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Differential topology from the point of view of simple homotopy theory

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CHAPTER 1

INTRODUCTION

It is striking (but not uncharacteristic) that the « first » question asked about higher dimensional geometry is yet unsolved:

Is every simply connected 3-manifold homeomorphic with S^3?

(Its original wording is slightly more general than this, and is false: H. Poincaré, *Analysis Situs* (1895).) The difficulty of this problem (in fact of *most* three-dimensional problems) led mathematicians to veer away from higher dimensional geometric homeomorphism-classificational questions.

Except for Whitney's foundational theory of differentiable manifolds and imbeddings (1936) and Morse's theory of Calculus of Variations in the Large (1934) and, in particular, his analysis of the homology structure of a differentiable manifold by studying critical points of C^∞ functions defined on the manifold, there were no classificational results about high dimensional manifolds until the era of Thom's Cobordisme Theory (1954), the beginning of " modern " differential topology.

(Classificational theorems did, however, exist for differentiable manifolds with " additional structure ", for example, homogeneous spaces and Lie groups.)

Instead, geometry was being approached with great success by the methods of algebraic topology which studies questions not of homeomorphism — but of homotopy-equivalence, a notion much looser than homeomorphism and amenable to algebraic techniques.

In contrast to the quiet of higher dimensional research, three-dimensional geometry had grown floridly, its most magnificent results being idiosyncratic to dimension three (for the most part ungeneralizeable). The goal of much of this research was to solve the problem stated above (the Poincaré conjecture).

J. H. C. Whitehead (1939) motivated by his work in three-dimensional geometry (and by the desire to strengthen algebraic-topological techniques so that they reach closer to homeomorphism questions), developed the notion of Simple Homotopy Equivalence, an equivalence more " geometrically rigid " than homotopy equivalence, yet still amenable to algebraic techniques. With the aid of this theory he succeeded in proving certain " stable " geometric results. Notably: If C^n is a compact contractible (combinatorial) manifold, then

$$C^n \times D^k \approx D^{n+k},$$

for some large k (the isomorphism being again combinatorial).

The classificational work of Smale (1961) reconsidered geometric questions from scratch. His researches began with thorough scrutiny of geometric information derivable from classical Morse theory. They revealed that in many ways geometry becomes " easier " beyond the embarrassing dimensions 3, 4. His analysis, remarkably, became fully applicable *only* when the dimension of the manifold was greater than four. The key idea of his work was to construct nicer and nicer functions on differentiable manifolds, according to an inductive scheme, and then to make novel geometric use of Poincaré duality. By means of these techniques he succeeded in proving

a) The Generalized Poincaré Conjecture in dimensions greater than four.

b) The h-cobordisme theorem: two simply connected manifolds (dimension greater than four) are h-cobordant if and only if they are diffeomorphic.

c) Existence of Morse functions: a function with prescribed non-degenerate critical points (i.e. data) exists on a simply connected manifold without boundary if and only if a certain homology-theoretic condition holds (the Morse inequalities).

This paper is an exercise in understanding and extending these geometric ideas of Smale ; it grew out of a seminar conducted by Serge Lang and myself on the Generalized Poincaré Conjecture at Harvard (1962).

Whereas Smale treats only simply connected issues, dealing with " single dimensions at a time ", one of the aims of this paper is to provide a direct analysis of the general case. The theory of simple homotopy types has shown itself to be natural and basic for these purposes, and by strikingly elementary geometric arguments, the general question of the existence of Morse functions may be reduced to questions in this theory. This needs some explanation:

The " fundamental theorem of Morse theory " says:

Let f be a C^∞ function on M^n, constant on ∂M with only nondegenerate critical points $p_1, \ldots, p_\nu$ such that

$$f(p_1) < f(p_2) < \ldots < f(p_\nu).$$

(Such a function will be called a *Morse function.*) And let $M_i \subseteq M$ be the submanifold

$$M_i = \{m \in M \mid f(m) \leq c_i\}$$

where c_i is a real number such that

$$f(p_i) < c_i < f(p_{i+1}).$$

Then, of course, $$M_1 \subset M_2 \subset \ldots \subset M_\nu = M$$

and $$M_i \approx M_{i-1} \cup_{\varphi_i} D^{k_i} \times D^{n-k_i}$$

where k_i is the index of f at the critical point p_i, and

$$\varphi_i : \partial D^{k_i} \times D^{n-k_i} \to \partial M_{i-1} \subseteq M_{i-1}$$

is a differentiable imbedding (i.e. " going past " a critical point of index k corresponds to attaching a thickened cell of dimension k).

Consequently, any Morse function f on M induces a very strict kind of geometric decomposition of M

$$M \approx D^{k_0} \times D^{n-k_0} \cup_{\varphi_1} D^{k_1} \times D^{n-k_1} \ldots \cup_{\varphi_\nu} D^{k_\nu} \times D^{n-k_\nu}$$

which provides much information concerning the geometric structure of M.

The attitude taken in this paper is that, at times, it is also useful to disregard some of the rigid information contained in such a decomposition. By " unthickening " each of the thickened cells $D^{k_i} \times D^{n-k_i}$, one may pass to a cell decomposition:

$$M_i \xrightarrow{s_i} X_i = D^{k_0} \cup_{\bar{\varphi}_1} D^{k_1} \cup_{\bar{\varphi}_2} \cup \ldots \cup_{\bar{\varphi}_i} D^{k_i}$$

so that

$$X_0 \subset X_1 \cup \ldots \subset X_\nu = X$$

and

$$X_i = X_{i-1} \cup_{\bar{\varphi}_i} D^{k_i}$$

where the

$$\bar{\varphi}_i : \partial D^{k_i} \to X_{i-1}$$

are just continuous maps.

The map $s : M \to X$ still preserves much structure of M; it is a simple homotopy equivalence (once Whitehead's notion is extended to be applicable to such cell decompositions, X).

Of course, the Morse function f doesn't give rise to a unique " differentiable decomposition "

$$M = \bigcup_i M_i$$

of M, nor is there a unique way of unthickening the decomposition $\bigcup_i M_i$ to yield the cell decomposition $X = \bigcup_i X_i$. However, there are natural equivalence notions for differentiable decompositions and for cell decompositions (equivalence classes of these I call differentiable cell filtrations and cell filtrations respectively), so that any differentiable filtration " unthickens " to yield a unique cell filtration, and any Morse function yields a unique differentiable filtration, which unthickens to a unique cell filtration.

The prime concern in this paper is to construct all geometric (differentiable) filtrations of an arbitrary differentiable manifold.

The problem will be solved by being reduced to simple-homotopy theory.

In the simply connected case, for example (where homotopy theory and simple homotopy theory coincide), our solution will yield (as corollaries) the theorems of Smale.

We will be faced with the following task: Given a simple homotopy equivalence $s : M \to X$ of a differentiable manifold M with a cell decomposition, when is there a differentiable filtration of M which unthickens to X and whose unthickening map is s?

This is a functorially refined version of the problem of constructing Morse functions with prescribed data.

Our first approach to this problem is oblique (chapter IV). Instead of fixing M and asking for differentiable filtrations of it, we fix X and ask for all differentiable filtrations which, when unthickened, yield X. Any such differentiable filtration will be called a *neighborhood* of X, and the set of n-dimensional neighborhoods of X will be denoted $\mathcal{N}^n(X)$. $\mathcal{N}^n(X)$ is a " functor " on cell decompositions and the crucial theorem proved is that $\mathcal{N}^n(X)$ is dependent only upon the simple homotopy type of X if $n \geq \dim X + 3$ (see Chapter VIII):

Non-Stable Neighborhood Theorem. — If $f: X \rightarrow Y$ is a simple homotopy equivalence of (properly ordered) cell decompositions, f induces an isomorphism

$$f^{(n)}: \mathcal{N}^n(X) \underset{\approx}{\rightarrow} \mathcal{N}^n(Y)$$

if $n \geq \max\{\dim X, \dim Y\} + 3$. (Isomorphism means a one-one correspondence where corresponding manifolds are diffeomorphic.)

The number 3 is crucial. By examples, the theorem may be seen to be best possible. Full application of this theorem is not made in this paper (i.e. the same result for $n \geq \max\{\dim X, \dim Y\} + 5$ is considerably easier to prove, and would suffice for uses here), however it is necessary in order to obtain best possible results in the " relative theory " (forthcoming), which generalizes Smale's h-cobordisme theory to nonsimply connected manifolds. The term neighborhood is justifiable to some extent by the results of Chapter VI: if $K \subseteq M$ is a simplicial complex nicely imbedded in a differentiable manifold, there is a neighborhood N of the cell decomposition induced by K such that $K \subseteq N \subseteq M$ (N being a differentiable submanifold of M). It is not, however, a neighborhood in the sense that X always admits a natural imbedding into $N \in \mathcal{N}^n(X)$. In some ways, a neighborhood of a cell decomposition behaves as if it were a generalization of a cell bundle — (actually it plays more of the role of a " co-neighborhood ").

It is useful to introduce the notion of geometric dimension. A differentiable manifold M has geometric dimension less than or equal to k (geom dim $M \leq k$) if M possesses a differentiable filtration which is a neighborhood of (unthickens to) a cell decomposition X such that $\dim X \leq k$.

A first application of the nonstable neighborhood theorem is the following existence theorem for n-manifolds whose geometric dimension is less than $n-2$.

Theorem. — Let $s: M^n \rightarrow X$ be a simple homotopy equivalence between the differentiable n-manifold M^n and the cell filtration X. Assume:

$$\dim X \leq n-3$$

$$\text{geom dim } M^n \leq n-3.$$

Then there exists a differentiable filtration $\mathscr{M}$ of M which is a neighborhood of X, with unthickening map s.

This is a best possible general statement ; for oriented manifolds without boundary, however, more can be said. Here, along with Smale, one makes use of Poincaré duality.

If M is a manifold without boundary, and $\mathscr{M}$ a differentiable filtration of M, by turning M upside-down one obtains a different filtration $\widetilde{\mathscr{M}}$ (the dual filtration). Thus, filtrations come in dual pairs, and they unthicken to pairs of cell filtrations X, $\widetilde{X}$, which obey a kind of homotopy theoretic duality, made explicit in Chapter X. They are said to be *paired into* M. The notion of two cell decompositions X, $\widetilde{X}$ being " paired into M " is a simple-homotopic-theoretic one. The existence theorem for oriented manifolds without boundary (chapter X) is:

Theorem. — Let M^n be an oriented manifold without boundary, $n \geq 7$.

Let
$$\begin{array}{ccc} M^n & \xrightarrow{\varphi} & X \\ {\scriptstyle\hat{\varphi}}\downarrow & & \\ X & & \end{array}$$

be a pairing of X, $\widetilde{X}$ to M^n. Then there are dual differentiable filtrations $\mathscr{M}, \widetilde{\mathscr{M}}$ of M^n which realize (up to similarity) the pairing $(\varphi, \widetilde{\varphi})$. This means that φ, $\widetilde{\varphi}$ are filtration- and orientation-preservation maps of $\mathscr{M}$ to X, $\widetilde{\mathscr{M}}$ to $\widetilde{X}$, respectively.

Consequently, cell differentiable filtrations of an oriented manifold without boundary may be " constructed " by simple-homotopy-theoretic means. As a result one gets existence of Morse functions with prescribed critical points if and only if a certain simple-homotopy-theoretic problem can be solved. The Smale theorem follows, for instance, that if M^n is k-connected $(n \geq 7)$ a Morse function on M^n may be constructed with a unique maximum and minimum and all other critical points of index greater than k, less than $n-k$. Because of its great interest a more direct proof of this is given (for $n \geq 6$) rather than obtaining it as a corollary of the general existence theorem (Chapter IX). From the above Smale theorem one immediately obtains the Generalized Poincaré conjecture for $n \geq 6$, of course. (The obstreperous dimension 5 may be treated separately.)

The stable theory of $\mathscr{N}^n(X)$ (that is, n large compared to dim X) is discussed fully in chapter XI.

It seems to me that the results of that theory may be of some interest (the methods are standard). In the stable situation, $\mathscr{N}^n(X)$ may be seen to be in natural one-one correspondence with $\widetilde{KO}(X)$, and the group of isotopy classes of automorphisms of $N \in \mathscr{N}^n(X)$ (which are homotopic to the identity) is isomorphic with $[X, O_\infty]$ (the group of homotopy classes of maps of X into O_∞, the infinite orthogonal group).

A " suspension theorem " is proved. There are natural suspension isomorphisms:

$$j : \mathscr{N}^n(X) \underset{\approx}{\rightarrow} \mathscr{N}^{n+1}(X) \qquad \text{if } n > 2 \dim X.$$

Independently of the nonstable theory, by quite trivial methods, a stable neighborhood theorem may be proved ; i.e. that $\mathcal{N}^n(X)$ depends only upon the simple homotopy type of X. As a corollary of this, one obtains that if

$$f : M_1^n \to M_2^n$$

is a simple homotopy equivalence of the differentiable manifolds, M_1^n, M_2^n, then any differentiable k-cell bundle E_1^{n+k} over M_1^n $(k>n)$ is diffeomorphic with a k-cell bundle E_2^{n+k} over M_2^n by a diffeomorphism $\varphi : E_1^{n+k} \underset{\approx}{\to} E_2^{n+k}$ such that

$$\begin{array}{ccc} E_1^{n+k} & \underset{\approx}{\xrightarrow{\varphi}} & E_2^{n+k} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1^n & \xrightarrow{f} & M_2^n \end{array}$$

is homotopy-commutative.

This last fact has been announced in [2], and has as corollary the following:

Let $f : M_1^n \to M_2^n$ be a simple homotopy equivalence of differentiable manifolds which is tangential (i.e. $f^*\tau_2 = \tau_1$, where τ_i is the stable vector bundle class of the tangent bundle of M_i $(i = 1, 2)$). Then there is a diffeomorphism

$$\begin{array}{ccc} M_1^n \times D^k & \underset{\approx}{\xrightarrow{\varphi}} & M_2^n \times D^k \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

such that the above diagram is homotopy commutative. This provides a complement to the " open " situation discussed in [2].

CHAPTER II

TERMINOLOGY AND FOUNDATIONAL THEOREMS FOR DIFFERENTIAL TOPOLOGY

Standard Spaces.

R^n: Euclidean space of n-dimension, taken with its vector space structure. It is endowed with its linear metric $|| \ ||$ given by: if $x=(x_1, \ldots, x_n)\in R^n$,

$$||x|| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$D^n(r) \subseteq R^n$: The closed unit cell (disc) of radius $r \in R$ in R^n. It is given by:

$$D^n(r) = \{x \in R^n \mid ||x|| \leq r\}.$$

Let $D^n = D^n(1)$ be the disc of radius 1. There is an identification

$$d_{n,m} : D^n \times D^m \to D^{n+m},$$

given by Douady (Séminaire Cartan, 1961-1962, Exp. 3, p. 5). It is obtained by smoothing the angles of $D^n \times D^m$ about $\partial D^n \times \partial D^m$.

In practice I will consider $D^n \times D^m$ and D^{n+m} " identified " by the correspondence $d_{n,m}$.

The 1-cell D^1 will be denoted by I. For typographical convenience I shall sometimes denote $D^n(r)$ by D^n_r where no confusion will arise.

S^n: The unit sphere in R^{n+1}:

$$S^n = \{x \in R^{n+1} \mid ||x|| = 1\}$$

$D^n_+(r) \subseteq S^n$: The " polar cap of " radius " r " in S^n:

$$D^n_+(r) = \left\{x = (x_0, \ldots, x_n) \in S^n \mid x_0 \geq \sqrt{1 - r^2}\right\} \text{ for } 1 \geq r \geq 0.$$

$$D^n_-(r) = \left\{x = (x_0, \ldots, x_n) \in S^n \mid x_0 \leq -\sqrt{1 - r^2}\right\} \text{ for } 1 \geq r \geq 0.$$

O_n is the Lie group of orthogonal transformations on R^n. There is a natural inclusion $j_{n,m} : O_n \to O_m$ for $m \geq n$ generated by the natural inclusion of R^n in R^m as an n-dimensional subspace.

Differential Topological Notions.

By differentiable manifold I shall mean of class C^∞, as will be made explicit in the section below, on " Creased Manifolds ". By map I shall mean differentiable map of class C^∞ on the interior of the manifold on which it is defined, and I am a bit more lenient concerning its nature on the boundary of the differentiable manifold (again as will be seen in the section on " creased manifolds "). If M, N are differentiable manifolds, $M \approx N$ connotes a differentiable isomorphism between M and N. ∂M will denote the boundary of M, considered again as a differentiable manifold. The tangent bundle of a differentiable manifold W will be denoted T(M). The tangent space at the point $p \in M$ will be called $T_p = T_p(M)$. If $f: M \to N$ is a differentiable map, its induced map on T(M) is denoted

$$df: T(M) \to T(N).$$

Simplicial Notions.

Δ^n will denote the n-simplex. Δ^n may be assumed " standardly " imbedded in R^n by associating its vertices $(d_0, \ldots, d_n)$ with the set of vectors $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ in R^n where

$$\varepsilon_0 = 0$$
$$\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \qquad i > 0,$$

(the nonzero entry of ε_i being at the i^{th} place).

Thus Δ^n may be thought of as possessing a metric.

Very often it will be desirable, given M, a differentiable manifold, to form the union

$$M \cup_{D_+^{n-1}} D^n = M^*$$

where $D_+^{n-1} \subseteq S^{n-1} = \partial D^n$ is identified with a patch on the boundary of D^n and $D_+^{n-1} \subseteq \partial M$ is identified (as well) with a patch on ∂M. It is easily seen that

$$M \approx M \cup_{D_+^{n-1}} D^n = M^*$$

and therefore the manifold M^* will be said to be *obtained from* M *by performing an " irrelevant addition "*.

Two simplicial complexes K_1, K_2 are equivalent if they have isomorphic refinements in the sense of barycentric subdivision. If K is a simplicial complex, $K^{(k)}$ is its k-skeleton. Let $\Delta^{(k)} = (\Delta^n)^{(k)}$ be the k-skeleton of Δ^n. Then

$$A_k^n = \left\{ x \in \Delta^n \mid \delta(x, \Delta^{(k)}) \leq \frac{1}{4} \right\}$$

where $\delta(x, \Delta^{(k)})$ is the distance from x to $\Delta^{(k)} \subset \Delta^n$ with respect to the canonical metric on Δ^n.

For the definition of regular neighborhood, see [14].

Any differentiable manifold possesses a C^1-compatible triangulation, unique up to combinatorial equivalence. See [15].

Topological Notions.

If $X \supseteq A$, $Y \supseteq B$ are topological spaces and $f: A \to B$ a continuous map, then $Y \cup_f X$ will denote the quotient space

$$Y \cup_f X = \frac{Y \cup X}{(\sim_f)}$$

of the disjoint union of X and Y with respect to the equivalence relation generated by $(\sim_f)$, which is defined as follows:

$$a \sim_f b, \quad \text{for } a \in A, \quad b \in B,$$

if and only if $f(a) = b$.

If $f: C \to D$ is a continuous map, then the mapping cylinder M is defined as follows:

$$M_f = Y \cup_{f \times \{0\}} X$$

where $Y = C \times I$, $B = C \times \{0\}$, $X = A = D$.

If $W \subseteq Y$ and $W \subseteq X$, then

$$Y \cup_W X = Y \cup_i X$$

where $i: W \to W$ is the identity map.

If X and Y are topological spaces, then $X \sim Y$ denotes the existence of a homotopy equivalence between X and Y. The identity map of a space X will be denoted $1 = 1_X : X \to X$. [X, Y] denotes the set of homotopy classes of continuous maps from X to Y. If $f: X \to Y$ is a continuous map, Im f will denote the image set and $[f]$ the homotopy class of f. If $\varphi : D^n \times D^m \to X$ is a map, I will use $\bar{\varphi} : D^n \to X$ to denote the restriction of φ to $D^n \times \{0\}$.

If M and N are differentiable manifolds, an *isotopy* $\varphi_t : M \to N$, $0 \leq t \leq 1$ is a differentiable map $\varphi : I \times M \to N$ such that φ_t is an imbedding for each $t \in I$. Then φ_0 and φ_1 are said to be *isotopic*, denoted $\varphi_0 \approx \varphi_1$.

Vector Bundle Notions.

For the definition and terminology regarding vector bundles, I refer to [6]. If η is a vector bundle over M, a differentiable manifold, there is (up to differentiable equivalence) a unique differentiable vector bundle η' over M such that η is equivalent to η' as topological vector bundles.

$V^k(M)$ is the set of equivalence classes of k-plane bundles over M. $\widetilde{KO}(M)$ is the Grothendieck group generated by vector bundles over M.

It is classical that $V^k(M) \underset{\beta}{\overset{\approx}{\to}} \widetilde{KO}(M)$ is a bijection for $k > \dim M$, where β is the natural map.

If η is a vector bundle, $E(\eta)$ is its total space, regarded as a differentiable manifold ;

any vector bundle η over a paracompact manifold possesses a Riemannian metric, with respect to which one may define the cell bundle:

$$E(r) = \{x \in E \mid ||x|| \leq r\} \qquad \text{for } r \geq 0.$$

Trivially, if $r_1, r_2 > 0$, then $E(r_1)$ is isomorphic with $E(r_2)$ as differentiable cell bundles, and the canonical class of differentiable cell bundles determined by η is denoted D_η.

If $U \subset M$ is a subset, the restriction of D_η to U is denoted $D_\eta(U)$.

The trivial k-plane bundle will be denoted 1_k. If $f: X \to Y$ is a continuous map and η a vector bundle over Y, then the pull-back of η, via f, considered as a vector bundle over X, will be denoted $f^* \eta$.

Creased Manifolds.

In the course of the theory to be presented, there will be perpetual need for the process of glueing two manifolds together on non-proper patches of their boundaries, of angle-straightening, of taking cartesian products of manifolds with boundary.

All three processes take one out of the realm of differentiable manifolds with smooth boundary, and there are a variety of way of legitimizing these processes.

My policy with regard to such questions which arise in the course of somewhat detailed proofs, is to ignore them. The most elaborate treatment of " angle smoothing " in the literature is: A. Douady, Séminaire Cartan, 1961-1962, Exposés 1, 2, 3. Douady's treatment suffices for the purposes of this paper. (One might eventually hope for a quite general existence and uniqueness theorem for smoothing creased manifolds in the direction of the Cairns-Hirsch smoothing theory [2].)

In general, the manifolds with which we will deal are *creased manifolds* (« variétés à bords anguleux » of Douady), and, by abuse of language, I will often refer to bijective mappings between manifolds which are diffeomorphisms outside " creases " as: isomorphisms.

Local Linearization.

The following lemma is well-known [8]:

Proposition I. — (Tubular Neighborhood Lemma.)

Let $E(r)$ be a differentiable cell bundle (endowed with Riemannian metric) over M, $r > 0$. Let A be a closed submanifold of M, $E_A(r)$ the part of the bundle over A. Let

$$f: E(r) \to W$$
$$g: E(r) \to W$$

be imbeddings such that if $i: M \to E(r)$ is the zero cross-section,

a) $\qquad f \circ i = g \circ i$

b) $\qquad f | E_A(r) = g | E_A(r)$.

Then, there is an $r_1>0$, an automorphism $\alpha : W \to W$ (isotopic to the identity), such that $\alpha | f E_A(r_1)$ is the identity map, and an orthogonal bundle map $\lambda : E \to E$ such that $\lambda | E_A$ is the identity, such that

$$\begin{array}{ccc} E(r_1) & \xrightarrow{f} & W \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\alpha} \\ E(r_1) & \xrightarrow[g]{} & W \end{array}$$

is commutative. This implies the existence of unique tubular neighborhoods about imbedded submanifolds.

General Position.

Proposition II. — (Stable Imbedding Theorem.)

If $k \geq 2n+2, f : M^n \to W^k$ a continuous map, there is a unique isotopy class of differentiable imbeddings of M^n in W^k, in the same homotopy class as f. See [16], [6].

Proposition III. — (Isotopy Extension Theory.)

Let $V \subset M$, be differentiable manifolds, compact without boundary, then any isotopy $h_t : V \to W$ is extendable to an isotopy $H_t : M \to W$. See [8].

In applying Propositions III, IV in the course of the proof, I shall often not give the precise bounds for k, and merely say: for k large enough. I shall often indicate that II, III are being invoked in the course of a proof by the phrase: " By General Positionality ". Similarly I shall invoke proposition I by the phrase: " By the Tubular Neighborhood Lemma ".

In Chapter IX a more delicate version of intersection removal will be needed, which is also foundational:

Proposition IV. — Let A, B, C be compact differentiable manifolds, $f : A \to C$, $g : B \to C$ imbeddings. Then there is an imbedding $h : B \to C$, homotopic to g $(h \sim g)$, such that h and f intersect transversally (i.e. for each point $p \in f(A) \cap h(B)$ the subspaces $df\{T_{f^{-1}(p)}(A)\}$ and $dh\{T_{h^{-1}(p)}(B)\}$ generate $T_p(C)$).

In the case where $\dim C = \dim A + \dim B$, the intersection set $f(A) \cap h(B)$ consists in a finite number of points, $p_1, \ldots, p_\nu$, the tangent planes $df\{T_{f^{-1}(p_i)}(A)\}$ and $dh\{T_{h^{-1}(p_i)}(A)\}$ being complementary subspaces in $T_{p_i}(C)$.

CHAPTER III

FILTRATIONS

As mentioned in the introduction, the geometric information about a differentiable manifold M given by a Morse function f on M may be summed up by the " recipe " for constructing M as a union of thickened cells (handles), each one corresponding to a critical point of f, as guaranteed by the Fundamental Theorem of Morse Theory. Hence any Morse function f gives rise to a decomposition of M,

$$M = \bigcup_{i=0}^{\nu} M_i$$

where M_i is built from M_{i-1} as follows:

$$M_i = M_{i-1} \cup_{\varphi_i} D^{n_i} \times D^{n-n_i}$$

and

$$\varphi_i : \partial D^{n_i} \times D^{n-n_i} \to \partial M_{i-1} \subset M_{i-1}$$

is a differentiable imbedding. Clearly, the φ_i are not uniquely determined by f, but they are determined up to a natural equivalence.

Again, in the spirit of the introduction, we may systematically ignore the differential-topological information obtained and pass from the Morse function f (or equivalently, from the decomposition $M = \cup_i M_i$) to a cell complex X which provides a description of the homotopy structure (in fact, the simple homotopy structure) of M.

The purpose of this chapter is to formalize the two brands of filtration (discussed in the introduction),

$$M = \bigcup_{i=0}^{\nu} M_i; \qquad X = \bigcup_{i=0}^{\nu} X_i$$

up to their " natural equivalence ", and to study the relations between them. Differential-geometric decompositions, $M = \bigcup_{i=0}^{\nu} M_i$ will be called *differentiable cell filtrations*, and equivalence classes of the homotopy decompositions, $X = \bigcup_{i=0}^{\nu} X_i$ will be called *cell filtrations*.

Definition **(3.1)**. — A differentiable cell decomposition, M^n, is a sequence of differentiable n-dimensional manifolds $(M = (M_0, M_1, \ldots, M)$ (i.e. " $M = \bigcup_i M_i$ ")) such that

$$M_i = M_{i-1} \cup_{\varphi_i} D^{n_i} \times D^{n-n_i}$$

where

$$\varphi_i : \partial D^{n_i} \times D^{n-n_i} \to \partial M_{i-1}$$

is an imbedding. I assume $M_0 = \emptyset$.

A cell decomposition, X, is a sequence of topological spaces,

$$X = (X_0, \ldots, X_\nu)$$

(i.e. " $X = \bigcup_i X_i$ ") such that $X_i = X_{i-1} \cup_{\varphi_i} D^{n_i}$ where $\varphi_i : \partial D^{n_i} \to X_{i-1}$ is a continuous map. Assume $X_0 = \emptyset$.

Maps of (differentiable) cell decompositions. — By a map of (differentiable) cell decompositions I shall mean a pair

$$(f, \omega) : X \to Y$$

for X, Y cell decompositions (differentiable or not), where

$$X = (X_0, \ldots, X_\nu)$$
$$Y = (Y_0, \ldots, Y_\mu)$$

such that

a) $$\omega : \{1, \ldots, \nu\} \to \{1, \ldots, \mu\}$$

is a monotonic function.

b) $$f : X_\nu \to Y_\mu$$

is a map such that $f(X_i) \subset Y_{\omega(i)}$.

By an *inclusion map* $\varphi : X \to Y$ will be meant a map $(\varphi, \omega) : X \to Y$ where $\omega(j) = j, j = 1, \ldots, \nu$. By a *filtration homotopy* of any filtered object $f_t : X \to Y$ will be meant a map f_t for $0 \leq t \leq 1$ such that f_t preserves filtrations at each stage for some monotonic function ω (i.e.

$$f_t(X_i) \subset Y_{\omega(i)} \qquad \text{for } i = 1, \ldots, \omega)$$

and f_t is a homotopy.

Let X be a cell decomposition,

$$X = (X_0, \ldots, X_\nu)$$
$$X_i = X_{i-1} \cup_{\varphi_i} D^{n_i}.$$

There is a natural injection

$$\pi_{n_i}(X_i, X_{i-1}) \leftarrow \pi_{n_i}(D^{n_i}, \partial D^{n_i})$$

by inclusion. Since D^{n_i} possesses a natural orientation, there is a natural isomorphism

$$\pi_{n_i}(X_i, X_{i-1}) \overset{\zeta_i}{\leftarrow} \mathbf{Z}$$

of the rational integers onto $\pi_{n_i}(X_i, X_{i-1})$.

