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MANIFOLDS WHICH ARE LIKE PROJECTIVE PLANES

by JAMES EELLS, Jr. (1) and NICOLAAS H. KUIPER (2)

INTRODUCTION

This paper presents a development of the results we announced in [6]. We deal with the following question, which we state here in rather broad terms: Given a *closed n-dimensional manifold* X *such that there exists a nondegenerate real valued function* $f: X \to \mathbf{R}$ *with precisely three critical points. In what way does the existence of f restrict* X*?* The problem will be considered from the topological, combinatorial, and differentiable point of view.

The corresponding question for a function with two critical points (which is the minimum number except in trivial cases) has been studied by Reeb [25], and Kuiper [17]. In that case X is homeomorphic to an n-sphere, and Milnor [19] used this fact in his discovery of inequivalent differentiable structures on the 7-sphere.

Qualitatively speaking, our results are of the following sort:

1) *Dimension and cohomology.* — The only values of n possible are $n = 2m = 0, 2, 4, 8, 16$. In these cases X has the integral cohomology structure of three points $(n = 0)$, the real $(n = 2)$, complex $(n = 4)$, quaternion $(n = 8)$, or the Cayley $(n = 16)$ projective plane.

2) *Homotopy type.* — If $n = 0$ the space X consists of three points. If $n = 2$, the space is the real projective plane. In the other dimensions X is connected and simply connected, and has a natural orientation. There is one homotopy type for $n = 4$, six homotopy types for $n = 8$, and sixty homotopy types for $n = 16$. These are all represented by certain combinatorial manifolds X_h^{2m} described in Section 2.

3) *Topologically,* X is a compactification of numerical $2m$-space $\mathbf{R}^{2m}$ by an m-sphere. Combinatorially and differentiably, X is obtained by attaching three cells to each other along the boundaries. In particular in the differentiable case with $n = 8$ or 16, X is homeomorphic to the Thom complex of a sphere bundle over a sphere, that is the one point compactification of the associated disc bundle.

4) *Differentiably,* there are infinitely many distinct cases for $n = 8$ (and quite possibly for $n = 16$). Their associated combinatorial structures (hence these manifolds themselves) are classified by their Pontrjagin classes. For $n = 2$ there is only the real

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projective plane. For $n=4$ we do not have complete results, partly due to the possible existence of knots.

5) All differentiable manifolds of the same dimension belong to the same unoriented *cobordism class*. In case $n=4$ the manifolds admit almost complex structures, all belonging to the same complex cobordism class. In cases $n=8$ and 16 the manifolds determine infinitely many oriented cobordism classes, classified by their Pontrjagin numbers.

6) *Combinatorially*, there are infinitely many distinct examples for $n=8$ and 16, distinguished by their Pontrjagin classes. Certain of the combinatorial manifolds admit no differentiable structure.

It seems plausible that the given combinatorial examples form the complete set of all combinatorial solutions of our problem for $n \neq 4$. We hope to come back to this problem in a later paper.

Our primary tools are: 1) Morse's theory of nondegenerate functions and deformations, also in a topological version, and in particular modified to get isotopic deformations. These are applied in order to obtain a convenient decomposition of the manifold X; 2) the precise knowledge of the structure of the appropriate orthogonal bundles over spheres; in particular their characteristic classes and Hopf invariants; 3) knowledge of certain homotopy groups of spheres; 4) (partial) knowledge of certain differentiable structures on S^7 and S^{15}.

I

ON THE TOPOLOGICAL STRUCTURE OF X

In this chapter we present those aspects of our problem which we can handle simultaneously in the topological, combinatorial, and differentiable cases. We will suppose that X is a *closed* (i.e. compact and without boundary) n-dimensional manifold; we will refer to X as a topological n-manifold or C^0-n-manifold. If moreover a combinatorial or a differentiable structure of class C^∞ is assumed, we refer to X as a combinatorial (C^{omb}) or a differentiable ($=C^\infty=$smooth) n-manifold.

1. Nondegenerate functions.

A) *The differentiable case* (C^∞).

If $f: X \to \mathbf{R}$ is a C^∞-function on the C^∞-n-manifold X, then the differential df and the second differential d^2f of f at a point $a \in X$ are the operators which assign to any differentiable map $g: \mathbf{R} \to X$ with $g(o)=a$ the values

$$df(a): g \to \left.\frac{(df(g(t)))}{dt}\right|t=o$$

$$d^2f(a): g \to \left.\frac{d^2(f(g(t)))}{dt^2}\right|t=o.$$

If $\varphi: (U, a) \to (\mathbf{R}^n, o)$ for $a \in U \subset X$ is an *a-centered* C^∞-coordinate system (φ, U), and $\varphi_1, \ldots, \varphi_n : U \to \mathbf{R}$ its C^∞-coordinates, then

$$df = \Sigma_{i=1}^n \frac{\partial f}{\partial \varphi_i} d\varphi_i$$

and

$$d^2f = \Sigma_{i,j=1}^n \frac{\partial^2 f}{\partial \varphi_i \partial \varphi_j} d\varphi_i d\varphi_j + \Sigma_{i=1}^n \frac{\partial f}{\partial \varphi_i} d^2\varphi_i.$$

The tangent space T_a is the dual of the vector space T_a^* of all differentials at a. The corresponding vector bundles over X are T(X) and $T^*(X)$.

The point a is said to be *ordinary* if $df(a) \neq o$, and *critical* if $df(a)=o$. In that latter case $d^2f(a)$ is a quadratic form. The critical point is called *nondegenerate* if the rank of this form is n. The *index of the critical point a* is the index of the quadratic form $d^2f(a)$. It is the minimum of the rank of those quadratic forms which added to $d^2f(a)$ give a nonnegative quadratic form, and it is equal to the number of negative eigen-values of the matrix of numbers $\left.\frac{\partial^2 f}{\partial\varphi_i\partial\varphi_j}\right|_a$. We say that f is *differentiably nondegenerate* if every critical point is nondegenerate.

There exist many nondegenerate functions on any C^∞-n-manifold X. For example, if $g : X \to \mathbf{R}^N$ is a C^∞-imbedding, then the composition of g with almost any linear function $h : \mathbf{R}^N \to \mathbf{R}$ on $\mathbf{R}^N$ is a nondegenerate function $h \circ g : X \to \mathbf{R}$ (Theorem of Sard).

Having in mind the graph

$$\{(x, f(x)) \mid x \in X\} \subset X \times \mathbf{R}$$

as a subset of the product space of X and a "vertical" real line $\mathbf{R}$, we think of f as a *height function on* X. We will accordingly refer to "higher" and "lower" points of X. This of course is only a matter of convenience of expression. Now let a C^∞-Riemannian metric be given in X. Let $*$ be for any $a \in X$ that isomorphism $* : T_a \to T_a^*$ which carries any ordered basis of orthogonal unit-vectors in T_a onto its ordered cobasis, or its inverse. Then the vector field $*(-df)$ (a crosssection in T(X)) defines an infinitesimal generator of a one parameter group of diffeomorphisms of X, whose fixed points are precisely the critical points. The images of any ordinary point form a trajectory, on which lower points correspond to higher values of the group parameter.

Morse used these tools in order to prove.

Proposition (*Morse* [23]). — *If* $a \in X$ *is a* C^∞*-ordinary point for the* C^∞*-function* $f : X \to \mathbf{R}$, *then there is an a-centered coordinate system* (φ, U) *on* X *and a number* $\lambda_a > 0$, *such that the n-th coordinate satisfies*

$$\varphi_n(x) = \lambda_a(f(x) - f(a)) \text{ for } x \in U. \tag{1}$$

If a is differentiably critical of index k, then there exists an a-centered C^∞*-coordinate system* (φ, U) *on* X *and a number* $\lambda_a > 0$ *such that*

$$-\Sigma_{i=1}^{k}\varphi_i^2(x) + \Sigma_{i=k+1}^{n}\varphi_i^2(x) = \lambda_a(f(x) - f(a)) \text{ for } x \in U. \tag{2}$$

We define *polar coordinates* $(r_1, \omega_1; r_2, \omega_2)$ *of type* k associated to given coordinates $\varphi_1 \dots \varphi_n$, by

$$r_1(x) = [\varphi_1^2(x) + \dots + \varphi_k^2(x)]^{\frac{1}{2}}, \qquad \omega_1(x) = \frac{1}{r_1(x)}(\varphi_1(x), \dots, \varphi_k(x))$$

$$r_2(x) = [\varphi_{k+1}^2(x) + \dots + \varphi_n^2(x)]^{\frac{1}{2}}, \qquad \omega_2(x) = \frac{1}{r_2(x)}(\varphi_{k+1}(x), \dots, \varphi_n(x)).$$

If $k = 0$ or n, we agree to use only one pair (r, ω). In terms of polar coordinates of type k, (2) is

$$-r_1^2(x) + r_2^2(x) = \lambda_a(f(x) - f(a)). \tag{3}$$

If the Riemannian metric is in these cases locally the Euclidean metric given by

$$ds^2 = \Sigma_{i=1}^{n} d\varphi_i^2$$

then the trajectories are the straight lines $(\varphi_1(x), \ldots, \varphi_{n-1}(x)) = \text{constant}$ in case (1), and the *hyperbola's*

$$[\omega_1(x), \omega_2(x), r_1(x) . r_2(x)] = \text{constant}$$

in case (2).

We let $\mu_k(X, f) = \mu_k(f)$ denote the number of critical points of index k of f on X, the so-called *k-th Morse number*. The polynomial $\Sigma_{k=0}^{n}\mu_k(f)t^k$ is the *Morse polynomial.*

B) *The topological case* (C^0).

Definition. — Let X be *a* topological n-manifold, and $f: X \to \mathbf{R}$ a continuous function. Say that $a \in X$ is a C^0-*ordinary point* or a C^0-*critical point of index k* if there is an a-centered C^0-coordinate system (φ, U) on X and a constant $\lambda_a > 0$, such that (1) or (2), (3) holds, respectively. Say that f is C^0-*nondegenerate* if every point $a \in X$ is either C^0-ordinary of C^0-critical of index k for some k.

As the critical points are clearly isolated, then f has only finitely many critical points on the closed manifold X.

From the definition we see immediately that a C^∞-*nondegenerate function f on a* C^∞-*n-manifold* X *is also* C^0-*nondegenerate.* But not every C^∞-function which is C^0-nondegenerate, is also C^∞-nondegenerate, as we see from the function $f(x) = \Sigma_{i=1}^{n}\varphi_i^4(x)$.

