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Corrections to "Differential topology from the point of view of simple homotopy theory" and further remarks

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CORRECTIONS TO :

DIFFERENTIAL TOPOLOGY FROM THE POINT OF VIEW OF SIMPLE HOMOTOPY THEORY AND FURTHER REMARKS

by BARRY MAZUR (1)

§ 1. Corrections to : Differential Topology from the Point of View of Simple Homotopy Theory and further remarks.

The purpose of this note is to atone for some of the sins committed in [1].

Namely: *Chapters III and IV:*

Lemma 3.2 page 19 is false, and even if it were true it would be a bad idea to pass to such equivalence classes of cell decompositions. In fact, the notion of equivalence given on page 18 is unnatural.

In expiation, I gave (hopefully!) the right definition of equivalence in [2]. It is that definition (Definition 11, sec. 5) which meshes well with the proofs of [1]. As a consequence, one must also modify the definition of neighborhood of chapter IV. This is done in sec. 8 of [2], and the terminology is changed from ". neighborhood " to " solid ", which is really more appropriate.

Here is a sketch of the new definitions:

1. D-*isotopy* or *Isotopy of differentiable cell decompositions:*

An object of the form

$$X = \{(X_0, \ldots, X_\nu);\ \varphi_i, i = 1, \ldots, \nu\}$$

and projection maps

$$\pi_i : X_i \to I \qquad (i = 0, \ldots, \nu)$$

where $X_i = X_{i-1} \cup_{\varphi_i} D^{m_i} \times D^{m-m_i} \times I$ and the φ_i are differentiable imbeddings such that

$$\begin{array}{ccc} \partial D^{m_i} \times D^{m-m_i} \times I & \xrightarrow{\varphi_i} & \partial X_{i-1} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_i} & \\ & I & \end{array}$$

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is commutative. (Thus X is a one-parameter family of cell decompositions). Restricting everything to any $t \in I$ we obtain a differentiable cell decomposition

$$X_t = \{(\pi_0^{-1}\{t\}, \pi_1^{-1}\{t\}, \ldots, \pi_\nu^{-1}\{t\}) ; \ \varphi_i | \pi^{-1}\{t\})\}.$$

Two cell decompositions X_0, X_1 linked by an isotopy will be called *isotopic.*

2. *Expansion Equivalence.* — Two cell decompositions M_0, M_1 are *expansion equivalent* if there are cell decompositions

$$M_i' = M_i \cup \text{irrelevant additions}$$
$$(i = 0, 1)$$

such that M_0' is isomorphic with M_1'.

3. D-*equivalence:* The equivalence relation generated by

a) isotopy;

b) re-ordering equivalence;

c) expansion equivalence.

Any D-equivalence gives rise to a unique isotopy class of differentiable isomorphisms which will again be called D-equivalences. (They are *free*: i.e. they do not preserve the decomposition, of course.)

2. Structure weakening: Consider the categories:

C_0 : differentiable manifolds (with boundary); differential imbeddings.

C_H : topological spaces; homotopy classes of continuous maps.

$$(\text{In } C_H \text{ set } \partial^* X = X).$$

Then we may form cell decompositions in each of the categories. A differentiable cell decomposition has already been defined; a cell decomposition of C_H is defined similarly where the maps are of C_H and ∂ is replaced by ∂^*. Thus we obtain two categories of cell decompositions (maps are inclusions), and a structure-weakening functor:

$$\rho : D_0 \to D_H.$$

Any notion in D_0 has its weak counterpart in D_H.

3. Let $X \in D_H$. An *n-dimensional solid over* X (*in* D_0) is a pair (M, π), where $M \in D_0$, $\pi : \rho M \to X$ is a D_H-equivalence.

An *isomorphism*

$$\gamma : (M_1, \pi_1) \underset{\approx}{\rightarrow} (M_2, \pi_2)$$

between two M-solids is a D-equivalence

$$\gamma : M_1 \dashrightarrow M_2$$

giving rise to the diagram

$$\begin{array}{ccc} \rho M_1 & \overset{\rho\gamma}{\dashrightarrow} & \rho M_2 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & X & \end{array}$$

(commutative up to C_H-isotopy). (Broken-line arrows mean that they do not preserve decompositions.)

Let $N^m(X)$ denote the set of isomorphism classes of m-solids over X.