Similarly, if M is a differentiable cell decomposition, there is a natural injection,

$$\begin{array}{ccc} \pi_{n_i}(\mathrm{M}_i, \mathrm{M}_{i-1}) & \leftarrow & \pi_{n_i}(\mathrm{D}^{n_i}\times \mathrm{D}^{n-n_i}, \partial\mathrm{D}^{n_i}\times\mathrm{D}^{n-n_i}) \\ {\scriptstyle \zeta_i}\nwarrow & & \swarrow {\scriptstyle \approx} \\ & \mathbf{Z} & \end{array}$$

Equivalence of (differentiable) cell decompositions. — Two cell decompositions X, Y

$$\mathrm{X}=(\mathrm{X}_0, \ldots, \mathrm{X}_\nu)$$
$$\mathrm{Y}=(\mathrm{Y}_0, \ldots, \mathrm{Y}_\mu)$$
$$\mathrm{X}_i=\mathrm{X}_{i-1}\cup_{\varphi_i}\mathrm{D}^{n_i}; \quad \mathrm{Y}_i=\mathrm{Y}_{i-1}\cup_{\psi_i}\mathrm{D}^{m_i}$$

will be called equivalent if

a) $\nu=\mu, \ n_i=m_i$

and there are maps $(f, 1) : \mathrm{X}\rightarrow\mathrm{Y}$

$(g, 1) : \mathrm{Y}\rightarrow\mathrm{X}$

b)

$$\begin{array}{ccc} \pi_{n_i}(\mathrm{X}_i, \mathrm{X}_{i-1}) & \xrightarrow{f_i} & \pi_{n_i}(\mathrm{Y}_i, \mathrm{Y}_{i-1}) \\ {\scriptstyle \zeta_i}\nwarrow & & \nearrow {\scriptstyle \zeta_i} \\ & \mathbf{Z} & \end{array}$$

The above diagram is commutative for all $i=1, \ldots, \nu$ where

$$f_i : \pi_{n_i}(\mathrm{X}_i, \mathrm{X}_{i-1}) \rightarrow \pi_{n_i}(\mathrm{Y}_i, \mathrm{Y}_{i-1})$$

is the homomorphism induced by f.

c) There are filtration preserving homotopies of gf to 1_X and fg to 1_Y.

If X and Y are equivalent, I shall write

$$\mathrm{X}\sim\mathrm{Y}.$$

To paraphrase the definition of equivalence given above, one might simply say that two cell decompositions are equivalent if there is a filtration-preserving homotopy equivalence between them, which " preserves " the homotopy class of attaching maps.

Two differentiable cell decompositions M, N will be called *equivalent* if there is a filtration-preserving diffeomorphism

$$f : \mathrm{M}\rightarrow\mathrm{N}$$

such that

$$\begin{array}{ccc} \pi_{m_i}(\mathrm{M}_i, \mathrm{M}_{i-1}) & \xrightarrow{f_*} & \pi_{n_i}(\mathrm{N}_i, \mathrm{N}_{i-1}) \\ \nwarrow & & \nearrow \\ & \mathbf{Z} & \end{array} \qquad i=1, \ldots, \nu$$

where the vertical maps are the natural ones (i.e. an equivalence between differentiable cell decompositions is a filtration-preserving diffeomorphism which " preserves " orientation of attached handles).

Very useful is the following easily proved lemma:

Lemma (**3.2**). — Any equivalence $(f, 1) : X \to X$ from an object X to itself is filtration-homotopic to the identity.

By virtue of Lemma 3.2 (which disclaims the existence of " automorphisms "), one may pass, functorially to filtration-homotopy equivalence classes of cell decompositions.

Definition (**3.3**). — A cell filtration will denote an equivalence class of cell decompositions. If X, Y are cell filtrations, a *map* $f: X \to Y$ will be a filtration-homotopy class of maps of representative cell decompositions.

A differentiable cell filtration will denote an equivalence class of differentiable cell decompositions.

Remarks. — Lemma 3.2 insures that the notion of *map*, above, is independent of the representative cell decompositions. Since the analogue of (3.2) is false for differentiable cell decompositions, one cannot talk, as easily, of maps from differentiable cell filtrations to differentiable cell filtrations (or if one does, one must be slightly careful).

After Definition 3.3, we may refer to the category of cell-filtrations whose maps are defined as in 3.3. Denote this category by the letter $\mathscr{F}$. I shall refer in the sequel also to $\mathscr{F}^{(n)}$, the subcategory of $\mathscr{F}$ consisting of objects X such that $\dim X \leq n$.

The category of *differentiable cell decompositions* I shall denote by $\mathscr{D}_*$.

A natural relation between cell filtrations which is slightly weaker than equivalence is the notion of *similarity* (this will be used only in Chapter X): A *similarity* $f: X \to Y$ between two cell filtrations is a filtration and " orientation "-preserving homotopy-equivalence. By orientation-preserving, I mean that the homology diagrams

$$\begin{array}{ccc} H_{n_i}(X_i, X_{i-1}) & \xrightarrow{f_{n_i}} & H_{n_i}(Y_i, Y_{i-1}) \\ & \nwarrow \quad \nearrow & \\ & H_{n_i}(D^{n_i}, \partial D^{n_i}) & \\ & \wr\wr & \\ & \mathbf{Z} & \end{array}$$

are commutative.

Unthickening a differentiable cell filtration to obtain a cell filtration. — A differentiable cell decomposition is a " recipe " for constructing a differentiable manifold by successively adding thickened handles

$$M = (M_0, \ldots, M_\nu)$$
$$M_i = M_{i-1} \cup_{\varphi_i} D^{n_i} \times D^{n-n_i}$$

Using the same recipe, we may construct a coarser object: a sequence of topological spaces $(X_0, \ldots, X_\nu)$ and compatible continuous maps $\pi_i : M_i \to X_i$, $i = 0, \ldots, \nu$, such that the sequence

$$X = (X_0, \ldots, X_\nu)$$

is a cell decomposition.

Inductively one may construct the (X_i, π_i) as follows: assume $\pi_{i-1} : M_{i-1} \to X_{i-1}$ given. Define

$$X_i = X_{i-1} \cup_{\psi_i} D^{n_i}$$

where $\psi_i : \partial D^{n_i} \to X_{i-1}$ is taken to be the composite map

$$\begin{array}{ccccccccc} \partial D^{n_i} & \underset{=}{\to} & \partial D^{n_i} \times \{0\} & \to & \partial D^{n_i} \times D^{n-n_i} & \underset{\varphi_i}{\to} & \partial M_{i-1} & \to & M_{i-1} \\ & & & & & & & & \downarrow \pi_{i-1} \\ & & & & \xrightarrow{\qquad \psi_i \qquad} & & & & X_{i-1} \end{array}$$

With this definition it is clearly seen that $\pi_i : M_i \to X_i$ may be defined uniquely up to homotopy class by requiring that

$$\pi_i : (M_0, \ldots, M_i) \to (X_0, \ldots, X_i)$$

is a homotopy equivalence of sequences that preserves homotopy class of attaching maps.

The eventual cell decomposition X, obtained by this unthickening process, is uniquely determined up to equivalence. Thus it is unique as a cell filtration. If M is a differentiable cell filtration, in this manner one may construct a cell filtration X (its unthickening) which I will denote by ρM, and a projection

$$\pi : M \to X = \rho M$$

which is a filtration-preserving homotopy equivalence which respects the orientation of attaching maps.

In more austere precision:

Proposition (3.4). — There is an " unthickening functor "

$$\rho : \mathcal{D}_* \to \mathcal{F}$$

assigning to any differentiable cell filtration M, a cell filtration $\rho M \in \mathcal{F}$ together with a filtration preserving homotopy equivalence

$$\rho_X : M \to X$$

where X is any cell decomposition in the equivalence class ρM. The map ρ_X is uniquely specified up to homotopy class by requiring that ρ_X respect the orientations of attaching cells.

Any such ρ_X is called a projection.

Reordering equivalence. — Two cell filtrations (X), (Y) are called reorderings of one another (denoted $(X)\equiv(Y)$) if and only if there are members $X\in(X)$, $Y\in(Y)$, X, Y cell decompositions satisfying:

$$X=(X_0, \ldots, X_\nu) \qquad X_i=X_{i-1}\cup D^{n_i}$$
$$Y=(Y_0, \ldots, Y_\nu) \qquad Y_i=Y_{i-1}\cup D^{m_i}$$

a) there is a permutation $\omega:\{1, \ldots, \nu\}\to\{1, \ldots, \nu\}$ such that $m_{\omega(i)}=n_i$, $i=1, \ldots, \nu$, and

b) there is a homeomorphism

$$f: X_\nu\to Y_\nu$$

such that $f(D^{n_i})\subset D^{m_{\omega(i)}}$.

Clearly the relation $(\equiv)$ is an equivalence relation.

We are forced to consider such an equivalence relation, since the ordering of a filtration is irrelevant for most of our purposes, and often does not arise naturally. Thus, let K be a simplicial complex. Then K gives rise to numerous objects of $\mathcal{F}$ depending upon how one orders the cells of K. The " Reordering Equivalence Class " of the cell filtered object determined by K, however, is unique.

Presentations of cell filtrations. — The maps φ_i used in the construction of X_i were not taken to be part of the explicit structure of a filtration. At times it is useful to emphasize them as follows:

Definition (**3.5**). — A presentation P is a filtration X together with an explicit collection $\Phi=\{\varphi_i\}_{i=1,\ldots,\nu}$ such that, if $X=(X_0, \ldots, X_\nu)$, then $X_i=X_{i-1}\cup_{\varphi_i}D^{n_i}$ or $X_i=X_{i-1}\cup_{\varphi_i}D^{n_i}\times D^{n-n_i}$ if X is a cell filtration or a differentiable filtration of dimension n.

P is called a *presentation* of X.

To suppress the filtration entirely, one might denote a presentation as follows: $P=\{\varphi_i, n_i\}$, $i=1, \ldots, \nu$, thereby giving " thorough bass " instructions for the reconstruction of the filtration X.

Proposition (**3.6**). — Let $f:X\to X'$ be an isomorphism in the category $\mathcal{F}$. Let P and P′ be presentations of X and X′ respectively : $P=\{\varphi_i, n_i\}$, $P'=\{\varphi'_i, n_i\}$. Then

$$\begin{array}{ccc} X_{i-1} & \xrightarrow{f} & X'_{i-1} \\ & {\scriptstyle\varphi_i}\nwarrow \quad \nearrow{\scriptstyle\varphi'_i} & \\ & \partial D^{n_i} & \end{array}$$

is homotopy commutative.

(Roughly: There is a unique presentation of any filtration up to homotopy).

Proposition (**3.6 bis**). — If $P=\{\varphi_i, n_i\}_{i=1,\ldots,\nu}$, $P'=\{\varphi'_i, n_i\}_{i=1,\ldots,\nu}$ are differentiable n-dimensional presentations such that $\varphi_i=\varphi'_i$ if $i\leq\nu-1$ and φ_ν is differentiably isotopic with φ'_ν, then P, P′ give rise to the same differentiable cell filtration.

Let X, Y, Z be cell filtrations such that $Y\subset X$, $Y\subset Z$, then

$$X\cup_Y Z=(X_0, \ldots, X_\nu, Z_{i_1}, \ldots, Z_{i_\mu})$$

is the cell filtration obtained by taking the cells of X first, and then adding the cells of $Z - Y$ in their order of occurrence in the filtration of Z where the attaching maps are suitably interpreted as having a larger range.

Definition **(3.7)**. — A (differentiable) cell filtration will be called " properly ordered " if X has a presentation

$$P = P(X) = \{\varphi_i, n_i\}_{i=1, \ldots, \nu}$$

where

$$n_i \leq n_j \qquad \text{for } 1 \leq i \leq j \leq \nu.$$

(That is to say, the cells are added according to increasing dimension.)

If X is a filtration $X = (X_0, \ldots, X_\nu)$, its *length* is ν.

Let $X = (X_0, \ldots, X_\nu)$. If X_ν is the " end space " of X, I shall speak of X as *a filtration of the space* X_ν.

The fundamental theorem of Morse Theory. — As mentioned in the introduction, by a *Morse function* on M is meant a real valued C^∞-function f on a differentiable manifold M taking a constant value on ∂M, whose critical points $p_1, \ldots, p_\nu$ are all nondegenerate and occur in the interior of M and

$$f(p_i) < f(p_j) \qquad \text{if } i < j.$$

(This last requirement is thrown in only for convenience and is no great restriction.)

The fundamental theorem relating Morse functions on M to the differential topology of M may be stated a follows:

Theorem. — To any Morse function f on M, one may associate a unique differentiable cell filtration $\mathcal{M}_f$ of M. If f has critical points $p_1, \ldots, p_\nu$ such that $f(p_1) < f(p_2) < \ldots < f(p_\nu)$, of index $k_1, \ldots, k_\nu$ respectively, then

$$\mathcal{M}_f = (M_0, M_1, \ldots, M_\nu)$$

where

$$M_i = M_{i-1} \cup_{\varphi_i} D^{k_i} \times D^{n-k_i} \qquad \text{for } i = 1, \ldots, \nu.$$

If $\mathcal{M}$ is a differentiable cell filtration of M, then there exists a Morse function f on M such that $\mathcal{M} = \mathcal{M}_f$.

The above theorem (in other language) is foundational to Morse theory, and shows the equivalence of Morse functions and geometric decompositions.

The categories of (differentiable) cell filtrations. — In dealing with filtered objects in subsequent chapters, very often, the filtration and the ordering is implicit in the defining expression for the object.

1) $$D^0 \times D^n \cup_{\varphi_1} D^{n_i} \times D^{n-n_i} \cup_{\varphi_2} \ldots \cup_{\varphi_\nu} D^\nu \times D^{n-\nu} = X$$

is unambiguously interpretable as referring to the differentiable cell filtration X with presentation $P = \{\varphi_0, 0; \varphi_1, n_1; \ldots; \varphi_\nu, n_\nu\}$. In such cases, without further talk, I may think of X as being a differentiable cell filtration.

I will generally abuse my formalism by speaking of a filtered object X as if it were a differentiable manifold (if $X \in \mathscr{D}_*$) or a topological space (if $X \in \mathscr{F}$), confusing the sequence X with its end-space X_ν. I might, for example, talk of the homotopy or homology groups of X, by which I would mean the homotopy or homology groups of X_ν. By ∂X, I would mean ∂X_ν, etc. The most grievous abuse of language of this kind occurs when $X \in \mathscr{D}_*$. If V is a differentiable manifold I may speak of an imbedding $\varphi : V \to X$ meaning an imbedding $\varphi : V \to X_\nu$; I may speak of a diffeomorphism $\varphi : X \to Y$ for $X, Y \in \mathscr{D}_*$ meaning a differentiable isomorphism $\varphi : X_\nu \to Y_\mu$ where X_ν, Y_μ are the end-spaces of X and Y, and φ need pay no attention to the filtrations of X_ν, Y_μ.

Whenever the " differentiable isomorphism " $\varphi : X \to Y$ is filtration-preserving, i.e. is an isomorphism in the category $\mathscr{D}_*$ I will say so explicitly. I reserve the word diffeomorphism for maps $\varphi : X \to Y$ which do not necessarily preserve filtration.

Another prevalent abuse of language will be the confusion of reordering-equivalent objects N_1, N_2 in situations where the ordering obviously doesn't matter. A notable example of this is the habit of considering a simplicial complex K as a cell filtration.

The category $\mathscr{F}$ of cell filtrations is very close to the category of finite CW-complexes, and to the category of finite simplicial complexes. In the sense that it takes a careful account of the precise filtration, it is a " stronger " category than the category of CW-complexes. In the sense that it takes account of the order of attaching maps, it is a " stronger " category than the category of simplicial complexes. Since the category $\mathscr{F}$ is much more careless about attaching maps than either of the other two categories (objects, defining, as they do, only equivalence classes of topological spaces), in this sense it is weaker than the other categories. It is precisely this " weakness " that is very convenient for the study of such spaces in the context of differential topology.

If $X \in \mathscr{F}$, and $\alpha \in \pi_q(X)$ then the expression $X^* = X \cup_\alpha D^{q+1}$ denotes a unique object $X^* \in \mathscr{F}$. This is useful for the definition of neighborhood (See Chapter IV) for example, where to be given a precise attaching map would be a great hindrance.

The category of finite simplicial complexes also has virtues. One may talk easily of subdivision, piece-wise linear, or piece-wise differentiable imbeddings, etc. When a simplicial complex " gives rise " to an element of $\mathscr{F}$, it always can be made to give rise to a properly ordered element of $\mathscr{F}$, the nicest kind of element that one could wish for.

There is, however, a certain unnaturality to simplical complexes as decomposition of topological spaces that would make it disagreeable to restrict attention to them exclusively, or to phrase general definitions in the language of simplicial complexes.

The compromise that I adopt is to use simplicial complexes only when one of their particular virtues makes them an irresistible convenience, and at those times I will be less interested in them, and more in the cell filtration to which they give rise.

Needless to say, all three categories are equally general. More precisely, if a homotopy class of spaces is representable in one of the three categories, it is representable in all three categories.

CHAPTER IV

NEIGHBORHOODS

The notion of neighborhood. — Since a differentiable cell filtration of a differentiable manifold $\mathscr{M}$ is a geometric description of M, it is of great geometric interest to know the totality of differentiable filtrations admitted by a fixed differentiable manifold M. This problem may be attacked obliquely by asking the following preliminary question: Given a cell filtration X, what is the totality of differentiable cell filtrations $\mathscr{M}$ which, when unthickened, yield X? (i.e. what are the differentiable cell filtrations $\mathscr{M}$ such that $\rho\mathscr{M}=X$?)

These differentiable filtrations play (in some loose sense) the role of " thickenings " of X.

Definition (4.1). — If X is a cell filtration, then a *neighborhood* of X is a differentiable cell filtration such that $\rho N=X$.

I should point out that the term neighborhood is used in a looser sense than usually meant. Since X is an equivalence class of topological spaces, if $\mathscr{M}$ is a neighborhood of X, it shouldn't be expected that X is imbedded in $\mathscr{M}$ in any particular manner. (It will turn out, however, that there is always a cell decomposition representing X which imbeds in $\mathscr{M}$ in some uncanonical way.)

A neighborhood N over X, a cell filtration, is a lifting of X into the realm of differential topology. I shall tend to think of N as a kind of generalized cell bundle over X, or as a thickening of X. The terminology that I choose will be suggestive of these two intuitions. Both attitudes towards neighborhoods are justifiable to some extent.

For example, smooth thickenings of finite simplicial complexes nicely imbedded in differentiable manifolds can always be given differentiable cell filtrations so as to exhibit them as neighborhoods over a cell filtration constructed by properly orderring the cells of the finite complex K.

In the stable theory, neighborhoods of a fixed cell filtration X are closely related to (in one-one correspondence with) stable vector bundles over X.

Let $\mathscr{N}^n(X)$ be the set of distinct n-dimensional neighborhoods of X. $\mathscr{N}^n$ is functorial in a sense to be more fully explained later. If $X \xrightarrow{f} Y$ is an inclusion map, then there is a restriction map $f^{(n)} : \mathcal{N}^n(Y) \to \mathcal{N}^n(X)$ obtained by " forgetting about $N \in \mathcal{N}^n(Y)$ except over the cells of X ". If

$$X = D^{n_0} \cup_{\varphi_1} D^{n_1} \cup_{\varphi_2} D^{n_2} \cup \ldots \cup_{\varphi_\nu} D^{n_\nu} \in \mathcal{F}$$

and $\quad N \in \mathcal{N}^n(X), \quad N = D^{n_0} \times D^{n-n_0} \cup_{\Phi_1} D^{n_1} \times D^{n-n_1} \cup_{\Phi_2} \ldots \cup_{\Phi_\nu} D^{n_\nu} \times D^{n-n_\nu}$

the cell $D^{n_j} \times D^{n-n_j} \subseteq N$ will be said to *lie over* the cell D^{n_j} of X. It may also be referred to as a *thickening* of D^{n_j}.

Definition (**4.2**). — If X, Y are cell filtrations, then by an imbedding

$$\varphi : \mathcal{N}^n(X) \to \mathcal{N}^n(Y)$$

is meant:

a) a map of sets, $|\varphi| : \mathcal{N}^n(X) \to \mathcal{N}^n(Y)$, and either of the following two possibilities obtain:

b) for each $N \in \mathcal{N}^n(X)$, a differentiable imbedding (isomorphism)

$$\varphi_N : N \to \varphi N$$

is given, in which case φ is called covariant.

b') for each $N \in \mathcal{N}^n(X)$, a differentiable imbedding

$$\varphi_N : \varphi N \to N$$

is given, in which case φ is called contravariant.

The imbedding $\varphi : \mathcal{N}^n(X) \to \mathcal{N}^n(Y)$ is called an *isomorphism* when $|\varphi|$ is one to one onto, and φ_N is an isomorphism for all $N \in \mathcal{N}^n(X)$.

The map $\varphi : \mathcal{N}^n(X) \to \mathcal{N}^n(Y)$ is called an *injection*, *surjection*, *bijection* when $|\varphi|$ is.

Roughly the set $\mathcal{N}^n(X)$ is looked upon as a disconnected differentiable manifold and the imbeddings

$$\varphi : \mathcal{N}^n(X) \to \mathcal{N}^n(Y)$$

are just maps which are imbeddings when restricted to connected components. Two maps $\psi, \varphi : \mathcal{N}^n(X) \to \mathcal{N}^n(Y)$ are called *isotopic* if $|\psi| = |\varphi|$ and $\varphi_N \approx \psi_N$ for all $N \in \mathcal{N}^n(X)$.

Generally, imbeddings $\varphi : \mathcal{N}^n(X) \to \mathcal{N}^n(Y)$ will be distinguished only up to isotopy class.

Definition (**4.3**). — If N is a neighborhood over $X \in \mathcal{F}$ or $\mathcal{D}_*$, then a filtration-preserving map $\pi : N \to X$ will be called a *projection*. A map $\beta : X \to N$ such that $\pi \circ \beta$ is filtration-homotopic to the identity will be called a *cross-section*.

It is easily seen that cross-sections and projections are unique up to homotopy class, and they are homotopy-inverses of one another.

Let X, Y be cell filtrations, and $f : X \to Y$ an inclusion map. Then

$$Y = (Y_0, \ldots, Y_\mu)$$
$$X = (Y_0, \ldots, Y_\nu) \qquad \text{for } \nu \leq \mu.$$

There is induced a contravariant differentiable imbedding $f^{(n)}:\mathcal{N}^n(\mathrm{Y})\rightarrow\mathcal{N}^n(\mathrm{X})$ by defining $f^{(n)}\mathrm{N}$, for $\mathrm{N}=(\mathrm{N}_0, \ldots, \mathrm{N}_\mu)\in\mathcal{N}^n(\mathrm{Y})$ as follows:

$$f^{(n)}\mathrm{N}=(\mathrm{N}_0, \ldots, \mathrm{N}_\nu)\in\mathcal{N}^n(\mathrm{X})$$

and

$$f^{(n)}_{\mathrm{N}}: f^{(n)}\mathrm{N}\rightarrow \mathrm{N}$$

is simply the inclusion map.

The assignation $f\rightarrow f^{(n)}$ is functorial for inclusion maps in the sense that if

$$\mathrm{X}\xrightarrow{f}\mathrm{Y}\xrightarrow{g}\mathrm{Z}$$

is a sequence of inclusions, then $(g\circ f)^{(n)}=f^{(n)}\circ g^{(n)}$, and if $1:\mathrm{X}\rightarrow\mathrm{X}$ is the identity map, then $1^{(n)}=1$.

The behavior of neighborhoods under reordering equivalence. — Let $\mathrm{X}_1, \mathrm{X}_2\in\mathcal{F}$ be two properly ordered cell filtrations which are reordering equivalent $(\mathrm{X}_1\equiv\mathrm{X}_2)$.

Proposition **(4.5)**. — If X, Y are reordering equivalent, and properly ordered, then there is a one-one correspondence $\gamma_{\mathrm{X,Y}}:\mathcal{N}^n(\mathrm{X})\underset{\approx}{\rightarrow}\mathcal{N}^n(\mathrm{Y})$ such that for each $\mathrm{N}\in\mathcal{N}^n(\mathrm{X})$, the pair N, γN are reordering equivalent. The correspondence γ enjoys the following functorial property:

a) If $\mathrm{X}\equiv\mathrm{Y}\equiv\mathrm{Z}$, then $\gamma_{\mathrm{X,Z}}=\gamma_{\mathrm{Y,Z}}\circ\gamma_{\mathrm{X,Y}}$

b) $\gamma_{\mathrm{X,X}}=1$ for all $\mathrm{X}\in\mathcal{F}$.

Thus if χ is a reordering equivalence class of properly ordered filtrations, one may talk of $\mathcal{N}^n(\chi)$, identifying all $\mathcal{N}^n(\mathrm{X})$ for $\mathrm{X}\in\chi$ via the γ's. (If you wish to be formal, you may take $\mathcal{N}^n(\chi)$ to be the collection of objects $\{\mathcal{N}^n(\mathrm{X})\mid\mathrm{X}\in\chi\}$, identified by maps

$$\gamma_{\mathrm{X,Y}}:\mathcal{N}^n(\mathrm{X})\underset{\approx}{\rightarrow}\mathcal{N}^n(\mathrm{Y}) \qquad \text{for X, Y}\in\chi).$$

Since a simplicial complex K gives rise to a unique reordering equivalence class of properly ordered cell filtrations, an application of Proposition 4.5 is that we may speak unambiguously of $\mathcal{N}^n(\mathrm{K})$, independent of chosen proper ordering. It is *not* true, however, that $\mathcal{N}^n(\mathrm{K})$ is independent of barycentric subdivision.

Proof of **(4.5)**. — Let there be a reordering equivalence relating X and Y.

Since both X and Y are properly ordered, and reorderings of one another, there is a sequence

$$\mathrm{X}=\mathrm{X}_0, \mathrm{X}_1, \ldots, \mathrm{X}_l=\mathrm{Y}$$

such that each X_i is properly ordered for $i=0, \ldots, l$ and $\mathrm{X}_i, \mathrm{X}_{i+1}$ are elementary reorderings of one another. A typical such elementary reordering is given by the transition from

$$\mathrm{X}_1=\widetilde{\mathrm{X}}\cup_{\varphi_1}\mathrm{D}^{n_1}\cup_{\varphi_2}\mathrm{D}^{n_2}\cup_{\varphi_3}\ldots\cup_{\varphi_\nu}\mathrm{D}^{n_\nu}$$

to $$X_2 = \widetilde{X} \cup_{\varphi_2'} D^{n_2} \cup_{\varphi_1'} D^{n_1} \cup_{\varphi_3'} D^{n_3} \ldots \cup_{\varphi_\nu'} D^{n_\nu}$$

If both X_1 and X_2 are properly ordered, then $n_1 = n_2 = n$.

Let $N_1 \in \mathcal{N}^k(X_1)$. Then:

$$N_1 = \widetilde{N} \cup_{\Phi_1} D^n \times D^{k-n} \cup_{\Phi_2} D^n \times D^{k-n} \cup_{\Phi_3} \ldots \cup_{\Phi_\nu} D^{n_\nu} \times D^{n-n_\nu}$$

$$\Phi_2 : \partial D^n \times D^{k-n} \to \partial(\widetilde{N} \cup_{\Phi_1} D^n \times D^{k-n}).$$

Since $\partial D^n \times D^{k-n}$ has virtual dimension $(n-1)$, and $\dim \partial(\widetilde{N} \cup_{\Phi_1} D^n \times D^{k-n}) = k-1$, $\dim \{0\} \times \partial D^{k-n} = k-n-1$, we may assume, possibly after an isotopy of Φ_2, by general positionality, that $\Phi_2(\partial D^n \times D^{k-n})$ does not meet $\{0\} \times \partial D^{k-n}$. Therefore it can be arranged (after a further isotopy) that $\Phi_2(\partial D^n \times D^{k-n})$ doesn't meet $D^n \times \partial D^{k-n}$. Hence, since $\operatorname{Im} \Phi_2$ and $\operatorname{Im} \Phi_1$ are disjoint, N_1 is reordering equivalent to

$$N_2 = \widetilde{N} \cup_{\Phi_2} D^n \times D^{k-n} \cup_{\Phi_1} D^n \times D^{k-n} \cup_{\Phi_3} \ldots \cup_{\Phi_\nu} D^{n_\nu} \times D^{k-n_\nu}$$

which is a neighborhood in $\mathcal{N}^k(X_2)$. Define $\gamma_{X_1,X_2}(N_1) = N_2$. Then define

$$\gamma_{X,Y} = \gamma_{X_l, X_{l-1}} \circ \ldots \circ \gamma_{X_1, X_0}.$$

The fact that $\gamma_{X,Y}$ is a one-one correspondence follows from functoriality.

Examples of neighborhoods. — Various examples of " non-trivial " ($(n+2)$-dimensional) neighborhoods over a specific filtration of the n-cell will be given in Chapter IX.

Let $\mathcal{L}^n \in \mathcal{F}$ be the filtration given by $\mathcal{L}^n = D^0 \cup_\varphi D^n$.

For every knot $\Gamma \subset S^3$, one may exhibit a neighborhood $N_\Gamma \in \mathcal{N}^4(\mathcal{L}^2)$:

$$N_\Gamma = D^4 \cup_\gamma D^2 \times D^2$$

where

$$\gamma : \partial D^2 \times D^2 \to \partial D^4$$

is some thickening of the knot Γ which has the property that the tangent bundle of the resulting N_Γ is trivial. Call N the *neighborhood of the knot* Γ. For inequivalent knots Γ_1, Γ_2, the neighborhoods N_{Γ_1} and N_{Γ_2} are inequivalent. This example should give an adequate idea of the great assortment of possible neighborhoods that exist.

CHAPTER V

J. H. C. WHITEHEAD'S CONCEPT : SIMPLE HOMOTOPY TYPE

In studying invariants finer than homotopy type, Whitehead [14] introduced the notion of simple homotopy type.

The conclusion of his theory is that although the notion of simple homotopy equivalence is quite " geometrically " defined, there is a purely " algebraic " criterion to determine whether any homotopy equivalence

$$f : X \to Y$$

is a simple homotopy equivalence.

More precisely, Whitehead defines a " torsion group " $W(G)$ which is an abelian group assignable to any group G. If X is a topological space he defines $W(X) = W(\pi_1(X))$. Whitehead then constructs an " obstruction " $\tau(f) \in W$ for any homotopy equivalence $f : X \to Y$ such that f is a simple homotopy equivalence if and only if $\tau(f) = 0$.

This clearly establishes the notion of simple homotopy type as an " algebraic " concept. In many cases, $W(G)$ may be computed to be trivial. (For example: $G = \{0\}, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}$). Thus, for spaces X having as fundamental group such a G, the concepts of simple homotopy equivalence and homotopy equivalence coincide.