In view of the combinatorial theory considered below, we mention another equivalent expression of a function near a C^0-critical point of index k. Let $(\varphi_1, \ldots, \varphi_n;$ $r_1, \omega_1; r_2, \omega_2)$ be a-centered coordinates as defined above for such a point a, covering the set

$$B = \{x \mid r_1(x) \leqslant 1 \text{ and } r_2(x) \leqslant 1\}.$$

In B we introduce new C^0-coordinates $(\varphi'_1, \ldots, \varphi'_n)$ and associated bipolar coordinates of type k $(r'_1, \omega'_1; r'_2, \omega'_2)$ such that for $x \in B$

$$\omega'_1(x) = \omega_1(x), \ \omega'_2(x) = \omega_2(x)$$
$$\max\{|\varphi'_1(x)|, \ldots, |\varphi'_k(x)|\} = r_1^2(x)$$
$$\max\{|\varphi'_{k+1}(x)|, \ldots, |\varphi'_n(x)|\} = r_2^2(x).$$

Then (2) (3) are equivalent to

$$-\max\{|\varphi'_1(x), \ldots, |\varphi'_k(x)|\} + \max\{|\varphi'_{k+1}(x)|, \ldots, |\varphi'_n(x)|\} = \lambda_a(f(x) - f(a)) \qquad (4)$$

Let
$$f^s = \{x \mid x \in X, f(x) \leqslant s\}$$
$$f^s_- = \{x \mid x \in X, f(x) < s\}.$$

Then, as Morse showed, the relative homology groups (any coefficients)

$$H_i(f_-^{f(a)} \cup a, f_-^{f(a)}) \quad i = 0, \ldots, n$$

vanish in case a is ordinary, and they all vanish except for $i = k$ in case a is a nondegenerate C^0-critical point of index k. Hence any point $a \in X$ has at most one of the properties: to be C^0-ordinary or to be nondegenerate C^0-critical of index k for $k = 0, \ldots, n$, and that property is a topological property of the triple (X, f, a).

Problem (M. Morse [24]). — Does there exist a topologically nondegenerate function on every closed n-manifold X *?*

If there exists a C^∞-structure on X, then (as we saw above) such a function exists. If X has dimension $n \leq 3$, then a C^∞-structure is known to exist and so again such a function exists. In general the answer is unknown, although the existence is assured in case X admits a combinatorial structure, as we will see below.

C) *The combinatorial case* (C^{omb}).

Definitions. — Let X be a topological n-manifold. A *triangulation* (K, h, X) of X is a finite simplicial complex K, together with a homeomorphism h of the geometric realisation $|K|$ of K onto X. For convenience we will occasionally consider X and $|K|$ as identical. A *combinatorial triangulation or Brouwer triangulation of* X is a triangulation such that the closed star of every vertex is isomorphic to a vertex star of a triangulation of $\mathbf{R}^n$. Recall that two simplicial polyhedra are said to be combinatorially equivalent if they have isomorphic rectilinear subdivisions.

A *combinatorial structure* on X is a maximal set of combinatorially equivalent combinatorial triangulations of X. We will say that X is a *combinatorial manifold,* if X is a topological manifold with a specific combinatorial structure.

Remark. — It is unknown whether every topological manifold admits a triangulation, whether a triangulated manifold admits a combinatorial structure, or whether some topological manifold may admit two or more different combinatorial structures. The Hauptvermutung for manifolds says that there is at most one.

Definition. — Let X be a combinatorial n-manifold and $f: X \to \mathbf{R}$ a continuous function; f is called a *combinatorial* (C^{omb}) *function* on X, if there exists a combinatorial triangulation $h: |K| \to X$ belonging to the C^{omb}-structure of X, such that the composition $f \circ h$ is a linear function on every affine simplex of $|K|$. Say that f is C^{omb}-*nondegenerate* if a triangulation h exists such that for every $a \in X$ there is an a-centered coordinate system (φ, U) whose coordinates $\varphi'_1, \ldots, \varphi'_n$ are simplexwise linear, and a real number $\lambda_a > 0$, such that either

$$\varphi'_n(x) = \lambda_a(f(x) - f(a)) \tag{1}$$

for $x \in U$; such an $a \in X$ is said to be C^{omb}-*ordinary*;
or

$$-\max\{|\varphi'_1(x)|, \ldots, |\varphi'_k(x)|\} + \max[|\varphi'_{k+1}(x)|, \ldots, |\varphi'_n(x)|] = \lambda_a(f(x) - f(a)) \tag{5}$$

for $x \in U$; such an $a \in X$ is a C^{omb}-*critical point of index* k. Other points are called C^{omb}-degenerate.

From the definitions in (2), (3), (4) we see immediately that a C^{omb}-*nondegenerate function on a* C^{omb}-*n-manifold is also* C^0-*nondegenerate on the underlying* C^0-*manifold.*

With respect to the problem of Morse we have the

Proposition. — On every closed C^{omb}-*manifold there exists a* C^{omb}-*non-degenerate function, hence a* C^0-*nondegenerate function.*

Indeed, such a function f can be obtained with the help of a Brouwer triangulation $h:|K|\to X$ and a barycentric subdivision $h':|K'|\to X$, by assigning the f-value k to the barycentre of each k-simplex of K, and by extending f linearly over each simplex of K'. We have to prove that f so obtained is a nondegenerate C^{omb}-function. As h is a Brouwer triangulation the star $\mathrm{St}(\sigma)$ of any k-simplex of K can be imbedded in $\mathbf{R}^k\times\mathbf{R}^{n-k}$ simplexwise affinely, and such that σ and its barycentre are mapped into $\mathbf{R}^k\times 0$ and 0×0, whereas all barycentres of simplices of $|K|$ different from σ that have σ as a face, are mapped into $0\times\mathbf{R}^{n-k}$ and onto the vertices of a polyhedron which is convex with respect to 0×0. f has value k on the barycentre of σ, value $<k$ in any other point of σ, values $>k$ in each of the last mentioned barycentres. Then, for a suitable subdivision $h'':K''\to X$ of K', we can obtain a simplexwise affine imbedding of $\mathrm{St}(\sigma)$ in $\mathbf{R}^k\times\mathbf{R}^{n-k}$, such that in some neighborhood of the barycentre of σ, f has the expression (5). For a suitable affine subdivision of $h':K'\to X$ the same applies to each of the finite number of barycentres of simplices of h, whereas of course f is C^{omb}-ordinary in any other point of X. Then f is a C^{omb}-nondegenerate function on X.

D) A combinatorial triangulation $h:|K|\to X$ on a C^∞-n-manifold is called a *differentiable triangulation* if the restriction of h to any closed simplex of K is a diffeomorphism.

A fundamental theorem of Cairns-Whitehead (for the latest proof compare [41]) asserts that *with each differentiable structure on a topological manifold there exist differentiable combinatorial triangulations, and any two of them are combinatorially equivalent.* Thus each differentiable structure on X determines a specific combinatorial structure, said to be *associated* with the differentiable structure. We say that a differentiable structure and a combinatorial triangulation are *compatible* if that triangulation is differentiable.

Proposition. — Let X *be a differentiable manifold, and* $f:X\to\mathbf{R}$ *a differentiably nondegenerate function.*

Then f is combinatorially nondegenerate with respect to the associated combinatorial structure.

Outline of proof. — There can be found a differentiable triangulation $h:|K|\to X$ of the C^∞-manifold X, with the following properties:

a) If $a\in X$ is a vertex of $|K|=X$ then the level hypersurface $\{x\,|\,x\in X, f(x)=f(a)\}$ is contained in the $(n-1)$-skeleton $|K_{n-1}|$ of $|K|=X$.

b) Every critical point $a\in X$ is a vertex of $|K|$.

c) For every critical point a of index k there exists a representation of f in C^∞-coordinates as in (2), (3), and a ball

$$B_a=\{x\,|\,x\in X,\ r_1(x)\leqslant 1,\ r_2(x)\leqslant 1\}$$

with boundary $\partial B_a\subset|K_{n-1}|$.

d) If σ is a simplex of $|K|$ not in the interior of any ball B_a about a critical point a of f, then f is linear on this affine simplex.

e) There exists a homeomorphism of $|K|=X$ which carries every simplex of $|K|$ onto itself. It determines a second combinatorially equivalent triangulation of X. This one also has the properties *a), b), c), d)*; moreover, for any critical point a and coordinates $(\varphi_1, \ldots, \varphi_n, \varphi_1', \ldots, \varphi_n')$ as in (2), (3), (5), the functions $\varphi_1', \ldots, \varphi_n'$ are linear with respect to the affine simplices of the new triangulation. Then f is C^{comb}-nondegenerate with respect to this second triangulation which belongs to the C^{comb}-structure of the C^∞-manifold X, although it is not a C^∞-triangulation.

Problem. — Which C^∞-functions on a C^∞-manifold are C^{comb}-functions?

2. Examples of manifolds and functions.

A) Let $\mathbf{F}$ be one of the following division algebras and m its dimension over $\mathbf{R}$: the real number field $\mathbf{R}$, with $m=1$; the complex number field $\mathbf{C}$, with $m=2$; the skew field of quaternions, with $m=4$; or the algebra of Cayley numbers (also called octaves), with $m=8$. In the cases $m=1, 2, 4$ the projective plane $P_2(\mathbf{F})$ can be defined as the totality of lines through the origin in $\mathbf{F}^3$. The homogeneous coordinates $(z_0, z_1, z_2) \neq (0, 0, 0)$ will be normalised by $z_0\bar{z}_0 + z_1\bar{z}_1 + z_2\bar{z}_2 = 1$.

A matrix M is called *Hermitian* if $M = {}^t\overline{M}$, where t denotes transposition and the bar conjugation in $\mathbf{F}$. The above projective plane can be analytically imbedded in the space $\mathbf{R}^3 \times \mathbf{F}^3$ or $\mathbf{R}^3 \times \mathbf{R}^{3m} = \mathbf{R}^{3+3m}$ of all Hermitian 3×3-matrices over $\mathbf{F}$ with coordinates $\xi_0, \xi_1, \xi_2, a_0, a_1, a_2$ and $\xi_i \in \mathbf{R}$, $a_i \in \mathbf{F}$, by assigning to the point (z_0, z_1, z_2) the Hermitian matrix

$$M = \begin{pmatrix} \xi_0 & a_2 & a_1 \\ \bar{a}_2 & \xi_1 & a_0 \\ \bar{a}_1 & \bar{a}_0 & \xi_2 \end{pmatrix} = \begin{pmatrix} z_0\bar{z}_0 & z_0\bar{z}_1 & z_0\bar{z}_2 \\ z_1\bar{z}_0 & z_1\bar{z}_1 & z_1\bar{z}_2 \\ z_2\bar{z}_0 & z_2\bar{z}_0 & z_2\bar{z}_2 \end{pmatrix}.$$

The image is the submanifold that consists of all Hermitian matrices M which are projections ($M^2 = M$) and have rank 1. *If we define* $P_2(\mathbf{F})$ *as this analytic submanifold, then this definition also makes sense in the case of octaves, as Freudenthal* [8] *showed.* The 16-manifold so obtained has the structure of a non-Desarguesian projective plane in which the projective lines are analytic 8-spheres. It is called the *Cayley projective plane.*

The analytic function on $P_2(\mathbf{F})$ which is defined in terms of the coordinates $(\xi_0, \xi_1, \xi_2, a_0, a_1, a_2)$ by

$$f = \xi_0 + 2\xi_1 + 3\xi_2 : P_2(\mathbf{F}) \to \mathbf{R}$$

is differentiably nondegenerate and has precisely three critical points with indices $0, m, 2m$.