With the understanding that we are always studying cell decompositions, up to this equivalence, we need never deal with " cell-filtrations ", that horror defined in [1]. (The reader should therefore skip Chapters III, IV and, rather, consult [2].)

Chapter VII:

In chapter VII, page 37 line 2, change the hypotheses to read $m>2$, $n>2$. Otherwise, for $n=0, 1, 2$, the existence of homotopy-isolation data would not insure the existence of isolation data. Consequently, one needs the extra hypothesis $n>2$ throughout chapter VII.

Chapter VIII:

(i) The key geometric result of the theory is lemma 8.3. Since the hypotheses of chapter VII have been strengthened, and all our definitions have been changed, we must take up the proof of this lemma again. There are a few things to notice. Namely, after our new definitions, we do not have the contravariant map i^k, and therefore the statement of the lemma must be changed; and with the new hypotheses for Chapter VII, we must exclude the case $k=5$ which therefore remains unsolved. Finally, in this corrected proof, a gap in the old one (pointed out to us by C. Zeeman) will be filled.

(To obtain a suitable notion of the essentiality of this lemma for our theory, one should notice that it, coupled with Prop. 5.4 of page 30, yields the nonstable neighborhood theorem for $k \geq \dim K + 5$ immediately.)

Let

$$i : K \to K^* = K \cup_{\bar{\varphi}} \Delta^m \cup_{\bar{\psi}} \Delta^{m+1}$$

be an elementary expansion, and let

$$i_{(k)} : N^k(K) \to N^k(K^*)$$

be the " irrelevant addition " map defined by lemma 8.1, p. 43.

Lemma (new 8.3). — *If*

(1) $k \geq \max\{\dim K + 1, M + 4\}$,

(2) $k>5$,

then i_k is a bijective isomorphism.

Proof. — *a)* If $i_k M_0$ is equivalent to $i_k M_1$, by definition, M_0 and M_1 are expansion-equivalent to equivalent cell decompositions. Hence they themselves are equivalent, and i_k is injective.

b) Let us show that i_k is surjective. [I assert on line 12, p. 45, that since

$$N \in \mathcal{N}^k(K^*), \quad N = (M_0 \cup_\varphi D^m \times D^{k-m}) \cup_\psi D^{m+1} \times D^{k-m-1}.$$

That needs proof. Therefore, to begin:]

Lemma 1. — *Assume:* (α) $k>m+2$; (β) $k>5$. *Then, if* $N\in\mathcal{N}^k(K^*)$, N *can be written (up to equivalence) as:*

$$N=(M_0\cup_\varphi D^m\times D^{k-m})\cup_\psi D^{m+1}\times D^{k-m-1}.$$

Assume Lemma 1 for the moment. Let us prove $i_{(k)}$ surjective in three cases:

I) *The case* $m>2$:

Then the techniques of chapter VII apply, and the argument of p. 45, 46 yield $N\approx i_{(k)}M_0$ (p. 46, line 10).

II) *The cases* $m=0, 1$:

Trivial for dimensional reasons.

III) *The case* $m=2$, $k\geq 6$:

Then (after Lemma 1) $N=(M_0^k\cup_\varphi D^2\times D^q)\cup_\psi D^3\times D^{q-1}$ where $k-2=q\geq 4$. Let $\pi=\pi_1(M_0)\approx\pi_1(\partial M_0)$. The map

$$\bar{\varphi}: S^1\rightarrow\partial M_0$$

is null-homotopic since it is null homotopic in M_0. Since $\dim\partial M_0\geq 5$, $\bar{\varphi}$ is an unknotted imbedding. Let $M_1=M_0\cup_\varphi D^2\times D^q$.

Consider the natural maps

$$\partial M_0\vee S^2\xleftarrow{\supset}(\partial M_0-\mathrm{im}\ \varphi)\vee S^2\xleftarrow[h]{}(\partial M_0-\mathrm{im}\ \varphi)\cup_\varphi(D^2\times\{x\})\subseteq\partial M_1$$

where $x\in\partial D^q$, and h is a homotopy equivalence which is the identity on $\partial M_0-\mathrm{im}\ \varphi$, and a map of degree $+1$ from $D^2\times\{x\}$ to S^2. Then these maps are all isomorphisms for π_2, and we obtain the following commutative diagram:

$$\begin{array}{ccc}\pi_2(\partial M_1) & \approx & \mathbf{Z}[\pi]\oplus\pi_2(\partial M_0)\\ \Big\downarrow{\scriptstyle i_1} & & \Big\downarrow{\scriptstyle 1\oplus i_2}\\ \pi_2(M_1) & \approx & \mathbf{Z}[\pi]\oplus\pi_2(M_0)\end{array}$$

($\mathbf{Z}[\pi]$ is the integral group ring of π, regarded as an abelian group). The vertical maps are the natural ones.

