrm{H}_2}$ and $\mathrm{J}_{\mathrm{H}_1}\subset\mathrm{J}_{\mathrm{H}_2}$.)

We observe that if H is a maximal subgroup such that $\mathrm{G}\supsetneq\mathrm{H}\supset\mathrm{B}$, then only the following cases are possible:

a) $\Omega_{\mathrm{H}}=\Omega$; then J_{H} is a maximal Ω-invariant subset of J.

b) $\Omega_{\mathrm{H}}\neq\Omega$; then $\mathrm{J}_{\mathrm{H}}=\mathrm{J}$ and Ω_{H} is a maximal subgroup of Ω.

Then we easily get the

Proposition **2.30**. — *The number of conjugacy classes of maximal subgroups of* G *containing a conjugate of* B *is equal to the sum of the number of* Ω*-orbits of* J *and the number of maximal subgroups of* Ω. *For simple Lie algebras over* **C** *these numbers are given by the following table* (I).

Table (I)

$(\mathrm{A}_l)_{l\geq 1}$: $1+s$, where s is the number of prime divisors of $l+1$.

$(\mathrm{B}_l)_{l\geq 2}$: $1+l$.

$(\mathrm{C}_l)_{l\geq 2}$: $2+\left[\frac{l}{2}\right]$.

$(\mathrm{D}_l)_{l\geq 3}$: $\frac{l+1}{2}$, if l is odd. $\frac{l}{2}+3$, if l is even.

(E_6) : 4.

(E_7) : 6.

(E_8) : 9.
(F_4) : 5.
(G_2) : 3.

Next let us consider the case where K is a locally compact field. Then $k=\mathfrak{O}/\mathfrak{P}$ is a finite field and G is an algebraic subgroup of $GL(\mathfrak{g}_K)=GL(n, K)$ where $n=\dim_{\mathbf{C}}\mathfrak{g}_{\mathbf{C}}$ (Ono [11]). It is seen easily then that U and B are open compact subgroups of G. Now we shall determine the number of conjugacy classes of maximal compact subgroups of G containing a conjugate of B for each simple Lie algebra $\mathfrak{g}_{\mathbf{C}}$ over $\mathbf{C}$. If H is a subgroup of G containing B, then by $H=\bigcup_{\sigma\in\tilde{W}_H} B\omega(\sigma)B$, H is compact if and only if $\widetilde{W}_H$ is a finite subgroup, i.e. if and only if $J_H \subsetneqq J$ (see the remark in § 1.9). Thus, in order to determine the number in question, we only have to determine the maximal ones in the subset $\mathfrak{S}_2=\{(\Omega', J')\in\mathfrak{S}_1; J'\subsetneqq J\}$ and then we have to determine the partition of $\mathfrak{S}_2$ by the equivalence relation given in Prop. 2.28. In this way, a simple computation using § 1.8 gives us the following

Proposition **2.31.** — *Let* K *be a locally compact field. Then the number of conjugacy classes of maximal compact subgroups of* G *containing a conjugate of* B *for simple Lie algebras* $\mathfrak{g}_{\mathbf{C}}$ *over* $\mathbf{C}$ *is given by the following table* (II).

Table (II)

$(A_l)_{l\geq 1}$: the number of positive divisors of $l+1$.
$(B_l)_{l\geq 2}$: $l+1$.
$(C_l)_{l\geq 3}$: $l+1$.
$(D_l)_{l\geq 4}$: $\begin{cases} l, & \text{if } l \text{ is odd.} \\ l+2, & \text{if } l \text{ is even.}\end{cases}$
(E_6) : 5.
(E_7) : 8.
(E_8) : 9.
(F_4) : 5.
(G_2) : 3.

For example, for type (D_l) $(l=2\nu)$, the representatives of the conjugacy classes of maximal compact subgroups H containing B (or conjugates of B) are given using (Ω_H, J_H) as follows (the notations being that of § 1.8):

Case (i) $\Omega_H=\Omega$. Then J_H is of the form $J_H=J-L$, where L is an orbit of Ω in J. There are ν orbits of Ω in J and we get ν conjugacy classes for this case.

Case (ii) $\Omega_H=\{1, \rho_1\}$. Then J_H is of the form $J_H=J-L'$, where L' is an orbit of $\{1, \rho_1\}$ and cannot contain any Ω-orbit. Thus we get $\nu-1$ conjugacy classes for this case.

Case (iii) $\Omega_H=\{1, \rho_{l-1}\}$. Then we get only one conjugacy class, e.g. $J_H=J-\{w_0, w_{l-1}\}$.
Case (iv) $\Omega_H=\{1, \rho_l\}$. We get only one conjugacy class, e.g. $J_H=J-\{w_0, w_l\}$.
Case (v) $\Omega_H=\{1\}$. We get only one conjugacy class, e.g. $J_H=J-\{w_0\}$.

Thus the total number of the conjugacy classes in question is $\nu+(\nu-1)+3=l+2$.