For $m=1, 2, 4$ this is easily calculated, whereas for $m=8$ it can be deduced from Freudenthal's paper.

Remark. — More generally, for $m=1, 2, 4$ we have a nondegenerate function $f : P_n(\mathbf{F}) \to \mathbf{R}$ with $n+1$ critical points of indices $0, m, \ldots, nm$ defined by

$$f(x) = \Sigma_{i=0}^{n} (i+1) z_i \bar{z}_i.$$

In order to prepare the way for our next construction, we note the following decomposition of $P_2(\mathbf{F})$: If we renove a suitable open differentiable $2m$-disc from $P_2(\mathbf{F})$, and we intersect the complement by the projective lines through the centre of the $2m$-disc, then we find a fibration of this complement such that it is expressible as an orthogonal m-disc bundle over a projective line $P_1(\mathbf{F})$ (which is the m-sphere). The boundary of this bundle is a $(2m-1)$-sphere, which is a S^{m-1}-bundle over S^m. It is the so-called *Hopf fibration* of S^{2m-1} corresponding to $\mathbf{F}$ [42, § 20]. Since the Thom complex of an S^{m-1}-bundle is by definition the one point compactification of the corresponding m-disc bundle, $P_2(\mathbf{F})$ *is homeomorphic to the Thom complex of that Hopf fibration.*

B) Let $S^{m,\sigma}$ be a topological m-sphere with some C^∞-structure σ and let $S^{m,0}$ have the usual C^∞-structure.

Consider the differentiable fibre bundles $p: E \to S^{m,\sigma}$ with fibre S^{m-1} and structural group the orthogonal group O_m; these will be called (O_m, S^{m-1})-*bundles.* We are particularly interested in those bundles for which E is homeomorphic to S^{2m-1}. Then the Euler-class W_m of the bundle is a generator of $H^m(S^m)$, and Milnor [22] has shown (as an application of a theorem of Bott) that consequently $m=1, 2, 4$ or 8.

In cases $m=1$, 2, σ can only take the value 0, and there is just one isomorphism class of such bundles, given by the Hopf fibrations. In cases $m=4$, 8 there are infinitely many isomorphism classes of bundles, studied in case $\sigma=0$ by Milnor [19] and Shimada [29].

All of the total spaces $Y_h^{2m,\sigma}$ of the associated (O_m, D^m)-bundles, where D^m denotes the closed unit disc in $\mathbf{R}^m$, can be represented as follows: Using right multiplication in $\mathbf{F}$ and letting D^m denote the unit m-disc in the Euclidean $\mathbf{R}^m$ underlying $\mathbf{F}$, we define for any integers h, j and diffeomorphism

$$\eta : \partial D^m \to \partial D^m$$

the diffeomorphism $$\eta \times f_{h,j} : \partial D^m \times D^m \to \partial D^m \times D^m,$$

by mapping

$$(u, v) \to (\eta u, u^h v u^j)$$

for all $$(u, v) \in \partial D^m \times D^m \subset \mathbf{F} \times \mathbf{F}. \tag{1}$$

The identification space $D^m \cup_\eta D^m$ is a topological m-sphere with a C^∞-structure $\sigma=\sigma(\eta)$; we denote it by $S^{m,\sigma}$.

The C^∞-identification space

$$Y_{h,j}^{2m,\sigma} = (D^m \times D^m) \cup_{\eta \times f_{h,j}} (D^m \times D^m)$$

is the total space of an (O_m, D^m)-bundle over $S^{m,\sigma}$. For $m=4,8$ it is an (SO_m, D^m)-bundle whose principal bundle we denote by $\zeta_{h,j}$. Its boundary (SO_m, S^{m-1})-*bundle has total space homeomorphic to* S^{2m-1} *if and only if* (Milnor [19], Shimada [29]), *the Euler number* $W_m(\zeta_{h,j}) . S^m = h+j=1$. We write $Y_h^{2m,\sigma} = Y_{h,1-h}^{2m,\sigma}$.

Furthermore, the Pontragin number is then

$$\begin{aligned} p_{m/4}(\zeta_{h,j}) . S^m &= \pm 2(2h-1) \quad \text{if} \quad m=4 \\ &\pm 6(2h-1) \quad \text{if} \quad m=8. \end{aligned} \tag{2}$$

The zero crosssection of the (SO_m, D^m) bundle is a C^0-imbedded usual m-sphere $S^{m,0}$. In view of a theorem of Haefliger [9; see our Section 8 B] it can be replaced by a homotopic C^∞-imbedding of $S^{m,0}$ in $Y_h^{2m,\sigma}$.

Then according to Smale's theory (in particular [43], Theorem A), it follows that $Y_h^{2m,\sigma}$ is diffeomorphic with $Y_h^{2m,0}$. Hence the C^∞ total space of the bundle does not depend on σ, although the given fibration does determine a specific C^∞-structure σ in the base space S^m. Hence we may write Y_h^{2m} instead of $Y_h^{2m,\sigma}$.

Choose any combinatorial triangulation compatible with the C^∞-structure of Y_h^{2m}. It is known that the topological space S^{2m-1} for $m=4,8$ admits only one combinatorial structure, so that the boundary ∂Y_h^{2m} has that usual combinatorial structure given by the induced triangulation. (As a matter of fact, the C^{comb}-structure on any topological S^m for $m \neq 4,5$ is known to be unique). Attach a cone to the boundary ∂Y_h^{2m}, forming a closed $2m$-manifold X_h^{2m}; the join of the given triangulation of ∂Y_h^{2m} with the vertex of the cone defines a triangulation of X_h^{2m} which is combinatorial.

Of course, the same construction is applicable to the manifolds obtained by attaching a cone to the above (O_m, D^m)-bundle for $m=1,2$. These manifolds are combinatorially equivalent to $P_2(\mathbf{R})$ and $P_2(\mathbf{C})$, respectively. In cases $m=4$ and 8 the manifolds X_1^8 and X_1^{16} are combinatorially equivalent to $P_2(\mathbf{H})$ and the Cayley plane, respectively.

It follows from Proposition 2D below that for every h the manifold X_h^{2m} admits a C^{comb}-nondegenerate function, hence a C^0-nondegenerate function with three critical points. Namely, we can always construct on the zero crosssection of the bundle Y_h^{2m}, S^m, a differentiably nondegenerate function f with two critical points of indices 0, m. Proposition 2D asserts that we can extend f to a C^{comb} nondegenerate function with three critical points of indices 0, m, $2m$.

C) In case $m=1$ in the above construction, the manifold obtained is 2-dimensional; therefore (as is well known) it possesses a unique differentiable structure. In case $m=2$ the manifold admits (by a theorem of Cairns) a compatible differentiable structure. We do not know whether that structure is unique. It would be unique if S^4 admits only one C^∞-structure. This is unknown.

In the cases $m=4$ and 8, if ∂Y_h^{2m} is diffeomorphic to $S^{2m-1,0}$ then we can attach a $2m$-disc D^{2m} differentiably to obtain a closed differentiable manifold (which will depend on the way of attaching D^{2m}). We have introduced in [7] a differential invariant which is useful in deciding for which values of h is ∂Y_h^{2m} diffeomorphic to $S^{2m-1,0}$. Granting that h is such a value, we obtain a closed differentiable manifold $X_h^{2m,\zeta}$ for each diffeomorphism $\zeta: S^{2m-1,0} \to \partial Y_h^{2m}$. See Section 9.

Again, Proposition 2D below shows that there is a differentiably non degenerate function on $X_h^{2m,\zeta}$ with three critical points.

D) *Proposition. — Let* $p: A \to B$ *be a differentiable* (O_m, D^m)*-bundle with* C^∞*-base* B. *Suppose that* ∂A *is combinatorially equivalent to* S^{n+m-1} *with the usual* C^{comb}*-structure, and attach an* $(n+m)$*-cone* C *to obtain a combinatorial manifold* $X = A \cup C$. *Any differentiably nondegenerate* *function f on* B (*considered as zero crosssection in* A), *admits an extension to a combinatorially nondegenerate function g on* X *with Morse numbers*

$$\begin{aligned}\mu_k(g) &= \mu_k(f) \quad \textit{if} \quad k<n+m\\ &= 1 \quad \textit{if} \quad k=n+m.\end{aligned}$$

If X *is differentiable, then we can extend f differentiably to have the analogous properties. If $f:\mathrm{B}\to\mathbf{R}$ is a $\mathrm{C}^{\mathrm{omb}}$-nondegenerate function on a $\mathrm{C}^{\mathrm{omb}}$-manifold then the same conclusion* ($\mathrm{C}^{\mathrm{omb}}$) *holds.*

Proof. — We consider B imbedded in $\mathrm{A}\subset\mathrm{X}$ as the zero section. We suppose that the fibres of A are Euclidean discs of radius 1. For each $y\in\mathrm{A}$ let $|y|$ denote the distance from y to the origin of the fibre through y. Then $\psi(y)=|y|^2-1$ defines a function on A such that

$$\begin{aligned}\psi(y) &= -1 \quad \text{if} \quad y\in\mathrm{B},\\ &= 0 \quad \text{if} \quad y\in\partial\mathrm{A},\end{aligned}$$

and ψ is quadratic on each fibre. We extend ψ over the cone C by defining ψ to be 1 on its vertex and extending linearly. Choose a C^∞-function $\varphi:\mathbf{R}\to\mathbf{R}$ such that

$$\varphi(t)=0 \quad \text{for} \quad t\geqslant 0, \qquad 0<\varphi(-1)<\frac{1}{2},$$

$$-\frac{1}{2}<\varphi'(t)=\frac{d\varphi}{dt}\leqslant 0 \qquad \text{for} \qquad t\in\mathbf{R}.$$

Define the function $h(y)$ by

$$f(y)=-1+\varphi(-1).h(y) \qquad \text{for} \quad y\in\mathrm{B}.$$

To construct an extension of f there is no loss of generality in assuming that

$$-1\leqslant h(y)\leqslant +1 \qquad \text{for all} \quad y\in\mathrm{B}.$$

Let

$$\begin{aligned}g(y) &= \psi(y) \quad \text{if} \quad y\in\mathrm{C},\\ &= \psi(y)+\varphi(\psi(y)).h(p(y)) \quad \text{if} \quad y\in\mathrm{A};\end{aligned}$$

then g has the required properties, as we now show.

It is clear that on C the function g is combinatorially nondegenerate, and has just one critical point at the vertex of C with index $n+m$. Furthermore, g is continuous on X, and every point of $\partial\mathrm{A}$ is ordinary. Also the restriction of g to B coincides with f.