Let $S\in\pi_2(\partial M_1)$ denote the homotopy class of $\bar{\psi}$. Then we have

$$i_1(S)=1.I\in\mathbf{Z}[\pi]\subset\mathbf{Z}[\pi]\oplus\pi_2(M_0).$$

Consequently: (*) $S=1.I\oplus x$ for $x\in\pi_2(\partial M_0)$.

Let $f:\partial D^3\rightarrow\partial M_1$ be a differentiable map representing the homotopy class S such that $f(\partial D^3)$ intersects the pole $\{0\}\times\partial D^q\subset\partial M_1$ exactly at one point p, and transversally at p. This is possible after (*). Since $\dim\partial M_1\geq 5$, f may be approximated by an imbedding g. If the approximation is sufficiently *close*, we may be sure that g has exactly one polar intersection also, which is transversal. Since g is homotopic to $\overline{\psi}$, Theorem 2 of § 2 below applies, yielding the following result: $\overline{\varphi}$ is differentiably isotopic to an imbedding $\overline{\varphi}$ which has exactly one transversal polar intersection. Thus the arguments of p. 45, 46 again apply.

Proof of lemma 1. — We may take N properly ordered,

$$N = M_0 \cup_\varphi (D^m \times D^{k-m}) \bigcup_{j=1}^{\nu} (D^m \times D^{k-m})_j \bigcup_{i=1}^{\mu} (D^{m+1} \times D^{k-m-1})_j \cup_\psi D^{m+1} \times D^{k-m-1} \qquad \text{(etc.)}$$

where (etc.) refers to the remaining handles. The handles $(D^m \times D^{k-m})_j$ may be removed from the vicinity of $(D^m \times D^{k-m}) \subset M_0 \cup D^m \times D^{k-m}$ by differentiable isotopy. Therefore we may reorder the attaching " $\cup_\psi D^m \times D^{k-m}$ " to come after all the $(D^m \times D^{k-m})_j$, which we may now " lump " into the M_0, and write:

$$N = M_0 \cup_\varphi D^m \times D^{k-m} \cup$$
$$(\cup_{f_1} D^{m+1} \times D^{k-m-1}_{(1)} \cup_{f_2} \dots \cup_{f_m} D^{m+1} \times D^{k-m-1}_{(m)}) \cup_\psi D^{m+1} \times D^{k-m-1} \qquad \text{(etc.)}$$

We may also regularize the f_j by isotopy so that

$$(*) \qquad f_j(\partial D^{m+1} \times D^{k-m-1}_{(j)}) \cap D^m \times \partial D^{k-m} \subset D^m \times \partial D^{k-m}_{-}$$

(where $\partial D^r_\pm \subset \partial D^r$ refers to the upper or lower hemisphere in ∂D^r).

Since K^* is an elementary expansion of K, we have that there is a continuous map,

$$\partial D^{m+1} \times I \xrightarrow{f} N \qquad \text{such that}$$

$$\text{(i)} \qquad f \mid \partial D^{m+1} \times 0 = \overline{\psi}$$

$$\text{(ii)} \qquad f(\partial D^{m+1} \times 1) \cap D^m \times D^{k-m} = D^m \times p$$

for some $p \in \partial D^{k-m}$ (which we may take in ∂D^{k-m}_{+}).

Lemma 2. — *After a differentiable isotopy* (P_t) *of* N, *we may find a continuous map* f *satisfying* (i), (ii) *above* (*for* $\psi = P_1 \psi$) *such that*

$$(**) \qquad f(\partial D^{m+1} \times I) \cap 0 \times D^{k-m-1}_{(j)} = \emptyset \qquad (j = 1, \dots, m).$$

For simplicity, denote $P_j = \{0\} \times D^{k-m-1}_{(j)}$ (the j^{th} " pole "). Call an element in $f(\partial D^{m+1} \times I) \cap P_j$ a " polar intersection ".