If M is a differentiable manifold, it is classical that there is a unique underlying combinatorial structure of M; that is, there exists a smooth triangulation of M which is unique as a simplicial complex up to rectilinear subdivision. Since Whitehead showed that the simple homotopy type of a finite complex is independent of subdivision, it follows that the simple homotopy type of a differentiable manifold is a well-defined notion.

" Unthickening " a differentiable cell filtration $\mathcal{M}$ loses much of the differential-topological structure of $\mathcal{M}$, however the simple homotopy type of $\mathcal{M}$ may still be recovered from the resulting cell filtration $\rho\mathcal{M}$. It turns out to be quite natural to define the simple homotopy type of a cell filtration.

In terms of this concept we may provide a strong necessary condition to the problem posed in Chapter IV; if $f: M \to X$ is a homotopy class of continuous maps of a differentiable manifold M to a cell filtration X: When does there exist a differentiable cell filtration $\mathcal{M}$ of M which is a neighborhood of X such that $f: M \to X$ is a projection map? A necessary condition on f for the existence of such an $\mathcal{M}$ is that f be a simple homotopy equivalence.

It is therefore to be expected that extensions of theorems of Smale to non simply connected situations will involve this notion.

This chapter is devoted to defining simple homotopy equivalence for cell filtrations and quoting results of Whitehead's theory to be used in the sequel.

Let K be a finite simplicial complex, Δ an n-simplex. Let J_0 be the union of all faces but one in the boundary of Δ. Let $\pi : \Delta \to J_0$ be a projection of Δ onto J_0 such that $\pi | J_0$ is the identity map.

Let K^* be the simplicial complex

$$K^* = K \cup_{J_0} \Delta$$

Then, in this circumstance,

$$i : K \to K^*,$$

the inclusion map is called an *elementary expansion*, and $\pi : K^* \to K$, the map defined by

a) $$\pi | \Delta = \pi$$

b) $$\pi | K = 1_K$$

is called an *elementary contraction*.

We shall recall Whitehead's well-known definition (*Formal deformation*, § 13, [14]):

Definition (**5.1**). — Let K, L be finite simplicial complexes. Then a simplicial map $f: K \to L$ is called a *simple homotopy equivalence* if there is a sequence

$$\mathcal{L} : K = K_0 \underset{f_1}{\to} K_1 \underset{f_2}{\to} K_2 \underset{f_3}{\to} \dots K_l = L$$

of simplicial complexes and maps $f_1, \dots, f_l$ such that f_i is either an elementary inclusion or contraction for $i = 1, \dots, l$, and the map f is the composite,

$$f = f_l \circ f_{l-1} \circ \dots \circ f_1.$$

Two simplicial complexes related in this way will be said to be of the same simple homotopy type.

The dimension of a simple homotopy equivalence, f, denoted $\dim f$, is defined to be:

$$\dim f \leq \max \dim K_i = \dim \mathcal{L} \qquad i = 0, \dots, l.$$

(It is taken to be the minimum dimension of all sequences $\mathcal{L}$ exhibiting f as a simple homotopy equivalence.)

As it stands, the simple homotopy type of a simplicial complex K is dependent upon the particular simplicial structure of $|K|$, the underlying topological space. It is unknown whether simple homotopy type is a topological invariant.

It is proved in [13] (Corollary to theorem 7) that the inclusion map $i : K' \to K$

of a subdivision K′ of K is a simple homotopy equivalence. (This proof apparently contains a flaw; cf. C. Zeeman, *Unknotting combinatorial balls*, but the result is nevertheless true. The reader is referred to Zeeman's paper (yet unpublished) for a variant to the Whitehead definition of simple homotopy equivalence (formal deformations) more suitable to general polyhedral spaces (i.e. not tied down to particular triangulations)).

I shall define an analogous notion for the category $\mathcal{F}$ of cell filtrations.

Let $X \in \mathcal{F}$. Let $X^* \in \mathcal{F}$ be any element of $\mathcal{F}$ which is reordering-equivalent to $X \cup_\varphi D^{n-1} \cup_\psi D^n$, where $\varphi : \partial D^{n-1} \to x_0 \in X$ is the constant map, and $\psi : \partial D^n \to X \cup_\varphi D^{n-1}$ is the "identification" $\psi : \partial D^n = S^{n-1} \to D^{n-1}/D^{n-1} \subseteq X \cup_\varphi D^{n-1}$.

Let $i : X \to X^*$ be the inclusion map. Then i is called an *elementary expansion*. Consider the filtration-*non*-preserving map $\pi : X^* \to X$ such that $\pi \circ i = 1$, and $\pi(D^n \cup D^{n-1}) = x_0 \in X$. π is called an *elementary contraction*. If $i : X \to X^*$ is an elementary expansion, then D^{n-1} is called the *free face of the expansion i*, and D^n is called the *cell of the expansion i*.

Again, if
$$X_0 \underset{f_1}{\to} X_1 \underset{f_2}{\to} X_2 \underset{f_3}{\to} \dots \underset{f_l}{\to} X_l$$
is a sequence of elementary expansions and contractions in $\mathcal{F}$, then the composite
$$f = f_l \circ f_{l-1} \circ \dots \circ f_1 : X_0 \to X_l$$
is called a *simple homotopy equivalence in the category $\mathcal{F}$*.

The definition of $\dim f$ is a duplication of the definition of the analogous concept in the category of simplicial complexes.

To relate the notion of simple homotopy type in the category of simplicial complexes to the notion of simple homotopy type in $\mathcal{F}$, one needs the following proposition (to be proved in Chapter XI):

(5.2) (Proposition 11.8) The simplicial complexes K, L are of the same simple homotopy type if and only if their induced cell filtrations $\widetilde{K}$, $\widetilde{L}$ are of the same simple homotopy type.

More exactly, as a consequence of Chapter XI, the underlying combinatorial manifold of a neighborhood of $X \in \mathcal{F}$ is some regular neighborhood of some simplicial complex K which gives rise to X.

Actually the nature of the proofs occurring in Whitehead's paper are such that they quite easily carry over to cell filtrations.

Proposition **(5.3)**. — If K, L are simplicial complexes which are simply connected, and $f : K \to L$ is a homotopy equivalence, then it is a simple homotopy equivalence.

Proof. — See [14] (As mentioned in the beginning of this chapter, this result is also valid if $\pi_1(K) = \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}$.)

Proposition **(5.4)**. — Let $f : K \to L$ be a simple homotopy equivalence of two simplicial complexes. Then f is homotopic to $g : K \to L$ such that:
$$\dim g \leq \text{Max}\{\dim K + 1, \dim L, 3\} + 1.$$

Proof. — See the " addendum " on Page 48 of [14].

Proposition **(5.5)**. — Let $f: X \to Y$ be a simple homotopy equivalence between two cell filtrations X, Y. Then there is a sequence

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_\nu} X = Y$$

such that

a) f_i is an elementary expansion for $0 \leq i < \mu$
b) f_i is an elementary contraction for $\mu \leq i < \nu$
c) $\dim f_0 \leq \dim f_1 \leq \dots \leq \dim f_{\mu-1}$
d) $\dim f_\mu \geq \dim f_{\mu+1} \geq \dots \geq \dim f_\nu$
e) $\dim f_i \leq \text{Max}\{\dim K + 1, \dim L, 3\} + 1$

The above proposition provides a convenient normal form for simple homotopy equivalences. Proposition (5.5) insures that one can find a sequence $\{f_i\}$ such that *e)* is satisfied.

Conditions *a)*, *b)*, *c)* and *d)* are brought by rearranging the order of occurrence of elementary expansions and contractions.

Clearly one may arrange it so that all elementary expansions come first in order of increasing dimension. This achieves *a)*, *b)* and *c)*. The last condition may be achieved by finding a " regular " cell decomposition X_k which represents X_k, up to rearrangement.

(A *regular cell decomposition* is a properly ordered cell decomposition Y such that $Y = (Y_0, \dots, Y_\nu)$; $Y_i = Y_{i-1} \cup_{\varphi_i} D^{n_i}$ where $\varphi_i(\partial D^{n_i})$ is contained in the $(n_i - 1)$-skeleton of Y, for $i = 1, \dots, \nu$. It is quite easy to see that any properly ordered cell filtration is represented by a regular cell decomposition.)

Clearly in the model X_k one may rearrange the sequence of elementary contractions to achieve *d)*.

CHAPTER VI

TRIANGULATIONS AND FILTRATIONS OF DIFFERENTIABLE MANIFOLDS

Definition **(6.1)**. — A differentiable triangulation, δ, of a manifold M is a homeomorphism $\delta : K \to M$ of a simplicial complex K onto M such that if Δ is any simplex of K, $\delta | \Delta$ is a differentiable imbedding of Δ into M.

An *imbedding* $f : K \to M$ of a simplicial simplex K in a differentiable manifold M is a map f which extends to a differentiable triangulation of M. Explicitly, it is a map for which there is a simplicial complex L and a triangulation $\delta : L \to M$ such that

$$\begin{array}{ccc} L & \searrow^{\delta} & \\ \cup\!| & & M \\ K & \nearrow_{f} & \end{array}$$

is commutative.

The following proposition guarantees that a finite complex imbedded in a differentiable manifold possesses a " smooth thickening ". It also shows that in certain circumstances neighborhoods *do* play the role of geometric neighborhoods.

Proposition **(6.2)**. — Let $f : K \to M^n$ be an imbedding. Then there exists an element $N \in \mathcal{N}^n(K)$, a " cross-section " $\beta_N : K \to N$ which is an imbedding, and an imbedding $F : N \to M^n$ such that

$$\begin{array}{ccc} K & \xrightarrow{f} & M^n \\ {}_{\beta_N}\searrow & & \nearrow_{F} \\ & N & \end{array}$$

f factors: $f = F \circ \beta_N$.

Proof. — We need prove (6.2) only in the case where f is a triangulation since the result would then follow for any imbedding by restriction.

Assume, inductively, that we have obtained a neighborhood extension to the k-skeleton

$$\begin{array}{ccc} K^{(k)} & \xrightarrow{f} & M^n \\ {}_{\beta_N^{(k)}}\searrow & & \nearrow_{F^{(k)}} \\ & N^{(k)} & \end{array}$$

which has the additional property that

(A) $F^{(k)}(\partial N^{(k)})$ intersects every simplex Δ^q of K transversally, where $q>k$.

(B) the pair $(\Delta^q, \Delta^q \cap F^{(k)}(N^{(k)}))$ is differentiably isomorphic with the pair (Δ^q, A^q_k). (See § 2 for definition of A^q_k.)

It shall be proven that one may construct a neighborhood extension of $K^{(k+1)}$ with the same properties. I shall construct a neighborhood extension of each $(k+1)$-cell, one at a time.

Let $M^{(k)} = F^{(k)}(N^{(k)})$. For each simplex Δ^{k+1} of the $(k+1)$-skeleton of K, let

$$B^{k+1} = \Delta^{k+1} - \Delta^{k+1} \cap \operatorname{int} M^{(k)}.$$

Since (by (A) of the inductive hypothesis) $\Delta^{k+1} \cap \operatorname{int} M^{(k)} = A^{k+1}_k$, B^{k+1} is isomorphic to a $(k+1)$-cell. If $W^{k+1} = M - \operatorname{int} M^{(k)}$, by (B), B^{k+1} is transversally imbedded in W^{k+1}.

(6.3) Normalization. — There is an open neighborhood U of Δ^{k+1} in M and an imbedding $\gamma : U \to R^n$ satisfying these properties:

(i) If $R^n = R^{k+1} \times R^{n-k-1}$,

$$\gamma : \Delta^{k+1} \to R^{k+1} \times \{0\} \subseteq R^n$$

is a linear imbedding of Δ^{k+1} in $R^{k+1} \times \{0\}$.

(ii) If $S^k = \partial B^{k+1}$, $O^{n-k-1} \subseteq R^{n-k-1}$ some neighborhood of $\{0\} \in R^{n-k-1}$, then

$$\gamma : \partial M^{(k)} \cap U \to S^k \times O^{n-k-1} \subseteq R^{k+1} \times R^{n-k-1} = R^n$$

The construction of such a normalization (γ, U) is foundational. Clearly a pair $(\bar{\gamma}, \bar{U})$ may be obtained satisfying only (i). Using (B) of the inductive hypothesis and the tubular neighborhood lemma, one may modify $(\bar{\gamma}, \bar{U})$ to a pair (γ, U) satisfying both (i) and (ii).

Take $N'_r(\Delta^{k+1}) = B^{k+1} \times D^{n-k-1}_r \subseteq R^{k+1} \times R^{n-k-1}$ for small $r>0$.

Let $N_r(\Delta^{k+1}) = \gamma^{-1} N'_r(\Delta^{k+1})$. For small enough r, the expression

$$N^{(k+1)}_r = N^{(k)} \cup \bigcup_{\Delta^{k+1} \in K} N_r(\Delta^{k+1}) \tag{6.4}$$

makes sense, for the $N_r(\Delta^{k+1})$'s are disjoint for distinct Δ^{k+1}'s. For arbitrary ordering of the $(k+1)$-simplices of K, (6.4) expresses $N^{(k+1)}_r$ as a neighborhood of $K^{(k+1)}$ (i.e. $N^{(k+1)}_r \in \mathcal{N}^n(K^{(k+1)})$.)

There is an obvious inclusion mapping

$$\beta^{(k+1)}_N : K^{(k+1)} \to N^{(k+1)}$$

and

$$N^{(k+1)} \xrightarrow{F^{(k+1)}} M.$$

We need only check that if r is chosen small enough, conditions (A), (B) needed to propagate the induction are still satisfied. These are, however, local questions which may be verified by going to the normalization $\gamma : U(\Delta^{k+1}) \to R^n$ of each Δ^{k+1} in $K^{(k+1)}$.

Therefore, any triangulation $\delta : K \to M$ induces a properly ordered filtration of M, which is intuitively a natural thickening of the triangulation. This demonstrates the existence of a properly ordered filtration of M, a fact necessary for later applications. Smale deduces the existence of such properly ordered filtrations by constructing (in his terms) a " nice function " [10] where stable and unstable manifolds of critical points intersect generically. Smale's method proves much more, however the above construction is direct, and considerably simpler.

CHAPTER VII

REMOVAL OF ZERO-DIMENSIONAL INTERSECTIONS

One of the objects of our analysis of neighborhoods is to determine how $\mathcal{N}^n(X)$ varies as X is replaced by another cell filtration of the same simple homotopy type This is reduced to a comparison of $\mathcal{N}^n(X)$ and $\mathcal{N}^n(Y)$ where

$$Y = X \cup_0 D^{k-1} \cup_1 D^k$$

(or a rearrangement).

In a particular situation we shall prove that there is an isomorphism

$$\mathcal{N}^n(X) \underset{\approx}{\rightarrow} \mathcal{N}^n(Y).$$

It is fairly easy to obtain a map from $\mathcal{N}^n(X)$ to $\mathcal{N}^n(Y)$ which associates to any neighborhood N_0 over X a neighborhood over Y which is diffeomorphic to N_0. The difficult part is to go in the reverse direction showing that any neighborhood N over Y " comes from " a neighborhood over X. (This is false unless $n > \max\{\dim X, \dim Y\} + 2$.)

Assume that a neighborhood N of Y may be written:

$$N = N_0 \cup_{\psi_0} D^{k-1} \times D^{n-k+1} \cup_{\psi_1} D^k \times D^{n-k}$$

(This is slightly incorrect in that the ordering of attached cells needn't be as written. However it is convenient for notational purposes, and anything to be said holds as well for more general orderings.)

In order to show that a neighborhood N actually comes from a neighborhood N_0 of X by " irrelevant additions " we must regularize the attaching maps ψ_0, ψ_1 in some way. The plan is to first study:

$$\overline{\psi}_1 : \partial D_k \times \{0\} \rightarrow N_0 \cup_{\psi_0} D^{k-1} \times D^{n-k+1}$$

The object of this chapter is to show that one may arrange it (by suitable isotopies) so that if $D_+^{k-1} \subseteq D^k \times \{0\} = S^{k-1}$ is the upper-hemisphere $\overline{\psi}_1 | D_+^{k-1}$ may be given by the composite

$$D_+^{k-1} \underset{K}{\rightarrow} D^{k-1} \times \{p_0\} \subseteq D^{k-1} \times D^{n-k+1} \subseteq N_0 \cup_{\psi_0} D^{k-1} \times D^{n-k+1}$$

where K is the standard identification and $\overline{\psi}_1 | D_-^{k-1} \subseteq N_0$.

It is elementary to arrange this if $\overline{\psi}_1(S^{k-1})$ intersects $\{0\} \times D^{n-k+1}$ in a single point. (The reader is urged to draw a picture !)

The bulk of this chapter is therefore concerned with the problem of simplifying, by isotopy, the intersection set of complementarily dimensioned submanifolds of a given manifold. The techniques are standard (Whitney, Haefliger) with one new difficulty: the given manifold needn't be simply connected. Therefore one shuffles between it and its universal covering space.

Throughout this section, the letters, V, W, Y will refer to a triple of differentiable manifolds satisfying these conditions:

a) $V \subseteq Y$, $W \subseteq Y$, both V and W are connected.

b) $\dim V = n$, $\dim W = m$, $\dim Y = n + m$.

c) If $S = V \cap W \subseteq Y$, then for $p \in S$,

(7.1) $$T_p(V) + T_p(W) = T_p(Y) \qquad \text{(in which case, by } b),$$
$$T_p(V) \cap T_p(W) = \{0\}).$$

d) $S \subset \text{int } V \cap \text{int } W$.

The object of this section is to " simplify " S by describing automorphisms $\alpha : Y \to Y$ such that $\alpha(V) \cap W$ is " smaller " than S and $\alpha \approx 1$.

Assume Y, V, W satisfy (7.1) and are all oriented. Let $p \in S = V \cap W$. Define I_p to be plus or minus one according as the orientation on $T_p(V) \oplus T_p(W)$ induced from the orientations of V and W coincides or conflicts with the orientation on $T_p(Y)$ induced by the orientation of Y.

If $p, q \in S = V \cap W$, define

$$I_{p,q} = I_p . I_q .$$

Lemma **(7.2)**. — Let Y, V, W be orientable manifolds satisfying (7.1). Let $p, q \in S$. Then the number $I_{p,q}$ obtained by choosing orientations of Y, V, W is independent of the orientations chosen.

In a certain context, geared for later application, the number $I_{p,q}$ may be invariantly defined regardless of the orientability of Y, V, W. It is in the case where one may isolate the intersections p, q to a small region by being supplied with the following data, which I'll call isolation data.

Definition **(7.3)**. — Let V, W, Y satisfy (7.1) and $p, q \in S$. Let $\gamma : I \to V$, $\lambda : I \to W$ be arcs in V, W such that

(i) $\gamma(\text{int } I) \cap S = \lambda(\text{int } I) \cap S = \emptyset$

(ii) $\gamma(0) = \lambda(1) = p$; $\lambda(0) = \gamma(1) = q$.

(iii) If $S^1 = \gamma(I) \cup \lambda(I)$ is the piece-wise differentiable closed curve formed by the union of both arcs, there is a nonsingular disc $D^2 \subset Y$ (creased at its boundary) such that

(iv) $\partial D^2 = S^1$

(v) $\text{int } D^2 \cap V = \text{int } D^2 \cap W = \emptyset$

(vi) D^2 meets V transversally at $\gamma(I)$.

In such a case, the triple $(\gamma, \lambda; D^2)$ will be called *isolation data* for the intersections $\{p, q\} \subseteq S$.

It should be mentioned that if $m>2$, and $n+m\geq 4$, the existence of arcs γ, λ satisfying (i), (ii) such that

(iii')
$$S^1 = \gamma(I) \cup \lambda(I)$$

is a closed curve homotopic to zero in Y insures the existence of isolation data $(\gamma, \lambda; D^2)$.

This is so by general positionality. A pair (γ, λ) satisfying (i), (ii), (iii') I'll call *homotopy-isolation data.*

If $(\gamma, \lambda; D^2)$ is a set of isolation data for $p, q \in S$ then, regardless of orientability of V, W, or Y, a number $I_{p,q}$ may be defined in terms of $(\gamma, \lambda; D^2)$.

Let $\widetilde{Y}$ be a tubular neighborhood of D^2 in Y which intersects V, W nicely. That is,

a) $\widetilde{Y} \cap V = \widetilde{V}$ is a tubular neighborhood, in V, of $\gamma(I) \subset V$.

(7.4) *b)* $\widetilde{Y} \cap W = \widetilde{W}$ is a tubular neighborhood, in W, of $\lambda(I) \subset W$.

Then the triple $\widetilde{Y}$, $\widetilde{V}$, $\widetilde{W}$ satisfies (7.1) and $\widetilde{S} = \widetilde{V} \cap \widetilde{W} = \{p, q\}$. Moreover, $\widetilde{Y}$, $\widetilde{V}$, $\widetilde{W}$ are orientable, for they are diffeomorphic with Euclidean cells. Define $I_{p,q} = I_{p,q}(\gamma, \lambda; D^2)$ to be the number $\widetilde{I}_{p,q}$ computed for $p, q \in \widetilde{S}$, with respect to the triple $\widetilde{Y}$, $\widetilde{V}$, $\widetilde{W}$.

Let, again, Y, V, W satisfy (7.1), $p, q \in S$ and $S^* = S - \{p, q\}$.

Lemma **(7.5)**. — *(Removal of Intersection.)*

Let $(\gamma, \lambda; D^2)$ be isolation data for $p, q \in S$. Let $I_{p,q} = I_{p,q}(\gamma, \lambda; D^2) = -1$.

Then there is an automorphism $\alpha : Y \to Y$ such that

(i) $\alpha \approx 1$.

(ii) If $\widetilde{Y}$ is some tubular neighborhood of D^2 satisfying (7.4), then the automorphism α is the identity in the complement of $\widetilde{Y}$.

(iii) If $\alpha(V) = V^*$

and
$$\widetilde{V}^* = V \cap \widetilde{Y}, \quad \widetilde{W} = W \cap \widetilde{Y},$$

then
$$\widetilde{V}^* \cap \widetilde{W} = \emptyset.$$

As a consequence of Lemma 7.5, the triple V^*, W, Y again satisfies (7.1), and $V^* \cap W = S^* = S - \{p, q\}$.

Lemma 7.5 is rather well-known. It is implicit in the work of Whitney [15] and Haefliger [1]. Roughly, one uses the disc D^2 as a " guide " to deform V along the vicinity of $\gamma(I)$ " through " $\lambda(I)$ by isotopy. The fact that $I_{p,q} = -1$ insures that in $\widetilde{Y}$, the final result of the deformation is to detach $\widetilde{V}^*$ from $\widetilde{W}$. This local isotopy may be extended to a global isotopy, the end product of which is the automorphism α which is sought. Therefore, the crux of the matter is to do things in $\widetilde{Y}$. But $\widetilde{Y}$ may be recoordinatized in a standard manner (using the fact: $I_{p,q} = -1$):

Let $(x, t, z, s) \in R^{n-1} \times R \times R^{m-1} \times R$ represent a general point of R^{n+m}. Then one may take $\widetilde{Y}$ to be the cube:

$$(7.6)\qquad \begin{cases} \widetilde{Y} = \{y = (x, t, z, s) \mid \frac{1}{2} \leq s \leq 2,\ |z| \leq 2,\ |t| \leq 2,\ |x| \leq 2\} \\ D^2 = \{d = (0, t, 0, s) \mid s \geq 0,\ s^2 + t^2 \leq 1\} \\ \widetilde{V} = \{v \in \widetilde{Y},\ v = (x, t, 0, s) \mid s^2 + t^2 = 1\} \\ \widetilde{W} = \{w \in \widetilde{Y},\ w = (0, t, y, 0)\} \end{cases}$$

In terms of this normalization, the α to be chosen is clear.

I shall deduce a corollary of 7.5 more explicitly suited for later application.

Let M_0 be a differentiable $(n+m+1)$-dimensional manifold with nonvacuous boundary. Let $M = M_0 \cup_\varphi D^n \times D^{m+1}$, where $\varphi : \partial D^n \times D^{m+1} \to \partial M_0$ is an imbedding. Let (S^n, s_0) be a base pointed sphere and

$$c : (M, M_0) \to (S^n, s_0)$$

the natural mapping which identifies M_0 to s_0, projecting $(D^n \times D^{m+1}, \partial D^n \times D^{m+1})$ onto (S^n, s_0) after the usual identification of $(D^n, \partial D^n)$ with (S^n, s_0). The mapping $c : (M, M_0) \to (S^n, s_0)$ is a homotopy equivalence and therefore induces an isomorphism:

$$c_* : \pi_n(M, M_0) \underset{\approx}{\to} \pi_n(S^n, s_0).$$

Corollary (7.7). — Let $M = M_0 \cup_\varphi D^n \times D^{m+1}$ as above, where $m > 2$ and $\dim M \geq 5$. Assume φ is homotopic to the constant map, so that there is a homotopy equivalence

$$\mu : M \underset{\sim}{\to} M_0 \vee S^n.$$

Let $f : S^n \to M$ be an imbedding such that if $\sigma : S^n \to M_0 \vee S^n$ is the inclusion mapping, then the triangle

$$(7.8)\qquad \begin{array}{ccc} S^n & \xrightarrow{f} & M \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & M_0 \vee S^n & \end{array}$$

is homotopy commutative.

Then there is an automorphism $\alpha : M \to M$ such that

(i) $\alpha \approx 1$.

(ii) $\alpha f(S^n)$ intersects $\{0\} \times \partial D^{m+1}$ transversally at precisely a single point $x \in \partial M$.

Proof. — By an initial automorphism $\alpha_0 : M \to M$, $\alpha_0 \approx 1$, it may be arranged that $\alpha_0 f(S^n) = V$ intersects $\{0\} \times \partial D^{m+1} = W$ transversally, so that the triple, $Y = \partial M$, V, W satisfies (7.1).

Let $\pi:\widetilde{M}\rightarrow M$ be the universal covering space of M. Let $\Pi=\pi_1(M)$. Then $f:S^n\rightarrow M$ extends to a lifting

$$\begin{array}{ccc} & \overset{\tilde f}{\nearrow} & \widetilde{M} \\ S^n & & \downarrow \pi \\ & \underset{f}{\searrow} & M \end{array}$$

since $n>1$. The inverse image of $W, \widetilde{W}=\pi^{-1}(W)$, consists in $[\Pi:1]$ disjoint copies of $W=\{0\}\times\partial D^{m+1}$

$$\widetilde{W}=\bigcup_{\omega\in\Pi} W_\omega.$$

It is clear that $\tilde f(S^n)$ intersects $\widetilde{W}$ transversally. Let $\widetilde{S}=\tilde f(S^n)\cap\widetilde{W}$. Since $\pi|\tilde f(S^n)$ is an isomorphism, the map $\pi:\widetilde{S}\rightarrow S$ establishes a one-one correspondence between $\widetilde{S}$ and S. Define $\widetilde{S}_\omega=\tilde f(S^n)\cap W_\omega$ for $\omega\in\Pi$; $S_\omega=\pi(\widetilde{S}_\omega)$. Then

$$\widetilde{S}=\bigcup_{\omega\in\Pi}\widetilde{S}_\omega;\ S=\bigcup_{\omega\in\Pi}S_\omega,$$

the unions being disjoint.

A more explicit description of $\widetilde{M}$ may be given as follows:

If $\widetilde{M}_0\overset{\pi_0}{\rightarrow}M_0$ is the universal covering space of M_0, then

$$\widetilde{M}=\widetilde{M}_0\cup\Big\{\bigcup_{\omega\in\pi_1(M_0)}(D^n\times D^{m+1})_\omega\Big\}.$$

The universal covering space $\widetilde{M_0\vee S^n}$ of $M_0\vee S^n$ can be written

$$\widetilde{M_0\vee S^n}=\widetilde{M}_0\vee\Big\{\bigvee_{\omega\in\pi_1(M_0)}(S^n)_\omega\Big\}.$$

There is a homotopy commutative diagram

(7.9)
$$\begin{array}{ccc} \widetilde{M} & \underset{\sim}{\overset{\tilde\mu}{\longrightarrow}} & \widetilde{M_0\vee S^n} \\ \pi\downarrow & & \downarrow\pi' \\ M & \underset{\sim}{\overset{\mu}{\longrightarrow}} & M_0\vee S^n \end{array}$$

where μ and $\tilde\mu$ are homotopy equivalences.

Combining (7.9) with (7.8) yields that

$$\widetilde{\mu\tilde f}:S^n\rightarrow\widetilde{M}_0\vee\Big\{\bigvee_{\omega\in\pi_1(M_0)}(S^n)_\omega\Big\}$$

must be homotopic to

$$i_{\omega_0}:S^n\underset{\approx}{\rightarrow}S^n_{\omega_0}\subseteq\widetilde{M}_0\vee\Big\{\bigvee_{\omega\in\pi_1(M_0)}(S^n)_\omega\Big\}$$

for some $\omega_0\in\pi_1(M_0)$.

This may be interpreted as saying that the homological intersection number I_ω of $\widetilde{f}(S^n)$ and W_ω is given by

$$I_\omega = \pm\delta_{\omega,\omega_0} = \begin{cases} \pm 1 & \text{if } \omega=\omega_0 \\ 0 & \text{otherwise} \end{cases}.$$

In terms of the intersection loci $\widetilde{S}_\omega$ themselves, the above fact can be taken to mean that

$$\widetilde{S}_\omega = \{p_1^\omega, q_1^\omega, p_2^\omega, q_2^\omega, \ldots, p_{t_\omega}^\omega, q_{t_\omega}^\omega\} \qquad \text{for } \omega \neq \omega_0$$

(7.10) $$\widetilde{S}_{\omega_0} = \{p_1^{\omega_0}, q_1^{\omega_0}, \ldots, p_{t_{\omega_0}}^{\omega_0}, q_{t_{\omega_0}}^{\omega_0}; x\} \qquad \text{where}$$

$$I_{p_j^\omega, q_j^\omega} = -1 \qquad \text{for all } \omega\in\pi_1(M_0),\ j\leq t_\omega.$$

The points of $\widetilde{S}_\omega$ for $\omega\neq\omega_0$ may be paired off into oppositely oriented couples of points, and the points of $\widetilde{S}_{\omega_0}$ may be so paired off, with a single point, x, left over.