To analyse the other critical points we use the differentiable structure on A and recall Proposition 1D. For any $y\in\mathrm{A}$ we compute the differential dg of g at y:

$$dg(y)=[1+\varphi'(\psi(y))h(p(y))]d\psi(y)+\varphi(\psi(y))dh(p(y)).$$

Since $-1\leqslant h(p(y))\leqslant +1$ and $-\frac{1}{2}<\varphi'(\psi(y))\leqslant 0$, we have $1+\varphi'(\psi(y))h(p(y))>\frac{1}{2}$ for all $y\in\mathrm{A}$.

The covectors $d\psi(y)$ and $dh(p(y))$ are linearly independent if they are different from zero, because they are in complementary subspaces due to the fibre structure of A. Thus $dg(y)=0$ implies that $d\psi(y)=0$ and $\varphi(\psi(y))dh(p(y))=0$. But the only critical points of ψ in A are in B, whence $\psi(y)=-1$ and $\varphi(\psi(y))>0$. It follows that if $y\in A$ is a critical point of g, then $y\in B$ and $df(p(y))=0$. Since $\psi=-1$ on B and ψ is semi-definite nonnegative of rank m at each point of B, we conclude that the index of g at y is the index of f at $p(y)$. Therefore, g and f have the same critical points of the same index k $(0\leqslant k\leqslant n)$ on B, and g has just one other critical point of index $n+m$.

If ∂A is diffeomorphic to $S^{n+m-1,0}$ and we form the differentiable manifold X^ζ by attaching a disc by the diffeomorphism $\zeta:\partial D^{n+m}\to\partial A$, then there is an extension of f to a differentiably nondegenerate function $g:X^\zeta\to\mathbf{R}$ with the same Morse number relations. In the above proof we take a differentiable extension of ψ from A to X^ζ having one nondegenerate maximum at the centre of D^{n+m}. If ∂A is not diffeomorphic to $S^{n+m-1,0}$, then at any rate it still is C^{comb}-equivalent to S^{n+m-1}, and D^{n+m} can be C^{comb}-attached to get the C^{comb}-manifold X. The C^{comb}-function ψ given on A, can be extended combinatorially over X in the required manner.

3. Deformations of X.

A homotopy $h:X\times I\to Y$ is called a *deformation* in case $X\subset Y$ and $h(x,0)=x$ for $x\in X$. If $h_t:X\times t\to Y$ is a homeomorphism for every $t\in I$ then h is called an *isotopy*. Let f be a topologically nondegenerate function on the topological n-manifold X. In this section we introduce general and isotopic deformations on X with certain attractive properties, and modeled on differentiable constructions given by Morse [23, VI; 6, 7]. We refer to Morse [24] for further properties of C^0-nondegenerate functions.

A) *The isotopic deformation* J_a *(local)*.

We take an oper cover of X indexed by the points of X, as follows: In each $x\in X$ let (φ_x, U_x) be an x-centered coordinate system with coordinates $\varphi_1,\ldots,\varphi_n$, such that

1) φ_x maps U_x onto $O^n(9)=\{y\in\mathbf{R}^n \mid |y|<9\}$;

$$\text{let } U_x(t)=\{y\in U_x \mid |\varphi_x(y)|<t\} \text{ for } 0\leqslant t\leqslant 9.$$

2) Each (φ_x, U_x) satisfies (2) or (3) of Section 1B for suitable numbers $\lambda_x>0$.

3) The coordinate systems (φ_x, U_x) indexed by the critical points of f are mutually disjoint.

Fix a point $a\in X$. We now define a deformation $J_a:X\times I\to X$ which is the identity outside the coordinate neighborhood U_a. For that purpose we choose a differentiable function $h:\mathbf{R}\to\mathbf{R}$ such that $h(t)=1$ for $|t|\leqslant 4$, $h(t)=0$ for $|t|>8$, $h(t)>0$ for $|t|<8$, and $-\frac{1}{2}<h'(t)<+\frac{1}{2}$ for all $t\in\mathbf{R}$.

If $a\in X$ is an *ordinary point*, then J_a is defined by $J_a(x,t)=x$ for $x\notin U_a(8)$,

$$\begin{aligned}\varphi_i(J_a(x,t))&=\varphi_i(x)\quad(1\leqslant i\leqslant n-1)\\ \varphi_n(J_a(x,t))&=\varphi_n(x)-th(|x|)\end{aligned}\tag{1}$$

for all $x\in U_a$, where $|x|$ is the polar coordinate $|x|=r(x)=|\varphi(x)|$. If $a\in X$ is a *critical* *point of index k*, then we define J_a in terms of polar coordinates of type k in (φ_a, U_a) by the formulas

$$J_a(x, t) = x \quad \text{for} \quad x \notin U_a(8),$$
$$\frac{d}{dt}\Big(\ln r_1(J_a(x, t))\Big) = -\frac{d}{dt}\Big(\ln r_2(J_a(x, t))\Big) = (\ln 2) h(|J_a(x, t)|) \tag{2}$$

$\omega_i(J_a(x_1 t)) = \omega_i(x)$ for all $x \in U_a$, where ln means logarithm to the basis e.

Remark. — If x satisfies $r_1(x) < 2$ and $r_2(x) = 0$, we have $r_1(J_a(x, t)) = 2^t$, whence for such x the map $J_a(x, 1)$ is given by doubling the r_1 coordinate in $O^n(2)$.

The properties listed below are immediate consequences of the definition of J_a:

1) For each $t \in I$ the transformation $x \to J_a(x, t)$ is a homeomorphism of X.

2) The point a is a fixed point for J_a, if and only if a is a critical point of f.

3) The restriction of f to each trajectory of J_a is a decreasing function; i.e. for each $x \in X$ and $t \leqslant t'$ we have $f(J_a(x, t')) \leqslant f(J_a(x, t))$. The inequality is strict for all $x \neq a$ for which $x \in U_a(8)$.

B) *The deformation* D_a *(local)*.

For each critical point $a \in X$ of index k we define another deformation D_a of X as follows: Using polar coordinates in $\mathbf{R}^n$ of type k, we define

$$D_a(x, t) = x \quad \text{for} \quad x \notin U_a(8),$$
$$(r_1, \omega_1; r_2, \omega_2)\,[D_a(x, t)] = (r_1(x), \omega_1(x); (1 - th(|x|) r_2(x), \omega_2(x)) \tag{3}$$

for all $x \in U_a$.

Again, the properties below are immediate:

1) f is decreasing on the trajectories of D_a.

2) The map $x \to D_a(x, 1)$ carries $U_a(4)$ onto the k-dimensional disc in U_a:

$$\varphi_a^{-1}\{y \in \mathbf{R}^n \mid r_1(y) < 4,\ r_2(y) = 0\}.$$

C) *The deformations* J *and* D *(global)*.

Let $a_1 \ldots a_r$ be an enumeration of the critical points of f. Let $X(t) = X - \bigcup_{i=1}^r U_{a_i}(t)$. Clearly the covering in 3A can be so chosen that for every $x \in X(5)$ the coordinate system (φ_x, U_x) satisfies $U_x \cap U_{a_i}(4) = \varnothing\ (1 \leqslant i \leqslant r)$. Since $X(7)$ is compact in $X(5)$ it follows that there are finite numbers of points $a_{r+1}, \ldots, a_s$ in $X(7)$, with coordinate neighbourhoods in $X(5)$, such that

1) $X = \bigcup_{i=1}^s U_{a_i}$,

2) $[\bigcup_{i=1}^r U_{a_i}(4)] \cap [\bigcup_{i=r+1}^s U_{a_i}] = \varnothing$.

Define the isotopic deformation J of X by taking the composition $J_{a_s} \circ \ldots \circ J_{a_r} \circ \ldots \circ J_{a_1}$ and adjusting the time parameter t to vary in I. Similarly we define the deformation D of X, starting with the composition $D_{a_r} \circ \ldots \circ D_{a_1} \circ J$. We have the following properties of these deformations:

1) For each t the map $x \to J(x, t)$ is a homeomorphism; similarly $x \to D(x, t)$ is a continuous surjective map.

2) $x = \mathrm{J}(x, 0) = \mathrm{D}(x, 0)$ for all $x \in \mathrm{X}$.

3) $\left.\begin{array}{l} f(\mathrm{J}(x, t')) \leqslant f(\mathrm{J}(x, t)) \\ f(\mathrm{D}(x, t')) \leqslant f(\mathrm{D}(x, t)) \end{array}\right\}$ for any $0 \leqslant t < t' \leqslant 1$

4) $f(\mathrm{J}(x, 1)) = f(x)$ if and only if $x = a_i$ $(i = 1 \ldots r)$
$f(\mathrm{D}(x, 1) = f(x)$ if and only if $x = a_i$ $(i = 1 \ldots r)$.

5) Except for a change in parameter, the restrictions of J and D to

$$\mathrm{U}_{a_i}(2) \times \mathrm{I} \ (1 \leqslant i \leqslant r)$$

are equal to the corresponding restrictions of J_{a_i} and D_{a_i}.

D) C^∞-*deformations and* $\mathrm{C}^{\mathrm{comb}}$-*deformations.*

If f is a C^∞-nondegenerate function on a C^∞-manifold X, then J and D can be chosen such that the map $x \to \mathrm{J}(x, t)$ (or, $x \to \mathrm{D}(x, t)$) for any $0 \leqslant t \leqslant 1$ is a diffeomorphism or a differentiable map respectively. This of course one finds in Morse [23].

If f is a $\mathrm{C}^{\mathrm{comb}}$-nondegenerate function on a $\mathrm{C}^{\mathrm{comb}}$-manifold X, then there exists a triangulation of $\mathrm{X} \times \mathrm{I}$, and J and D can be chosen such that the mappings of $\mathrm{X} \times \mathrm{I}$ onto itself defined by

$$(x, t) \to (\mathrm{J}(x, t), t)$$

and

$$(x, t) \to (\mathrm{D}(x, t), t)$$

are simplexwise affine. Such deformations we call $\mathrm{C}^{\mathrm{comb}}$-deformations. The existence follows from the existence of the corresponding local $\mathrm{C}^{\mathrm{comb}}$-deformations J_a and D_a for any $a \in \mathrm{X}$.

We have the

Proposition. — It can be assumed that the deformations J *and* D *defined in Section* 3C *are* C^∞ *or* $\mathrm{C}^{\mathrm{comb}}$, *in case* (X, f) *is* C^∞ *or* $\mathrm{C}^{\mathrm{comb}}$ *respectively.*

4. Cellular decomposition of X.

A) In the theorem below we obtain for a topologically nondegenerate function f a decomposition of X, which takes the place of a decomposition obtained from gradient lines in the differentiable case. The present construction depends on the deformations J given in 3C, and in particular on a coordinate covering of X as in 3A, which we now suppose given.