Assume Lemma 2, for the moment. If (**) is true, then by an " expansion isotopy " centered at the pole P_j, we may obtain:

$$(***) \qquad f(\partial D^{m+1} \times I) \cap D^{m+1} \times D^{k-m-1}_{(j)} = \emptyset.$$

Since $f \mid \partial D^{m+1} \times \{0\} = \overline{\psi}$, we may then arrange

$$\psi(\partial D^{m+1} \times D^{k-m-1}) \cap D^{m+1} \times D^{k-m-1}_{(j)} = \emptyset.$$

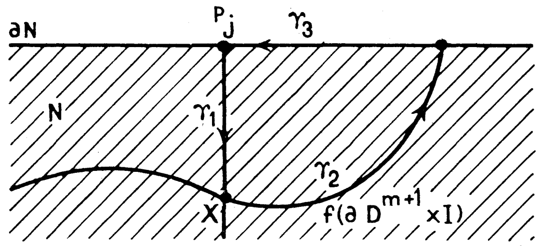
Then we may reorder N, having the " ψ "-handle glued prior to the " f_j "-handles. Moreover (***) assures us that we have a map

$$f : \partial D^{m+1} \times I \to M_0 \cup_\varphi D^m \times D^{k-m} \cup_\psi D^{m+1} \times D^{k-m-1}$$

This means that we finally find ourselves in the situation I blithely took to be the " given " in my original proof.

Proof of Lemma 2. — By choosing f in general position we may suppose the intersection $f(\partial D^{m+1}\times I)\cap P_j$ transversal. Therefore there are only a finite number of polar intersections. Let us " remove " them, one at a time

Removal of a Polar intersection $\{x\}$:



Choose nonsingular paths:

$\gamma_1\cdots$	From ∂P_j to x	along P_j
$\gamma_2\cdots$	From x to $f(\partial D^{m+1}\times\{0\})$	along $f(\partial D^{m-1}\times I)$
$\gamma_3\cdots$	From $\gamma_2(1)$ to $\gamma_1(0)$	along ∂N

with these properties:

1. γ_1 should contain no other polar intersections (possible since $\dim P_j > 1$ by (α)).

2. γ_2 should be disjoint from the image of the singular set $S\subset\partial D^{m+1}\times I$ under the map f (possible since $\dim S \leq 2(m+1)-k \leq (m+1)-2$ by (α) and therefore S cannot separate x from $\partial D^{m+1}\times\{0\}$).

3. The path γ_3 should be extendable to a path defined for $0\leq t\leq 2$ and should be so that the circuit $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3$ is null-homotopic in N. (Possible since $\pi_1(N, \partial N)=0$ by hypotheses (1), (2) of the main lemma).

Let $\sigma\subset N$ be a nonsingular 2-disc whose boundary is γ (possible by (β)). Orient everything in a neighborhood of $\sigma\subset N$. Let $P_t : 0\leq t\leq 2$ be a differentiable isotopy of (N, ∂N) possessing these properties:

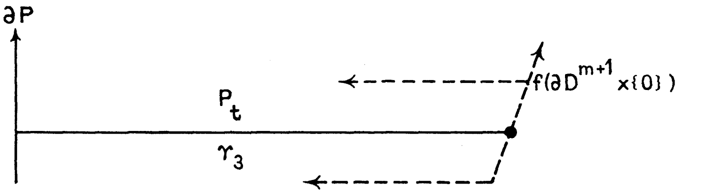
(i) $P_0=1$.

(ii) P_t has support in some small neighborhood of γ_3.

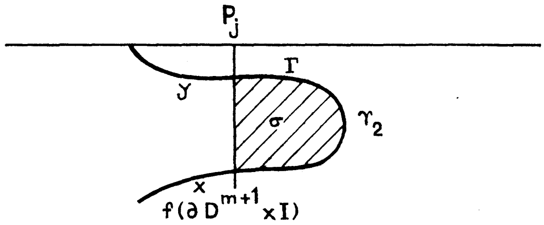
(iii) $P_t(\gamma_3(0))=\gamma_3(t)$, $0\leq t\leq 2$.