The situation is much simpler when we consider the group G′B or G′. Namely, a subgroup H of G′B (resp. of G′) containing B (resp. B′) is determined by a subset J_H of J, where J_H is the intersection of J and the subgroup $\widetilde{W}_{H'}$ of D′W defined by $\widetilde{W}_{H'}=\{\sigma\in D'W; B\omega(\sigma)B\subset H\}$ (resp. by $\widetilde{W}_{H'}=\{\sigma\in D'W; B'\omega(\sigma)B'\subset H\}$). Hence H is maximal if and only if J_H is a maximal subset of J, i.e. if and only if $|J_H|=|J|-1$ where $|J_H|$, $|J|$ mean the cardinalities of the finite sets J_H, J respectively. Thus if K is locally compact, every proper subgroup H of G′B (resp. of G′) with $H\supset B$ (resp. with $H\supset B'$) consists of finite double cosets of the open, compact subgroup B (resp. B′), hence H is compact. Therefore we have the

Proposition **2.32**. — *Let* K *be a locally compact field. Then the number of conjugate classes of maximal compact subgroups of* G′B (*resp. of* G′) *containing a conjugate of* B (*resp. of* B′) *is equal to* $|J|=l+r$, *where* l *is the rank of* $\mathfrak{g}_{\mathbf{C}}$ *and* r *is the number of simple ideals of* $\mathfrak{g}_{\mathbf{C}}$.

We shall now give an " Iwasawa decomposition " of G.

Proposition **2.33**. — $G=U\mathfrak{H}\mathfrak{U}=UD\mathfrak{U}$.

Proof. — Take the following system of generators of G: $\mathfrak{X}_\alpha$ $(\alpha\in\Delta^+)$, $\mathfrak{H}$, $\mathfrak{X}_{-\alpha_i}$ $(\alpha_i\in\Pi)$. We then can show without difficulty that $z\mathfrak{U}\mathfrak{H}U\subset\mathfrak{U}\mathfrak{H}U$ for any z in the system of generators, by using Prop. 2.3.

Finally we shall give the decomposition of G into double cosets of the form H_1xH_2 $(x\in G)$, where H_1 and H_2 are subgroups of G containing B. As before, we fix a map ω from $\widetilde{W}=DW$ into $\mathfrak{W}$ such that $\widetilde{\zeta}(\omega(\sigma))=\sigma$ for any $\sigma\in\widetilde{W}$.

Proposition **2.34**. — *Let* H_1 *and* H_2 *be subgroups of* G *containing* B *and* $\widetilde{W}_{H_1}$, $\widetilde{W}_{H_2}$ *be the subgroups of* W *associated with* H_1, H_2 *respectively*.

(i) *Let* $\sigma\in\widetilde{W}$. *Then* $H_1\omega(\sigma)H_2=\bigcup_{\tau\in\widetilde{W}_{H_1}\sigma\widetilde{W}_{H_2}} B\omega(\tau)B$.

(ii) *Let* $\sigma, \tau\in\widetilde{W}$. *Then* $H_1\omega(\sigma)H_2=H_1\omega(\tau)H_2$ *if and only* $\widetilde{W}_{H_1}\sigma\widetilde{W}_{H_2}=\widetilde{W}_{H_1}\tau\widetilde{W}_{H_2}$.

(iii) *Let* $\widetilde{W}=\bigcup_\lambda \widetilde{W}_{H_1}.\sigma_\lambda.\widetilde{W}_{H_2}$ *be any partition of* $\widetilde{W}$ *into double cosets* mod. $\widetilde{W}_{H_1}:\widetilde{W}_{H_2}$. *Then* $G=\bigcup_\lambda H_1\omega(\sigma_\lambda)H_2$ *is a disjoint union.*

Proof. — (i) Let $A=\bigcup_{\tau\in\widetilde{W}_{H_1}\sigma\widetilde{W}_{H_2}} B\omega(\tau)B$. Then clearly we have $A\subset H_1\omega(\sigma)H_2$. Since $\omega(\sigma)\in A$, to show $A=H_1\omega(\sigma)H_2$, it is enough to show $H_1A\subset A$ and $AH_2\subset A$. Since H_1 is generated by

$$\omega(\rho)\ (\rho\in\Omega_{H_1}),\quad \omega(\tau)\ (\tau\in\widetilde{W}'_H),\ \text{and } B,$$

to show $H_1A\subset A$, it is sufficient to see that $zA\subset A$ for any z in the above system of generators of H_1. For $z\in B$, $zA\subset A$ is trivial. For $z=\omega(\rho)$, $\rho\in\Omega_{H_1}$, z is in N(B) and we have $\omega(\rho)B\omega(\xi)B=B\omega(\rho\xi)B$; hence $zA\subset A$. Now let $\tau\in\widetilde{W}'_{H_1}$. Then τ can be written as $\tau = w_{i_1} \ldots w_{i_r}$ with $w_{i_1}, \ldots, w_{i_r} \in J_{H_1}$. Thus we have only to show that $\omega(w_i)A \subset A$ for any $w_i \in J_{H_1}$. However this is easily seen, because if $\xi \in \widetilde{W}_{H_1}\sigma\widetilde{W}_{H_2}$, then by Prop. 2.8, we have $\omega(w_i)B\omega(\xi)B \subset B\omega(\xi)B \cup B\omega(w_i\xi)B \subset A$. Similarly we have $AH_2 \subset A$ and the proof of (i) is complete.