Theorem. — Let X *be a topological n-manifold, and* $f : \mathrm{X} \to \mathbf{R}$ *a topologically nondegenerate unction with three critical points. Then* X *is a compactification of* $\mathbf{R}^n$ *by an m-sphere.*

Otherwise said, X contains a topologically imbedded m-sphere S^- such that $\mathrm{X} - \mathrm{S}^-$ is homeomorphic to $\mathbf{R}^n$. *We will see in Section* 5A *that n is even, and in fact* $n = 2m$ *and moreover* (*Section* 6) $n = 0, 2, 4, 8$ *or* 16.

Definition. — If X_i $(i = 0, \ldots, n)$ is a sequence of closed subsets of a closed n-manifold $\mathrm{X}_n = \mathrm{X}$, and for $i = 1, \ldots, n$ we have $\mathrm{X}_{i-1} \subset \mathrm{X}_i$, and $\mathrm{X}_i - \mathrm{X}_{i-1}$ is the disjoint union of a finite number γ_i of i-dimensional open cells, then the sequence X_i is called a *cellular presentation* of X.

Problem. — If the topological n-manifold X admits a C^0-nondegenerate function f with Morse numbers μ_i, $i=0, \ldots, n$, does there exist a cellular presentation with invariants $\gamma_i = \mu_i$? The above theorem asserts this in case f has three critical points.

Problem. — Can X be obtained by attaching an n-disc D^n to S^- by a map $h : \partial D^n \to S^-$? We will see in Proposition 6A that X has the homotopy type of such a CW-complex; compare also Section 7C.

We exclude now and henceforth the case $n=0$, for which X consists of three points. *Thus the assumptions on* f *imply that* X *is connected.*

B) *Proof of the theorem.*

It is clear that f has a minimum and a maximum, since X is compact; furthermore, the corresponding critical points have indices 0 and n. Let a_0, a_1, a_2 be the three critical points of f, of indices 0, m, n respectively.

Take the a_1-centered coordinate system (φ_{a_1}, U_{a_1}) described in 3A, and introduce in U_{a_1} polar coordinates of type m.

For each $t (0 \leqslant t < 9)$ set

$$D^-(t) = \{x \in U_{a_1} \mid r_1(x) < t,\ r_2(0) = 0\}$$
$$D^+(t) = \{x \in U_{a_1} \mid r_1(x) = 0,\ r_2(x) < t\}.$$

If we define the homeomorphism τ of X by $\tau(x) = J(x, 1)$, then $\tau(D^-(2)) = D^-(4)$, for by Remark 3A any $\tau(x)$ with $x \in D^-(2)$ has polar coordinates $(2r_1(x), \omega_1(x))$ with $r_2 = 0$. Letting $\tau^i = \tau \circ \ldots \circ \tau$ (i-fold iterate of τ), it follows by induction that $\tau^i(D^-(2)) \subset \tau^{i+1}(D^-(2))$, whence we can define the injective limit space

$$\tau^\infty(D^-(2)) = \lim_{i=\infty} \tau^i(D^-(2)). \tag{1}$$

Observe that we do *not* define τ^∞.

We now construct a homeomorphism ψ of $\tau^\infty(D^-(2))$ onto $\mathbf{R}^m$. For any $x \in \tau^\infty(D^-(2))$ there is an integer i_x such that $x \in \tau^i(D^-(2))$ for all $i \geqslant i_x$. Define $\psi(x) \in \mathbf{R}^m$ in terms of polar coordinates by

$$\psi(x) = \{2^i r_1(\tau^{-i}(x)),\ \omega_1(\tau^{-i}(x))\}. \tag{2}$$

That representation is independent of the choice of $i \geqslant i_x$, because

$$\{2^j r_1(\tau^{-j}(y)),\ \omega_1(\tau^{-j}(y))\} = \{r_1(y),\ \omega_1(y)\}$$

for every $y \in D^-(2)$ and integer $j \geqslant 0$. Clearly ψ maps $\tau^j(D^-(2))$ homeomorphically onto $O^m(2^{j+1}) \subset \mathbf{R}^m$ for all $j \geqslant 0$, and hence the injective limit $\tau^\infty(D^-(2))$ onto the injective limit $O^m(2^\infty) = \mathbf{R}^m$. Compare Stallings [33] for the generalisation in terms of categories of this Mazurean argument.

We next introduce a continuous function $\delta : X \to \mathbf{R}$ by setting $\delta(x) = f(x) - f(\tau(x))$. Thus δ is a nonnegative function which measures the amount that x is dropped by τ; moreover, $\delta(x) = 0$ when and only when $x = a_0, a_1, a_2$. Fix a number ε $(0 < \varepsilon < 1)$, and set $\inf\{\delta(x) \mid x \in X - U_{a_0}(\varepsilon) - U_{a_1}(\varepsilon) - U_{a_2}(\varepsilon)\} = \delta_\varepsilon$.

Then $\delta_\varepsilon > 0$, for in this definition x varies in a compact subset of X.

Any $x \in \tau^\infty(D^-(2)) - U_{a_0}(\varepsilon) - U_{a_1}(\varepsilon)$ is lower than a_1, and τ drops it at least δ_ε units. Let i_ε be an integer such that $i_\varepsilon \delta_\varepsilon > f(a_1) - f(a_0)$. Then $\tau^i(x) \in U_{a_0}(\varepsilon)$ for $i = i_\varepsilon$; that also holds for $i \geq i_\varepsilon$, because the restriction of τ to $U_{a_0}(\varepsilon)$ is multiplication by $\frac{1}{2}$; in particular, $\lim_{i=\infty} \tau^i(x) = a_0$. Then for sufficiently large i we have

$$\tau^\infty(D^-(2)) - \tau^i(D^-(2)) \subset U_{a_0}(\varepsilon).$$

Because ε is arbitrary, it follows that $\tau^\infty(D^-(2)) \cup a_0$ *is the* 1*-point compactification of* $\tau^\infty(D^-(2))$; compare Kuiper [17]. Therefore, the subspace $S^- = \tau^\infty(D^-(2)) \cup a_0$ is homeomorphic to the m-sphere, and S^- is invariant under τ.

Finally we consider the injective limit set $\tau^\infty(U_{a_2}(2))$, which is seen to be homeomorphic to $\mathbf{R}^n$ by the above argument. We will show, using the function $-f$ instead of f, that

$$\tau^\infty(U_{a_2}(2)) = X - S^-. \tag{3}$$

On the one hand, if $x \in \tau^\infty(U_{a_2}(a))$ then $x \notin S^-$, for otherwise $\tau^{-i}(x)$ is in S^- for all i. That is impossible, because $\tau^{-i}(x)$ is in $U_{a_2}(2)$ for suitably large i, and yet $S^- \cap U_{a_2}(2) = \emptyset$. Thus $\tau^\infty(U_{a_2}(2)) \subset X - S^-$.

On the other hand, if $x \in X - S^-$, we will show that there is an integer i such that $\tau^{-i}(x) \in U_{a_2}(2)$. There is a number $\varepsilon\,(0 < \varepsilon < 1)$ such that

$$x \in X - S^- - U_{a_0}(\varepsilon) - U_{a_1}(\varepsilon).$$

The transformation τ^{-1} leaves invariant the critical points of f_1 and raises every other point of X to a higher level. Arguing as before, we conclude that there is an i for which either

1) $\tau^{-i}(x) \in U_{a_2}(2)$; or

2) $x' = \tau^{-i}(x) \in U_{a_1}(\varepsilon)$.

In case 2) we note that $x' \notin D^-(\varepsilon) \subset S^-$. Inside $U_{a_1}(2)$ the action of τ^{-1} is explicitly given by equations (2) of Section 3; we see that there is an integer j for which $\tau^{-i-j}(x) = \tau^{-j}(x')$ is higher than a_1. It follows that for some $k > i+j$ we have $\tau^{-k}(x) \in U_{a_2}(2)$.

Therefore, in both cases 1) and 2) we have $x \in \tau^\infty(U_{a_2}(a))$; i.e. $X - S^- \subset \tau^\infty(U_{a_2}(2))$. This completes the proof of the theorem.

Corollary. — If $m > 1$, *then* X *is simply connected.*

5. The homology of X.

A) Given any topologically nondegenerate function $f: X \to \mathbf{R}$, set

$$f^s = \{x \in X \mid f(x) \leq s\}.$$

For any coefficient field $\mathbf{F}$ we let $\beta_k(f^s, \mathbf{F}) = \dim H_k(f^s; \mathbf{F})$. Then using powers of the deformation D (compare the proof of Proposition 6A), Morse theory shows that

$$\Sigma_{k=0}^n \{\mu_k(f^s) - \beta_k(f^s, \mathbf{F})\} t^k / (1+t)$$

is a polynomial in t with coefficients which are nondecreasing integervalued functions of s.

These coefficients are nonnegative, for they are zero for $s=-\infty$. For $s=+\infty$ this concerns the space $f^{+\infty}=X$ and we have the *Morse relations*, to the effect that

$$\Sigma_{k=0}^{n}\{\mu_k(f)-\beta_k(X, \mathbf{F})\}t^k/(1+t)$$

is a polynomial with nonnegative integral coefficients.

In detail: *For every integer* p $(0 \leqslant p < n)$ *we have*

$$\Sigma_{k=0}^{p}(-1)^{p-k}\mu_k(f) \geqslant \Sigma_{k=0}^{p}(-1)^{p-k}\beta_k(X, \mathbf{F}),$$
$$\Sigma_{k=0}^{n}(-1)^{n-k}\mu_k(f) = \Sigma_{k=0}^{n}(-1)^{n-k}\beta_k(X, \mathbf{F}) = \chi(X), \tag{1}$$

the Euler characteristic of X. Consequently in particular,

$$\mu_k(f) \geqslant \beta_k(X, \mathbf{F}). \tag{2}$$

Remark. — Another inequality concerning μ_1 is $\mu_1(f) \geqslant \rho(\pi_1(X))$, where $\rho(\pi_1(X))$ is the minimal number of elements of the fundamental group $\pi_1(X)$ that can generate this group. We will not need this here, however.

Lemma. — *If f is a topologically nondegenerate function on* X *with three critical points, then n is even. If we set $n=2m$, then the Morse numbers satisfy* $\mu_0(f)=\mu_m(f)=\mu_{2m}(f)=1$.

Proof. — Since we have excluded the case $n=0$, we see that (1) implies $n>1$. Taking $\mathbf{F}=\mathbf{Z}_2$ (the field with two elements), we have $\beta_0(X, \mathbf{Z}_2)=\beta_n(X, \mathbf{Z}_2)=1$, because X is closed and connected. Taken with the equation $\Sigma_{k=0}^{n}\mu_k(f)=3$, the Morse relations now imply

$$\mu_k(f)=\beta_k(X, \mathbf{Z}_2) \text{ for all } k.$$

The same argument applied to the function $-f$ yields

$$\mu_k(-f)=\mu_{n-k}(f)=\beta_k(X, \mathbf{Z}_2),$$

and the lemma follows.