Let $\widetilde{p}, \widetilde{q}\in\widetilde{S}_\omega$ refer to an arbitrary pair $p_j^\omega, q_j^\omega\in\widetilde{S}_\omega$ for some $\omega\in\pi_1(M_0)$. Let $\widetilde{\lambda}=\widetilde{\lambda}_{\widetilde{p},\widetilde{q}}$ be an arc in W_ω, such that $\widetilde{\lambda}(0)=\widetilde{p}$, $\widetilde{\lambda}(1)=\widetilde{q}$. Let $\widetilde{\gamma}=\widetilde{\gamma}_{\widetilde{p},\widetilde{q}}$ be an arc in $\widetilde{f}(S^n)$ such that $\widetilde{\gamma}(0)=\widetilde{q}$, $\widetilde{\gamma}(1)=\widetilde{p}$.

Since $\widetilde{M}$ is simply connected, the pair $(\widetilde{\gamma}_{\widetilde{p},\widetilde{q}}, \widetilde{\lambda}_{\widetilde{p},\widetilde{q}})$ is homotopy isolation data for $\widetilde{p}, \widetilde{q}$. Applying the projection map π to $\widetilde{f}(S^n)$ and W_ω, we see that if $\gamma=\pi\widetilde{\gamma}$, $\lambda=\pi\widetilde{\lambda}$, $\pi\widetilde{p}=p$, $\pi\widetilde{q}=q$, then the pair (γ,λ) provides homotopy isolation data for $p, q\in S$.

Since $\widetilde{p}$, $\widetilde{q}$ are oppositely oriented intersections in $\widetilde{M}$, $I_{p,q}=-1$, and since $m>2$, $n+m\geq 4$, the existence of homotopy isolation data shows the existence of isolation data. Therefore Lemma 7.5 applies and we may " remove " the intersections p, q. This may be repeated for all couples p_j^ω, q_j^ω, eventually reducing the intersection set S to the single point $\{x\}$, proving Corollary 7.7.

Let $D_+^n\subset S^n$ be the upper hemisphere

$$D_+^n=\{(x_0,\ldots,x_n)\in S^n,\ x_n\geq 0\}.$$

Let $K: D_+^n \underset{\approx}{\rightarrow} D^n$ be the identification

$$K(x_0,\ldots,x_n)=(x_0,\ldots,x_{n-1})\in D^n.$$

Let $K': D^n\rightarrow D_+^n$ be the identification by stereographic projection.

Let $D_+^n(r)\subseteq D_+^n$ be the polar cap of radius r,

$$D_+^n(r)=K^{-1}(D_r^n) \qquad \text{for } 0\leq r\leq 1.$$

Let $f: S^n\rightarrow\partial M=\partial(M_0\cup_\varphi D^n\times D^{m+1})$ be as in the situation of Corollary 7.7. That is,

$$\begin{array}{ccc} S^n & \xrightarrow{f} & M \\ & {}_\sigma\searrow \quad \swarrow_\mu & \\ & M_0\vee S^n & \end{array}$$

is homotopy-commutative. By the conclusion of 7.7, f is isotopic to an imbedding $g: S^n \to M$, $g = \alpha f$, such that $g(S^n) \cap \{o\} \times \partial D^{m+1}$ consists of a single point, say $x = (o, p_0)$ for $p_0 \in \partial D^{m+1}$.

There is no loss of generality in assuming that $g^{-1}(x) = q_0 = (o, \ldots, o, 1) \in S^n \subseteq R^{n+1}$, the " north pole ".

It is clear that there is a sufficiently small $\varepsilon > o$ such that

$$g(S^n) \cap \{p\} \times \partial D^{m+1}$$

consists of a single point for $p \in D^n_\varepsilon$, in fact, after isotopy, it may be assumed that

$$g(s) = (K(s), p_0)$$

for $$s \in D^n_+(\varepsilon) \subseteq S^n, \quad K(s) \in D^n_\varepsilon, \quad p_0 \in \partial D^{m+1}.$$

Let $\zeta_t : S^n \to S^n$ be an isotopy of S^n through itself, with the properties

(i) $$\zeta_0 = 1,$$

(ii) $$\zeta_t(x) = K^{-1} \circ \{(1 - t + t/\varepsilon) \circ K(x)\} \qquad \text{for } x \in D^n_+(\varepsilon).$$

Therefore ζ_t stretches $D^n_+(\varepsilon)$ over D^n_+.

Let $\zeta'_t : M \to M$ be an isotopy with the properties

(i)' $$\zeta'_0 = 1,$$

(ii)' $$\zeta'_t(q, p) = (\{(1 - t + t/\varepsilon) \circ q\}, p) \qquad \text{for } (q, p) \in D^n \times D^{m+1}.$$

Clearly there exist isotopies ζ_t, ζ'_t satisfying (i), (ii) and (i)', (ii)'.

Define $h : S^n \to \partial M$ by $h = (\zeta'_1) g \zeta_1^{-1}$. Then h is isotopic to g, and

a) $$h(s) = (K(s), p_0) \qquad \text{for } s \in D^n_+.$$

b) $$h(s) \in M_0 \qquad \text{for } s \in D^n_-.$$

Thus Corollary 7.7 may be improved as follows:

Corollary **(7.11)**. — Let $f : S^n \to \partial M$ be precisely as in Corollary 7.7. Then there is an automorphism $\alpha : M \to M$ isotopic to the identity such that if $g = \alpha f$, then

(i) $$g(s) = (K(s), p_0) \in \partial M \qquad \text{for } s \in D^n_+,$$

and p_0 a fixed point of ∂M^{m+1};

(ii) $$g(s) \in M_0 \qquad \text{for } s \in D^n_-.$$

It is this form of " removal of intersections " that will be useful later on.

CHAPTER VIII

THE NONSTABLE NEIGHBORHOOD THEOREM

The object of this section is to show that the set $\mathcal{N}^n(X)$, considered as a set of differentiable manifolds is dependent only upon the simple homotopy type of X if $n > \dim X + 2$. It will be shown that this is no longer true if $n = \dim X + 2$. Roughly, the reason for this is that the range of dimensions $n = \dim X + 2$ is the " knot range ". The existence of knots destroys the possibility of $\mathcal{N}^n(*)$ being dependent only upon simple homotopy type. This result is considerably stronger than the stable version of the theorem, below, which says that $\mathcal{N}^n(X)$ is dependent only upon the simple homotopy type of X if $n \geq 2 \dim X + 1$. The methods used in the proof are quite definitely nonstable methods. An interesting thing is that the only general positionality result employed is (7.11) which is a statement going only one dimension into the nonstable range (i.e. it concerns the simplification of intersects of complementary dimensioned manifolds in an ambient space).

The theorem is proven in two parts. The first part is Lemma 8.3 which says that if $f : X \to X^*$ is an elementary expansion and $k > \dim f + 2$ then f induces a bijective isomorphism denoted $f_{(k)} : \mathcal{N}^{(k)}(X) \underset{\approx}{\to} \mathcal{N}^{(k)}(X^*)$. This follows from the fact that in this range of dimensions the normalization (7.11) may be applied.

The second part is a discussion of the elementary expansions and contractions f_q comprizing a simple homotopy equivalence $f = f_l \circ \ldots \circ f_1 : X \to Y$ such that $\dim f_q > \dim X, \dim Y$. Each expansion cell D^{n_q} with $n_q > \max\{\dim X, \dim Y\}$ must eventually be deleted, and each contraction cell D^{n_q} for $n_q > \max\{\dim X, \dim Y\}$ must have been at some prior time an expansion cell. The problem of the second part is to define $(f_q) : \mathcal{N}^k(X_q) \to \mathcal{N}^k(X_{q+1})$ for all the troublesome indices q such that $n_q > \max(\dim X, \dim Y)$. If q represents an expansion, any neighborhood N over X_q may be " turned into " a neighborhood over X_{q+1} in a trivial way (See (8.1) and Figure 1). If q represents a contraction: $X_q = X_{q+1} \cup D_1^{n_q - 1} \cup D^{n_q}$ where $D_1^{n_q-1}$ was the expansion free face with respect to which D^{n_q} was previously added, then any neighborhood N_q over X_q which came from a neighborhood N over X (i.e. $N_q = \hat{f}_{q-1} \circ \ldots \circ \hat{f}_1(N)$) may be " turned into " a neighborhood N_{q+1} over X_{q+1} simply by " cancelling " the thickened cells lying over $D_1^{n_q-1}$ and D^{n_q}. This cancellation is possible because those thickened cells had previously been introduced in a particularly trivial manner by one of the prior $\hat{f}_p$

(for p an expansion index less than q). Moreover, N_{q+1} is diffeomorphic with N_q because N_q is obtainable from N_{q+1} by an irrelevant addition (of a k-cell).

Complications arise in defining

$$(f_q) : \mathcal{N}^{(k)}(X_q) \to \mathcal{N}^{(k)}(X_{q+1})$$

especially when the index q represents a contraction of a cell D^{n_q} via a contraction free face $D_1^{n_q-1}$ different from the expansion free face $D_0^{n_q-1}$ with respect to which D^{n_q} " came in ".

In that case, some elaborate isotopies are needed before cancellation is possible. [See Figure 3]. The letters K, X, Y will stand for cell filtrations in Chapter VIII. Because of the profusion of different cells D^m, I shall reserve the letter Δ^m to designate cells occurring in elementary expansions or contractions.

Lemma **(8.1)**. — Let $i : K \to K^* = K \cup_\varphi \Delta^n \cup_\psi \Delta^{n+1}$ be an elementary expansion. There is a differentiable injection

$$i_{(k)} : \mathcal{N}^k(K) \to \mathcal{N}^k(K^*)$$

induced if $k \geq n+1$, such that

(i) $i^{(k)} \circ i_{(k)} = 1$

(ii) the pair $(i_{(k)}N, N)$ is differentiably isomorphic to $(N \cup_{D_+^{k-1}} D^k, N)$, for all $N \in \mathcal{N}^k(K)$. (In particular, $i_{(k)}N \approx N$ for all $N \in \mathcal{N}^k(K)$).

Idea of Proof. — Roughly, the manifold $i_{(k)}N$ consists in N with a thickened n-cell added in a " trivial " way, with a thickened $(n+1)$-cell added which " caps " the hole produced, the capping also done in a standard way; this makes $i_{(k)}N$ into a neighborhood over K^*. Further, the net result of the process is to have added a k-cell D^k to N via an attaching map of a $(k-1)$-cell on ∂D^k to a $(k-1)$-cell on N (i.e. *an irrelevant addition*). This doesn't alter the diffeomorphy type of N. The details follow. To each $N \in \mathcal{N}^{(k)}(K)$, let $D^{k-1} \subset \partial N$ be some differentiable cell. Let $A^{k-1} \subset D^{k-1}$ be the solid n-dimensional ring,

$$A^{k-1} = \left\{ (x, y) \in R^{k-1} = R^n \times R^{k-n-1},\ 1/2 \leq ||x|| \leq 1\frac{1}{2},\ 0 \leq ||y|| \leq \frac{1}{2} \right\},$$

$$A^{k-1} = S^{n-1} \times I \times D^{k-n-1} = S^{n-1} \times D^{k-n}.$$

Define $N_1 = N \cup_\varphi D^n \times D^{k-n}$, where

$$\varphi : \partial D^n \times D^{k-n} \to A^{k-1} \subset \partial N$$

is the identity map. N_1 is taken to be a differentiable cell filtration with the obvious filtration.

$$\partial N_1 = D^n \times \partial D^{k-n} \cup \{\partial N - A^{k-1}\}.$$

Define $N_2 = N_1 \cup_\psi D^{n+1} \times D^{k-n-1}$

where $$\psi : \partial D^{n+1} \times D^{k-n-1} \to \partial N_1$$

is given as follows:

(i) $$\psi(x,y)=(\mathrm{K}(x),\ \mathrm{K}'(y))\in \mathrm{D}^n\times\partial\,\mathrm{D}^{k-n}$$

if $$x\in \mathrm{D}^n_+\subseteq \partial \mathrm{D}^{n+1},\quad y\in \mathrm{D}^{k-n-1}$$

(ii) $$\psi(x,y)=(\mathrm{K}(x),y)\in \mathrm{D}^n\times \mathrm{D}^{k-n-1}\subseteq \mathrm{D}^{k-1}\subseteq\partial \mathrm{N}_1$$

if $x\in \mathrm{D}^n, y\in \mathrm{D}^{k-n-1}$ (for definitions of $\mathrm{K}(x)$, $\mathrm{K}'(y)$, see p. 40)

Then $i_{(k)}\mathrm{N}=\mathrm{N}_2\in\mathcal{N}^k(\mathrm{K}^*)$, and property (i) of (8.1) is immediate. Define $\mathscr{E}^k$ to be the set-theoretic union

$$\mathscr{E}^k=\mathrm{D}^n\times \mathrm{D}^{k-n}\cup \mathrm{D}^{n+1}\times \mathrm{D}^{k-n-1}\subseteq \mathrm{N}_2.$$

For ease of reference, let

$$\mathscr{H}^{(n)}=\mathrm{D}^n\times \mathrm{D}^{k-n}\subseteq\mathscr{E}^k;\quad \mathscr{C}^{(n+1)}=\mathrm{D}^{n+1}\times \mathrm{D}^{k-n-1}\subseteq\mathscr{E}^k$$

then $$\mathscr{E}^k=\mathscr{H}^{(n)}\cup\mathscr{C}^{(n+1)}$$

($\mathscr{H}$ is for handle and $\mathscr{C}$ is for cap). The cell $\mathscr{H}^{(n)}=\mathrm{D}^n\times \mathrm{D}^{k-n}$ is a thickening of Δ^n, the free face of the expansion, and the cell $\mathscr{C}^{(n+1)}=\mathrm{D}^{n+1}\times \mathrm{D}^{k-n-1}$ is a thickening of Δ^{n+1}. The most commodious parametrization of $\mathscr{E}^k$ is given by

$$\mathscr{E}^k=\{d=(x,y,r)\in \mathrm{R}^n\times \mathrm{R}^{k-n-1}\times \mathrm{R}\mid \|d\|\leq 2,\ r\geq 0\}$$

where

$$\mathrm{D}^n\times \mathrm{D}^{k-n}=\{d\in\mathscr{E}^k\mid \|x\|\geq 1\}=\mathscr{H}^{(n)}$$

$$\mathrm{D}^{n+1}\times \mathrm{D}^{k-n-1}=\{d\in\mathscr{E}^k\mid \|x\|\leq 1\}=\mathscr{C}^{(n+1)}.$$

Let $$\mathscr{E}^{k-1}=\{d\in\mathscr{E}^k\mid r=0\}.$$

Then

(8.2) $$i_{(k)}\mathrm{N}=\mathrm{N}\cup_{\mathscr{E}^{k-1}}\mathscr{E}^k$$

(See Figure 1).

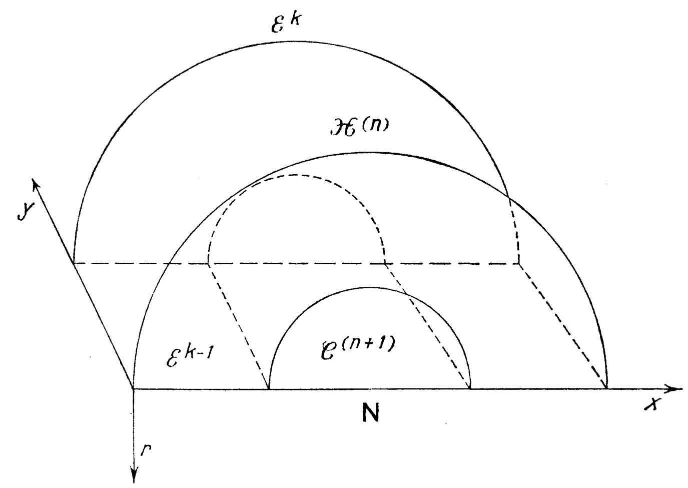


Fig. 1

Since $(\mathscr{E}^k, \mathscr{E}^{k-1}) \approx (D^k, D_+^{k-1})$, $i_{(k)}N$ is obtainable from N by an irrelevant addition, and (ii) is verified.

Lemma **(8.3)**. — Let $i : K \to K^* = K \cup_{\bar\varphi} \Delta^n \cup_{\bar\psi} \Delta^{n+1}$ be an elementary expansion. Let $k \geq \max(\dim K^*, n+4)$. Then the maps

$$i^{(k)} : \mathscr{N}^{(k)}(K^*) \underset{\approx}{\to} \mathscr{N}^{(k)}(K)$$

$$i_{(k)} : \mathscr{N}^{(k)}(K) \underset{\approx}{\to} \mathscr{N}^{(k)}(K^*)$$

are isomorphisms and inverses of each other.

Proof. — After Lemma (8.1), we have that $i^{(k)} \circ i_{(k)} = 1$. Lemma (8.3) would follow if it could be shown that $i_{(k)} \circ i^{(k)} = 1$.

Let, then, $M_0 = i^{(k)}N$. It shall be demonstrated that $i_{(k)}M_0 = N$.

Since $N \in \mathscr{N}^k(K^*)$,

$$N = (M_0 \cup_\varphi D^n \times D^{k-n}) \cup_\psi D^{n+1} \times D^{k-n-1}.$$

Calling $M = M_0 \cup_\varphi D^n \times D^{k-n}$, and $f : S^n \to \partial M$ the imbedding $f = \bar\psi : \partial D^{n+1} \times \{0\} \to \partial M$, letting $m = k - n - 1$, we see that we are in the situation of (7.11), for

$$m = k - (n+1) \geq k - \dim K^* > 2.$$

Applying (7.11) we see that the map $f = \bar\psi$ may be assumed to have these properties:

(i) $$f(x) = (K(x), p_0) \in D^n \times \partial D^{k-n} \subseteq \partial M \qquad \text{if } x \in D_+^n$$

(for $p_0 \in D^{k-n}$ any fixed point, which I take to be $p_0 = (0, 0, \ldots, 0, 1) \in R^{k-n}$).

(ii) $$f(x) \in M_0 \qquad \text{if } x \in D_-^n.$$

An application of the tubular neighborhood lemma gives us that $\psi : \partial D^{n+1} \times D^{k-n-1} \to \partial M$ may be assumed (after isotopy) to have these properties:

(i) $$\psi(x, y) = (K(x), K'(y)) \qquad \text{if } x \in D_+^n.$$

(8.4)

(ii) $$\psi(x, y) \in M_0 \qquad \text{if } x \in D_-^n.$$

Rewriting the equation

$$N = (M_0 \cup_\varphi D^n \times D^{k-n}) \cup_\psi D^{n+1} \times D^{k-n-1}$$

as $$N = (M_0 \cup_{\partial D^n \times D^{k-n}} D^n \times D^{k-n}) \cup_{\partial D^{n+1} \times D^{k-n-1}} D^{n+1} \times D^{k-n-1},$$

in the light of (8.4) we may interchange the order of union as follows:

$$N = (M_0 \cup_{D_-^{n+1} \times D^{k-n-1}} D^{n+1} \times D^{k-n-1}) \cup_{B^{k-1}} (D^n \times D^{k-n})$$

where $$B^{k-1} = \partial D^n \times D^{k-n} \cup D^n \times D_+^{k-n-1}.$$

It is easily seen that $B^{k-1} \approx D^{k-1} \subseteq \partial(D^n \times D^{k-n})$.

(Since $D^{k-n} \approx D_+^{k-n-1} \times I$,

$$\begin{aligned} B^{k-1} &\approx \partial D^n \times D_+^{k-n-1} \times I \cup D^n \times D_+^{k-n-1} \\ &\approx (\partial D^n \times I \cup D^n) \times D_+^{k-n-1} \\ &\approx D^n \times D_+^{k-n-1} \\ &\approx D^{k-1}). \end{aligned}$$

Since $D^{n+1} \times D^{k-n-1} = D^n \times D^{k-n} = D^k$, and

$$D_-^{n+1} \times D^{k-n-1} \approx D_+^{k-1} \subseteq \partial D^k,$$

the above decomposition of N becomes:

$$\textbf{(8.5)} \qquad N \approx (M_0 \cup_{D_+^{k-1}} D^k) \cup_{D_+^{k-1}} D^k.$$

It follows easily that $N = i_{(k)} M_0$, as was to be shown, proving (8.3).

Lemma **(8.6)**. — Let $f: K \to L$ be a simple homotopy equivalence, and $k \geq \max\{\dim K, \dim L, \dim f + 3\}$ then f induces a bijective differentiable isomorphism,

$$\hat{f}: \mathcal{N}^{(k)}(K) \underset{\approx}{\to} \mathcal{N}^{(k)}(L).$$

Proof. — Let

$$f: K = K_0 \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \dots \xrightarrow{f_l} K_l = L$$

be the sequence of elementary maps. The map $\hat{f}$ is defined to be the composite, $\hat{f} = \hat{f}_l \circ \hat{f}_{l-1} \circ \dots \circ \hat{f}_1$ where in case *a)*,

$$K_i = K_{i-1} \cup \Delta^{k_i - 1} \cup \Delta^{k_i}$$

$f_i: K_{i-1} \to K_i$ is an elementary expansion, then define

$$\hat{f}_i = (f_i)_{(k)}: \mathcal{N}^{(k)}(K_{i-1}) \underset{\approx}{\to} \mathcal{N}^k(K_i)$$

which is a bijective differentiable isomorphism by (8.3).

In case *b)*,

$$K_{i-1} = K_i \cup \Delta^{k_i - 1} \cup \Delta^{k_i}$$

$f_i: K_i \to K_{i-1}$ is an elementary contraction, and define

$$\hat{f}_i = (f_i)^{(k)}: \mathcal{N}^{(k)}(K_{i-1}) \underset{\approx}{\to} \mathcal{N}^{(k)}(K_i),$$

which is again a bijective differentiable isomorphism. This proves (8.6).

The assignation $f \to \hat{f}$ is functorial in the sense that $\hat{1} = 1$ and $\widehat{f \circ g} = \hat{f} \circ \hat{g}$ as is easily seen.

Let
$$K_0 \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \dots \xrightarrow{f_l} K_l$$

be a sequence of elementary expansions and contractions so that the map

$$f = f_l \circ \dots \circ f_2 \circ f_1 : K_0 \to K_l$$

is a simple homotopy equivalence, and let either

a) $$K_i = K_{i-1} \cup \Delta^{k_i-1} \cup \Delta^{k_i}$$

or *b)* $$K_{i-1} = K_i \cup \Delta^{k_i-1} \cup \Delta^{k_i} \qquad \text{for } i = 1, \ldots, l.$$

If *a)* holds, the index i will be called an *expansion index*, if *b)* holds the index i will be called a *contraction index*. Let

$$m = \max\{\dim K_0, \dim K_l\}.$$

Let $k \geq m+3$. According to (5.6), if K_0, K_l are simple-homotopy-equivalent, there is such a simple homotopy equivalence for which all $k_i \leq m+2$ (therefore $k_i \leq k$), and all expansion indices occur in the range

$$1 \leq i \leq \mu.$$

All contraction indices occur for

$$\mu < i \leq l.$$

Moreover,
$$k_1 \leq k_2 \leq \ldots \leq k_\mu$$
$$k_{\mu+1} \geq k_{\mu+2} \geq \ldots \geq k_l.$$

Theorem (Nonstable Neighborhood Theorem). — For all simple homotopy equivalences,

$$f : K \to L,$$

K, L properly ordered filtrations, and integers $k \geq \max\{\dim K, \dim L\} + 3$, there are imbeddings

$$\hat{f} : \mathcal{N}^{(k)}(K) \to \mathcal{N}^{(k)}(L)$$

characterized by the properties:

1) $\hat{f}$ is " functorial " for simple homotopy equivalences and integers k satisfying the above property

2) if $f : K \to K^*$ is an elementary expansion, $\hat{f}$ is isotopic to $f_{(k)}$.

The (unique) imbedding described above is an isomorphism

$$\hat{f} : \mathcal{N}^{(k)}(K) \underset{\approx}{\to} \mathcal{N}^{(k)}(L).$$

Proof. — Assume the sequence of elementary contractions and expansions from $K = K_0$ to $L = K_l$ has the above form.

Let ν_1, ν_2 be indices such that (α) $k_\nu > m$ for all indices $\nu_1 \leq \nu \leq \mu$, $\mu > \nu \geq \nu_2$. (β) $k_\nu \leq m$ for all indices $\nu < \nu_1$, $\nu_2 > \nu$.

Since each elementary expansion f_ν ($\nu < \nu_1$) and elementary contraction f_ν ($\nu > \nu_2$) satisfies the dimension restriction of Lemma 8.6, the conclusion of (8.6) applies, yielding an isomorphism

$$\hat{f}_\nu^{(k)} : \mathcal{N}^k(K_{\nu-1}) \underset{\approx}{\to} \mathcal{N}^k(K) \qquad \text{for } \nu \leq \nu_1 \text{ and } \nu > \nu_2.$$

Letting $g = f_{\nu_2} \circ f_{\nu_2 - 1} \circ \ldots \circ f_{\nu_1 + 1} : K_{\nu_1} \to K_{\nu_2}$ it suffices, to prove the nonstable neighborhood theorem, to obtain a (functorial) isomorphism,

$$\hat{g}^{(k)} : \mathcal{N}^k(K_{\nu_1}) \to \mathcal{N}^k(K_{\nu_2}).$$

Having said this, we may rename

$$K_0 = K_{\nu_1};\ K_{\nu_2} = K_l$$

and we may assume that our simple homotopy sequence,

$$K_0 \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \xrightarrow{f_3} \ldots \xrightarrow{f_l} K_l$$

obeys the following condition:

$$\dim f_\nu > m \qquad \text{for } 1 \leq \nu \leq l.$$

(I may also assume that f_ν is an expansion for $1 \leq \nu \leq \mu$ and a contraction for $\mu < \nu \leq l$:

$$f_\nu : K_{\nu-1} \to K_{\nu-1} \cup_0 D^{n_\nu - 1} \cup_1 D^{n_\nu} = K_\nu \qquad \text{for } \nu \leq \mu,$$

and

$$f_\nu : K_\nu = K_{\nu-1} \cup_0 D^{n_\nu - 1} \cup_1 D^{n_\nu} \to K_{\nu-1} \qquad \text{for } \nu > \mu.)$$

Since $n_\nu > m = \max\{\dim K_0, \dim K_l\}$ for all $\nu = 1, \ldots, l$, it is clear the every expansion n_ν-cell that is added must be removed by some later elementary contraction, for there are no cells of dimension greater than m in K_l. For the same reason, every contraction n_ν-cell removed must have previously been added as an elementary expansion, for there were no cells of dimension greater than in K_0.

A complication that might arise is when the expansion cell $\Delta^n = \Delta^{n_q}$ is added " via " the free face of expansion Δ_0^{n-1}

$$K_\nu \xrightarrow{f_\nu} K_{\nu+1} = K_\nu \cup_{\varphi_0} \Delta_0^{n-1} \cup_{\psi_0} \Delta^n$$

and is deleted via the free face of contraction Δ_1^{n-1} (i.e. occurs as a contraction cell as below for some index $\nu' > \nu$)

$$K_{\nu'-1} = K_{\nu'} \cup_{\varphi_1} \Delta_1^{n-1} \cup_{\psi_1} \Delta^n \xrightarrow{f_{\nu'}} K_{\nu'}$$

where $\Delta_0^{n-1} \neq \Delta_1^{n-1}$. That is to say, the expansion cell $\Delta^n = \Delta^{n_q}$ does not go out via the same free face with respect to which it came in. (See Figure 2.)

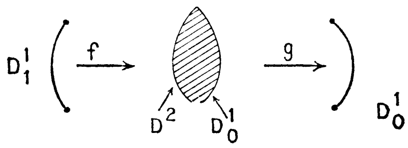


FIG. 2

I shall now describe the imbedding

$$\hat{f}^{(k)} : \mathcal{N}^{(k)}(K_0) \to \mathcal{N}^{(k)}(K_l).$$

To every neighborhood $N \in \mathcal{N}^{(k)}(K_0)$ we may ascribe, in an elementary manner, a neighborhood $N_\mu \in \mathcal{N}^{(k)}(K_\mu)$. Since $f_\nu : K_\nu \to K_{\nu+1}$ $(\nu < \mu)$ is an elementary expansion, Lemma 8.1 applies, yielding an imbedding

$$(f_\nu)_{(k)} : \mathcal{N}^{(k)}(K_\nu) \to \mathcal{N}^{(k)}(K_{\nu+1}).$$

Iteration yields an imbedding

$$(f_{\mu-1})_{(k)} \circ \ldots \circ (f_1)_{(k)} : \mathcal{N}^{(k)}(K_0) \to \mathcal{N}^{k}(K_\mu).$$

Let $N_\mu = (f_{\mu-1})_{(k)} \circ \ldots \circ (f_1)_{(k)} N$. It is geometrically quite trivial to visualize N. In the terminology of formula (8.2),

$$N_\mu = N \cup_{\mathscr{E}_1^{k-1}} \mathscr{E}_1^k \cup_{\mathscr{E}_2^{k-1}} \cup \mathscr{E}_2^k \ldots \cup_{\mathscr{E}_{\mu-1}^{k-}} \mathscr{E}_{\mu-1}^k.$$

That is, N_μ is N, with $\mu - 1$ irrelevant additions. This battery of additions, $\mathscr{E}_1^k, \ldots, \mathscr{E}_{\mu-1}^k$ we visualize as microscopic disjoint cells added to ∂N by the patches $\mathscr{E}_1^{k-1}, \ldots, \mathscr{E}_{\mu-1}^{k-1}$. I shall refer to the subset $\mathscr{E}_1^k \cup \mathscr{E}_2^k \cup \ldots \cup \mathscr{E}_{\mu-1}^k$ as the *battery*.