(ii), (iii) are immediate consequences of (i).

Corollary **2.35**. — (i) $G = \bigcup_{d \in D} BdU = \bigcup_{d \in D} UdB$ *(disjoint unions).*

(ii) *Let* $D_+ = \{d \in D;\ (d, \alpha_i) \geq 0$ *for* $1 \leq i \leq l\}$. *Then* $G = \bigcup_{d \in D_+} UdU$ *is a disjoint union.*

Proof. — (i) Since $U = \bigcup_{w \in W} B\omega(w)B$, we have $\widetilde{W}_U = W$. Now since $\widetilde{W} = DW = WD$ is a semi-direct product, we get (i).

(ii) This is immediate since $DW = \bigcup_{d \in D_+} WdW$ is a disjoint union.

§ 3. On the structure of the Hecke ring $\mathscr{H}(G, B)$.

Through this section we assume that $k = \mathfrak{O}/\mathfrak{P}$ is a finite field consisting of q elements. (But we assume nothing about the completeness of K, thus K need not be locally compact.) We use the notations of §§ 1, 2. Also for the convenience of description, we assume that $\mathfrak{g}_C$ is simple through § 3.

3.1. Let $x \in G$. We denote by $\mathrm{ind}(x)$ the index $[B : B \cap x^{-1}Bx]$.

$$\mathrm{ind}(bxb') = \mathrm{ind}(x) \quad \text{for any} \quad x \in G;\ b, b' \in B.$$

Let $\Gamma = B \cap x^{-1}Bx$. Then the map $\Gamma y \to Bxy$ $(y \in B)$ from the coset space $\Gamma\backslash B = \{\Gamma y; y \in B\}$ into the coset space $B\backslash BxB = \{Bxy, y \in B\}$ is bijective. Hence

$$\mathrm{ind}(x) = |B\backslash BxB|$$

where $|B\backslash BxB|$ means the cardinality of the set $B\backslash BxB$.

Suppose $\mathrm{ind}(x) < \infty$, $\mathrm{ind}(y) < \infty$. Then we have $\mathrm{ind}(xy) < \infty$. In fact, we have $BxyB \subset BxByB$. Moreover there exist finite subsets $\{x_1, \ldots, x_r\}, \{y_1, \ldots, y_s\}$ of B such that $BxB = \bigcup_i Bx_i$, $ByB = \bigcup_j By_j$. Hence $BxByB = \bigcup_{i,j} Bx_iBy_j = \bigcup_j BxBy_j = \bigcup_{i,j} Bx_iy_j$. Now, by Prop. 2.6, we have

$$\mathrm{ind}(\omega(w_i)) = q \qquad \text{for } i = 0, 1, \ldots, l$$

where ω is a map from DW into $\mathfrak{W}$ such that $\widetilde{\zeta}(\omega(\sigma)) = \sigma$ for any $\sigma \in DW$. Hence we have $\mathrm{ind}(x) < \infty$ for any $x \in G'B$ by Prop. 2.20 and the proof of Lemma 2.14. Also it is clear that we have $\mathrm{ind}(x) = 1$ for every $x \in N(B)$. Thus we get by Cor. 2.21 that

Proposition **3.1**. — *We have* $\mathrm{ind}(x) < \infty$ *for any* $x \in G$.

Thus B is commensurable with any conjugate of it and we can consider the Hecke ring $\mathscr{H}(G, B)$ (see e.g. [10, § 1]). $\mathscr{H}(G, B)$ is defined as follows: let $\mathfrak{M}$ be the free **Z**-module generated by the double cosets $B\omega(\sigma)B$, $\sigma \in DW$. We denote by S_σ the double coset $B\omega(\sigma)B$ regarded as an element in $\mathfrak{M}$. Then the multiplication between the basic elements S_σ ($\sigma \in DW$) of $\mathfrak{M}$ is defined by

$$S_\sigma S_\tau = \sum_\mu m^\mu_{\sigma,\tau} S_\mu$$

where the structure constants $m^\mu_{\sigma,\tau}$ are defined as the number of cosets of the form Bx in the set $B\omega(\sigma)^{-1}B\omega(\mu) \cap B\omega(\tau)B$:

$$m^\mu_{\sigma,\tau} = |B\backslash B\omega(\sigma)^{-1}B\omega(\mu) \cap B\omega(\tau)B|.$$

Then, for any fixed $\sigma, \tau \in DW$, there is only a finite number of $\mu \in DW$ such that $m^\mu_{\sigma,\tau} \neq 0$, because $m^\mu_{\sigma,\tau} \neq 0$ is equivalent to

$$B\omega(\mu)B \subset B\omega(\sigma)B\omega(\tau)B.$$

Provided with this multiplication law, $\mathscr{H}(G, B)$ forms a ring with the unit element $1 = S_1$ (see e.g. [10, § 1]).