Corollary. — *If $n=2$, then* X *is homeomorphic to the real projective plane.*

For in that case $\chi(X)=1$, and we apply the topological classification of closed 2-manifolds (Seifert-Threlfall [27, Kap. 6]).

Remark. — These Morse relations also show that for $n \geqslant 4$ the manifold X has no torsion and has integral homology groups $H_i(X)=\mathbf{Z}$ for $i=0, m, 2m=n$; 0 otherwise. However, our next result gives more precision.

B) *Theorem.* — *Let* X *be a topological n-manifold which admits a nondegenerate function with three critical points. Then for $n \neq 2$ the integral cohomology ring of* X *is a truncated polynomial ring in one generator σ of height three*:

$$H^*(X)=\mathbf{Z}[\sigma]/(\sigma^3).$$

Proof. — First of all, Corollary 4B shows that X is orientable; we suppose that a definite orientation has been chosen (but note the definition below). For a given

$f: \mathrm{X} \to \mathbf{R}$ we consider the sphere S^- in X, described in Theorem 4A. By choosing an orientation of S^-, that imbedding determines an isomorphism

$$\varphi : \mathrm{H}^i(\mathrm{S}^-) \to \mathrm{H}^{m+i}(\mathrm{X}, \mathrm{X}-\mathrm{S}^-)$$

for all i, by combining Poincaré duality D of S^- with the Alexander-Pontrjagin duality α of S^- in X:

$$\begin{tikzcd} \mathrm{H}_{m-i}(\mathrm{S}^-) \arrow[r, "\alpha"] & \mathrm{H}^{n-m+i}(\mathrm{X}, \mathrm{X}-\mathrm{S}^-) \\ \mathrm{H}^i(\mathrm{S}^-) \arrow[u, "\mathrm{D}"] \arrow[ur, "\varphi=\alpha\circ\mathrm{D}"'] & \end{tikzcd}$$

Furthermore, if we set $\varphi(1)=\sigma$ then for any $u \in \mathrm{H}^i(\mathrm{S}^-)$ we have $\varphi(u)=u \cup \sigma$. For this description of φ compare Thom [35, Introduction].

Now Theorem 4A implies that there is a canonical isomorphism

$$\mathrm{H}^{m+i}(\mathrm{X}, \mathrm{X}-\mathrm{S}^-) \approx \mathrm{H}^{m+i}(\mathrm{X}, a_2),$$

from which we conclude that we have a canonical isomorphism (also called φ) of $\mathrm{H}^i(\mathrm{S}^-)$ onto $\mathrm{H}^{m+i}(\mathrm{X})$ for all i. Interpreting $\sigma \in \mathrm{H}^m(\mathrm{X})$ shows that $\sigma^2=\sigma\cup\sigma$ generates $\mathrm{H}^n(\mathrm{X})$; the theorem follows.

Definitions. — *Say that the orientations in* X *and* S^- *are compatible, if* σ^2 *is the orientation generator of* X. *Note that for* $n \neq 2$ *there is a natural orientation* ξ *on* X, *given by*

$$\xi = \sigma\cup\sigma = (-\sigma)\cup(-\sigma).$$

In his work on the Hopf invariant Adams [1] proved a fundamental theorem on the vanishing of Steenrod squares, a special case of which is the following; *Let* X *be a space, and* m *an integer for which* $\mathrm{H}^i(\mathrm{X}, \mathbf{Z}_2)=0$ *for* $m<i<2m$. *Then the operation*

$$\mathrm{S}q^m : \mathrm{H}^m(\mathrm{X}; \mathbf{Z}_2) \to \mathrm{H}^{2m}(\mathrm{X}; \mathbf{Z}_2)$$

which is just the cup product square, *is zero, except perhaps for* $m=1, 2, 4, 8$. Applying that to our theorem, we obtain the

Corollary. — *If* X *is a topological n-manifold which admits a nondegenerate function with three critical points, then* X *has the cohomology ring (for any coefficient ring) of a projective plane over the real, complex, quaternion, or Cayley numbers.*

Remark. — It follows from this corollary that the Lusternik-Schnirelmann category of X is 3.

C) The Stiefel-Whitney classes $w_k(\mathrm{X}) \in \mathrm{H}^k(\mathrm{X}; \mathbf{Z}_2)$ have been defined by Thom and Wu (See Thom [35, Ch. III]; they write W_k) for any closed topological n-manifold.

Namely, for each j $(0 \leqslant j \leqslant n)$ let $\mathrm{V}^j \in \mathrm{H}^j(\mathrm{X}; \mathbf{Z}_2)$ be Wu's cohomology realisation of $\mathrm{S}q$ defined by

$$\mathrm{S}q^j(x) = x \cup \mathrm{V}^j \text{ for all } x \in \mathrm{H}^{n-j}(\mathrm{X}; \mathbf{Z}_2)$$

then $w_k(\mathrm{X})$ is defined by the formula

$$w_k(\mathrm{X}) = \sum_{i+j=k} \mathrm{S}q^i(\mathrm{V}^j).$$

For our manifolds X the only significant value of j is $j=m$, in which case $Sq^m(x)=x\cup x$. Thus V^m is the modulo 2 reduction of σ, and $w_m(X)=Sq^0V^m=V^m$, so that $w_m(X)$ *is the modulo two reduction of* σ, whence $w_n(X)=w_m(X)\cup w_m(X)$. Thus we obtain the

Proposition. — All manifolds as in Theorem 5B *have the same Stiefel-Whitney numbers, which include the Euler characteristic* $\chi(X)=1$ *or* 3.

Restricting to C^∞-manifolds, the application of the fundamental theorem [36, Th. IV, 10] of Thom that nonoriented cobordism classes are characterised by their sets of Stiefel-Whitney numbers leads to the

Corollary. — All differentiable 2*m-manifolds* $m=1$, 2, 4, *or* 8, *which admit a nondegenerate function with three critical points, belong to the same nontrivial* O_n*-cobordism class of* $P_2(\mathbf{F})$.

We will see in Chapter 2 below that for $m=4$, 8 such C^∞-manifolds belong to many different SO_n-cobordism classes, for their Pontrjagin numbers differ.

Problem. — Are all C^0-2m-manifolds which admit a C^0-nondegenerate function with three critical points cobordant with $P_2(\mathbf{F})$?

6. The homotopy type of X.

A) Recall that a (continuous) map $\varphi: A\to B$ between any spaces is a *homotopy equivalence* if there exists a map $\psi: B\to A$ for which $\psi\circ\varphi$ (resp. $\varphi\circ\psi$) is homotopic to the identity map of A (resp. of B). If A and B are oriented n-manifolds, we say that φ *is an oriented homotopy equivalence* if the induced cohomology isomorphism φ^* preserves orientation generators.

Lemma. — (See Hilton [10, Theorem 6.6]). *Let* A *and* B *be spaces, and* $\varphi: A\to B$ *a homotopy equivalence. If* $\alpha:\partial D^r\to A$ *and* $\beta:\partial D^r\to B$ *are maps such that* β *is homotopic to* $\varphi\circ\alpha$, *then there is an extension of* φ *to a homotopy equivalence of the identification spaces.*

$$A\cup_\alpha D^r\to B\cup_\beta D^r.$$

We will sketch the proof. It suffices to prove two special cases:

1) $A=B$, φ is the identity, and $\alpha_0=\alpha$ is homotopic to $\alpha_1=\beta$ by the homotopy α_t, $t\in I$. Expressing $D^r=\{(y,t)\mid y\in\partial D^r \text{ and } 0\leqslant t\leqslant 1\}$, we define the map

$$\begin{aligned}\tilde\varphi: A\cup_{\alpha_0}D^r&\to A\cup_{\alpha_1}D^r \text{ by}\\ \tilde\varphi(x)&=x \quad \text{for all} \quad x\in A,\\ \tilde\varphi(y,t)&=(y,2t) \quad \text{for} \quad 0\leqslant t\leqslant \tfrac{1}{2},\\ &=\alpha_{2-2t}(y) \quad \text{for} \quad \tfrac{1}{2}\leqslant t\leqslant 1.\end{aligned}$$

Then φ is a homotopy equivalence, with homotopy inverse $\tilde\psi$ given by

$$\tilde\psi(x)=x \quad \text{for all} \quad x\in A,$$

$$\widetilde{\psi}(y, t) = (y, 2t) \quad \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2},$$
$$= \alpha_{2t-1}(y) \quad \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1.$$

This case reduces the lemma to the next case:

2) $\varphi : A \to B$ *is arbitrary*, but $\beta = \varphi \circ \alpha$.

We define $\widetilde{\varphi} : A \cup_\alpha D^r \to B \cup_{\varphi\circ\alpha} D^r$ by

$$\widetilde{\varphi}(x) = \varphi(x) \ \text{for} \ x \in A$$
$$\widetilde{\varphi}(y, t) = (y, t) \ \text{for interior points} \ t < 1 \ \text{of} \ D^r.$$

If $\psi : B \to A$ is a homotopy inverse of φ, then we define $\widetilde{\psi} : B \cup_{\varphi\circ\alpha} D^r \to A \cup_{\psi\circ\varphi\circ\alpha} D^r$ analogously. Now take a homotopy equivalence $\theta : A \cup_{\psi\circ\varphi\circ\alpha} D^r \to A \cup_\alpha D^r$ as in Case 1). The composition $\theta \circ \widetilde{\psi}$ is a homotopy inverse of $\widetilde{\varphi}$.

Proposition. — *If the topological* $2m$*-manifold* X *admits a nondegenerate function with three critical points, then there is a map* $g : \partial D^{2m} \to S^m$ *such that* X *has the homotopy type of the space* $S^m \cup_g D^{2m}$.

Proof. — We resume the notation of Section 3, and choose a deformation D as in 3C. Then D carries every $x \in X - U_{a_0}(2) - U_{a_1}(2) - U_{a_2}(2)$ into a lower point, and deforms S^- into itself. It carries $U_{a_1}(2)$ into $D^-(2)$ and $U_{a_0}(2)$ into a_0. Some power D_i of D will deform $X - U_{a_2}(2)$ onto S^-.

Now X is obtained from $X - U_{a_2}(2)$ by attaching the $2m$-disc $D = \overline{U}_{a_2}(2)$ by a homeomorphism α of its boundary. If we define the homotopy equivalence

$$\varphi : X - U_{a_2}(2) \to S^- \quad \text{by} \quad \varphi(x) = D^i(x, 1),$$

then the proposition follows from Lemma 6A with $g = \varphi \circ \alpha$.

B) For any map $g : \partial D^{2m} \to S^m$ the space $X(g) = S^m \cup_g D^{2m}$ has integral cohomology groups given by $H^i(X(g)) = \mathbf{Z}$ for $i = 0, m, 2m$; otherwise 0.

Namely, the imbedding $i : S^m \to X(g)$ induces a cohomology isomorphism in dimension m, and $g : (D^{2m}, \partial D^{2m}) \to (X(g), S^m)$ induces the isomorphism

$$g^* : H^{2m}(X(g), S^m) \to H^{2m}(D^{2m}, \partial D^{2m});$$

but $H^{2m}(X(g), S^m)$ is canonically isomorphic to $H^{2m}(X(g))$.