The problem ahead of us is to produce a neighborhood N_l over K_l from the differentiable cell filtration N_μ. This will be done by constructing a sequence N_μ, $N_{\mu+1}, \ldots, N_l$ where N_ν is a neighborhood over K_ν, $\mu \leq \nu \leq l$, and each N_ν will have a specified "battery" of irrelevant additions, denoted $\mathscr{E}_1^k, \mathscr{E}_2^k, \ldots, \mathscr{E}_{q_\nu}^k$; where the number q_ν may vary. Thus N_ν is a reordering of:

$$N_\nu \equiv \widetilde{N}_\nu \cup \mathscr{E}_1^k \cup \mathscr{E}_2^k \cup \ldots \cup \mathscr{E}_{q_\nu}^k.$$

(Recall that the symbol $M \equiv M'$ denotes that M' is a reordering of M.) The differentiable cell filtration N_ν and its battery will be so constructed that $\widetilde{N}_\nu$ will be a neighborhood of $\widetilde{K}_\nu$ a cell filtration such that $\dim \widetilde{K}_\nu \leq m$. (That is, the battery lies over all cells of K_ν of dimension greater than m.)

Inductive lemma. — Assume the situation is as above; N_ν is a neighborhood of K_ν with a battery of irrelevant additions: $\mathscr{E}_1, \ldots, \mathscr{E}_{q_\nu}$ which lie over all cells of K_ν of dimension greater than m. Let

$$K_\nu \equiv K_{\nu+1} \cup_{\varphi_0} \Delta^{k_\nu - 1} \cup_1 \Delta^{k_\nu} \qquad \text{where } k_\nu > m.$$

Then

$$\textbf{(8.7)} \qquad N_\nu \equiv N_{\nu+1} \cup_{\varphi_0} \Delta^{k_\nu-1} \times D^{k-k_\nu+1} \cup_{\varphi_1} \Delta^{k_\nu} \times D^{k-k_\nu}$$

where $N_{\nu+1}$ is a neighborhood over some reordering $K_{\nu+1}$ and

$$\Delta^{k_\nu-1} \times D^{k-k_\nu+1} \cup \Delta^{k_\nu} \times D^{k-k_\nu}$$

forms an irrelevant addition $\mathscr{E}_{(\nu)}$. So

$$N_\nu = N_{\nu+1} \cup \mathscr{E}_{(\nu)}.$$

The differentiable cell filtration $N_{\nu+1}$ also possesses a battery of irrelevant additions $\mathscr{E}_1, \ldots, \mathscr{E}_{q_{\nu+1}}$ lying over all cells of $K_{\nu+1}$ of dimension greater than m. To prepare the reader for the nature of the proof of this inductive lemma (which is geometrically quite elementary) several facts should be recalled. The object of the inductive lemma is to construct a particular differentiable cell decomposition possessing the form of the right hand side of (8.7). One begins with an arbitrary differentiable cell decomposition representing N_ν which possesses a battery as described above and one must construct an equivalent differentiable cell decomposition possessing the form of the right hand side of (8.7). This will be done by performing differentiable isotopies of the attaching maps of the original differentiable cell decomposition. We are assured, then, by a foundational proposition (3.6 bis), that two differentiable cell decomposition differing by isotopies of their attaching maps are equivalent (by a filtration and orientation-preserving diffeomorphism) and therefore give rise to the same differentiable cell filtration.

The essential fact, necessary for the proof of this inductive lemma, is that (by previous assumption) $k_\nu > m$, and therefore the " thickened cell " in N_ν, lying above Δ^{k_ν} is part of the battery of irrelevant additions of N_ν, and therefore its attaching map is regular enough so that we may arrange to attain (8.7) by isotopy.

Of course, if $\Delta^{k_\nu-1} \times D^{k-k_\nu+1}$ is part of the battery of N_ν, things are even simplert I shall first indicate how to achieve (8.7) in this case. I assume $\Delta^{k_\nu-1} \times D^{k-k_\nu+1}$ and $\Delta^{k_\nu} \times D^{k-k_\nu}$ do not belong to the same irrelevant addition $\mathscr{E}_i^k$, otherwise there would be nothing to do (i.e. $N_\nu = N_{\nu+1} \cup \Delta^{k_\nu-1} \times D^{k-k_\nu+1} \cup \Delta^{k_\nu} \times D^{k-k_\nu}$ where $N_{\nu+1}$ is just N_ν with $\mathscr{E}_i^k$ deleted, its battery being $\mathscr{E}_1^k, \ldots, \mathscr{E}_{q_\nu}^k$ again with $\mathscr{E}_i^k$ deleted.) Thus let $\Delta^{k_\nu-1} \times D^{k-k_\nu+1}$ belong to one irrelevant addition $\mathscr{E}_1^k$, say, and $\Delta^{k_\nu} \times D^{k-k_\nu}$ belong to another, $\mathscr{E}_2^k$. There are two possibilities. Either $\Delta^{k_\nu-1} \times D^{k-k_\nu+1}$ is the cap on the handle of $\mathscr{E}_1^k$. In either case it is an elementary matter to " move " $\mathscr{E}_1^k$ around (by isotopies) so that $\Delta^{k_\nu-1} \times D^{k-k_\nu+1} \cup \Delta^{k_\nu} \times D^{k-k_\nu}$ forms an irrelevant addition $\mathscr{E}_{(\nu)}$ and the remaining cap of $\mathscr{E}_1^k$ and cap-or-handle of $\mathscr{E}_2^k$ form an irrelevant addition $\mathscr{E}_{1,2}^k$. (The reader sould notice that this may be viewed as a problem entirely in Euclidean space and independen. of the nature of N_ν, for $\mathscr{E}_1^k$, $\mathscr{E}_2^k$ may be thought of as microscopic irrelevant additions added to a small disc in N_ν, the problem being then a local one.) This yields a differentiable cell decomposition representing N_ν such that

$$N_\nu \equiv N_{\nu+1} \cup \mathscr{E}_{(\nu)}^k$$

where $N_{\nu+1}$ is N_ν with the cells $\mathscr{E}_{(\nu)}^k$ omitted. Clearly $N_{\nu+1}$ is a neighborhood of some reordering of $K_{\nu+1}$ and its " battery " may be taken to be $\mathscr{E}_{1,2}^k, \ldots, \mathscr{E}_3^k, \ldots, \mathscr{E}_{q_\nu}^k$.

After this digression, I can assume that the free face of contraction $\Delta^{k_\nu-1} \times D^{k-k_\nu+1}$ is not contained in the battery of N_ν. This, of course, implies that $k_\nu - 1 \leq m$. Since, by assumption, $k_\nu > m$, it follows that $k_\nu = m+1$. We have:

$$\Delta^{k_\nu-1} \times D^{k-k_\nu+1} = \Delta^m \times D^{k-m},$$

$$\Delta^{k_\nu} \times D^{k-k_\nu} = \Delta^{m+1} \times D^{k-m-1}.$$

Let $\Delta^{m+1} \times D^{k-m-1}$ be part of the irrelevant addition $\mathscr{E}_1^k$ and

$$\mathscr{E}_1^k = \Delta_1^m \times D^{k-m} \cup \Delta^{m+1} \times D^{k-m-1}.$$

The reader will be able to follow the details of the remainder of the proof more easily by referring to Figure 3.

Changing the attaching maps of thickened cells by five isotopies we shall eventually represent the differentiable cell filtration N_ν by the differentiable cell decomposition pictured in (V). It will clearly be of the form required for the inductive lemma. Being given

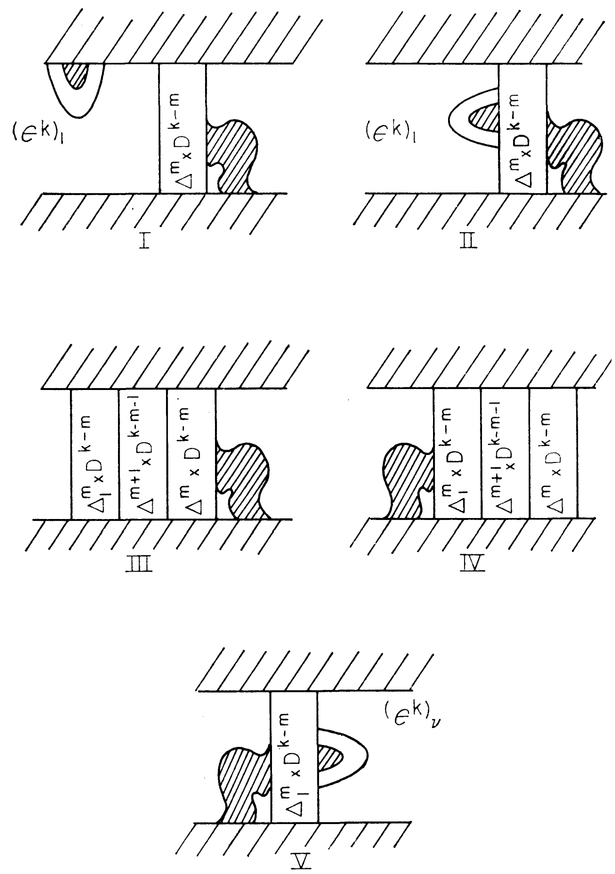


FIG. 3

a differentiable cell decomposition (I) with $\Delta^{m+1} \times D^{k-m-1}$ and $\Delta^m \times D^{k-m}$ being cap and handle (respectively) of an irrelevant addition we shall construct the differentiable cell decomposition (V), equivalent to (I), in the manner schematically described by Figure 3.

We may represent N_ν, up to reordering, by:

$$N_\nu \equiv D^{n_1} \times D^{k-n_1} \cup_{\varphi_2} \ldots \cup \Delta^m \times D^{k-m}$$
$$\cup \ldots \cup_{\varphi_q} D^{n_q} \times D^{k-n_q} \cup \mathscr{E}_1^{(k)}$$

(i.e. (I) of Figure 3; the cells of relevance to our discussion are $\Delta^m \times D^{k-m}$ and those of $\mathscr{E}_1^k$. Since $\mathscr{E}_1^k$ is an irrelevant addition, after reordering N_ν, I may have it come last.)

Since $n_i < \dim N_\nu = k$ for all $i = 1, \ldots, q$, it may be arranged by isotopy that $\Delta^m \times \partial D^{k-m}$ intersects ∂N_ν in an open set (a patch):

$$\Delta^m \times \partial D^{k-m} \cap \partial N_\nu \neq \varnothing.$$

(This is so because $\bigcup_{j=1}^{q} \operatorname{Im} \varphi$ may be deformed by isotopy into an arbitrarily small neighborhood of

$$\bigcup_{j=1}^{q} \varphi_j(\partial D^{n_j} \times \{0\})$$

which is definitely of smaller dimension than $\Delta^m \times \partial D^{k-m}$).

Once one has some non-empty open subset of $\Delta^m \times \partial D^{k-m}$ contained in ∂N_ν, one may as well normalize things by another isotopy, and assume, say, that $\Delta^m \times D_+^{k-m-1} \subseteq \Delta^m \times \partial D^{k-m}$ is contained in ∂N_ν.

Let $\rho : \mathscr{E}_1^{k-1} \to \Delta^m \times D_+^{k-m-1}$ be the differentiable isomorphism

$$\rho(x, y, 0) = (x/2, y/2)$$

where
$$\mathscr{E}_1^{k-1} = \{d = (x, y, 0) \in R^m \times R^{k-m-1} \times R\}$$

is given by its standard parametrization. Then, after isotopy (see the transition from (I) to (II) in Figure 3), the neighborhood N_ν may be represented by

$$N_\nu \equiv D^{n_1} \times D^{k-n_1} \cup_{\varphi_2} \ldots \cup \Delta^m \times D^{k-m} \cup \ldots \cup_{\varphi_q} D^{n_q} \times D^{k-n_q} \cup_\rho \mathscr{E}_1^k.$$

Let $\gamma'_t : \Delta^m(1/4) \times D^{k-m} \to \Delta^m \times D^{k-m}$ be the isotopy $\gamma'_t(x, y) = ((1 + 3t).x, y)$. Let $\gamma_t : N_\nu \to N_\nu$ be an isotopy which extends γ'_t so that $\gamma_0 = 1$. Then the neighborhood N_ν may be represented by

(8.8)
$$N_\nu = D^{n_1} \times D^{k-n_1} \cup_{\varphi_2} \cup \ldots \cup_{\varphi_q} D^{n_q} \times D^{k-n_q} \cup_{\gamma_1 \circ \rho} \mathscr{E}_1^k$$

(see the transition from (II) to (III) in Figure 3).

Let $B = \Delta^m \times D^{k-m} \cup \mathscr{E}_1^k \subseteq N_\nu$ in the representation (8.3). Then B might be parametrized by

$$B = \{b = (x, y, r) \in \Delta^m \times D^{k-m-1} \times 1\}$$

where
$$\mathscr{H}_1^{(m)} = \Delta_1^m \times D^{k-m} \subseteq B$$

is given by:

$$\Delta_1^m \times D^{k-m} = \left\{ b = (x, y, r) \in B \mid 0 \leq r \leq \frac{1}{3} \right\},$$

and $\mathscr{C}^{(m+1)} = \Delta^{m+1} \times D^{k-m-1} \subseteq \mathscr{E}_1^k \subseteq B$ is given by

$$\Delta^{m+1} \times D^{k-m-1} = \{b = (x, y, r) \in B \mid 1/3 \leq r \leq 2/3\},$$

and
$$\mathscr{H}^{(m)} = \Delta^m \times D^{k-m} \subseteq (\mathscr{E}^k)_0 \subseteq B$$

is given by
$$\Delta^m \times D^{k-m} = \{b = (x, y, r) \in B \mid 2/3 \leq r \leq 1\}.$$

Let $\lambda_t : B \to B$ be the isotopy which rotates B 180^0 about the (y_1, r)-plane so that $\lambda_1(\Delta^m \times D^{k-m}) = \Delta_1^m \times D^{k-m}$. Then N_ν may be represented, up to reordering as follows:

(8.9) $$N_\nu = D^{n_1} \times D^{k-n_1} \cup_{\varphi_2} \cup \ldots \cup_{\lambda_1 \gamma_1 \rho} \mathscr{E}_1^k \cup_{\lambda_1 \varphi} \Delta^m \times D^{k-m} \ldots .$$

(see the transition from (III) to (IV) in Figure 3).

Applying the isotopy γ_t^{-1} to the attaching maps of $\lambda_1(\Delta^m \times D^{k-m})$ and $\lambda_1(\Delta^{m+1} \times D^{k-m-1})$ one may obtain a representation of N_ν as in (V) of Figure 3. The cells $\Delta^m \times D^{k-m} \cup \Delta^{m+1} \times D^{k-m-1}$ form an irrelevant addition $\mathscr{E}_{(\nu)}^k$ added to a " patch " of $\Delta_1^m \times D^{k-m}$. Moreover, $\mathscr{E}_{(\nu)}^k$ is free of further attaching maps, so we may reorder N_ν to obtain such a representation: $N_\nu \equiv N_{\nu+1} \cup \mathscr{E}_{(\nu)}^k$, and $N_{\nu+1}$ is easily seen to be a neighborhood over some reordering of $K_{\nu+1}$. The new battery of $N_{\nu+1}$ is simply $\mathscr{E}_2^k, \ldots, \mathscr{E}_{q_\nu}^k$ (i.e. it is the old battery with $\mathscr{E}_1^k$ deleted).

Since N_ν is $N_{\nu+1}$ with an " irrelevant addition ", there is a unique isotopy class of diffeomorphisms

$$\hat{f}_{N_\nu} : N_\nu \underset{\approx}{\to} N_{\nu+1}.$$

This establishes the inductive lemma, and hence the theorem, for given any $N_0 \in \mathscr{N}^k(K_0)$, taking N_μ to be as above, with its battery of irrelevant additions, repeated application of the inductive lemma yields for us a neighborhood N_l of some reordering of K_l. By a foundational proposition any neighborhood of a cell filtration K′ may be reordered to yield a neighborhood of any properly ordered cell filtration K which is a reordering of K′.

Since K_l is properly ordered, N_l may be reordered to yield a neighborhood of K_l. I shall denote that neighborhood by N_l, as well. Define

$$\hat{f} : \mathscr{N}^k(K_0) \to \mathscr{N}^k(K_l)$$

by $\hat{f}(N_0) = N_l$, and to each N_0 one may define $\hat{f}_N : N_0 \underset{\approx}{\to} N_l$ to be the composite diffeomorphism. This yields an imbedding

$$\hat{f} : \mathscr{N}^k(K_0) \to \mathscr{N}^k(K_l).$$

It is easily seen that $\hat{f}$ is functorial (i.e. $\widehat{f \circ g} = \hat{f} \circ \hat{g}$; $\hat{1} = 1$) which implies that $\hat{f}$ is an isomorphism. Therefore the nonstable neighborhood theorem is proved.

As an application of the nonstable neighborhood theorem, I shall compute the set of k-dimensional neighborhoods over a filtration of the n-disc for various values of n and k.

Let $C^n \in \mathscr{F}$ be the following filtered object:

$$C^n = (C_0, C_1, C_2, C_3)$$

where $$C_0 = \varnothing;\ C_1 = D^0;\ C_2 = D^0 \cup D^{n-1};\ C_3 = D^0 \cup D^{n-1} \cup_\varphi D^n$$

where $D^0 \cup D^{n-1}$ is to be identified with $S^{n-1} = \partial D^n$ and $\varphi : \partial D^n \to D^0 \cup D^{n-1}$ is the identity map. C^n is of the same simple homotopy type as D^0 (i.e. the map $f : D^0 \to C^n$ is a simple homotopy equivalence) and $\dim f = n$.

Call $1_k \in \mathscr{N}^k(C^n)$ the trivial neighborhood over C^n with $k \geq n$.

Corollary **(8.10)**. — $\mathcal{N}^k(C^n) = \{1_k\}$ for $k \geq n+3$.

It is trivial to see that $\mathcal{N}^n(C^n) = \{1_n\}$. This leaves two undetermined cases: $\mathcal{N}^{n+1}(C^n)$, and $\mathcal{N}^{n+2}(C^n)$.

It is easily seen that $\mathcal{N}^{n+2}(C^n)$ can be woefully large. In particular, $\mathcal{N}^4(C^2)$ contains a large number of neighborhoods $N^4 \in \mathcal{N}^4(C^2)$ with non-simply connected boundaries. (Figure 1 of [3] gives rise to a filtration of W^4 (in [3]) which exhibits W^4 as an element of $\mathcal{N}^4(C^2)$).

Of course, $\mathcal{N}^3(C^1) = \{1_3\}$, since the dimension is too low for non-trivial happenings.

I know nothing about $\mathcal{N}^{n+1}(C^n)$, in general. $\mathcal{N}^3(C^2) = \{1_3\}$, for reasons of low dimension. If $\partial C^n = (C_0, C_1, C_2) \subseteq C^n$, then assuming that a neighborhood $N \in \mathcal{N}^{n+1}(C^n)$ restricts to the trivial neighborhood 1_{n+1} over ∂C^n, it can be proved (using a differentiable version of the Schoenflies theorem extended to cover the case of an " annulus " (see [4])), that N is combinatorially filtration-equivalent to $1_{n+1} \in \mathcal{N}^{n+3}(C^n)$. (It would probably follow by recent results of Cairns and Munkres that $N = 1_{n+1}$.)

Differentiable cell filtrations of manifolds. — Let M be a compact differentiable manifold.

Definition: The *geometric dimension* of M is less than or equal to k (written: geom dim $M \leq k$) if there is a differentiable cell filtration $\mathcal{M}$ of M, which is a neighborhood of a cell filtration X such that $\dim X \leq k$.

Thus geom dim $M \leq k$ if and only if there is a Morse function f on M all of whose critical points have index less than or equal to k.

An application of the nonstable neighborhood theorem is the following:

Existence theorem. — Let M^n be an n-dimensional differentiable manifold such that geom dim $M^n \leq n-3$. Let $f: M^n \to X$ be a simple homotopy equivalence between M^n and a properly ordered cell filtration X such that $\dim X \leq n-3$. Then there is a differentiable cell filtration $\mathcal{M}$ of M which is a neighborhood of X such that $f: M^n \to X$ is a projection map for $\mathcal{M}$.

This theorem provides a partial answer to the question of existence of differentiable cell filtrations of manifolds.

Proof. — Since geom dim $M^n \leq n-3$, there is a differentiable cell filtration $\mathcal{M}'$ of M which is a neighborhood of a cell filtration X′ such that $\dim X' \leq n-3$, and hence a reordering $\mathcal{M}''$ of $\mathcal{M}'$ is a neighborhood of a properly ordered cell filtration X″, $\dim X'' \leq n-3$. Thus there is a simple homotopy equivalence $f'': X'' \to X$ obtained from the homotopy commutative diagram:

$$\begin{array}{ccc} M^n & \xrightarrow{f} & X \\ {\scriptstyle \pi''}\downarrow & \nearrow_{f''} & \\ X'' & & \end{array}$$

where π'' is a projection map for $\mathcal{M}''$.

According to the nonstable neighborhood theorem,

$$\hat{f}'' : \mathcal{N}^k(X'') \underset{\approx}{\rightarrow} \mathcal{N}^k(X)$$

is an isomorphism if

$$k \geq \max\{\dim X'', \dim X\} + 3 \qquad \text{or } k \geq n.$$

Considering $\mathcal{M}''$ as an element of $\mathcal{N}^n(X'')$, let $\mathcal{M} = (\hat{f}'')(\mathcal{M}'')$. This differentiable cell filtration $\mathcal{M}$ possesses the desired properties. Since $\hat{f}''$ is an isomorphism and $\mathcal{M}''$ is a differentiable filtration of M^n, $\mathcal{M}$ is also a differentiable filtration of M^n. Moreover, $\mathcal{M} \in \mathcal{N}^n(X)$, and it is easily seen that $f : M^n \rightarrow X$ is a projection map for $\mathcal{M}$. Notice that the hypotheses of the existence theorem are never satisfied for manifolds without boundary. Also, some restriction concerning geometric dimension is necessary for the conclusion to hold. For example, there are compact contractible manifolds not diffeomorphic with a cell. (Any such manifold C is of the simple homotopy type of the cell filtration consisting of a single point. It is not a neighborhood of the cell filtration, however.) An example of a class of manifolds that obey the hypotheses of the theorem is the class of differentiable k-cell bundles over differentiable manifolds ($k \geq 3$).

CHAPTER IX

DUALITY AND THE EXISTENCE OF DIFFERENTIAL CELL FILTRATIONS FOR MANIFOLDS WITHOUT BOUNDARY

The main theorem proved in this paper, which gives a strong sufficient condition for existence of a differentiable cell filtration $\mathcal{M}$ of a differentiable manifold M such that $\mathcal{M}$ is neighborhood of a given cell filtration X, is the existence theorem of Chapter VIII. In general (for manifolds with boundary), it is easily seen that one cannot dispense with any the conditions of the theorem without the conclusion being false. If the manifold M possesses no boundary, there are ways of strengthening the existence theorem to apply more generally by making use of Poincaré duality. This idea is originally due to Smale, and was used in his proof of the Poincaré conjecture.

Duality. — The geometric manifestation of Poincaré duality is that if you take a filtered manifold and turn it upside-down it still looks like a filtered manifold, with a rather different (dual) filtration.

From the point of view of Morse theory, this fact can be expressed even more simply: If f is a Morse function, so is $(-f)$.

In particular, if f had critical points $p_1, \ldots, p_\nu$ such that

(i) $f(p_1) < f(p_2) < \ldots < f(p_\nu)$

(ii) the index of p_i is n_i,

then the Morse function $g(x) = -f(x)$ has critical points $p_\nu, p_{\nu-1}, \ldots, p_1$ such that

(i) $g(p_\nu) < g(p_{\nu-1}) < \ldots < g(p_1)$

(ii) the index of p_i is $n - n_i$ (where $n = \dim M$).

By virtue of the " equivalence " of Morse functions and differentiable cell filtrations, one may expect an analogous duality.

Let $M = (M_0, \ldots, M_\nu)^n$ be a differentiable filtration, and let

$$\varphi_i : \partial D^{n_i} \times D^{n-n_i} \to \partial M_{i-1}$$

be the " attaching maps ", $i = 1, \ldots, \nu$.

Let $M_i^* = \overline{M_\nu - M_{\nu-i}}$ be the closures of the complements of the $M_{\nu-i}$, $i = 0, \ldots, \nu$. Then

$$M_i^* = M_{i-1}^* \cup_{\varphi_i^*} D^{n_{(\nu-i)}} \times D^{n-n_{(\nu-i)}}$$

where

$$\varphi_i^* : D^{n_{(\nu-i)}} \times \partial D^{n-n_{(\nu-i)}} \to \partial M_{i-1}^*$$

is the inclusion map in M_i^*. Therefore

$$M^* = (M_0^*, \ldots, M_\nu^*)^n$$

is a filtration called the dual filtration to M, where

$$n_i^* = n - n_{(\nu-i)},$$

$$\varphi_i : \partial D^{n_i^*} \times D^{n-n_i} \to \partial M_{i-1}^*.$$

Of course, if the Morse function f corresponds to the differentiable cell filtration M, the Morse function $(-f)$ will correspond to M^*.

Geometric k-skeletons. — Let M be a differentiable manifold without boundary. By a *geometric k-skeleton of* M, I will mean a differentiable submanifold $U \subset M$ such that U is a compact manifold obtainable as the closure of an open set in M, and if $V \subseteq M$ is the closed complement of U in M, then

$$\text{geom dim } U \leq k$$

$$\text{geom dim } V \leq n - k - 1 \qquad \text{where } n = \dim M.$$

It is easily seen that geometric k-skeletons always exist; they are not unique.

If X is a properly ordered cell filtration, then $X^{(k)}$, the *k-skeleton of* X, will mean the subfiltration of X consisting of all cells of X of dimension less than or equal to k. If the differentiable cell filtration $\mathscr{M}$ is a neighborhood of X, then $\mathscr{M}^{(k)}$, the *k-skeleton of* $\mathscr{M}$, will denote the part of $\mathscr{M}$ lying above $X^{(k)}$.

If $\mathscr{M}$ is a properly ordered differentiable filtration of M, then the submanifold $U \subset M$ cut out by the k-skeleton $\mathscr{M}^{(k)}$ of $\mathscr{M}$ is a geometric k-skeleton of M.

To see this, first notice that geom dim $U \leq k$, by definition. Further, if V is the closed complement of U in M, then V is easily seen to be cut out by $(\widetilde{\mathscr{M}})^{(n-k-1)}$, the $(n-k-1)$-skeleton of $\widetilde{\mathscr{M}}$, the dual filtration of $\mathscr{M}$. Thus geom dim $V \leq n-k-1$, and U is therefore a geometric k-skeleton of M.

Construction of differentiable cell filtrations for manifolds without boundary. — Any geometric k-skeleton $U \subset M$ decomposes M as the union of two submanifolds

$$M = U \cup V.$$

If $2 \leq k < n-2$, then

$$\text{geom dim } U \leq n-3$$

$$\text{geom dim } V \leq n-3,$$

which means that both U and V satisfy the requirements of existence theorem, and one may construct differentiable cell filtrations of $\mathcal{U}$, $\mathcal{V}$ by " simple-homotopy-theoretic means ". Therefore, to obtain differentiable cell filtrations of M one must have a method for compounding differentiable filtrations $\mathcal{U}$, $\mathcal{V}$ of U, V into a differentiable filtration $\mathcal{M}$ of M. This is an easy matter:

Proposition **(9.1)**. — Let M be a differentiable manifold without boundary, and

$$M = U \cup V$$
$$U \cap V = \partial U = \partial V$$

where U, V are submanifolds of M. If $\mathcal{U}$ is a differentiable cell filtration of U, and $\mathcal{V}$ is a differentiable cell filtration of V, then there is a differentiable cell filtration $\mathcal{M}$ of M such that $\mathcal{U}$ is a subfiltration of $\mathcal{M}$, and $\mathcal{V}$ is a subfiltration of $\widetilde{\mathcal{M}}$.

The proof of (9.1) may be given most succinctly in the language of Morse functions. The differentiable filtrations $\mathcal{U}$, $\mathcal{V}$ give rise to Morse functions f_U, f_V on U, V respectively and if we take $I \times (U \cap V) \subset M$ as a tubular neighborhood of $U \cap V = \partial U = \partial V$ in M

$$(\text{with } \{-1\} \times (U \cap V) \subset U, \quad \{+1\} \times (U \cap V) \subset V, \quad \{0\} \times (U \cap V) = U \cap V)$$

we may assume the normalization:

$$f_U(t, x) = t \qquad \text{if } t \leq 0, \ x \in U \cap V$$
$$f_V(t, x) = -t \qquad \text{if } t \geq 0, \ x \in U \cap V.$$

Let $f : M \to R$ denote the following differentiable function:

a) $\qquad f|U = f_U$

b) $\qquad f|V = -f_V$.

By virtue of the normalizations, f is again a Morse function. It gives rise to a differentiable cell filtration $\mathcal{M}$ of M, which has the properties desired (i.e. it is easily verified that $\mathcal{M}$ satisfies *a)*, *b)*, *c)*, *d)*).

Proposition (9.1) succeeds in constructing differentiable cell filtrations of a manifold without boundary, solely on the basis of simple homotopy theoretic information, once one is provided with some geometric k-skeleton $(2 \leq k < n-2)$. It is much weaker than the most general existence theorem that one might conjecture: Let $f : M \to X$ be a simple homotopy equivalence between the differentiable manifold M (without boundary) and the cell filtration X. Then there is a differentiable cell filtration $\mathcal{M}$ of M which is a neighborhood of X such that $f : M \to X$ is a projection map.

A weakness of (9.1) is that so far, no homotopy-theoretic criterion has been given for the existence of geometric k-skeletons. (Of course, once a geometric k-skeleton has been given, all other information necessary is simple-homotopy-theoretic.) The next paragraph will provide certain homotopy criteria for the construction of nice geometric k-skeletons.

Construction of geometric k-skeletons of k-connected manifolds.

Lemma **(9.2).** — Let X be a cell filtration which is k-connected and $(k+1)$-dimensional. Then X is simple-homotopy equivalent with

$$S_1^{k+1} \vee \ldots \vee S_q^{k+1}$$

where q is the rank of $H_{k+1}(X)$.

Proof. — *a)* If $k=0$, then X is a 1-dimensional connected complex which is clearly of the simple homotopy type of $S_1^1 \vee \ldots \vee S_q^1$.

b) If $k \geq 1$, then X is simply connected and of the homotopy type of $S_1^{k+1} \vee \ldots \vee S_q^{k+1}$, and one may apply (5.4).