The map $\sum_\sigma \lambda_\sigma . S_\sigma \to \sum_\sigma \lambda_\sigma . \mathrm{ind}(\omega(\sigma)) \in \mathbf{Z}$ is a ring homomorphism from $\mathscr{H}(G, B)$ onto $\mathbf{Z}$ (cf. e.g. [10, § 1]). We denote this homomorphism also by ind:

$$\mathrm{ind}(\sum_\sigma \lambda_\sigma . S_\sigma) = \sum_\sigma \lambda_\sigma . \mathrm{ind}(\omega(\sigma)).$$

Now let $\sigma \in DW$, $\sigma = \rho\tau$, $\rho \in \Omega$, $\tau \in D'W$. Then, since $\omega(\rho)$ is in the normalizer $N(B)$ of B, we easily have

$$S_\sigma = S_\rho S_\tau.$$

Let $\tau = w_{i_1} \dots w_{i_r}$ be a reduced expression of τ. Then $\lambda(\tau') < \lambda(\tau)$ where $\tau' = w_{i_1}\tau$ and we get by Prop. 2.8 (i) and Cor. 2.9

$$S_\tau = S_{i_1} S_{\tau'}, \qquad \text{where we put} \quad S_i = S_{w_i}.$$

Continuing this, we get finally

$$S_\tau = S_{i_1} \dots S_{i_r}.$$

Therefore, by applying the homomorphism $\mathrm{ind} : \mathscr{H}(G, B) \to \mathbf{Z}$, we see that

$$\mathrm{ind}(S_\tau) = q^r = q^{\lambda(\tau)}.$$

Now, since $\mathrm{ind}(S_\rho) = 1$, we have proved the

Proposition **3.2**. — $\mathrm{ind}(\omega(\sigma)) = \mathrm{ind}(S_\sigma) = q^{\lambda(\sigma)}$ *for any* $\sigma \in DW$.

Corollary **3.3**. — $\mathrm{ind}(x) = \mathrm{ind}(x^{-1})$ *for any* $x \in G$.

Also, by what we have shown above, we have the

Theorem **3.3**. — $\mathscr{H}(G, B)$ *is generated by* S_ρ ($\rho \in \Omega$), $S_0, S_1, \dots, S$ *where* $S_i = S_w$ ($0 \leq i \leq l$). *Moreover, let* $\sigma = \rho\tau$, $\rho \in \Omega$, $\tau \in D'W$, *and* $\tau = w_{i_1} \dots w_{i_r}$ *a reduced expression of* τ. *Then*

$$S_\sigma = S_\rho S_\tau = S_\rho S_{i_1} \dots S_{i_r}.$$

Now let us consider the Hecke ring $\mathscr{H}(G'B, B)$. $\mathscr{H}(G'B, B)$ can be regarded in an obvious way as a subring of $\mathscr{H}(G, B)$ with the common unit element. By Prop. 2.20 and Th. 3.3, $\mathscr{H}(G'B, B)$ is generated by $1, S_0, S_1, \dots, S_l$. Now we shall characterize the ring $\mathcal{H}(G'B, B)$ by giving the defining relations among the generators $1, S_0, \ldots, S_l$. Let us denote by $\theta_{ij}=\theta_{ji}$ $(1 \leq i \neq j \leq l)$ the angle between the fundamental roots α_i, α_j. Also we denote by $\theta_{0i}=\theta_{i0}$ $(1 \leq i \leq l)$ the angle between α_i and $-\alpha_0$, where α_0 is the highest root of the root system Δ.

Proposition 3.4. — (i) $S_i^2=q.1+(q-1)S_i$ *for* $i=0, \ldots, l$.

$$\text{(ii)} \quad \begin{cases} S_iS_j=S_jS_i, & \text{if} \quad \theta_{ij}=\pi/2, \\ S_iS_jS_i=S_jS_iS_j, & \text{if} \quad \theta_{ij}=2\pi/3, \\ (S_iS_j)^2=(S_jS_i)^2, & \text{if} \quad \theta_{ij}=3\pi/4, \\ (S_iS_j)^3=(S_jS_i)^3, & \text{if} \quad \theta_{ij}=5\pi/6. \end{cases}$$

Proof. — (i) By Prop. 2.8, $B\omega(w_i)B\omega(w_i)B=B\omega(w_i)B \cup B$. Hence $S_i^2=\lambda.1+\mu.S_i$ with some positive integers λ, μ. Furthermore, λ, μ are given by

$$\lambda=|B\backslash B\omega(w_i)^{-1}B \cap B\omega(w_i)B|=|B\backslash B\omega(w_i)B|=q,$$
$$\mu=|B\backslash B\omega(w_i)^{-1}B\omega(w_i) \cap B\omega(w_i)B|.$$

However the value of μ is easily obtained by applying the homomorphism $\text{ind}: \mathcal{H}(G, B) \to \mathbf{Z}$ to the equality $S_i^2=\lambda.1+\mu.S_i$: we get $q^2=\lambda+\mu.q$. Since $\lambda=q$, we get $\mu=q-1$.