Since D^{2m} and S^m have orientation induced from that of the ambient spaces $\mathbf{R}^{2m}$ and $\mathbf{R}^{m+1}$, we have distinguished generators $\sigma_g \in H^m(X(g))$ and $\xi_g \in H^{2m}(X(g))$.

Definition. — The *Hopf invariant* of the map $g : \partial D^{2m} \to S^m$ is the integer $\gamma(g)$ such that $\sigma_g \cup \sigma_g = \gamma(g)\xi_g$. Actually, $\gamma(g)$ is the negative of the invariant originally defined by Hopf. It follows from Case 1) of Lemma 6A that $\gamma(g)$ depends only on the homotopy class of g, because the cohomology ring (of $D^{2m} \cup_g S^m$) is an invariant of homotopy type.

With the notation of Proposition 6A we have the

Proposition. — *If* X *and* S^- *are compatibly oriented, then there is a map* $g : \partial D \to S^-$ *and an oriented homotopy equivalence* $\theta : X \to X(g)$ *mapping* S^- *into itself. The Hopf invariant* $\gamma(g) = 1$, *and it can be interpreted as the self intersection of* S^- *in* X.

Proof. — Proposition 6A shows that a homotopy equivalence θ exists which on S^- is a deformation of the identity. Let ξ and σ denote the orientation generators of X and S^-, and use the subscript g to refer to $X(g) = S^- \cup_g D$. Since $D = \overline{U}_{a_2}(2)$ has orientation induced from X, we have $\theta^*(\xi_g) = \xi$, i.e., θ preserves orientations.

Now $\theta \circ i$ is homotopic to i_g ($i : S^- \to X$ is again the inclusion map), whence $i^*(\sigma) = i_g^*(\sigma_g) = i^* \circ \theta^*(\sigma_g)$; we conclude that $\sigma = \theta^*(\sigma_g)$. Therefore,

$$\gamma(g)\xi = \theta^*(\gamma(g)\xi_g) = \theta^*(\sigma_g \cup \sigma_g) = \sigma \cup \sigma = \xi,$$

and thus $\gamma(g) = 1$. The self intersection property of S^- is a translation into homology of these multiplicative relations. Geometrically it follows (but for a sign) from the fact that S^- and S^+, analogously defined with $(X, -f)$ instead of (X, f), meet in one point a_1.

Corollary. — *If* X *is a topological* 4*-manifold admitting a nondegenerate function with three critical points, then* X *has the oriented homotopy type of the complex projective plane.*

For the Hopf invariant defines an isomorphism $\gamma : \pi_3(S^2) \to \mathbf{Z}$, whence there is only one homotopy class of maps g with $\gamma(g) = 1$. Furthermore, that class is represented by the Hopf fibration $p : S^3 \to S^2$, whose Thom complex is $P_2(\mathbf{C})$.

Remark. — This corollary can also be obtained from Corollary 4B and Theorem 3 of Milnor [21], which states that the oriented homotopy types of simply connected 4-manifolds are classified by their quadratic forms.

C) Henceforth we restrict attention to the dimensions $n = 2m = 8, 16$. From Proposition 6B we see that to determine the homotopy types of our manifolds it suffices to consider the spaces $X(g) = S^m \cup_g D^{2m}$ for maps g with $\gamma(g) = 1$. We use the following knowledge of the structure of $\pi_{2m-1}(S^m)$, as interpreted by Shimada [29]:

The sequence

$$0 \to \pi_{2m-2}(S^{m-1}) \xrightarrow{E} \pi_{2m-1}(S^m) \xrightarrow{\gamma} \mathbf{Z} \to 0 \tag{1}$$

is exact, where E denotes the Freudenthal suspension. The homomorphism $\lambda : \mathbf{Z} \to \pi_{2m-1}(S^m)$ which assigns to 1 the map of the Hopf fibration, determines a splitting of (1).

The corresponding projection

$$\alpha : \pi_{2m-1}(S^m) \to \pi_{2m-2}(S^{m-1})$$

is defined by

$$\alpha(g) = E^{-1}\{[g] - \lambda\gamma(g)\}.$$

Then the direct sum decomposition

$$\gamma \oplus \alpha : \pi_{2m-1}(S^m) \to \mathbf{Z} \oplus \pi_{2m-2}(S^{m-1}) \tag{2}$$

carries $[g]$ onto $\gamma(g) \oplus \alpha(g)$.

We recall that

$$\pi_{2m-2}(S^{m-1}) = \mathbf{Z}_{12} \text{ for } m=4$$
$$\mathbf{Z}_{120} \text{ for } m=8.$$

For any integer h let $g_h : \partial y_h^{2m} \to S^m$ be the fibre map of Section 2B associated with the bundle $\xi_{h,j}$ with $h+j=1$. We orient its Thom complex X_h^{2m} by taking as orientation generator $\xi \in H^{2m}(X_h^{2m})$ the element $\xi = \sigma \cup \sigma$, where σ is either generator of $H^m(X_h^{2m})$. Then $\gamma(g_h)=1$. (In general the Euler class, i.e. the self intersection of the zero section in the associated (SO_m, D^m)-bundle is $W_m(\xi_{h,j}) = h+j$.)

Shimada [29, § 4] has remarked that if the bundle $\xi_{h,j}$ is represented by $[\xi_{h,j}] \in \pi_{m-1}(SO_m)$ and $J : \pi_{m-1}(SO_m) \to \pi_{2m-1}(S^m)$ is the J-homomorphism of G. W. Whitehead, then the corresponding homotopy type of mapping cylinder is characterised by

$$[g_{h,j}] = J[\xi_{h,j}] = \lambda(h+j) - jE(\tau_{m-1})$$

for a suitable generator $\tau_{m-1} \in \pi_{2m-2}(S^{m-1})$.

In particular in our case $j = 1-h$ we may write

$$\alpha(g_h) = r_{12}(h-1) \text{ if } m=4$$
$$r_{120}(h-1) \text{ if } m=8 \tag{3}$$

where $r_k : \mathbf{Z} \to \mathbf{Z}_k$ denotes reduction modulo k. We now recall the effect on Y_h^{2m} of reversing the orientation of S^m :

Lemma (compare Shimada [29]). — *Fix an integer h. The effect of reversing the orientation of S^m in the construction of Y_h^{2m} and X_h^{2m} is to obtain Y_{1-h}^{2m} and X_{1-h}^{2m}.*

Proof. — We use the notation of Section 1.2 (see equation (1)) and in the C^∞-case we only consider the case that $\eta =$ identity. Now we introduce new coordinates $v' = (v)^{-1}$ in each fibre of the sphere bundle space ∂Y_h^{2m}, and then use them as polar coordinates in each fibre of the 4-disc bundle space Y_h^{2m}. This means a reversing of the orientation of Y_h^{2m} but keeping the orientation of the zero crosssection S^m fixed. We next reverse the roles of the two parts from which Y_h^{2m} was constructed by an identification. With that the orientation of Y_h^{2m} is again reversed, and also that of S^m.

Thus we have kept the orientation of Y_h^{2m} fixed and we have reversed the orientation of S^m. Instead of (1) of Section 1.2 we now have (taking $\eta =$ identity) the relations in (u, v') :

$$(u, (v')^{-1}) \leftarrow (u, u^h(v')^{-1}u^j)$$

or

$$(u, u^{-h}(v')^{-1}u^{-j}) \leftarrow (u, (v')^{-1})$$
$$(u, v') \to (u, u^j v' u^h). \tag{4}$$

But $h+j=1$, $j=1-h$, and this defines Y_{1-h}^{2m}; the lemma is proved.

D) For our next step we need the

Lemma. — *Let $g_i : \partial D^{2m} \to S^m (i=0, 1 : m=4$ or $8)$ be maps. Form $X(g_i) = S^m \cup_{g_i} D^{2m}$, and let $\theta : (X(g_0), S^m) \to (X(g_1), S^m)$ be an oriented homotopy equivalence. Suppose $\theta | S^m$ is homotopic to the identity on S^m. Then g_0 and g_1 are homotopic.*

In particular, $\gamma(g_0) = \gamma(g_1)$.

Remark. — The same conclusions and the same proof apply if we only assume that $\theta : X(g_0) \to X(g_1)$ is a continuous map such that its dual carries the natural orientation cohomology classes in dimensions m and $2m$ of $X(g_2)$ onto those of $X(g_1)$ and $\gamma(X(g_0)) = \gamma(X(g_1)) = 1$. So *these conditions already imply that* θ *is an oriented homotopy equivalence.*

Proof. — For any map $g : \partial D^{2m} \to S^m$ we have

$$\begin{aligned} \pi_i(X(g), S^m) &= 0 \text{ for } i < 2m, \\ &= \mathbf{Z} \text{ for } i = 2m, \end{aligned}$$

and the relative homotopy class (g) of g is a generator. This follows from Hurewicz' theorem, because $H_i(X(g), S^m) = 0$ for $i < 2m$ by 6B, whence the Hurewicz' map h is an isomorphism in dimension $2m$:

$$\pi_{2m}(X(g), S^m) \xrightarrow{h} H_{2m}(X(g), S^m) \xleftarrow{j_*} H_{2m}(X(g)) = \mathbf{Z} \tag{5}$$

$$\theta_* : \pi_i(X(g_0), S^m) \to \pi_i(X(g_1), S^m)$$

is an isomorphism for all i, and $\theta_*(g_0) = (g_1)$. This is a consequence of the 5-lemma, a segment of which is

$$\begin{array}{ccccccc} \pi_{2m}(X(g_0), S^m) & \xrightarrow{\partial_0} & \pi_{2m-1}(S^m) & \to & \pi_{2m-1}(X(g_0)) & \to & 0 \\ \downarrow{\scriptstyle\theta_*} & & \downarrow & & \downarrow{\scriptstyle\theta_*} & & \\ \pi_{2m}(X(g_1), S^m) & \xrightarrow{\partial_1} & \pi_{2m-1}(S^m) & \to & \pi_{2m-1}(X(g_1)) & \to & 0; \end{array}$$

we use here the fact that θ preserves orientations. Since θ induces the identity map on $\pi_{2m-1}(S^m)$, we have $[g_0] = \partial_0(g_0) = \partial_1\theta_*(g_0) = \partial_1(g_1) = [g_1]$.

Proposition. — Let $m = 4$ *or* 8, *and consider the elements* $[g] \in \pi_{2m-1}(S^m)$ *with* $\gamma(g) = 1$. *The homotopy classes of spaces* $X(g)$ *correspond one to one to the unordered pairs of elements* $\{\alpha(g), \alpha(-g)\}$ *in* $\pi_{2m-2}(S^{m-1})$ *where* $-g$ *is the composition of the map* $g : S^{2m-1} \to S^m$ *and a reflection* $S^m \to S^m$ *with respect to an equator* $S^{m-1} \subset S^m$.