(iv) The intersection of $A=P_1 f(\partial D^{m+1}\times I)$ and ∂P_j in ∂N at $\gamma_3(1)$ is transversal.

Now up to isotopy there are precisely *two* such P_t's (*since* ∂P_j *has positive dimension* by (α)), corresponding to plus and minus intersection indices between A and ∂P_j at $\gamma_3(1)$.



Choose that P_t which yields an intersection index different from the intersection index of P_j and $f(\partial D^{m+1} \times I)$ at x. After that isotopy P_t, we have introduced a new polar intersection y.



It has an opposite index to x, and σ yields " homotopy-isolation data ".

Now we may replace f by a map g which differs from f only in a small neighborhood of Γ, the segment between $f^{-1}(x)$ and $f^{-1}(y)$ in $f^{-1}\gamma_2$ such that

a) $f=g$ on $\partial D^{m+1} \times \partial I$ (therefore g obeys (i), (ii));

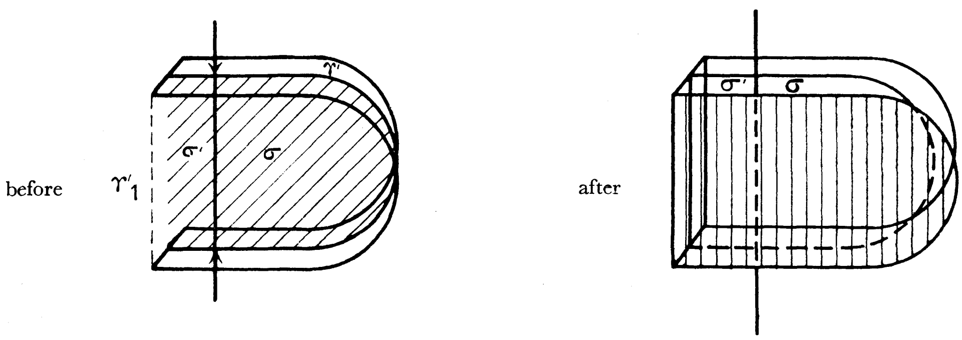
b) g has precisely two fewer polar intersections than f (namely x and y).

One constructs g by means of σ, but first we make a few remarks about how σ meets $f(\partial D^{m+1} \times I)$ at Γ. Again by (α), if σ is chosen in " general position with respect to $f(\partial D^{m+1} \times I)$ ", the normal directions along from Γ will be transverse to $f(\partial D^{m+1} \times I)$. (Remark: int σ *may* intersect $f(\partial D^{m+1} \times I)$ at a finite number of unavoidable points.) Now we shall modify f in a neighborhood R of $f^{-1}\Gamma$ " guided by σ ".

Let σ' be an enlargement of σ, as visualized below. Let $\partial\sigma' = \gamma_1' \cup \gamma'$. Let $\gamma_1 \times Q \subset P$ be a tubular neighborhood of γ_1 in P, and choose some product coordination $\sigma' \times Q$ of a neighborhood of σ' so that the image of f in a neighborhood of γ' may be described as $\gamma' \times Q = f(R)$. As indicated below, modify f as follows:

(i) $g=f$ outside R.

(ii) $g(R) = \gamma_1' \times Q \cup \sigma' \times \partial Q$.



Notice that this g may have more *self*-intersections than f because of the parenthetical remark above. The important thing is that g satisfies *a)*, *b)* above, and is continuous.

(ii) Since we have included a new hypothesis (i.e. (2): $k>5$) in lemma 8.3, we must include a new hypothesis: $k>5$ in the nonstable neighborhood theorem (p. 47).

§ 2. Low Dimensional Intersections:

Let A, B be compact differentiable manifolds, and f_0, $f_1 : A \to B$ differentiable imbeddings.

Consider the following weakening of the ordinary notion of differentiable isotopy:

Definition 1. — Let $L \subset A$, $K \subset B$ be finite subcomplexes. Then f_0 *is congruent to* f_1 *mod*$(L \to K)$ if:

Given any regular neighborhood of K, $M \subset B$, there exists a regular neighborhood of L, $N \subset A$, and a differentiable isotopy $\varphi_t : A \to B$ such that

a) $\varphi_0 = f_0$;

b) $\varphi_1(N) \subset M$;

c) $\varphi_1 | A - N = f_1 | A - N$.