(ii) Let $\theta_{ij}=\pi/2$. Then $w_iw_j=w_jw_i$. Now if we can show that $\lambda(w_iw_j)=2$, then we have also $\lambda(w_jw_i)=2$. Thus w_iw_j, w_jw_i are both reduced expressions of some element $\sigma \in D'W$. Hence we get $S_\sigma=S_iS_j$ and $S_\sigma=S_jS_i$ by Th. 3.3. So let us prove that $\theta_{ij}=\pi/2$ implies $\lambda(w_iw_j)=2$. Firstly, we have $\lambda(w_iw_j)=l(w_iw_j)$ (Prop. 1.10), hence $\lambda(w_iw_j) \leq 2$. If $\lambda(w_iw_j)=0$, then we have $w_iw_j \in \Omega \cap D'W=\{1\}$, hence $w_i=w_j$, which contradicts $\theta_{ij}=\pi/2$. If $\lambda(w_iw_j)=1$, then we get a contradiction by Prop. 1.5. Thus we have $\lambda(w_iw_j)=2$.

Next let $\theta_{ij}=2\pi/3$. Then we get $w_iw_jw_i=w_jw_iw_j$ and by the same reason as above, it is enough to show that $\lambda(w_iw_jw_i)=3$ in order to prove that $S_iS_jS_i=S_jS_iS_j$. Firstly we obviously have $\lambda(w_iw_jw_i) \leq 3$. On the other hand, by $\theta_{ij}=2\pi/3$, we get (with the notation of § 1) that the hyperplanes P_i, $w_i(P_j)$, $w_iw_j(P_i)$ are all distinct. Then we have $\{P_i, w_i(P_j), w_iw_j(P_i)\} \subset \widetilde{\Delta}(w_iw_jw_i)$ by Cor. 1.4 and $\lambda(w_iw_jw_i) \leq 3$. Thus we have $\lambda(w_iw_jw_i)=3$, hence $S_iS_jS_i=S_jS_iS_j$. The remaining cases are also proved in a similar manner.

Theorem 3.5. — *Let $\mathfrak{F}$ be the free ring over* $\mathbf{Z}$ *generated by* $\Delta_0, \Delta_1, \ldots, \Delta_l$ *together with the unit element* 1. *Let* φ *be the ring homomorphism from* $\mathfrak{F}$ *onto* $\mathcal{H}(G'B, B)$ *defined by* $\varphi(\Delta_i)=S_i$ $(0 \leq i \leq l)$. *Then the kernel of* φ *coincides with the ideal* $\mathfrak{a}$ *of* $\mathfrak{F}$ *generated by the following elements*:

$$\begin{array}{ll} \Delta_i^2-(q.1+(q-1)\Delta_i) & (0 \leq i \leq l), \\ \Delta_i\Delta_j-\Delta_j\Delta_i & (\textit{for} \quad \theta_{ij}=\pi/2), \\ \Delta_i\Delta_j\Delta_i-\Delta_j\Delta_i\Delta_j & (\textit{for} \quad \theta_{ij}=2\pi/3), \\ (\Delta_i\Delta_j)^2-(\Delta_j\Delta_i)^2 & (\textit{for} \quad \theta_{ij}=3\pi/4), \\ (\Delta_i\Delta_j)^3-(\Delta_j\Delta_i)^3 & (\textit{for} \quad \theta_{ij}=5\pi/6). \end{array}$$

Proof. — We have $\mathfrak{a}\subset\mathrm{Ker}(\varphi)$ by Prop. 3.4. Thus φ induces a ring homomorphism $\overline{\varphi}$ from $\overline{\mathfrak{F}}=\mathfrak{F}/\mathfrak{a}$ onto $\mathscr{H}(\mathrm{G'B, B})$ such that $\overline{\varphi}(\overline{\Delta}_i)=\mathrm{S}_i$ $(0\leq i\leq l)$, where $\overline{\Delta}_i$ is the image of Δ_i under the canonical homomorphism $\mathfrak{F}\to\overline{\mathfrak{F}}$. The $\overline{\Delta}_i$ satisfy the relations (i), (ii) of Prop. 3.4 (replacing there each S_i by $\overline{\Delta}_i$ respectively). Now we have to show that $\overline{\varphi}$ is bijective. Let Θ be the set of all finite sequences $(i_1, i_2, \ldots, i_r)$ of integers $i_1, \ldots, i_r$ with $0\leq i_1, \ldots, i_r\leq l$. For each element σ in D'W let us choose a reduced expression $\sigma=w_{j_1}\ldots w_{j_s}$ of σ and denote by $\theta(\sigma)$ the element of Θ defined by