Proposition **(9.3).** — Let M^n be a k-connected manifold without boundary, such that

$$k \leq n-4.$$

Then a differentiably imbedded n-cell, $D^n \subset M^n$ is a geometric k-skeleton of M^n.

Proof. — Let $\mathscr{M}$ be a properly ordered differentiable cell filtration of M, and let $\mathscr{M}^{(k+1)}$ be its $(k+1)$-skeleton. Let $\mathscr{M}^{(k+1)}$ be a filtration of the submanifold $U \subseteq M$. Let $\mathscr{M}^{(k+1)}$ be a neighborhood of $X^{(k+1)}$, the $(k+1)$-skeleton of X, the cell filtration of which $\mathscr{M}$ is a neighborhood. Then

$$\pi_q(X^{(k+1)}) \underset{\approx}{\rightarrow} \pi_q(X) \approx \pi_q(M) = \{0\}$$

if $q \leq k$. The first isomorphism can be seen from the homotopy sequence for the pair $(X, X^{(k+1)})$ and the fact that $X - X^{(k+1)}$ is made up only of cells of dimension greater than $k+1$.

Therefore $X^{(k+1)}$ satisfies the requirements of Lemma 9.2 and is of the simple homotopy type of $S_1^{k+1} \vee \ldots \vee S_q^{k+1}$. Thus U, the submanifold of M cut out by $\mathscr{M}^{(k+1)}$, is a geometric $(k+1)$-skeleton of M, $k+1 < n-2$, and there is a simple homotopy equivalence $\xi : U \rightarrow S_1^{k+1} \vee \ldots \vee S_q^{k+1}$. The existence theorem applies, yielding a differentiable cell filtration $\mathscr{U}$ of U such that $\mathscr{U} \in \mathscr{N}^n(S_1^{k+1} \vee \ldots \vee S_q^{k+1})$. (By $S_1^{k+1} \vee \ldots \vee S_q^{k+1}$ I refer to the cell filtration:

$$D^0 \cup_0 D_1^{k+1} \cup_0 D_2^{k+1} \cup_0 \ldots \cup_0 D_q^{k+1}.)$$

Let $\mathscr{V} = \widetilde{\mathscr{M}}^{(n-k-2)}$ be the $(n-k-2)$-skeleton of $\widetilde{\mathscr{M}}$, the dual filtration of $\mathscr{M}$. Then $\mathscr{V}$ is a filtration of V, the closure of the complement of U.

Proposition 9.1 applies and we may conglomerate the filtrations $\mathscr{U}$ on U, $\mathscr{V}$ on V to obtain a differentiable filtration $\mathscr{W}$ on M. It follows that $\mathscr{W}^{(k)} = \mathscr{U}^{(k)}$ is a neighborhood of D^0. That is,

$$\mathscr{W}^{(k)} = D^0 \times D^n.$$

Therefore the submanifold $D^n = D^0 \times D^n \subset M^n$ is a geometric k-skeleton of M^n, proving Proposition 9.2.

Another way of stating (9.2) is:

Corollary **(9.3)**. — Let M^n be a k-connected differentiable manifold without boundary ($k \leq n-4$). Let M_* be the bounded differentiable manifold obtained by removing the interior of an n-cell from M^n. Then $\text{geom dim } M^n_* \leq n-k-1$.

Corollary **(9.4)**. — Let M^n be a 2-connected manifold without boundary, $n \geq 6$. Let M_* be as in (9.3). Let $f: M_* \to X$ be a (simple) homotopy equivalence between M_* and a properly ordered q-dimensional cell filtration X ($q \leq n-3$). Then there is a differentiable filtration $\mathcal{M}_*$ of M_* which is a neighborhood of X such that $f: M_* \to X$ is a projection.

Proof. — Since $\text{geom dim } M^n_* \leq n-k-1$, the existence theorem applies. From these considerations one may prove a theorem of Smale's:

Corollary **(9.5)**. — Let M^n be a k-connected manifold without boundary, $k \leq 4$. Then there exists a Morse function on M^n with a unique maximum, a unique minimum, and all of whose critical points p_i have indices j_i such that

$$k < j_i < n-k.$$

Or, equivalently, if M_* is M^n with the interior of an n-cell removed, then M_* admits a differentiable cell filtration $\mathcal{M}_*$ which is a neighborhood of a cell filtration X such that

(i) $\dim X < n-k$.

(ii) X has a unique 0-cell, D^0.

(iii) X possesses no cells of dimension less than or equal to k.

Proof. — We shall prove the second version of the above proposition. By Proposition 9.3, $\text{geom dim } M_* \leq n-k-1$. Therefore there is a differentiable filtration $\mathcal{W}_*$ of M_* such that $\mathcal{W}_*$ is a neighborhood of a properly ordered cell filtration Y of dimension less than $n-k$. Giving D^n the trivial differentiable filtration, and expressing $M = M_* \cup D^n$, (9.1) applies to give a differentiable filtration $\mathcal{W}$ of M. Then $\mathcal{W}^{(k+1)}$ cuts out a submanifold U of M, and by the same reasoning as in (9.3), $\mathcal{W}^{(k+1)}$ is a neighborhood of $X^{(k+1)}$, a $(k+1)$-dimensional k-connected cell filtration. By (9.2) $X^{(k+1)}$ is of the simple homotopy type of $S_1^{k+1} \vee \ldots \vee S_q^{k+1}$ for some q. Again, applying the existence theorem, U admits a differentiable cell filtration $\mathcal{U}$ which is a neighborhood of $S_1^{k+1} \vee \ldots \vee S_q^{k+1}$. If V is the closed complement of U in M, then

$$\mathcal{V} = \widetilde{\mathcal{W}}^{(n-k-2)}$$

is a differentiable cell filtration of V. Combining the differentiable filtrations $\mathcal{U}$ of U, $\mathcal{V}$ of V to obtain a filtration $\mathcal{M}$ of M (by means of (9.1)), it is immediately seen that $\mathcal{M}$ possesses the required properties for 9.5. More precisely, $\mathcal{M}$ possesses a unique " thickened " n-cell:

$$\mathcal{M} = \mathcal{M}_* \cup D^n \times D^0$$

and $\mathcal{M}_*$ is a differentiable cell filtration of M_* obeying (i), (ii), (iii).

Manifolds of the same homotopy type as S^n. — The general Poincaré conjecture for differentiable manifolds of the same homotopy type as the n-sphere was originally proved by Smale [11] for $n \geq 5$:

Theorem of Smale. — Let M^n be a differentiable manifold such that $n \geq 5$; $M^n \sim S^n$. Then

$$M^n \approx D_1^n \cup_\varphi D_2^n$$

where $\varphi : \partial D_2^n \to \partial D_1^n$ is a differentiable isomorphism.

Other equivalent statements of Smale's theorem are:

A) There is a function f on M^n possessing precisely two non-degenerate critical points (a maximum and a minimum).

B) $M^n \in \Gamma^n$ which is the group of differentiable n-spheres defined by Milnor [7].

C) There is a piece-wise differentiable homeomorphism $g : M^n \to S^n$.

D) M^n possesses a differentiable cell filtration $\mathscr{M}$ which is a neighborhood of $D^0 \cup_0 D^n$.

Corollary 9.5 implies Smale's theorem for $n \geq 6$. This may be seen since M^n, by hypothesis, is k-connected for $k = n - 4$. According to 9.5, M_* admits a differentiable filtration $\mathscr{M}_*$ which is a neighborhood of a cell filtration X containing a single 0-cell, no cells of dimension less than $k+1$, and $\dim X \leq n-k-1$. It follows that $X = D^0$, and consequently $\mathscr{M}_* = D^0 \times D^n$ if

$$n-k-1 < k+1.$$

This happens if $n > 6$. If $n = 6$, $X = D^0 \cup_0 D_1^3 \cup_0 D_2^3 \cup \ldots \cup D_q^3$. Since

$$H_3(X) \approx H_3(M_*) \approx \{0\}$$

it follows that $q = 0$, $X = D^0$, and $M_* = D^0 \times D^n$. The last isomorphims may be seen, for instance, using the Mayer-Vietoris sequence for the decomposition $M^n = M_* \cup D^n$, making use of the fact:

$$H_*(M^n) = H_*(S^n).$$

Smale's theorem for $n = 5$ cannot be directly obtained from Corollary 9.5. The best that one may easily obtain from (9.5) is that there is a differentiable filtration of $M^5 \times I$, $\mathscr{M}$, which is a neighborhood of $X = D^0 \cup D^5$. From this fact, and the differentiable Schoenflies theorem (in dimension 5) [4], [9], one may obtain that M^5 is homeomorphic with S^5.

Homotopy Skeletons. — The problem of the existence of geometric k-skeletons will be reduced to a homotopy question (see Proposition 9.8). For this, we will need a lemma of simple homotopy theory (Lemma 15, page 46 of [14]).

Lemma **(9.6).** — Let K be a subfiltration of the cell filtration X. Let $\pi_n(X, K) = 0$, $n = 1, \ldots, r$. Then there is a cell filtration Y satisfying these properties:

a) There is an inclusion $\beta : K \subseteq Y$.

b) All cells of $Y - K$ are of dimension greater than r.

c) There is a simple homotopy equivalence

$$f: X \to Y$$

which is the identity on K.

d) $\dim Y \leq \max\{r+2, \dim X\}$.

Definition **(9.7)**. — By a *homotopy k-skeleton* of the differentiable manifold M, I shall mean a continuous map

$$f: X \to M$$

of the cell filtration X into M such that

a) $\dim X \leq k$

b) $\pi_q(M, X) = 0 \qquad q = 1, \ldots, k.$

If $f: X \to M^n$ is an arbitrary continuous map of the cell filtration X into the differentiable manifold M, I will say that the compact submanifold $U^n \subseteq M^n$ is a *neighborhood of the mapping f* if

a) $f(X) \subseteq U$.

b) There is a differentiable cell filtration $\mathcal{U}$ of U which is a neighborhood of X, for which $f: X \to U$ is a cross-section.

Proposition **(9.8)**. — Let M^n be a compact differentiable manifold without boundary. Let $k \leq n-5$ and

$$f: X \to M^n$$

a homotopy k-skeleton for M^n; assume X properly ordered. Then there is a neighborhood $U^n \subseteq M^n$ of the mapping f, which is a geometric k-skeleton of M^n. Conversely, every geometric k-skeleton of M^n comes from a homotopy k-skeleton, $f: X \to M^n$, in this manner.

After Proposition 9.8, the construction of geometric k-skeletons in manifolds of high enough dimension is a homotopy theoretic matter.

Let $M^{(n-5)}$ be some geometric $(n-5)$-skeleton of M; $M = M^{(n-5)} \cup M^{(5)}$. Thus there is a differentiable cell filtration of $M^{(n-5)}$, $\mathcal{M}^{(n-5)}$, which is a neighborhood of Y, a cell filtration, $\dim Y \leq n-5$.

Since $\dim X = k \leq n-5$, we may deform $f: X \to M$ up to homotopy, and assume that $f(X) \subseteq M^{(n-5)}) \subseteq M$.

Let $g: X \to Y$ be the continuous map

$$\begin{array}{ccc} X & \xrightarrow{f} & M^{(n-5)} \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

where π is a projection map for the differentiable cell filtration $\mathcal{M}^{(n-5)}$. Let W be the cell filtration consisting in the mapping cylinder of g:

$$W = X \times I \cup Y / \{(x, 1) \sim g(x)\}$$

(the right-hand side may naturally be given the structure of a cell filtration). There is a " projection map " $p: \mathrm{W} \rightarrow \mathrm{Y}$ which is a simple homotopy equivalence, $\mathrm{X} \subset \mathrm{W}$ as a subfiltration, and the diagram:

$$\begin{array}{ccc} \mathrm{X} & \subset & \mathrm{W} \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle p} \\ & & \mathrm{Y} \end{array}$$

is commutative. (This is standard.)

Since $$\pi_q(\mathrm{M}, \mathrm{X}) = 0, \qquad q = 1, \ldots, k$$

we have:

$$\pi_q(\mathrm{W}, \mathrm{X}) = 0 \qquad q = 1, \ldots, k.$$

$$\dim \mathrm{W} \leq \max\{k+1, n-5\} \leq n-4.$$

Consequently, according to Lemma 9.6, there is a cell filtration Y', such that $\mathrm{X} \subseteq \mathrm{Y}'$, $\dim \mathrm{Y}' \leq n-3$ and all cells of $\mathrm{Y}' - \mathrm{X}$ are of dimensions greater than k, and there is a simple homotopy equivalence $f: \mathrm{W} \rightarrow \mathrm{Y}'$ which is identity on X. The cell filtration Y' may be taken to be properly ordered. Let $\varphi: \mathrm{Y}' \rightarrow \mathrm{Y}$ be a simple homotopy equivalence making

$$\begin{array}{ccc} \mathrm{W} & \xrightarrow{p} & \mathrm{Y} \\ \downarrow & \nearrow{\scriptstyle \varphi} & \\ \mathrm{Y}' & & \end{array}$$

homotopy-commutative.

Since $\mathscr{M}^{(n-5)} \in \mathscr{N}^n(\mathrm{Y})$ and $\max\{\dim \mathrm{Y}, \dim \mathrm{Y}'\} < n-2$ the nonstable neighborhood theorem applies, yielding a differentiable cell filtration $\mathscr{M}' \in \mathscr{N}^n(\mathrm{Y}')$ corresponding to $\mathscr{M}^{(n-5)}$ under $\hat{\varphi}$. Hence $\mathscr{M}'$ is also a filtration of the differentiable manifold $\mathrm{M}^{(n-5)}$. By means of the decomposition,

$$\mathrm{M} = \mathrm{M}^{(n-5)} \cup \mathrm{M}^{(5)},$$

one may compound a differentiable cell filtration $\mathscr{W}$ on M using $\mathscr{M}'$ on $\mathrm{M}^{(n-5)}$ and $\mathscr{M}^{(5)}$ on $\mathrm{M}^{(5)}$. Then $\mathscr{W}$ is a differentiable cell filtration of M such that $\mathscr{M}' \subseteq \mathscr{W}$. If $\mathscr{W}^{(k)}$ is the k-skeleton of $\mathscr{W}$, it is immediate that $\mathscr{W}^{(k)}$ is a neighborhood of X and U, the submanifold cut out by $\mathscr{W}^{(k)}$, is a neighborhood of the mapping $f: \mathrm{X} \rightarrow \mathrm{M}$. Clearly U is a geometric k-skeleton of M, proving 9.8.

CHAPTER X

ORIENTED MANIFOLDS WITHOUT BOUNDARY

For the general case of compact differentiable manifolds with boundary, we have seen that the existence theorem of Chapter VIII is the " best possible ".

If the manifold possesses no boundary, then Poincaré Duality enforces a very strong symmetry enabling us to deduce stronger geometric statements. In this case, any differentiable filtration $\mathscr{M}$ possesses a dual filtration $\widetilde{\mathscr{M}}$. In this chapter, it will be seen that the very rigid duality between $\mathscr{M}$ and $\widetilde{\mathscr{M}}$ ensures a similar duality between X, $\widetilde{\mathrm{X}}$ (the cell filtrations of which, $\mathscr{M}$, $\widetilde{\mathscr{M}}$ are neighborhoods). In particular, in the terminology of this chapter, X, $\widetilde{\mathrm{X}}$ are *paired* into M. The notion of being " paired into M " is a simple-homotopy-theoretic notion, and the main theorem of this chapter (The Existence Theorem for Oriented Manifolds without boundary) says that if $n \geq 7$ any pairing of cell filtrations X, $\widetilde{\mathrm{X}}$ may be realized (in some natural sense) by dual differentiable cell filtrations $\mathscr{M}$, $\widetilde{\mathscr{M}}$ of M.

In this manner, the question of existence of differentiable cell filtrations of an oriented manifold M^n is reduced to the simple-homotopy-theoretic question of constructing " pairings " of cell filtrations, X, $\widetilde{\mathrm{X}}$ into M^n.

Pairings of cell filtrations into Oriented Manifolds. — Recall that a regular cell decomposition is a properly ordered one, X, such that

$$\varphi_i(\partial \mathrm{D}^{n_i}) \subseteq \mathrm{X}^{(n_i-1)}$$

where φ_i is the i^{th} attaching map. Any properly ordered cell filtration X may be represented by a regular cell decomposition, and for the ensuing discussion, I will identify X with some regular cell decomposition representing X. There will be no harm in this, since the situations arising from two choices of regular representatives will be canonically isomorphic.

If X is properly ordered, $\mathrm{C}^*(\mathrm{X})$ $(\mathrm{C}_*(\mathrm{X}))$ will denote the cellular cochain (chain) complex with respect to the filtration X. That is, $\mathrm{C}^q(\mathrm{X}) = \mathrm{H}^q(\mathrm{X}^{(q)}, \mathrm{X}^{(q-1)})$, $\mathrm{C}^*(\mathrm{X}) = \Sigma_{q=0}^{\infty} \mathrm{C}^q(\mathrm{X})$, endowed with the coboundary operators obtained in a natural way by considering the cohomology exact sequences of the couples $(\mathrm{X}^{(q)}, \mathrm{X}^{(q-1)})$.

If X, Y are properly ordered cell filtrations, there is a natural cell filtration $\mathrm{X} \times \mathrm{Y}$

(the cartesian product) whose cells are $D^{n_i} \times D^{m_j}$, products of cells D^{n_i} of X, D^{m_j} of Y. The ordering of cells is lexicographical in each dimension, thus $X \times Y$ is properly ordered. It is easily seen that $C^*(X \times Y) \approx C^*(X) \times C^*(Y)$.

If X is a properly ordered cell filtration, $C^*(X)$ may be defined by choosing a representative cell decomposition; it is independent (up to unique isomorphism) of the representative chosen.

If $X = (X_0, \ldots, X_\nu)$ is a regular cell decomposition, $X_i = X_{i-1} \cup D^{n_i}$, each cell D^{n_i} determines an element $\xi_i \in C^*(X)$. Intuitively, ξ_i is the cochain which takes the value $+1$ on the chain $+D^{n_i}$, and the value 0 on all other n_i-dimensional cells. The set $\xi = \{\xi_i\}$, $i = 1, \ldots, \nu$ forms a linearly independent set of generators of $C^*(X)$. This basis I will refer to as the *chosen basis* of $C^*(X)$. Then if M is a differentiable manifold, by $C^*(M)$ I shall mean a simplicial cochain complex of M (for any triangulation of M).

Let M^n be an oriented differentiable manifold without boundary. Let $\mathcal{M}$ be a properly ordered differentiable cell filtration of M^n. Let $\widetilde{\mathcal{M}}$ be the dual filtration to $\mathcal{M}$. Let $\mathcal{M}$, $\widetilde{\mathcal{M}}$ be neighborhoods of the cell filtrations X, $\widetilde{X}$ (which for convenience, we identify with regular cell decompositions representing them) having

$$\varphi : M \to X$$
$$\widetilde{\varphi} : M \to \widetilde{X}$$

as respective projection maps.

Under this situation, a rather strict " Poincaré Duality " may be seen to hold between X and $\widetilde{X}$.

Given any two cell filtrations X, Y there is a natural map $\lambda : C^n(X \times Y) \to \mathbf{Z}$ obtained by extending the pairing

$$\lambda(\xi_i \otimes \eta_j) = \delta_{ij} \qquad i, j = 1, \ldots, \nu$$

by linearity to all of $C^*(X \times Y) \approx C^*(X) \otimes C^*(Y)$, and then by restricting to

$$C^n(X \times Y) \approx \sum_{q=0}^{n} C^q(X) \otimes C^{n-q}(Y),$$

where $\{\xi_i\}$, $\{\eta_j\}$ are the chosen bases of X, Y.

If X, $\widetilde{X}$ are the two cell filtrations coming, as above, from dual differentiable cell filtrations, the maps:

$$\lambda : C^q(X) \otimes C^{n-q}(\widetilde{X}) \to \mathbf{Z}$$

are non-degenerate pairings for all $q = 0, \ldots, n$. This statement is obviously equivalent to the fact that if:

$$X = (X_0, \ldots, X_\nu); \quad X_i = X_{i-1} \cup D^{n_i}$$
$$\widetilde{X} = (\widetilde{X}_0, \ldots, \widetilde{X}_{\tilde{\nu}}); \quad \widetilde{X}_i = \widetilde{X}_{i-1} \cup D^{\tilde{n}_i}$$

then

a) $$\nu = \tilde{\nu}$$

b) $$n_i + \tilde{n}_i = n \qquad \text{for } i = 1, \ldots, \nu.$$

Suggested by this, two properly ordered cell filtrations X, Y will be called *n-compatible* if

$$\lambda : C^q(X) \otimes C^{n-q}(Y) \to \mathbf{Z}$$

is a non-degenerate pairing for $q = 0, \ldots, n$.

Moreover, if X, $\tilde{X}$ are the cell filtrations coming from dual differentiable filtrations $\mathcal{M}$, $\tilde{\mathcal{M}}$ of M, with projection maps φ, $\tilde{\varphi}$, the following commutative diagram is easily checked:

(10.1)
$$\begin{array}{ccc} C^n(X \times \tilde{X}) \supseteq Z^n(X \times \tilde{X}) & \to & H^n(X \times \tilde{X}) \\ \searrow^{\lambda} & & \swarrow^{(\varphi \times \tilde{\varphi})^{(n)}} \\ & \mathbf{Z} \approx H^n(M) & \end{array}$$

and if $f : X \to \tilde{X}$ is the simple homotopy equivalence making

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & X \\ {\scriptstyle\tilde{\varphi}}\downarrow & \swarrow_f & \\ \tilde{X} & & \end{array}$$

commutative, $\Delta : X \to X \times X$ the diagonal map,

(10.2)
$$\begin{array}{ccc} H^n(X \times \tilde{X}) & \xleftarrow[\approx]{(1 \times f)^n} & H^n(X \times X) \\ \Big\downarrow \lambda & & \Big\downarrow \Delta^n \\ \mathbf{Z} & \xleftarrow[\approx]{\varphi^n} & H^n(X) \end{array}$$

is also commutative, where λ is now the homomorphism induced on cohomology by the natural pairing $\lambda : C^n(X \times \tilde{X}) \to \mathbf{Z} \approx H^n(M)$.

These definitions are suggested by (10.2):

An *n-dimensional homology-orientation* of a cell filtration X, is an isomorphism $\theta : H^n(X) \underset{\approx}{\to} \mathbf{Z}$. A *pairing* of two n-dimensional n-compatible homology-oriented cell filtrations X, $\widetilde{X}$ is given by an orientation-preserving simple homotopy equivalence $f : X \to \widetilde{X}$ making:

$$\begin{array}{ccc} H^n(X \times \widetilde{X}) & \xleftarrow[\approx]{(1\times f)^n} & H^n(X \times X) \\ \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\Delta^n} \\ \mathbf{Z} & \xleftarrow{\ \ \theta\ \ } & H^n(X) \end{array}$$

commutative.

Thus any differentiable cell filtration $\mathcal{M}$ of an oriented differentiable manifold M^n, without boundary, yields a pairing of the two cell filtrations X, $\widetilde{X}$ obtained as the unthickenings of the dual filtrations $\mathcal{M}$, $\widetilde{\mathcal{M}}$.

If M is an oriented differentiable n-manifold, a *pairing of* X *and* $\widetilde{X}$ *into* M will denote a pair $(\varphi, \widetilde{\varphi})$, of simple homotopy equivalences,

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & X \\ {\scriptstyle\widetilde{\varphi}}\downarrow & & \\ \widetilde{X} & & \end{array}$$

such that the simple homotopy equivalence, $f = \widetilde{\varphi}\varphi^{-1} : X \to \widetilde{X}$ is a pairing of X to $\widetilde{X}$ where X is given the homology orientation induced by φ and $\widetilde{X}$ that induced by $\widetilde{\varphi}$.

The notion of a pairing of cell filtrations is a simple-homotopy-theoretic notion. I shall say that a differentiable cell filtration $\mathcal{M}$ *realizes* the pairing $(\varphi, \widetilde{\varphi})$ of X, $\widetilde{X}$ to M if the differentiable cell filtration $\mathcal{M}$ gives rise to that pairing in the manner described above. (i.e., if $\mathcal{M}$, $\widetilde{\mathcal{M}}$ are neighborhoods of X, $\widetilde{X}$ with $\varphi : M \to X, \widetilde{\varphi} : M \to \widetilde{X}$ as projection maps.)

I shall say that a subfiltration $\widetilde{K} \subseteq \widetilde{X}$ is *complementary* to a subfiltration $K \subseteq X$ if $C^*(\widetilde{K}) \subseteq C^*(\widetilde{X})$ is the annihilator of $C^*(K) \subseteq C^*(X)$ with respect to the natural (non-degenerate) pairing, λ. The following technical lemma will be useful:

Lemma (**10.3**). — Let

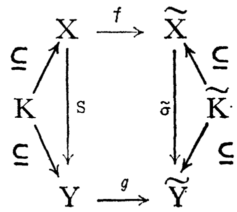
$$(10.4) \qquad \begin{array}{ccccc} & X & \xrightarrow{f} & \widetilde{X} & \\ \subseteq\nearrow & \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow{\scriptstyle\widetilde{\sigma}} & \nwarrow\subseteq \\ K & & & & \widetilde{K} \\ \subseteq\searrow & & & & \swarrow\subseteq \\ & Y & \xrightarrow{g} & \widetilde{Y} & \end{array}$$

be a commutative diagram and let the horizontal arrows f, g be pairings of properly ordered cell filtrations, the vertical arrows, simple homotopy equivalences. Let K be a subfiltration of both X, Y and $\widetilde{K}$ of $\widetilde{X}$, $\widetilde{Y}$ and let K and $\widetilde{K}$ be complementary. Assume, further, that K contains the 2-skeleton of X and of Y. Then there is a cell filtration similarity-mapping

$$S : X \to Y$$

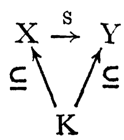
(i.e. S is orientation and filtration-preserving)

such that



is commutative.

Proof. — We must show that $\sigma : X \to Y$ may be replaced by an orientation and filtration-preserving map,



Such a map will be constructed inductively, by modifying σ successively so that it is filtration-preserving on increasingly larger subfiltrations, L,

$$K \subseteq L \subseteq X.$$

Assume, then that $\sigma : X \to Y$ is filtration and orientation-preserving on L, so that, for convenience of notation we may identify L with its image and write:

$$X = L \cup D^{n_1} \cup \ldots \cup D^{n_\nu}$$

$$Y = L \cup D^{m_1} \cup \ldots \cup D^{m_\mu}.$$

It is seen at once by commutativity of (10.4) that $n_1 = m_1 = k$. Again, for convenience, call $D^{n_1} = D^k$ and let $Y^{(k)} = L \cup D_1^k \cup \ldots \cup D_\lambda^k$. Our first aim will be to show that $\sigma : L \cup D^k \to Y$ may be homotopically modified to a map,

$$\sigma_1 : L \cup D^k \to L \cup D_1^k \subseteq Y.$$

For elementary reasons, we may immediately assume $\sigma(L \cup D^k) \subseteq Y^{(k)}$. Since σ is the identity on L,

$$\sigma : (D^k, \partial D^k) \to (Y^{(k)}, L)$$

gives rise to an element

$$[\sigma] \in \pi_k(Y^{(k)}, L).$$

To show that a map σ_1 exists such that

$$\begin{array}{ccc} L\cup D^k & \xrightarrow{\sigma_1} & L\cup D_1^k \\ & \searrow & \cap\!\! \mid \\ & & Y^{(k)} \end{array}$$

is commutative, one need only show that $[\sigma]$ (see chap. I) is in the image of the natural homomorphism j:

$$\begin{array}{ccc} \pi_k(L\cup D_1^k, L) & \xrightarrow{j} & \pi_k(Y^{(k)}, L) \\ {\scriptstyle h}\big\downarrow\approx & & {\scriptstyle h}\big\downarrow\approx \\ H_k(L\cup D_1^k, L) & \xrightarrow{j} & H_k(Y^{(k)}, L) \\ \big\downarrow\approx & & \big\uparrow \\ \mathbf{Z} & & H_k(Y^{(k)}, Y^{(k-1)}) \\ & & \big\uparrow\approx \\ & & C_k(Y) \end{array}$$

Using the definition of pairing and commutativity of (10.4), it is easily seen that

$$h([\sigma]) = j(+1)$$

where h is the Hurewicz map and $+1$ is the natural generator of $H_k(L\cup D_1^k, L)$. Since the Hurewicz maps are isomorphisms modulo the natural action of $\pi_1(L)$ the result follows.

Complements. — Given a homotopy k-skeleton

$$f : K \to M^n$$

we shall need to know simple-homotopy-theoretic information about the closed complement of a neighborhood of the mapping f. This is done by " imbedding " K in X, a cell filtration paired into M, and the proposition reads:

Proposition **(10.5)**. — If M^n is an oriented n-manifold without boundary, $(\varphi, \tilde{\varphi})$ a pairing into M,

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & X \\ {\scriptstyle\tilde{\varphi}}\downarrow & & \\ \tilde{X} & & \end{array}$$

and $K \subset X$, $\tilde{K} \subseteq \tilde{X}$ are complementary subfiltrations such that

a) $$X^{(2)} \subseteq K \subseteq X$$

b) $$\dim K \leq n-5$$

then there are homotopy-inverses to $\varphi, \tilde{\varphi}$:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & M \\ & & \uparrow{\scriptstyle\tilde{\psi}} \\ & & \tilde{X} \end{array}$$

such that there are neighborhoods $U, V \subseteq M$ of the maps

$$\psi : K \to M$$

$$\tilde{\psi} : \tilde{K} \to M$$

which are closed complements of one another in M. (i.e.,

$$M = U \cup V, \qquad U \cap V = \partial U = \partial V).$$

For the proof of Proposition 10.5 we shall need the following technical lemma:

Lemma **(10.6)**. — Let X, Y be cell decompositions and C(X), C(Y) their cellular chain complexes, with chosen bases. Let $f : X \to Y$ be a continuous map such that $f(X^{(q)}) \subseteq Y^{(q)}$ for all q, which is a simple homotopy equivalence of X and Y. Assume, further, that

$$f_{(q)} : C_q(X) \to C_q(Y)$$

is an isomorphism for all $q > k$ and f_q takes the chosen basis of $C_q(X)$ to the chosen basis of $C_q(X)$ $(q > k)$. Then

$$f^k : X^{(k)} \to Y^{(k)}$$

is a simple homotopy equivalence.