This is a weakening of differentiable isotopy and taking $L = \emptyset$, $K = \emptyset$ one gets exactly differentiable isotopy.

Definition 2. — f_0 *is congruent to* f_1 *mod* q (written: $f_0 \equiv f_1 (\text{mod } q)$) for $q \geq 0$ an integer, if there are complexes K^q, L^{q-1} (of dimension q, $q-1$ respectively) such that $f_0 \equiv f_1 \text{ mod}(L^{q-1} \to K^q)$.

(Is this an equivalence relation? We have introduced this notion to obtain the following theorem:)

Theorem 1. — *Assume* $\dim A = 2$, $\dim B = 5$, *and that* f_0 *is homotopic to* f_1. *Then*

$$f_0 \equiv f_1 (\text{mod } 1)$$

Proof. — In this range of dimensions, if f_0 and f_1 are homotopic, then they are regularly homotopic.

Let f_t $(0 \leq t \leq 1)$ be the regular homotopy, $f : A \times I \to B$. Then we may assume that there are exactly a finite number of points $p_1, \ldots, p_{2n}$ at which f fails to be a differentiable isotopy, and the immersions f_t possess only double points. Consider a pair of double points $\{(p_1, t_0)(p_2, t_0)\}$ and for simplicity of notation assume this to be the only pair. (Our proof works as well in general.) Set $P_j = \{p_j\} \times [0, t_0] \subset A \times I$ for $j = 1, 2$. Find f', a C^1-approximation to f which is equal to f except in a small neighborhood of $P_1 \cup P_2$, and such that:

a) $f' | P_j$ is a differentiable imbedding possessing a nonsingular jacobian for $j = 1, 2$.

b) $f'(p_1, t_0) = f'(P_1) \cap f'(P_2) = f'(p_2, t_0)$.

To obtain f' a differentiable imbedding on P_j is easy. To insure that it have a nonsingular jacobian involves a slight calculation: If $G_{n,m}$ is the Grassman manifold of n-planes in m-space, then $\dim G_{2,5}=6$, $\dim G_{1,4}=3$. Consequently a path of 2-planes in 5-space may be C^1 approximated by one such that no 2-plane of that path contains a given line. To obtain *b)* is also easy.

Then f' will be, again, a regular homotopy with only one pair of singular points: (p_1, t_0), (p_2, t_0). Let $S_j = D_j^2(\varepsilon)\times[0, t_0-\delta]$ $(j=1, 2)$ be tubular neighborhoods of P_j, small enough so that f is a differentiable imbedding on S_j (possible by implicit function theorem, since the jacobian of f is nonsingular on P_j).

Now modify f' to f'' which is C^1-close, has all the nice properties of f' and the further property:

$f''(A\times I - \text{int}\, S_1 \cup S_2)$ does not intersect the lines $f''(P_j)$. (Possible since $3+1<5$.)

By compactness there are differentiable tubular neighborhoods R_j of $f''(P_j)$ such that $f''(A\times I - \text{int}\{S_1\cup S_2\})$ does not intersect the R_j.

Since f'' is a differentiable imbedding on S_j we may cut R_j, S_j down to smaller tubular neighborhoods $R'_j \subset R_j$, $S'_j \subset S_j$ where $S'_j = D_j^2(\varepsilon')\times[0, t_1]$, $t_1 = t_0-\delta'$, which are adapted to one another in the following sense:

$$f''(S'_j, \partial S'_j)\subset(R'_j, \partial R'_j) \qquad j=1, 2.$$

To do this, a suitable version of the tubular neighborhoods lemma must be used.

We may conclude:

(*) $$f''(A\times I - S'_1\cup S'_2) \quad \text{does not intersect } R'_1\cup R'_2.$$

Notice:

1. f''_t is a differentiable isotopy as t ranges in $[t_1, 1]$ $(t_1 = t_0+\delta')$.

For simplicity of notation, set

$$D' = D_1^2(\varepsilon')\cup D_2^2(\varepsilon'); \qquad R' = R'_1\cup R'_2.$$

Then:

2. $f''_t : (A - \text{int}\, D', \partial D') \to (B - \text{int}\, R', \partial R')$

is a differentiable isotopy for $t\in[0, t_1]$.

3. $$f''_t : (D', \partial D') \to (R', \partial R')$$

is a regular homotopy, which is a differentiable on $\partial D'$ and a differentiable imbedding for $t=t_1$.