$$\theta(\sigma)=(j_1, \ldots, j_s).$$

Let $\Theta_0=\{\theta(\sigma);\ \sigma\in\mathrm{D'W}\}$. Now, for each $\theta\in\Theta$, let us denote by $\overline{\Delta}(\theta)$ the element of $\overline{\mathfrak{F}}$ defined by $\overline{\Delta}(\theta)=\overline{\Delta}_{i_1}\ldots\overline{\Delta}_{i_r}$ where $\theta=(i_1, \ldots, i_r)$, and by $\overline{\Delta}(\theta)=1$ if θ is empty. Let $\overline{\mathfrak{F}}_0$ be the submodule of $\overline{\mathfrak{F}}$ spanned by $\overline{\Delta}(\theta(\sigma))$, $\sigma\in\mathrm{D'W}$. Then we have $\overline{\varphi}(\overline{\Delta}(\theta(\sigma)))=\mathrm{S}_\sigma$. Since $\{\mathrm{S}_\sigma;\ \sigma\in\mathrm{D'W}\}$ form a base of the free $\mathbf{Z}$-module $\mathscr{H}(\mathrm{G'B, B})$, $\{\overline{\Delta}(\theta(\sigma));\ \sigma\in\mathrm{D'W}\}$ are linearly independent over $\mathbf{Z}$ and $\overline{\varphi}|\overline{\mathfrak{F}}_0$ is a bijective map from $\overline{\mathfrak{F}}_0$ onto $\mathscr{H}(\mathrm{G'B, B})$. Hence we shall get $\mathrm{Ker}(\varphi)=\mathfrak{a}$ if we can show that $\overline{\mathfrak{F}}=\overline{\mathfrak{F}}_0$. Therefore we claim that $\overline{\mathfrak{F}}_0$ is a subring of $\overline{\mathfrak{F}}$. (Then, since $1, \overline{\Delta}_0, \ldots, \overline{\Delta}_l\in\overline{\mathfrak{F}}_0$, we get immediately $\overline{\mathfrak{F}}=\overline{\mathfrak{F}}_0$). Thus we have only to show that $\overline{\Delta}(\theta(\sigma)).\overline{\Delta}(\theta(\tau))\in\overline{\mathfrak{F}}_0$ for any σ, $\tau\in\mathrm{D'W}$. However this will be the case if we have $\overline{\Delta}_i.\overline{\Delta}(\theta(\tau))\in\overline{\mathfrak{F}}_0$ for any i with $0\leq i\leq l$ and for any $\tau\in\mathrm{D'W}$. Let $\theta(\tau)=(j_1, \ldots, j_s)$. We distinguish two cases:

Case 1. — Suppose that $\lambda(w_i w_{j_1}\ldots w_{j_s})=s+1$. Then, by Prop. 1.15, we have $\overline{\Delta}(\theta(\sigma))=\overline{\Delta}_i\overline{\Delta}_{j_1}\ldots\overline{\Delta}_{j_s}$, where $\sigma=w_i w_{j_1}\ldots w_{j_s}$. Hence $\overline{\Delta}_i.\overline{\Delta}(\theta(\tau))\in\overline{\mathfrak{F}}_0$.

Case 2. — Suppose that $\lambda(w_i w_{j_1}\ldots w_{j_s})=s-1$. Then, by Cor. 1.11 and Lemma 1.5, there exists a reduced expression $w_{k_1}\ldots w_{k_s}$ of τ such that $i=k_1$. Then by Prop. 1.15, we have $\overline{\Delta}(\theta(\tau))=\overline{\Delta}_i\overline{\Delta}_{k_2}\ldots\overline{\Delta}_{k_s}$. Hence

$$\begin{aligned}\overline{\Delta}_i\overline{\Delta}(\theta(\tau))&=\overline{\Delta}_i^2(\overline{\Delta}_{k_2}\ldots\overline{\Delta}_{k_s})\\&=q\overline{\Delta}_{k_2}\ldots\overline{\Delta}_{k_s}+(q-1)\overline{\Delta}_i\overline{\Delta}_{k_2}\ldots\overline{\Delta}_{k_s}\\&=q\overline{\Delta}(\theta(\rho))+(q-1)\overline{\Delta}(\theta(\tau)),\end{aligned}$$

where $\rho=w_{k_2}\ldots w_{k_s}=w_i\tau$. Thus $\overline{\Delta}_i\overline{\Delta}(\theta(\tau))\in\overline{\mathfrak{F}}_0$, which completes the proof.

Corollary **3.6.** — *Let* $\sigma\in\mathrm{DW}$, $0\leq i\leq l$. *Then*

$$\begin{aligned}&\mathrm{S}_i\mathrm{S}_\sigma=q\mathrm{S}_{w_i\sigma}+(q-1)\mathrm{S}_\sigma, && \textit{if}\quad \lambda(w_i\sigma)<\lambda(\sigma),\\&\mathrm{S}_\sigma\mathrm{S}_i=q\mathrm{S}_{\sigma w_i}+(q-1)\mathrm{S}_\sigma, && \textit{if}\quad \lambda(\sigma w_i)<\lambda(\sigma),\\&\mathrm{S}_i\mathrm{S}_\sigma=\mathrm{S}_{w_i\sigma}, && \textit{if}\quad \lambda(w_i\sigma)>\lambda(\sigma),\\&\mathrm{S}_\sigma\mathrm{S}_i=\mathrm{S}_{\sigma w_i}, && \textit{if}\quad \lambda(\sigma w_i)>\lambda(\sigma).\end{aligned}$$

Proof. — This is obvious from above if $\sigma\in\mathrm{D'W}$. When $\sigma\in\mathrm{DW}$, let $\sigma=\tau\rho$, $\tau\in\mathrm{D'W}$, $\rho\in\Omega$. Then, by $\mathrm{S}_\sigma=\mathrm{S}_\tau\mathrm{S}_\rho$, we get the desired formulas easily.