Proof. — It is easily seen that f^k is an isomorphism of fundamental group and homology groups. Therefore, by the classical Whitehead theorem, f^k is a homotopy equivalence. Using the fact that the Whitehead torsion of the map f is zero, and that $f_{(q)} : C_q(X) \underset{\approx}{\to} C_q(Y)$ is a correspondence of chosen bases, one obtains that the Whitehead torsion of f^k is zero, proving the lemma.

The only case of Proposition 10.5 that we will use will be where $K = X^{(k)}$ for some k. For simplicity, I shall prove it only in this case; the general result would then follow quite easily. So we may set $K = X^{(k)}$. Then $\widetilde{K} = \widetilde{X}^{(n-k-1)}$, and $2 \leq k \leq n-5$.

It is immediately seen that

$$\psi^k : X^{(k)} \to M^n$$

is a homotopy k-skeleton, and therefore, if $k \leq n-5$, Proposition 9.7 applies, and we have that there is a neighborhood $U \subseteq M^n$ of the mapping ψ^k which is a geometric k-skeleton of M^n. Explicitly, there is a differentiable cell filtration $\mathscr{W}$ of M^n such that $\mathscr{W}^{(k)}$ is a neighborhood of $X^{(k)}$ cutting out the submanifold $U \subseteq M^n$. The closed complement of U, $V \subseteq M^n$ may then be given as the submanifold of M^n cut out by $\widetilde{\mathscr{W}}^{(n-k-1)}$.

We may choose a homotopy-inverse to $\widetilde{\varphi} : M \to \widetilde{X}$, $\widetilde{\psi}$, so that

$$\widetilde{\psi}(\widetilde{X}^{(n-k-1)}) \subseteq V:$$

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\psi}} & M^n \\ \cup| & & \cup| \\ \widetilde{X}^{(n-k-1)} & \overset{\tilde{\psi}}{\to} & V. \end{array}$$

Let Z be the cell filtration which is the " unthickening " of $\mathscr{W}$ (i.e. $\mathscr{W} \in \mathscr{N}^n(Z)$) and let $g : M \to Z \times \widetilde{Z}$ be the resulting pairing. Then we may interpret $\widetilde{\psi}$ as a map:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\psi}} & \widetilde{Z} \\ \cup| & & \cup| \\ \widetilde{X}^{(n-k-1)} & \underset{\tilde{\psi}}{\to} & \widetilde{Z}^{(n-k-1)} \end{array}$$

where $\psi : X^{(k)} \to Z^{(k)}$ is the identity map. The map $\widetilde{\psi}$ may be seen to satisfy the hypotheses of lemma (10.6) showing that

$$\widetilde{X}^{(n-k-1)} \underset{\tilde{\psi}}{\to} V$$

is a simple homotopy equivalence.

Since U is a geometric k-skeleton of M^n, $k \geq 2$, we have: geom dim $V \leq n-3$ and therefore, by the existence theorem of Chapter VIII, V is a neighborhood of the mapping

$$\widetilde{\psi} : \widetilde{K} = \widetilde{X}^{(n-k-1)} \to V.$$

The existence theorem for oriented manifolds without boundary. — The object of this chapter is to demonstrate an existence theorem showing that purely on the basis of simple-homotopy-theoretic information, one may conclude the existence of differentiable cell filtrations of M.

The essential statement of the existence theorem is that if M^n is an oriented manifold without boundary, $n \geq 7$, any pairing of $X, \widetilde{X}$ to M^n is *realizable* by dual cell filtrations, $\mathcal{M}$, $\widetilde{\mathcal{M}}$ of M. But not quite. Something a bit weaker than that will be proved, and in order to make the statement of the theorem succinct, I shall prepare the way for its entrance by a definition.

Definition (**10.7**). — The pair of dual differentiable cell filtrations $\mathcal{M}$, $\widetilde{\mathcal{M}}$ of the oriented manifold M^n will be said to be a *realization up to similarity* of the pairing

$$\begin{array}{ccc} M^n & \xrightarrow{\varphi} & X \\ {\scriptstyle\tilde\varphi}\downarrow & & \\ \widetilde{X} & & \end{array}$$

if the pair $\mathcal{M}, \widetilde{\mathcal{M}}$ is a realization of a pairing $(\psi, \widetilde{\psi})$

$$\begin{array}{ccccc} & & Y & & \\ & & \uparrow{\scriptstyle\psi} & \searrow{\scriptstyle\sigma} & \\ \widetilde{Y} & \xleftarrow{\tilde\psi} & M^n & \xrightarrow{\varphi} & X \\ & \searrow{\scriptstyle\tilde\sigma} & \downarrow{\scriptstyle\tilde\varphi} & & \\ & & \widetilde{X} & & \end{array}$$

such that there are similarity mappings (i.e., filtration and orientation-preserving homotopy equivalences) $\sigma, \widetilde{\sigma}$ making the above diagram homotopy-commutative.

More succinctly: the pair $\mathcal{M}, \widetilde{\mathcal{M}}$ is a realization up to similarity of $X, \widetilde{X}$ if there are filtration and orientation-preserving homotopy equivalences $\pi, \widetilde{\pi}$,

$$\begin{array}{ccccc} \mathcal{M} & & & & \widetilde{\mathcal{M}} \\ & \diagdown\!\diagdown & & /\!/ & \\ {\scriptstyle\pi}\downarrow & & M^n & & \downarrow{\scriptstyle\tilde\pi} \\ & \swarrow{\scriptstyle\varphi} & & \searrow{\scriptstyle\tilde\varphi} & \\ X & & & & \widetilde{X} \end{array}$$

such that the above diagram is homotopy commutative.

Existence theorem for oriented manifolds. — Let M^n be an oriented manifold without boundary, $n \geq 7$. Let

$$\begin{array}{ccc} M^n & \xrightarrow{\varphi} & X \\ {\scriptstyle\tilde\varphi}\downarrow & & \\ \widetilde{X} & & \end{array}$$

be a pairing. Then there exists a realization (up to similarity) of the pairing $(\varphi, \widetilde{\varphi})$ by dual differentiable cell filtrations $\mathcal{M}$, $\widetilde{\mathcal{M}}$ of M^n.

Proof. — Let $\psi : X \to M$ be a homotopy inverse to φ and let $K = X^{(2)} \subseteq X$. Then $\widetilde{K} = \widetilde{X}^{(n-3)} \subseteq \widetilde{X}$.

Since $2 \leq n-5$, Proposition 10.5 applies, and there is a homotopy inverse to $\widetilde{\varphi}$,

$$\widetilde{\psi} : \widetilde{X} \to M^n$$

and complementary submanifolds $U, V \subseteq M^n$ such that

$$\psi : K \to U$$
$$\widetilde{\psi} : \widetilde{K} \to V$$

and U is a neighborhood of ψ, V is a neighborhood of $\widetilde{\psi}$.

Thus U is cut out by the differentiable filtration $\mathscr{U}$, V by $\mathscr{V}$ such that $\mathscr{U}$ is a neighborhood of K, $\mathscr{V}$ of $\widetilde{K}$. By Proposition 9.1 we may compound the two differentiable filtrations $\mathscr{U}$ on U, $\mathscr{V}$ on V to obtain a single differentiable filtration $\mathscr{W}$ on M such that $\mathscr{W}$ contains $\mathscr{U}$ as a subfiltration, $\widetilde{\mathscr{W}}$ contains $\mathscr{V}$ as a subfiltration. Consequently the pair of differentiable filtrations $\mathscr{W}$, $\widetilde{\mathscr{W}}$ give rise to a pairing

$$\begin{array}{ccc} M & \xrightarrow{\varphi'} & Y \\ {\scriptstyle \widetilde{\varphi}'}\downarrow & & \\ Y & & \end{array}$$

and $K \subseteq Y$, $\widetilde{K} \subseteq \widetilde{Y}$ are complementary subfiltrations giving rise to a commutative diagram

$$\begin{array}{ccccc} & & Y & \supseteq & K \\ & & \uparrow {\scriptstyle \varphi'} & & \cap \\ \widetilde{Y} & \xleftarrow{\widetilde{\varphi}'} & M & \xrightarrow{\varphi} & X \\ \cup & & \downarrow {\scriptstyle \widetilde{\varphi}} & & \\ \widetilde{K} & \subseteq & \widetilde{X} & & \end{array}$$

But, in this situation, lemma 10.3 applies, yielding similarity-equivalences σ, $\widetilde{\sigma}$ such that

$$\begin{array}{ccccc} & & Y & \supseteq & K \\ & & \uparrow {\scriptstyle \varphi'} & \nwarrow {\scriptstyle \widetilde{\sigma}} & \cup \\ \widetilde{Y} & \xleftarrow{\widetilde{\varphi}'} & M & \xrightarrow{\varphi} & X \\ \cup & \nwarrow {\scriptstyle \sigma} & \downarrow {\scriptstyle \widetilde{\varphi}} & & \\ \widetilde{K} & \subseteq & \widetilde{X} & & \end{array}$$

is commutative. This proves that $\mathscr{W}$, $\widetilde{\mathscr{W}}$ is a realization (up to similarity) of the pairing,

$$\begin{array}{ccc} M^n & \xrightarrow{\varphi} & X \\ \downarrow {\scriptstyle \widetilde{\varphi}} & & \\ \widetilde{X} & & \end{array}$$

proving the theorem.

The existence of Morse functions with prescribed data. — The set of integers

$$J = (j_0, \ldots, j_n)$$

will be called the *data* of a Morse function f on the manifold M^n if f has precisely j_k critical points of index k.

The *data* of a cell filtration X will be a set of integers $J = \{j_0, \ldots, j_n\}$ such that X has precisely j_k cells of dimension k.

An application of the existence theorem for oriented manifolds is the following:

Proposition (**10.8**). — Let M^n be an oriented manifold without boundary, $n \geq 7$. Then there exists a Morse function f on M^n with prescribed data J, if and only if there is a pairing of X, $\widetilde{X}$ to M^n such that the data of the cell filtration X is J.

(Thus: the question of existence of Morse functions on M^n with prescribed data, J, is a simple-homotopy-theoretic question; it is rephraseable as a question involving only the simple-homotopy type of M^n.)

This generalizes the theorem of Smale regarding existence of Morse functions with prescribed data on simply connected manifolds to the non-simply connected case. It follows immediately from the existence theorem.

CHAPTER XI

THE STABLE THEORY

In the case where n is large compared to $\dim X$, the theory of n-dimensional neighborhoods over X admits a certain amount of completeness using only standard " general position " techniques of stable differential topology. One may obtain these results (the notations being these of chap. I):

1) The map $t : \mathcal{N}^n(X) \to \widetilde{KO}(X)$ which assigns to each neighborhood over X its stable tangent bundle (considered as a differentiable manifold, every neighborhood of X has a tangent bundle which may be regarded as a stable vector bundle over X) is an isomorphism of sets.

2) If N is a stable neighborhood of X (i.e. an n-dimensional neighborhood for n large compared to $\dim X$) and aut(N) is the group of isotopy classes of differentiable automorphisms of N homotopic to the identity, then

$$\mathrm{aut}(N) \approx [X, O_n] \ (n \text{ large}).$$

3) Fairly easily, using only stable techniques, one may get the " stable " neighborhood theorem:

If $f : X \to Y$ is a simple homotopy equivalence of cell filtrations, then f induces an isomorphism:

$$f^{(n)} : \mathcal{N}^n(X) \underset{\approx}{\to} \mathcal{N}^n(Y).$$

4) If $f : M^k \to X$ is a simple homotopy equivalence of a differentiable manifold M^k with a cell filtration X, and n is large compared to k and $\dim X$, then any stable neighborhood $N \in \mathcal{N}^n(X)$ is diffeomorphic with a differentiable $(n-k)$-cell bundle over M^k, and conversely, any differentiable $(n-k)$-cell bundle over M^k is diffeomorphic with a stable neighborhood of X.

5) One of the main applications of 3), 4) is a strengthening of a theorem of Whitehead so that it applies to differentiable manifolds:

Theorem. — Let $f : M_1^k \to M_2^k$ be a simple homotopy equivalence of the two differentiable manifolds M_1^k, M_2^k. Then if n is large compared to k, the map f induces a differentiable isomorphism:

$$f^{(n)} : D^n(M_1^k) \underset{\approx}{\to} D^n(M_2^k)$$

where $D^n(M)$ is the set of differentiable n-cell bundles over M, considered as a collection of differentiable manifolds.

This chapter is independent of the nonstable theory and has, as objective, the proof of the five statements above. Many of the results obtained are extendable to the " metastable range " (i.e. $n >> 3/2 \dim X$).

Existence of stable filtration. — Compared to the nonstable situation it is an utter triviality to show that enough stable filtrations exist:

Proposition (**11.1**). — Let $n \geq 2k+2$. If $f: X \to M^n$ is a continuous map of a k-dimensional cell filtration X into an n-dimensional differentiable manifold M^n, there is a differentiable cell filtration $N \in \mathcal{N}^n(X)$, an imbedding $\beta : N \to M^n$ and a projection map $\pi : N \to X$ such that

$$\begin{array}{ccc} N & \overset{\beta}{\subseteq} & M^n \\ {\scriptstyle\pi}\downarrow & \nearrow_{f} & \\ X & & \end{array}$$

is homotopy commutative. The differentiable submanifold $U = \beta(N)$ is unique up to differentiable isotopy of M^n (dependent only upon the homotopy class of f).

Proof. — N^n is defined inductively in terms of the length of $X \in \mathcal{F}^{(k)}$. Assume, then

$$X = \widetilde{X} \cup_\varphi D^s$$

and $n \geq 2k+2$. Assume that $l(X) = \nu$ and N^n has been defined on all elements of $\mathcal{F}^{(k)}$ of length less than ν.

Therefore $N^n(\widetilde{X}) = \widetilde{N}$ is defined, and we may assume $\widetilde{N} \subseteq M^n$ uniquely up to domain automorphism and range isotopy. Let $\varphi : \partial D^s \to \widetilde{X}$ represent the element $[\varphi] \in \pi_{s-1}(\widetilde{X})$.

But

$$\text{(11.2)} \qquad \pi_{s-1}(\widetilde{X}) \approx \pi_{s-1}(\widetilde{N}) \approx \pi_{s-1}(\partial\widetilde{N}).$$

These isomorphisms follow from standard homotopy considerations and the inequalities $k \geq s$, $n \geq 2k+1$.

By general positionality and (11.2) it follows that there is a differentiable imbedding

$$\overline{\varphi} : \partial D^n \to \partial\widetilde{N}$$

representing $[\varphi] \in \pi_{s-1}(\partial\widetilde{N})$ and the differentiable isotopy class of $\overline{\varphi}$ is unique. Since $\widetilde{N} \approx C_{\tilde{\sigma}}$ is a mapping cylinder over X and $\dim X + s < n$, there is a continuous mapping $\overline{\varphi}' : D^s \to M^n$ extending $\overline{\varphi} : \partial D^s \to \partial\widetilde{N} \subseteq M^n$ such that if W is the closure of $M^n - \widetilde{N}$, then $\overline{\varphi}' : D^s \to W \subseteq M^n$ and $\overline{\varphi}'$ is homotopic to f considered as maps,

$$f, \overline{\varphi}' : (D^s, \partial D^s) \to (W, \partial W).$$

Since s is small compared with n, there is a differentiable imbedding $\overline{\psi} : (D^s, \partial D^s) \to (W, \partial W)$ which is homotopic to $\overline{\varphi}'$ and such that

(i) $\bar{\psi} \mid \partial D^s = \bar{\varphi}$.

(ii) $\bar{\psi}(D^s) \cap \partial W = \varphi(\partial D^s)$.

(iii) $\bar{\psi}(D^s)$ meets ∂W transversally.

This imbedding $\bar{\psi} : D^s \to W$ is unique up to isotopy class, by general positionality.

Let $\pi : U \to D^s$ exhibit $U \subset W$ as a transverse tubular neighborhood of D^s in W. By the tubular neighborhood lemma, up to range isotopy, U is unique.

Since any cell bundle over D^s is trivial, there is a natural isomorphism

$$\begin{array}{ccc} \gamma : D^s \times D^{n-s} & \xrightarrow{\approx} & U \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & D^s & \end{array}$$

where p is projection onto the first factor.

Defining $N = \widetilde{N} \cup U = \widetilde{N} \cup_\gamma D^s \times D^{n-s}$ expresses $N \subseteq M^n$ as a neighborhood of X imbeddable uniquely in M^n up to domain automorphism and range isotopy. Clearly this neighborhood N satisfies the stipulations of (11.1).

There is one special case of (11.1) to be singled out. That is, when $M^n = R^n$. Then all maps $f : X \to R^n$ are homotopic, so we may as well choose f to be the constant map. Then the neighborhood of X obtained via (11.1) will be called the *canonical neighborhood of* X, denoted $N^n(X)$. $N^n(X)$ plays the role of a canonical tubular neighborhood of X " imbedded " in n-dimensional Euclidean space. Its underlying combinatorial structure will be isomorphic with a " regular neighborhood " of K rectilinearly imbedded in Euclidean space (in the sense of Whitehead [13]) if X is a filtration coming from a finite simplical complex K.

If X is a filtration coming from a C^1-compatible triangulation of a differentiable manifold M, then $N^n(X)$ is differentiably isomorphic with a compact cell-bundle neighborhood of M imbedded differentiably in R^n. This will show, for instance, that the differentiable structure on such a cell bundle neighborhood is dependent only on the underlying combinatorial structure of M. (In fact, it is dependent only upon the simple homotopy type of M.)

By N(X) I shall mean $N^n(X)$ for some n for which it is defined.

In dealing with the canonical neighborhood, these properties are useful and easily proven:

(11.3) If $W = X \cup_Y Z$ are cell filtrations, then

$$N^k(W) = N^k(X) \cup_{N^k(Y)} N^k(Z).$$

(11.4) If X is a reordering of Y, $N^k(X)$ is a reordering of $N^k(Y)$.

(11.5) Let $N \in \mathcal{N}^n(X)$ such that $n \geq 2 \dim X + 2$. If N is differentiably imbeddable in R^n, then $N = N^n(X)$.

(11.6) If K is a finite simplicial complex, $\widetilde{K}$ the cell filtration induced by K, U the simplicial regular neighborhood of K as rectilinearly imbedded in R^n (n large), then U is diffeomorphic with $N^n(\widetilde{K})$.

The compatibility of the two definitions of simple homotopy type for simplicial complexes.

Proposition **(11.7)**. — If K, L are simplicial complexes, $\widetilde{K}$, $\widetilde{L}$ the cell filtrations to which they give rise, $f: K \to L$ a continuous map, then $f: K \to L$ is a simple homotopy equivalence (of simplicial complexes) if and only if $f: K \to L$ is a simple homotopy equivalence (of cell filtrations).

Proof. — One way is essentially immediate. I shall concentrate on proving that if $f: \widetilde{K} \to \widetilde{L}$ is a simple homotopy equivalence, so is $f: K \to L$.

The above proposition follows from the three results immediately following (combined with the classical theorem asserting the uniqueness of the combinatorial equivalence class of smooth triangulations of a differentiable manifold).

Lemma **(11.8)**. — If $i: X \to X^*$ is an elementary expansion, then

$$N^n(i) : N^n(X) \to N^n(X^*)$$

is isotopic to a diffeomorphism.

Proof. — Standard use of general positionality.

Proposition **(11.9)**. — If $X, Y \in \mathscr{F}$ are of the same simple homotopy type, then $N^n(X) \approx N^n(Y)$ " as differentiable manifolds ".

Proposition 11.9 follows by repeated application of Lemma 11.8.

(11.10) If $\widetilde{K}$ is a cell filtration coming from a smooth triangulation, K, of the differentiable manifold N which is a neighborhood of the cell filtration X, then $f: \widetilde{K} \to X$, the continuous map induced by a projection, $\pi : N \to X$, is a simple homotopy equivalence of cell filtrations.

Stable neighborhoods and simple homotopy type. — In contrast to the difficulty of proof of the nonstable neighborhood theorem, the stable version is trivial:

Stable Neighborhood Theorem. — Let $f: X \to Y$ be a simple homotopy equivalence of cell filtrations; let $n > 2 \dim f + 1$. Then there is induced a bijective differentiable isomorphism

$$\hat{f} : \mathscr{N}^n(X) \to \mathscr{N}^n(Y)$$

which is functorial for simple homotopy equivalences, and characterized by:

If $f: X \to X^*$ is an elementary expansion, then $(\hat{f})$ is isotopic to $f^{(n)}$.

Proof. — Prove it for an elementary expansion, and then use functoriality.

The relationship between neighborhoods and vector bundles. — Let $V^k(X)$ be the set of k-plane bundles over X, a cell filtration. There is a map

$$\sigma : V^k(X) \times \mathcal{N}^n(X) \to \mathcal{N}^{k+n}(X).$$

The neighborhood $\sigma(\eta, N) \in \mathcal{N}^{k+n}(X)$, for $\eta \in V^k(X)$, $N \in \mathcal{N}(X)$ may be obtained as follows:

Let $\pi : N \to X$ be a projection map, and $\pi^* : V^k(X) \to V^k(N)$ represent pull-back of vector bundles. Then $\pi^*(\eta)$ is equivalent to a differentiable vector bundle ξ over N, and ξ is unique up to differentiable vector bundle isomorphism. Let $p : D(\eta) \to N$ be the unique differentiable k-cell bundle over N associated to ξ. Let M be the filtered object,

$$M = (M_0, \ldots, M_\nu)$$

where $$M_i = p^{-1}(N_i) \subseteq D(\eta)$$

if $N = (N_0, \ldots, N_\nu)$, $N_i = N_{i-1} \cup_{\varphi_i} D^{n_i} \times D^{n-n_i}$.

Since $p : D(\eta) \to N$ is a differentiable cell bundle,

$$p^{-1}(D^{n_i} \times D^{n-n_i}) = (D^{n_i} \times D^{n-n_i}) \times D^k$$
$$= D^{n_i} \times D^{n+k-n_i}.$$

Therefore M is a differentiable cell bundle and an $(n+k)$-dimensional neighborhood of X. Define $\sigma(\eta, N) = M$. I shall sometimes write $M = N^\eta$.

Suspensions, and automorphisms of stable filtrations. — In this section a " Suspension Theorem " will be proved.

Natural " suspension " maps:

$$j : \eta^n(X) \to \eta^{n+1}(X)$$
$$j_\alpha : \alpha\{M\} \to \alpha\{jM\}$$

for $M \in \mathcal{N}^n(X)$ will be defined, where $\alpha\{M\}$ is the group of isotopy classes of filtration-preserving differentiable automorphisms of M. It will then be shown that these maps are bijections if

$$n \geq 2 \dim X + 1.$$

All arguments used are elementary, and the matter is methodically reduced to the stability of the homotopy groups of orthogonal groups.

Let $f_t : A \to B$ be an isotopy and $V \subset A$; the isotopy f_t is called V-*rigid* if $f_t | V$ is the identity for all t. If $\varphi : \partial D^s \times D^{n-s} \to W^{n-1}$ is an imbedding, let $\overline{\varphi} = \varphi | \partial D^s \times \{0\}$, and denote by $Is(\varphi)$ the set of $(\partial D^s \times \{0\})$-rigid isotopy classes of imbeddings $\psi : \partial D^s \times D^{n-s} \to W^{n-1}$ such that $\overline{\varphi} = \overline{\psi}$.

Let M be a differentiable cell decomposition. Then $\alpha\{M\}$ will denote the set of isotopy classes of filtration-preserving automorphisms of M.

(11.11) (The " suspension " maps j.)

If φ is as above, $\psi \in \mathrm{Is}(\varphi)$, let

$$\psi \times 1 : \partial D^s \times D^{n-s} \times D^1 \to W^{n-1} \times D^1$$

be the map $(\psi \times 1)(x, y) = (\psi(x), y)$ for $x \in \partial D^s \times D^{n-s}$, $y \in D^1$.

We have then a map

$$j : \mathrm{Is}(\varphi) \to \mathrm{Is}(\varphi \times 1)$$

given by $j(\psi) = \psi \times 1$. It clearly behaves well with respect to isotopy equivalence.

If X is a cell filtration, define a map

$$j : \mathcal{N}^n(X) \to \mathcal{N}^m(X) \qquad m \geq n$$

as follows:

$$j(N) = N^{1_{m-n}} \qquad \text{if } N \in \mathcal{N}^n(X).$$

Clearly

$$\begin{array}{ccc} \mathcal{N}^p(X) & \xrightarrow{j} & \mathcal{N}^q(X) \\ & {\scriptstyle j}\searrow \quad \swarrow{\scriptstyle j} & \\ & \mathcal{N}^r(X) & \end{array}$$

is commutative.

If $M \in \mathcal{N}^n(X)$, and

$$j : \mathcal{N}^n(X) \to \mathcal{N}^m(X),$$

there is a map

$$j_\alpha : \alpha\{M\} \to \alpha\{jM\}$$

given as follows:

The total space jM is a trivial bundle over M. Choosing a particular trivialization of jM, any filtration preserving automorphism of M gives rise to such an automorphism of jM (extending it via the identity automorphism of the fibre).

Let $i : O_n \to O_m$ be the inclusion map of the n-dimensional orthogonal group O_n in O_m.

Let $i_* : \pi_*(O_n) \to \pi_*(O_m)$ denote the induced map on homotopy groups.

Let $f : \partial D^s \to O_{n-s}$ be a differentiable map. Then

$$a_f : \partial D^s \times D^{n-s} \to \partial D^s \times D^{n-s}$$

is the differentiable automorphism

$$a_f(x, y) = (x, f(x) . y) \qquad \text{for } x \in \partial D^s, y \in D^{n-s},$$

where multiplication denotes the action of O_{n-s} on D^{n-s} considered as the unit cell in R^{n-s}.

It is elementary that the differentiable isotopy class of a_f is dependent only upon the homotopy class of f, giving rise to a map of $\pi_{s-1}(O_{n-s})$ onto linear isotopy classes of bundle-automorphisms of $\partial D^s \times D^{n-s}$ (denoted $\mathrm{Is}_{\text{linear}}(\partial D^s \times D^{n-s})$)

$$a : \pi_{s-1}(O_{n-s}) \to \mathrm{Is}_{\text{linear}}(\partial D^s \times D^{n-s})$$

$$a([f]) = a_f.$$

The map a is a bijection.

More generally:

If A is a differentiable manifold and E a differentiable k-plane bundle over A. Let $\mathrm{Is}_l(\mathrm{E})$ be the linear isotopy classes of bundle automorphisms of E, covering the identity. Similarly, as above, a map $a : [\mathrm{A}, \mathrm{O}_k] \to \mathrm{Is}_l(\mathrm{E})$ may be defined.

Lemma (**11.12**). — The map a is a bijection.

Let $\varphi : \partial \mathrm{D}^s \times \mathrm{D}^{n-s} \to \mathrm{W}^{n-1}$ be as above. Define a map

$$\rho : \pi_{s-1}(\mathrm{O}_{n-s}) \to \mathrm{Is}(\varphi)$$

by $\rho([f]) = \varphi \circ a_f \in \mathrm{Is}(\varphi)$.

Let $\psi : \partial \mathrm{D}^s \times \mathrm{D}^{n-s} \to \mathrm{W}^{n-1}$ represent an element of $\mathrm{Is}(\varphi)$. Then by the tubular neighborhood lemma, there is an automorphism $\gamma : \mathrm{W}^{n-1} \to \mathrm{W}^{n-1}$ isotopic to the identity, and a linear automorphism

$$\lambda : \partial \mathrm{D}^s \times \mathrm{D}^{n-s} \to \partial \mathrm{D}^s \times \mathrm{D}^{n-s},$$

such that

$$\begin{array}{ccc} \partial \mathrm{D}^s \times \mathrm{D}^{n-s} & \xrightarrow{\varphi} & \mathrm{W}^{n-1} \\ \downarrow \lambda & & \downarrow \gamma \\ \partial \mathrm{D}^s \times \mathrm{D}^{n-s} & \xrightarrow{\psi} & \mathrm{W}^{n-1} \end{array}$$

is commutative. It is immediate that such a λ is unique up to linear isotopy, defining a map

$$\eta : \mathrm{Is}(\varphi) \to \pi_{s-1}(\mathrm{O}_{n-s}),$$
$$\eta(\psi) = a^1(\lambda).$$

Lemma (**11.13**). — The maps

$$\eta : \mathrm{Is}(\varphi) \to \pi_{s-1}(\mathrm{O}_{n-s})$$
$$\rho : \pi_{s-1}(\mathrm{O}_{n-s}) \to \mathrm{Is}(\varphi)$$

are inverses of one another, and are therefore bijections.

Proof. — Immediate.

Lemma (**11.14**). — The diagram

$$\begin{array}{ccc} \mathrm{Is}(\varphi) & \xrightarrow{\eta} & \pi_{s-1}(\mathrm{O}_{n-s}) \\ \downarrow j & & \downarrow i_* \\ \mathrm{Is}(\varphi \times 1) & \xrightarrow{\eta} & \pi_{s-1}(\mathrm{O}_{n-s+1}) \end{array}$$

is commutative.

Let $+1 : \partial D^s \to \partial D^s$ refer to the identity map, and $-1 : \partial D^s \to \partial D^s$ the map induced from the map $-1 : R^s \to R^s$ which is defined as

$$(-1)(x_1, x_2, \ldots, x_s) = (-x_1, x_2, \ldots, x_s).$$

The maps $\pm 1 : \partial D^s \to \partial D^s$ extend to maps $\pm 1 : \partial D^s \times D^{n-s} \to \partial D^s \times D^{n-s}$ by having them be the identity on the D^{n-s} factor.

Lemma **(11.15)**. — Let $n > 2s - 1$. If $f : \partial D^s \times D^{n-s} \to \partial D^s \times D^{n-s}$ is an automorphism which is the restriction to $\partial D^s \times D^{n-s}$ of an automorphism

$$F : D^s \times D^{n-s} \to D^s \times D^{n-s}$$

then f is isotopic to ± 1 (one of the two maps $+1$ or -1).