After (3) we may apply the isotopy extension theorem (relative version) to obtain a differentiable isotopy

$$g_t : (D', \partial D') \to (R', \partial R') \qquad t\in[0, t_1]$$

such that

a) $$g_{t_1} = f''_{t_1}$$

b) $$g_t \mid \partial D' = f''_t \mid \partial D'$$

Now set $h'_t : A \to B$ to be the (not yet differentiable) isotopy $0 \leq t \leq 1$ given by:

a) $h'_t(a) = f'_t(a)$ if $a \notin D$ or $t \geq t_1$

b) $h'_t(a) = g_t(a)$ if $a \in D'$, $t \leq t_1$.

This isotopy is not yet differentiable at $\partial D' \times [0, t_1]$, but it may be smoothed. Let h_t be a differentiable isotopy which is C^0-close to h'_t and C^1-close except in a small neighborhood about $\partial D'$.

Set $K^1 = f''(P_1) \cup f''(P_2)$. Then R' could have been chosen small enough so as to be contained in a regular neighborhood of K^1.

The final differentiable isotopy h_t is C^1-close to the original f_t except in some small neighborhood of D' (i.e. some small neighborhood of $L^0 = \{p_1\} \cup \{p_2\} \subset A$) and $h_1(D') \subset R'$. Consequently h_t may be approximated by a differentiable isotopy φ which is a congruence $(\mathrm{mod}[L : K])$ between f_0 and f_1. Thus theorem 1 is proved. It will be used in the following application:

Let $Y^3 \subset Z^5$ be a compact submanifold. Let $f, g : X^2 \to Z^5$ be homotopic imbeddings of the compact differentiable 2-manifold X^2 in Z^5.

Theorem 2. — *Suppose* $g(X^2)$ *meets* Y^3 *transversally at k points* $(0 \leq k < +\infty)$. *Then there is a differentiable isotopy* $f_t : X^2 \to Z^5$ *such that* $f_0 = f$ *and* $f_1(X^2)$ *meets* Y^3 *transversally at exactly k points.*

Proof. — By theorem 1, $f \equiv g \pmod 1$. Thus $f \equiv g \,\mathrm{mod}(L^0 \to K^1)$. We may assume first that L^0 does not intersect $g^{-1}(g(X^2) \cap Y^3)$ since these are both zero-dimensional sets which may be moved about by differentiable isotopy. We may also assume that K^1 doesn't intersect Y^3, after a slight C^1-perturbation of Y^3, say. Let M, N be regular neighborhoods K^1, L^0 respectively such that

a) $M \cap Y^3 = \emptyset$

b) $N \cap g^{-1}(g(X) \cap Y) = \emptyset$.

Applying theorem 1 we obtain a differentiable isotopy f_t such that

(i) $f_0 = f$;

(ii) $f_1(N) \subset M$;

(iii) $f_1 | (X - N) = g | (X - N)$.

We obtain the following string of equalities:

$$f_1(X) \cap Y = f_1(X - N) \cap Y = g(X - N) \cap Y = g(X) \cap Y$$

(the first because $f_1(N) \cap Y = \emptyset$, after (ii) and *a)*; the second after (iii); the third because $g(N) \cap Y = \emptyset$, after *b)*).

Theorem 2 is therefore proved.

§ 3. **Correction to : Definition of Equivalence of Combinatorial Imbeddings.**

Let me take this opportunity to warn the reader of an error in [3]. Namely: p. 11, condition (iii) in § 9 is impossible to obtain in general. Rather, one gets a union of intervals. The proof of the main theorem, however, can still be carried out. One should do it in a more direct way, however. The function spaces introduced in § 15 are unnecessary.

The reader is referred to the recent I.H.E.S. seminar of C. Zeeman for a complete theory of combinatorial isotopy, which makes [3] unnecessary.

REFERENCES

[1] B. MAZUR, *Differential topology from the point of view of simple homotopy theory*, *Publ. math.*, I.H.E.S., n° 15 (1963).
[2] —, *Morse theory in three categories*, Symposium in Honor of Marston Morse, (1963), Princeton.
[3] —, *Definition of equivalence of combinatorial imbeddings*, *Publ. math.*, I.H.E.S., n° 3 (1959).

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