Now by Th. 3.5, the defining relations for the generators $\mathrm{S}_0, \ldots, \mathrm{S}_l$ of $\mathscr{H}(\mathrm{G'B, B})$ are given. Thus the structure of $\mathscr{H}(\mathrm{G'B, B})$ is determined only by the structures of $\mathfrak{g}_{\mathbf{C}}$ and $k=\mathfrak{D}/\mathfrak{P}$. Hence, for example, $\mathscr{H}(\mathrm{G'B, B})\simeq\mathscr{H}(\overline{\mathrm{G}}'\overline{\mathrm{B}}, \overline{\mathrm{B}})$ where bar means the corresponding groups for the Chevalley group associated with $\mathfrak{g}_{\mathbf{C}}$ over the completion $\overline{\mathrm{K}}$ of K.

It is almost obvious that for the Hecke ring $\mathscr{H}(\mathrm{G}', \mathrm{B}')$, Th. 3.5 is also true, and in fact, it is shown quite analogously using the properties of G′, B′ in § 2. More precisely we shall give the following proposition. (We may omit the proof.)

Proposition **3.7.** — *Let* S'_σ *denote the double coset* $\mathrm{B}'\omega(\sigma)\mathrm{B}'$ $(\sigma \in \mathrm{D}'\mathrm{W})$ *regarded as an element of* $\mathscr{H}(\mathrm{G}', \mathrm{B}')$. *Then* $\mathrm{ind}(\mathrm{S}'_\sigma) = q^{\lambda(\sigma)}$. *If* $\sigma = w_{i_1} \ldots w_{i_r}$ *is a reduced expression of* $\sigma \in \mathrm{D}'\mathrm{W}$, *then* $\mathrm{S}'_\sigma = \mathrm{S}'_{i_1} \ldots \mathrm{S}'_{i_r}$ *where* $\mathrm{S}'_i = \mathrm{S}'_{w_i}$. $\mathscr{H}(\mathrm{G}', \mathrm{B}')$ *is isomorphic to* $\mathscr{H}(\mathrm{G}'\mathrm{B}, \mathrm{B})$ *by the map* $\mathrm{S}'_\sigma \to \mathrm{S}_\sigma$ $(\sigma \in \mathrm{D}'\mathrm{W})$.

Now let us consider $\mathscr{H}(\mathrm{G}, \mathrm{B})$. Let $\mathbf{Z}[\Omega]$ be the integral group ring of Ω. Then it is easy to see that $\rho \to \mathrm{S}_\rho$ $(\rho \in \Omega)$ defines an injective ring homomorphism from $\mathbf{Z}[\Omega]$ into $\mathscr{H}(\mathrm{G}, \mathrm{B})$ since $\mathrm{S}_\rho \mathrm{S}_\tau = \mathrm{S}_{\rho\tau}$ for any $\rho \in \Omega$, $\tau \in \mathrm{DW}$. We shall identify the ring $\mathbf{Z}[\Omega]$ with its image $\mathscr{H}(\mathrm{N}(\mathrm{B}), \mathrm{B})$ in $\mathscr{H}(\mathrm{G}, \mathrm{B})$. Now by $\Omega(\mathrm{D}'\mathrm{W}) = \mathrm{DW}$ and $\Omega \cap \mathrm{D}'\mathrm{W} = \{1\}$, $\mathscr{H}(\mathrm{G}, \mathrm{B})$ is identified as $\mathbf{Z}$-module with the tensor product $\mathscr{H}(\mathrm{N}(\mathrm{B}), \mathrm{B}) \otimes_{\mathbf{Z}} \mathscr{H}(\mathrm{G}'\mathrm{B}, \mathrm{B}) = \mathbf{Z}[\Omega] \otimes_{\mathbf{Z}} \mathscr{H}(\mathrm{G}'\mathrm{B}, \mathrm{B})$ by $\rho \otimes \mathrm{S}_\sigma = \mathrm{S}_\rho \mathrm{S}_\sigma$ $(\rho \in \Omega, \sigma \in \mathrm{D}'\mathrm{W})$. Now for any $\rho \in \Omega$, S_ρ is invertible in $\mathscr{H}(\mathrm{G}, \mathrm{B})$: $\mathrm{S}_\rho \mathrm{S}_{\rho^{-1}} = \mathrm{S}_{\rho^{-1}} \mathrm{S}_\rho = 1$. Hence Ω acts on $\mathscr{H}(\mathrm{G}'\mathrm{B}, \mathrm{B})$ as an automorphism group through the setting $\rho(\mathrm{S}_\sigma) = \mathrm{S}_\rho \mathrm{S}_\sigma \mathrm{S}_\rho^{-1} = \mathrm{S}_{\rho\sigma\rho^{-1}}$ $(\rho \in \Omega, \sigma \in \mathrm{D}'\mathrm{W})$. Thus the multiplication law in the tensor product $\mathbf{Z}[\Omega] \otimes_{\mathbf{Z}} \mathscr{H}(\mathrm{G}'\mathrm{B}, \mathrm{B})$ is given by