The proof of (11.11) is quite easy. Recall the following elementary lemma:

Lemma **(11.16)**. — The map

$$i_* : \pi_q(O_n) \to \pi_q(O_{n+1})$$

is a bijection for $q \leq n-2$, and surjective for $q = n-1$.

Lemma **(11.17)**. — If $n \geq 2s+1$, $\varphi : \partial D^s \times D^{n-s} \to W^{n-1}$ an imbedding then $j : \mathrm{Is}(\varphi) \to \mathrm{Is}(\varphi \times 1)$ is a bijection.

Proposition **(11.18)**. — *(Suspension Theorem.)* Let $X \in \mathscr{F}$, $n \geq 2 \dim X + 1$. Then the maps

$$j : \mathscr{N}^n(X) \to \mathscr{N}^{n+1}(X)$$

$$j_\alpha : \alpha\{M\} \to \alpha\{jM\} \qquad \text{for } M \in \mathscr{N}^n(X)$$

are bijections.

Proof. — *a)* Injectivity of j and surjectivity of j_α will be proved first, by induction on the length of X.

Let $M_1, M_2 \in \mathscr{N}^n(X)$ and suppose $g : j(M_1) \underset{\approx}{\to} j(M_2)$ is a filtration preserving isomorphism. It will be shown that there is an isomorphism $g_0 : M_1 \underset{\approx}{\to} M_2$ such that $j_\alpha(g_0) \approx g$.

This will establish both injectivity of j and surjectivity of j_α.

Let $X = (X_0, \ldots, X_\nu)$ and assume the proposition already verified for all filtrations of length less than ν. Let

$$\widetilde{X} = (X_0, \ldots, X_{\nu-1}).$$

Then $X = \widetilde{X} \cup_\varphi D^s$, for some integer s. Let $\widetilde{M}_1, \widetilde{M}_2$ be the restrictions of M_1, M_2 to $\widetilde{X}$, so

$$M_1 = \widetilde{M}_1 \cup_{\varphi_1} D^s \times D^{n-s}$$

$$M_2 = \widetilde{M}_2 \cup_{\varphi_2} D^s \times D^{n-s}$$

and

$$jM_1 = j\widetilde{M}_1 \cup_{j\varphi_1} D^s \times D^{n-s} \times D^1$$

$$jM_2 = j\widetilde{M}_2 \cup_{j\varphi_2} D^s \times D^{n-s} \times D^1.$$

Let

$$\widetilde{g} : j\widetilde{M}_1 \underset{\approx}{\to} j\widetilde{M}_2$$

be the restriction of g to $j\widetilde{M}_1$. By the inductive assumption

$$j_\alpha : \alpha\{\widetilde{M}_1\} \to \alpha\{j\widetilde{M}_1\}$$

is surjective, and therefore there is an isomorphism

$$\widetilde{g}_0 : \widetilde{M}_1 \underset{\approx}{\to} \widetilde{M}_2$$

such that "$j_\alpha(\widetilde{g}_0) \approx \widetilde{g}$".

The diagram

(II.19)
$$\begin{array}{ccc} j\partial\widetilde{M}_1 & \overset{j\varphi_1}{\longleftarrow} & \partial D^s \times D^{n-s} \times D^1 \subseteq D^s \times D^{n-s} \times D^1 \\ \downarrow \widetilde{g} & & \downarrow \widetilde{g} \qquad\qquad \downarrow \widetilde{g} \\ j\partial\widetilde{M}_2 & \overset{j\varphi_2}{\longleftarrow} & \partial D^s \times D^{n-s} \times D^1 \subseteq D^s \times D^{n-s} \times D^1 \end{array}$$

is, of course, commutative.

By (II.15),

$$\widetilde{g} : \partial D^s \times D^{n-s} \times D^1 \to \partial D^s \times D^{n-s} \times D^1$$

is isotopic (within $\partial D^s \times D^{n-s} \times D^1$) to $\pm 1 : \partial D^s \times D^{n-s} \times D^1 \to \partial D^s \times D^{n-s} \times D^1$.

Therefore, the diagram

(II.20)
$$\begin{array}{ccc} j\partial\widetilde{M}_1 & \overset{j\varphi_1}{\longleftarrow} & \partial D^s \times D^{n-s} \times D^1 \\ \downarrow j_\alpha(\widetilde{g}_0) & & \downarrow \pm 1 \\ j\partial\widetilde{M}_2 & \underset{j\varphi_2}{\longleftarrow} & \partial D^s \times D^{n-s} \times D^1 \end{array}$$

is commutative up to differentiable isotopy.

By virtue of (II.17), in the range $n \geq 2s+1$, the diagram

(II.21)
$$\begin{array}{ccc} \partial\widetilde{M}_1 & \overset{\varphi_1}{\leftarrow} & \partial D^s \times D^{n-s} \\ \downarrow \widetilde{g}_0 & & \downarrow \pm 1 \\ \partial\widetilde{M}_2 & \overset{\varphi_2}{\leftarrow} & \partial D^s \times D^{n-s} \end{array}$$

is also commutative up to differentiable isotopy. (In order to apply (11.17) one must first perform an isotopy so that

(11.22)
$$\begin{array}{ccc} \partial\widetilde{M}_1 & \overset{\overline{\varphi}_1}{\leftarrow} & \partial D^s \times \{0\} \\ \downarrow \tilde{g}_0 & & \downarrow \pm 1 \\ \partial\widetilde{M}_2 & \overset{\overline{\varphi}_2}{\leftarrow} & \partial D^s \times \{0\} \end{array}$$

is actually commutative. This may be done by virtue of the fact that (11.22) is homotopy commutative, and we are in a range of dimensions where any homotopy may be approximated by an isotopy, for $2(s-1)+2 \leq n-1$.)

It is immediate, from (11.21) that there exists an isomorphism

$$g_0 : M_1 \underset{\approx}{\rightarrow} M_2$$

extending $\widetilde{g}_0 : M_1 \underset{\approx}{\rightarrow} M_2$.

This isomorphism is unique up to isotopy, and clearly

$$j_\alpha(g_0) \approx g.$$

This proves injectivity of j and surjectivity of j_α.

b) Injectivity of j_α is easy, and I shall omit it. It remains to prove surjectivity of j.

(11.23) Let $\varphi : \partial D^s \times D^{n-s+1} \rightarrow \partial j M^n$ be an imbedding, where $M \in \mathscr{D}_*$, and $n \geq 2s+1$.

Then φ may be changed by differentiable isotopy to φ' so that

$$\varphi'(\partial D^s \times D^{n-s+1}) \subseteq j\partial M^n \subset \partial j M^n.$$

Proof.
$$jM = M \times I.$$
$$\partial jM = \partial M \times I \cup M \times \partial I.$$

To prove (11.23), it suffices to " remove "

$$\overline{\varphi} : \partial D^s \rightarrow \partial M \times I \cup M \times \partial I$$

from the interior of $M \times \partial I$ ($M \times \partial I$ consists, of course, in two disjoint copies of M). This removal may be done, inductively with respect to the filtration of M, by general position arguments.

Let X be again of length ν, and assume that surjectivity of j has been proved for filtrations of length less than ν. Let $M \in \mathscr{N}^{n+1}(X)$,

$$M = \widetilde{M} \cup_\varphi D^s \times D^{n-s+1}$$

where
$$\widetilde{M} \in \mathscr{N}^{n+1}(\widetilde{X}).$$

So, by induction, $\widetilde{M} = j(\widetilde{M}_0)$,

$$M = j\widetilde{M}_0 \cup_\varphi D^s \times D^{n-s+1}$$

and
$$\varphi : \partial D^s \times D^{n-s+1} \rightarrow \partial j \widetilde{M}_0$$

is an imbedding. By (11.23), there is an imbedding φ' such that

$$\varphi' : \partial D^s \times D^{n-s+1} \to j\partial\widetilde{M}_0$$

and $\varphi' \approx \varphi$. Equivalently,

$$\varphi' : \partial D^s \times D^{n-s} \times D^1 \to \partial\widetilde{M}_0 \times D^1.$$

Another change via isotopy will produce a map

$$\varphi'' : \partial D^s \times D^{n-s} \times D^1 \to \partial\widetilde{M}_0 \times D^1$$

such that $$\bar{\varphi}'' : \partial D^s \times D^{n-s} \times \{0\} \to \partial\widetilde{M}_0 \times \{0\}.$$

Since $\varphi'' \in \mathrm{Is}(\bar{\varphi}'' \times 1)$, by (11.17), $\varphi'' \approx j(\psi)$ for

$$\psi : \partial D^s \times D^{n-s} \times \{0\} \to \partial\widetilde{M}_0 \times \{0\}.$$

Hence $\varphi \approx j(\psi)$.

Denoting $M_0 = \widetilde{M}_0 \cup_\psi D^s \times D^{n-s}$, clearly

$$jM_0 \approx M,$$

proving (11.18).

Prompted by (11.18), I shall call an element $M \in \mathscr{D}_*$ a *stable filtration* if M is a neighborhood of $X \in \mathscr{F}$ where

$$\dim M \geq 2 \dim X + 1.$$

The sets $\alpha\{M\}$ have natural group structures where the group law is given by composition.

Let M^n be a stable filtration, and $j^m_\alpha : \alpha\{M\} \to \alpha\{j^m M\}$ for some large $m > n+1$.

Any element $\gamma \in [M, O_m]$ induces an automorphism of $j^m M = M \times D^m$, $a(\gamma)$, defined as in (5.1).

Therefore, there is a map

$$b : [M, O_m] \to \alpha\{j^m M\}.$$

It is easily seen that b is a homomorphism of the two groups. Define $u : [M, O_m] \to \alpha\{M\}$ to be $u = (j^m_\alpha)^{-1} b$.

Proposition **(11.24).** — If M is a stable filtration, $u : [M, O_m] \underset{\approx}{\to} \alpha\{M\}$ is an isomorphism.

The proof of (11.24) is easy.

Stable neighborhoods.

If N is a neighborhood of X, let $\pi_N : N \to X$ be a projection of N, and $\beta_N : X \to N$ a homotopy inverse for π_N.

Define $t : \mathscr{N}^n(X) \to V^n(X)$ to be the map $t(N) = \beta_N^* T(N)$ where $T(N)$ is the tangent bundle of N.

Let $\lambda_m : V^n(X) \to V^{n+m}(X)$ (for m large) be the map $\lambda_m(\xi) = \sigma(N^m(X), \xi)$.

Proposition (**11.25**). — Let X be a cell filtration. If n, m are sufficiently large, then

$$\begin{array}{ccc} \mathcal{N}^n(X) & \xrightarrow{t} & V^n(X) \\ \downarrow j & \swarrow \lambda_m & \\ \mathcal{N}^{n+m}(X) & & \end{array} \tag{11.26}$$

is commutative.

Proof. — Let $N \in \mathcal{N}^n(X)$. Let ν_k be a normal bundle of N as imbedded in R^{k+n} for large k. Then ν can be considered as an element in $V^k(X)$. If τ^n is the tangent bundle of N, then $\nu \oplus \tau = 1_{k+n}$. Set $m = k+n$. Then

$$j(N) = N^{1_{k+n}} = N^{\nu \oplus \tau} = \sigma(N^\nu, t(N)).$$

Since N^ν is imbeddable in R^{k+n}, by (11.6), $N^\nu \approx N^{k+n}(X)$. Therefore, $j(N) = \lambda_m t(N)$ proving (11.25).

Lemma (**11.27**). — Let $X \in \mathcal{F}$. If n is sufficiently large,

$$t : \mathcal{N}^n(X) \to V^n(X)$$

is surjective.

Proof. — Let p, q be integers, large compared with dim X. Let $n = p+q$. Then the map $j : V^p(X) \to V^n(X)$ defined by $j(\xi) = \xi \oplus 1_q$ is a bijection.

Choose any $\eta \in V^n(X)$. Then $\eta = \xi \oplus 1_q$ for some ξ. Let

$$N = \sigma(N^q(X), \xi).$$

Then $t(N) = \eta$, proving (11.27).

Proposition (**11.28**). — Let $X \in \mathcal{F}$; n, m large integers compared with dim X. Then all maps of the triangle

$$\begin{array}{ccc} \mathcal{N}^n(X) & \xrightarrow[\approx]{t} & V^n(X) \\ \searrow^{\approx}_{j} & & \swarrow^{\approx}_{\lambda_m} \\ & \mathcal{N}^{n+m}(X) & \end{array}$$

are bijections.

Proof. — The map j is bijective by the " Suspension theorem ". The map t is surjective by (11.27), and since $t \circ \lambda_m = j$, t is also injective. Therefore t is a bijection, and it follows that λ_m is bijective as well.

If $\mathcal{N}(X)$ is defined to be the direct limit of the sequence

$$\mathcal{N}^n(X) \underset{j}{\to} \mathcal{N}^{n+1}(X) \underset{j}{\to} \mathcal{N}^{n+2}(X) \underset{j}{\to} \dots$$

then, by (11.28), the limit is achieved " finitely " (i.e. as soon as $n > 2 \dim X + 1$) and the map

$$t : \mathcal{N}(X) \to \widetilde{KO}(X)$$

induced by the maps

$$t : \mathcal{N}^n(X) \to V^n(X)$$

is a bijection. (A stable neighborhood is characterized up to filtration-preserving differentiable isomorphism by its tangent bundle class in $\widetilde{KO}(X)$.)

Stable neighborhoods of smooth triangulations of differentiable manifolds. — Consider, now, the special case where X is a cell filtration obtained by ordering the simplices of a smooth triangulation of some differentiable manifold, W. By $t(X)$ I shall mean the stable tangent bundle of the differentiable manifold W. If $n \geq 2 \dim X + 1$, then I may identify $V^n(X)$ with $\widetilde{KO}(X)$. Since X is obtained as a triangulation of W, there is a differentiable cell filtration $\mathcal{W}$ of W, $\mathcal{W} \in \mathcal{N}^w(X)$ where $w = \dim W$. (This is proved in Chapter VI.)

Let the map

$$v : V^n(X) \to \mathcal{N}^n(X)$$

be given by:

$$v(\eta) = j^{-n}(\mathcal{W}^\eta)$$

(the map j^{-1} exists by the dimension restriction and the suspension theorem).

Lemma (**11.29**). — Let X be as above, $n \geq 2 \dim X + 1$, then $v : V^n(X) \to \mathcal{N}^n(X)$ is a bijection. The map

$$v^{-1} : \mathcal{N}^n(X) \to V^n(X) \approx \widetilde{KO}(X)$$

is given by:

$$v^{-1}(N) = t(N) - t(X) \qquad \text{for } N \in \mathcal{N}^n(X)$$

where $t(N)$ is the stable vector bundle class of the tangent bundle of N considered as a differentiable manifold. Subtraction makes sense since $\widetilde{KO}(X)$ is a commutative group.

Proof. — Clearly the map v^{-1} defined in the statement of (11.29) provides a right-inverse for v. It suffices to prove that v is surjective. This is done by induction. Let $N \in \mathcal{N}^n(X)$ and we must show that N is in the image of v, or equivalently, that there is a projection $\pi : N \to X$ which exhibits N as a cell bundle over X.

Let $\pi : N_{i-1} \to X_{i-1}$ be a cell bundle map for which β is the zero cross-section. It suffices to show that after isotopy, π may be extended to a π' which is a cell bundle map on N_i, for which (again) β is the zero cross-section. Thus

$$\begin{array}{ccc} \partial D^{n_i} \times D^{n-n_i} \times D^k & \xrightarrow[\Phi_i]{} & \partial N_{i-1} \subseteq N_{i-1} \\ \uparrow & & \downarrow \pi \\ \partial D^{n_i} \times D^{n-n_i} & \xrightarrow[\varphi_i]{} & \partial X_{i-1} \subseteq X_{i-1} \end{array}$$

It is easily seen that changing Φ_i by an isotopy of N_{i-1} gives:

$$\begin{array}{ccc} \partial D^{n_i} \times D^{n-n_i} \times D^k & \xrightarrow[\Phi'_i]{} & \pi^{-1}(\partial X_{i-1}) \\ \uparrow & & \downarrow \pi \\ \partial D^{n_i} \times D^{n-n_i} & \xrightarrow[\varphi_i]{} & \partial X_{i-1} \end{array}$$

Furthermore, the statement of the tubular neighborhood lemma assures us that Φ'_i may be changed by a further isotopy to Φ''_i such that if

$$\pi_0 : \partial D^{n_i} \times D^{n-n_i} \times D^k \to \partial D^{n_i} \times D^{n-n_i}$$

is the projection map exhibiting $\partial D^{n_i} \times D^{n-n_i} \times D^k$ as a trivial k-cell bundle over $\partial D^{n_i} \times D^{n-n_i}$,

$$\textbf{(11.30)} \qquad \begin{array}{ccc} \partial D^{n_i} \times D^{n-n_i} \times D^k & \xrightarrow[\Phi''_i]{} & \pi^{-1}(\partial X_{i-1}) \\ \downarrow \pi_0 & & \downarrow \pi \\ \partial D^{n_i} \times D^{n-n_i} & \xrightarrow[\varphi_i]{} & \partial X_{i-1} \end{array}$$

the map Φ''_i is a cell bundle. Thus define

$$\pi'' : N_{i-1} \cup_{\Phi''_i} D^{n_i} \times D^{n+k-n_i} \to X_{i-1} \cup_{\varphi_i} D^{n_i} \times D^{n-n_i}$$

to be

a) $$\pi'' | N_{i-1} = \pi$$

b) $$\pi'' | D^{n_i} \times D^{n+k-n_i} = \pi_0$$

and π'' is a cell bundle map with β as zero cross-section. Since N_i is filtration isomorphic with $N_{i-1} \cup_{\Phi''_i} D^{n_i} \times D^{n+k-n_i}$, (11.29) follows.

Setting $D^{n-w}(W)$ to be the set of inequivalent bundle isomorphism classes of differentiable $(n-w)$-cell bundles over W, by the dimension restrictions already imposed, we get:

Corollary **(11.31)**. — The imbedding $\delta : D^{n-w}(W) \to \mathcal{N}^n(X)$ which assigns to each $\eta \in D^{n-w}(W)$ the neighborhood $\delta(\eta) = \mathcal{W}^n \in \mathcal{N}^n(X)$ is an isomorphism (i.e., corresponding manifolds are diffeomorphic, as well).

Proof. — Any differentiable cell bundle $\eta \in D^{n-w}(W)$ is diffeomorphic (via the "identity map") with its image $\delta(\eta)$.

By Lemma 11.29, δ is a bijection.

Definition (**11.32**). — Let (M_i, δ_i) be triangulated manifolds $i=0, 1$. Then $f: M_1 \to M_2$ is a *simple homotopy equivalence* if and only if $\delta_2^{-1} f \delta_1 : K_1 \to K_2$ is a simple homotopy equivalence.

If $f: M_1^n \to M_2^n$ is a simple homotopy equivalence, and $q \geq 2n+1$, let

$$\widetilde{f} : D^{q-n}(M_2) \to D^{q-n}(M_1)$$

be the map given by $\widetilde{f} D_\xi = D_\eta$ where $\eta \in \widetilde{KO}(M_1) \approx D^{q-n}(M_1)$ is given as

$$\eta = f^* \xi + f^* \tau_2 - \tau_1$$

where τ_i is the stable class of the tangent bundle of M_i, $i=1, 2$. Clearly

$$\widetilde{f} : D^{q-n}(M_2) \underset{\approx}{\to} D^{q-n}(M_1)$$

is a bijection.

Corollary (**11.33**). — If $f: M_1^n \to M_2^n$ is a simple homotopy equivalence, $q \geq 2n+1$, and $\widetilde{f} : D^{q-n}(M_2) \underset{\approx}{\to} D^{q-n}(M_1)$ defined as above, then for each $\xi \in \widetilde{KO}(M_2)$ there is a diffeomorphism

$$f^\xi : D_\xi \underset{\approx}{\to} \widetilde{f} D_\xi. \tag{11.34}$$

Proof. — Apply Corollary 11.31 and the Stable Neighborhood Theorem. Thus the differentiable structure of stable cell bundles over manifolds is "independent of the simple homotopy type" of the manifold.

Corollary (**11.35**). — Let M_1^m, M_2^m be differentiable manifolds which are combinatorially equivalent, i.e. there is a simplicial complex K and triangulations $\delta_1 : K \to M_1$ and $\delta_2 : K \to M_2$. Further, let $\tau_i \in \widetilde{KO}(M_i)$ be the stable class of the tangent bundle of M_i for $i=1, 2$, and

$$\delta_1^*(\tau_1) = \delta_2^*(\tau_2). \tag{11.36}$$

Then $$M_1^m \times D^k \approx M_2^m \times D^k \qquad \text{for } k \geq m+1.$$

For example, if Σ^m is a differentiable manifold of the homotopy type of S^m for $m \neq 0, 1$, mod 8, then it is well known that the manifolds Σ^m, S^m satisfy the hypotheses of Corollary 11.31, and therefore $\Sigma^m \times D^k \approx S^m \times D^k$.

M. Hirsch has also obtained this result by another method.

Corollary (**11.37**). — Let M_1^m be of the same simple homotopy type as M_2^m. Let M_1^m be imbedded in R^k for $k \geq 2m+1$ and U_i^k a closed tubular neighborhood of M_i^m, as imbedded $(i=1, 2)$. Then

$$U_1^k \approx U_2^k.$$

CHAPTER XII

THE MAPPING CYLINDER LEMMA

In this chapter I prove that neighborhoods are in some topological sense, mapping cylinders.

I have relegated to this chapter the geometric lemma which shows that neighborhoods as defined in Chapter IV possess desired geometric properties.

The best general statement is the following:

Lemma (**12.1**). — Let X be a cell filtration, $N \in \mathcal{N}^n(X)$. Then there is a cell decomposition X_1 which is a member of X such that there exists a map

$$\sigma : \partial N \to X$$

and a homeomorphism

$$\Sigma : C_\sigma \to N$$

(where C_σ is the mapping cylinder of σ) such that the natural projection η

$$C_\sigma \overset{\eta}{\to} X$$

induces a projection

$$\eta \circ \Sigma^{-1} = \pi : N \to X.$$

There is a simplicial subdivision K of X such that if ∂N is given a compatible triangulation, σ can be taken to be a simplicial map, and

$$\Sigma : C_\sigma \to N$$

a triangulation of N, where C_σ is given its induced combinatorial structure.

If C_f is the mapping cylinder of $f : X \to Y$, let $w : I \times X \to C_f$ be the natural mapping and let

$$C_f(t) = \omega([t, 1] \times X) \subseteq C_f$$

so that $C_f(1) = Y$, $C_f(0) = C_f$.

(12.1) shall be proved inductively on the length of neighborhoods N.

Let N be a neighborhood of $X \in \mathcal{F}$, and $X = \widetilde{X} \cup_\varphi D^s$. Let N restrict to a neighborhood $\widetilde{N}$ over $\widetilde{X}$, and assume given

$$\widetilde{\sigma} : \partial \widetilde{N} \to \widetilde{K} \qquad \widetilde{K} \in \widetilde{X}$$

and $\widetilde{\Sigma} : C_{\widetilde{\sigma}} \to \widetilde{N}$ a topological isomorphism.

We must construct maps

$$\sigma : \partial N \to K$$
$$\Sigma : C_\sigma \to N$$

where $K \in X$, and Σ is a topological isomorphism.

If $N = \widetilde{N} \cup_\Phi D^s \times D^{n-s}$, then N may be written as:

$$N \approx C_{\tilde{\sigma}} \cup_\psi D^s \times D^{n-s}$$

where $\psi = \Sigma^{-1} \circ \Phi$.

For ease of terminology if $Z \subseteq \partial C_{\tilde{\sigma}}$ is a subset, I shall call $[Z \times I] \subseteq C_{\tilde{\sigma}}$ the subset

$$[Z \times I] = \omega\{\omega^{-1}(Z) \times I\} \subseteq C_{\tilde{\sigma}}.$$

Let $A = [\psi(\partial D^s \times \{0\}) \times I] \subseteq C_{\tilde{\sigma}}$. Let

$$D_0^s = D^s \cup_h \psi(\partial D^s \times \{0\}) \times I$$

where $$h : \partial D^s \to \psi(\partial D^s \times \{0\}) \times I$$

is given by $$h(x) = \psi(x, 0) \times \{0\}, \qquad x \in \partial D^s.$$

Then D_0^s is again an s-cell.

Let $g : D_0^s \to D^s \times \{0\} \cup A \subseteq N$ defined by:

(i) $g(x) = (x, 0)$ if $x \in D^s$.

(ii) $g(x) = \omega(x)$ if $x \in \psi(\partial D^s \times \{0\}) \times I$.

Let $K = \widetilde{K} \cup D^s \times \{0\} \cup A \subseteq N$. Therefore $K = \widetilde{K} \cup_{g_0} D_0^s$ where $g_0 = g | \partial D^s$. Clearly, given that filtration, $K \in X \in \mathscr{F}$.

I must now give maps

$$\sigma : \partial N \to K$$

and $$\Sigma : C_\sigma \to N.$$

It is most convenient to describe first the subsets

$$N_t = \Sigma(C_\sigma(t)) \subseteq N \qquad \text{for all } t \in I.$$

By means of these N_t's, σ, and Σ will be defined. (See Figure 4).

(12.2) $$N_t = D^s \times D_{1-t}^{n-s} \cup [\psi(\partial D^s \times D_{1-t}^{n-s}) \times [0, t]] \cup C_{\tilde{\sigma}}(t).$$

From (12.2) it is evident that

$$N_1 = K.$$

The proof of lemma 12.1 will be completed by describing a map

$$\omega : \partial N \times I \to N$$

so that

(i) $$\omega_t : \partial N \times \{t\} \to N_t$$

is a topological isomorphism for $t > 1$;

(ii) $$\omega_1 : \partial N \times \{1\} \to K.$$

Setting $\sigma : \partial N \to K$ to be

$$\sigma(x) = \omega(x, 1) \in N_1 = K$$

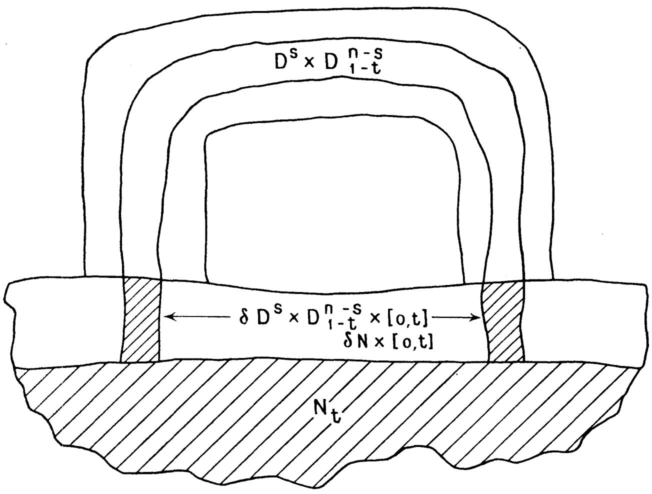


FIG. 4

there is clearly a map $\Sigma : C_\sigma \to N$ which is a topological homeomorphism, making the diagram

$$\begin{array}{ccc} \partial N \times I & \xrightarrow{\omega} & N \\ \downarrow & \nearrow_{\Sigma} & \\ C_\sigma & & \end{array}$$

commutative.

Explicit formulae for ω:

I) Definition of ω on $(D^s \times \partial D^{n-s}) \times I$:

$$\omega_t : (D^s \times \partial D^{n-s}) \times \{t\} \to D^s \times \partial D^{n-s}_{1-t} \cup (\psi(\partial D^s \times \partial D^{n-s}_{1-t}) \times [0, 1])$$

(i) $\omega(x, y, t) = \left(\frac{1}{1 - t/2} x, (1-t)y\right) \in D^s \times \partial D^{n-s}_{1-t}$ if $x \in D^s_{1-t/2}$, $y \in \partial D^{n-s}$

(ii) $\omega(x, y, t) = \psi\left(\frac{x}{||x||}, (1-t)y\right) \times \left\{2t\left(||x|| - 1 + \frac{t}{2}\right)\right\}$ if $x \in D^s - D^s_{1-t/2}$, $y \in \partial D^{n-s}$.

II) Definition of ω on $\{\partial N - \psi(\partial D^s \times D^{n-s})\} \times I$:

Let $\partial D^s \times D^{n-s}_2$ represent a nice neighborhood of $\psi(\partial D^s \times D^{n-s})$ in N, where $\psi(\partial D^s \times D^{n-s})$ is identified with $\partial D^s \times D^{n-s}_1$ in the obvious manner.

We must define

$$\omega : (\partial N - \psi(\partial D^s \times D^{n-s})) \times \{t\} \to \partial N \times \{t\} - \partial D^s \times D^{n-s}_{1-t}$$

(i) $\omega(x, t) = [(x, t)]$ if $x \notin \partial D^s \times D_2^{n-s}$.

(ii) $\omega(x_1, x_2, t) = \left(x_1, \left\{\frac{(1+t)\,||x_2|| - 2t}{||x_2||}\right\} x_2, t\right)$ if $x_1 \in \partial D^s$, $x_2 \in D_2^{n-s} - D_1^{n-s}$.

(iii) $\omega(x_1, x_2, t) = (x_1, tx_2, t)$ if $x_1 \in \partial D^s$, $x_2 \in D_t^{n-s}$.

It is easily seen that $\omega \,|\, \partial N \times [0, 1]$ is a homeomorphism.

This concludes (12.1).

Remark. — The maps involved above may be taken to be simplicial.

Corollary (**12.3**). — Let $X \in \mathscr{F}$, $\dim X = k$, $N \in \mathscr{N}^n(X)$. If $i : \partial N \to N$ is the injection map,

$$i_q : \pi_q(\partial N) \to \pi_q(N)$$

is a bijection for $q < n-k-1$.

Proof. — Since $N \approx C_\sigma$, ∂N is a strong deformation retract of $N - K$

$$(N - K \approx \partial N \times [0, 1]),$$

both surjectivity and injectivity of i_q follow by general positionality (removing generators and relations of $\pi_q(N)$ from K).

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