$$(\rho \otimes \mathrm{S}_\sigma) \, . \, (\rho' \otimes \mathrm{S}_{\sigma'}) = \rho\rho' \otimes' \rho^{-1}(\mathrm{S}_\sigma)\mathrm{S}_{\sigma'}$$

for any $\rho, \rho' \in \Omega$ and $\sigma, \sigma' \in \mathrm{D}'\mathrm{W}$.

Let us call in general such a ring structure of $\mathbf{Z}[\Gamma] \otimes_{\mathbf{Z}} \mathfrak{R}$, where $\mathfrak{R}$ is a ring over $\mathbf{Z}$ and Γ is a group acting on $\mathfrak{R}$ as an automorphism group, the *twisted tensor product* and denote by $\mathbf{Z}[\Gamma] \widetilde{\otimes}_{\mathbf{Z}} \mathfrak{R}$ the ring thus obtained Then we have the following proposition by what we have observed above:

Proposition **3.8.** — $\mathscr{H}(\mathrm{G}, \mathrm{B}) = \mathbf{Z}[\Omega] \widetilde{\otimes}_{\mathbf{Z}} \mathscr{H}(\mathrm{G}', \mathrm{B}')$.

For example, if $\mathfrak{g}_{\mathbf{C}}$ is of type (A_l), then $\mathscr{H}(\mathrm{G}, \mathrm{B})$ is generated by $1, \rho, \mathrm{S}_0, \ldots, \mathrm{S}_l$ together with the following defining relations:

$$\begin{aligned}
&\rho^{l+1} = 1, \ \rho \mathrm{S}_i \rho^{-1} = \mathrm{S}_{i+1} && (0 \leq i \leq l; \ \mathrm{S}_{l+1} = \mathrm{S}_0), \\
&\mathrm{S}_i^2 = q \, . \, 1 + (q-1)\mathrm{S}_i, && (0 \leq i \leq l). \\
&\mathrm{S}_i \mathrm{S}_j \mathrm{S}_i = \mathrm{S}_j \mathrm{S}_i \mathrm{S}_j, && \text{if} \quad j \equiv i \pm 1 \ (\mathrm{mod.}\ l+1), \\
&\mathrm{S}_i \mathrm{S}_j = \mathrm{S}_j \mathrm{S}_i, && \text{if} \quad j \not\equiv i + 1 \ (\mathrm{mod.}\ l+1).
\end{aligned}$$

For the other complex simple Lie algebras, similar relations are easily obtained by considering the extended Dynkin diagram (with $-\alpha_0$ attached) and the action of Ω on $\mathrm{J} = \{w_0, \ldots, w_l\}$ (cf. § 1.8).

3.2. As an application of Th. 3.5 and Prop. 3.8, we see that $\mathrm{S}_i \to -1$ $(0 \leq i \leq l)$ $\mathrm{S}_\rho \to 1$ $(\rho \in \Omega)$ can be extended uniquely to a homomorphism from $\mathscr{H}(\mathrm{G}, \mathrm{B})$ into $\mathbf{Z}$. We shall denote this homomorphism by sgn. Then, as in [10, § 5] an involutive automorphism $\xi \to \hat{\xi}$ of $\mathscr{H}(G, B)$ is defined by

$$\hat{S}_i = (q-1) . 1 - S_i \qquad (0 \leq i \leq l)$$

which satisfies the following properties:

(i) S_i is invertible in $\mathscr{H}_{\mathbf{Q}}(G, B) = \mathscr{H}(G, B) \underset{\mathbf{Z}}{\otimes} \mathbf{Q}$ and $S_i^{-1} = \frac{1}{q}(S_i(q-1) . 1)$. Then every S_σ ($\sigma \in DW$) is also invertible in $\mathscr{H}_{\mathbf{Q}}(G, B)$ and we have

$$\hat{S}_\sigma = \operatorname{sgn}(S_\sigma) . \operatorname{ind}(S_\sigma) S_\sigma^{-1}$$

(ii) $\operatorname{ind}(\hat{\xi}) = \operatorname{sgn}(\xi)$, $\operatorname{sgn}(\hat{\xi}) = \operatorname{ind}(\xi)$ for any $\xi \in \mathscr{H}(G, B)$.

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Added in Proof. For an abstract approach to Prop. 1. 15 and its consequences see

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