Each homotopy type is represented by some C^{omb}*-manifold which is the Thom complex of a fibre bundle* $\zeta_{h,j}(h+j=1)$.

Proof. — Each $g \in [g]$ defines both $X(g)$ and $\alpha(g)$; if g_0 and g_1 are two such (homotopic) maps, then $X(g_0)$ and $X(g_1)$ have the same homotopy type by Lemma 6A (take $A = B = S^m$ and φ the identity map). If we take two spaces in the homotopy type of $X(g)$, then there is a homotopy equivalence θ between them which maps S^m into itself and is homotopic to the identity on S^m or to a reflection with respect to an $S^{m-1} \subset S^m$. In this last case we take instead of the second manifold (say X_h^{2m}) the manifold X_{1-h}^{2m}. In the first case the above lemma implies that the two spaces $X(g)$ of the same homotopy type determine the same $\alpha(g)$.

Finally, from (3) above we see that every element of $\pi_{2m-2}(S^{m-1})$ corresponds to some (in fact, infinitely many) $X(g_h) = X_h^{2m}$.

When considering the oriented homotopy types of X_h^{2m} we permit either orientation of S^m. Thus taking into account (3) and (4) and Lemma 6C we obtain the

Corollary. — *Two* C^{omb}*-manifolds* $X_{h_i}^{2m}$ $(i=0, 1)$ *with the natural orientation* σ^2, *have the same oriented homotopy type if and only if*

$$\begin{aligned} h_0 - h_1 &\equiv 0 \quad \textit{or} \\ h_0 + h_1 &\equiv 1 \end{aligned} \tag{6}$$

modulo 12 $(m=4)$ *or modulo* 120 $(m=8)$.

We summarize the results of this section in the

Theorem. — *The following is a classification by homotopy types of the topological n-manifolds* X *which admit a nondegenerate function with three critical points:*

If $n=2$ *or* 4, *then* X *has the homotopy type of the real or complex projective plane.*

If $n=2m=8$ *(resp.* 16), *then* X *belongs to one of six (resp. sixty) homotopy types, each of which can be represented by a combinatorial manifold* X_h^{2m}. *These are numerically classified according to the congruences* (6).

Remark 1. — From the remark following Lemma 6D it can be deduced that if the continuous map

$$\theta : X_{h_0}^{2m} \to X_{h_1}^{2m}$$

induces an isomorphism

$$\theta^* : H^{2m}(X_{h_0}^{2m}) \leftarrow H^{2m}(X_{h_1}^{2m})$$

then θ is a homotopy equivalence.

Remark 2. — The invariants of the above classification coincide with those of James' classification of the Hopf space structures on S^{m-1}, which are (roughly speaking) given by the elements of the group $\pi_{2m-2}(S^{m-1})$; see James [14]. There is also a close relationship to the classification of (SO_m, S^{m-1})-bundles over S^m by fibre homotopy type; see Dold [4, Staz 4, 6].

II

CONSEQUENCES OF ADDITIONAL STRUCTURE ON X

7. Transverse foliations.

We are not able to obtain more specific information about our manifold without imposing further structure. In the case of a C^∞-manifold X and a C^∞-nondegenerate function f one can introduce a Riemannian metric, and then the gradient lines provide a convenient tool. They give an example of a 1-dimensional leaved structure on $X-(a_1\cup\ldots\cup a_r)$ in the sense of Ehresmann and Reeb [25, p. 101], transverse to the levels of f, and with special regularity properties in the neighbourhood of the critical points a_i $(1\leqslant i\leqslant r)$ of f. We will see that the existence of such a transverse foliation (even in the C^0-case), permits us to draw conclusions that reach further than those of Chapter I.

A) *Definition.* — Let X be a closed topological n-manifold and $f: X\to \mathbf{R}$ a nondegenerate function. Given a point $a\in X$, a continuous map

$\alpha: U\to\mathbf{R}^{n-1}$ of a neighbourhood U of a is a *local foliation transverse relative* f if the product map

$\alpha\times f: U\to\mathbf{R}^{n-1}\times\mathbf{R}$, defined by $(a\times f)x=\alpha(x)\times f(x)$ for all $x\in U$, is a coordinate system on U. For any $x\in U$ we define *the trajectory in* U *through* x as the connected component of $\alpha^{-1}(\alpha(x))$ in U containing x.

Two local transverse foliations $\alpha_1, \alpha_2: U\to\mathbf{R}^{n-1}$ are *compatible* if there is a homeomorphism $r:\alpha_1(U)\to\alpha_2(U)$ such that $r\circ\alpha_1=\alpha_2$. Two local transverse foliations α_1, α_2, with different domains U_1, U_2 are *compatible* if every $x\in U_1\cap U_2$ has a neighbourhood $U\subset U_1\cap U_2$ such that the restrictions $\alpha_1|U$ and $\alpha_2|U$ are compatible.

If $a\in X$ is an ordinary point of f and (φ, U) is an a-centered coordinate system as in Section 1, then $\alpha(x)=(\varphi_1(x), \ldots, \varphi_{n-1}(x))$ defines a local transverse foliation in U. We denote by $\mathscr{T}(\varphi, U)$ *the topology* in U for which the open sets are the (1-dimensional) ordinary open sets on the trajectories in U.

If on the other hand a is a critical point of index k, then no local transverse foliation exists. Let (φ, U) be as in (3) of Section 1. In this case we denote by $\mathscr{T}(\varphi, U)$ the topology in $U-a$, for which the open sets are the 1-dimensional ordinary open sets on the orthogonal trajectories of the level manifolds of f with respect to the Euclidean metric in $U-a$.

Finally we give for the global case (see Reeb [25, p. 100]) the

Definition. — Let $f: X\to\mathbf{R}$ be a nondegenerate function and $A=\{a_1, \ldots, a_r\}$ the set of critical points of f. *A transverse foliation of* X *relative to* f is a topology $\mathscr{T}$ on $X-A$

such that for any $a \in X$ there is an a-centered coordinate system (φ, U) satisfying either (1) or (2) of Section 1, with a topology $\mathcal{T}(\varphi, U)$ *which is equal to the restriction to the point set* U *or* $U - a$ *respectively of the topclogy* $\mathcal{T}$.

Clearly any two local transverse foliations obtained by restriction from $\mathcal{T}$ are compatible.

Problem. — We do not know whether there exists a transverse foliation for every C^0-nondegenerate function on every C^0-manifold; however, we will see below that such can be constructed in the differentiable and combinatorial cases.

Proposition. — Let X *be a* C^∞ *n-manifold, and* $f: X \to \mathbf{R}$ *a* C^∞*-nondegenerate function. Then* X *admits a transverse foliation relative to* f.

Proof. — By means of a differentiable partition of unity we introduce a differentiable Riemannian metric on X which has the representation

$$ds^2 = \Sigma_{i=1}^n d\varphi_i^2$$

in some neighbourhood of each critical point a (recall that A is a finite set), in terms of some a-centered differentiable coordinate system as in (1) of Section 1. In terms of that metric the differential of f determines a differentiable contravariant vector field $*(-df)$ on X which in turn defines a one parameter group of diffeomorphisms. The trajectories of that group (i.e., the gradient lines of f) define the required topology on $X - A$.

B) *Proposition. — Let* X *be a closed* C^{comb}*-n-manifold and* $f: X \to \mathbf{R}$ *a* C^{comb}*-nondegenerate function. Then* X *admits a transverse foliation relative to* f.

Proof. — For any set $W \subset X$ let $L(W, \varepsilon)$ be the set of all $x \in X$ for which there exists $w \in W$ such that

$$|f(x) - f(w)| < \varepsilon.$$

We take a combinatorial triangulation (K, h, X) relative to f, as in Section 1 C. Let $K^{(0)} \subset X$ be the zero skeleton of K. Of course $A \subset K^{(0)}$ where A is the set of critical points. We first define for any $\varepsilon > 0$ (to be fixed later) a transverse foliation on

a) $$X - L(K^{(0)}, \varepsilon).$$

Any point x of this set lies in the interior of a unique affine r-simplex σ_r of dimension $r \geqslant 1$ of K. The restriction of f to σ_r is a linear function (height). σ_r has $p+1 \geqslant 1$ vertices higher and the remaining $q+1 = r-p \geqslant 1$ vertices lower then x. These sets of vertices are the vertices of two simplices σ_p and σ_q of which σ_r is the join. The point x then is contained in a unique straight line segment which connects σ_p and σ_q. The connected part of x in $X - L(K^{(0)}, \varepsilon)$ is the one dimensional leaf which contains x. Applying the same for all points we find the transverse foliation in part *a)*.

Next we define a transverse foliation in

b) $$L(K^{(0)}, \varepsilon) - L(A, \varepsilon).$$

We introduce a new triangulation (K_1, h_1, X), for which f is also combinatorial; for example, by a small change of (K, h, X), but such that no vertex of $K_1^{(0)}$ lies in *b)*.

As $K^{(0)}$ and A are finite sets, this is certainly possible for all ε sufficiently small, say $\varepsilon<\varepsilon_0$. But then we can use (K_1, h_1, X), as in the former paragraph, to construct a transverse foliation in part *b)*. There remains the construction in $L(A, \varepsilon)$.

For each critical point $a\in A$ we use coordinates $(\varphi_1, \ldots, \varphi_n, \varphi'_1, \ldots, \varphi'_n)$ as in (2), (3) and (5) of Section 1. The coordinate neighbourhoods in X so obtained are assumed disjoint. $U_a(t)$ will have the same meaning as in Section 3A, and

$$U(A, t) = \bigcup_{a\in A} U_a(t).$$

We choose a preliminary transverse foliation in $U(A, t_3)$ for some $t_3>0$, by taking as leaves the orthogonal trajectories of the level sets of f with respect to the Euclidean metric $\Sigma_i^n d\varphi_i^2$ in $U_a(t_3)$ for each $a\in A$ (Cf. Section 1A).

We leave this foliation unaltered, for some $0<t_1<t_2<t_3$, $0<\varepsilon<\varepsilon_0$, in the set

c) $$L(A, \varepsilon)\cap U(A, t_1),$$

but we change it outside, such that in

d) $$L(A, \varepsilon)\cap [U(A, t_3)-U(A, t_2)]$$

it can be obtained from some subdivision (K_0, h_0, X) of (K, h, X) for which f is also combinatorial, with the methods described for the sets *a)*. Here it will be assumed, but this is no restriction of the argument, that no vertex of K_0 (hence no vertex of K), lies in

$$L(A, \varepsilon)-U(A, t_2).$$

Observe that the construction of the foliation in *c)* and *d)* is a local affair !

The same method used for *a)*, but with the complex K_0, applies to the remaining part of X (with part *d)* included):

e) $$L(A, \varepsilon)-U(A, t_2).$$

The transverse foliation obtained in this way has the properties required in the definition of Section 7A.

C) *Proposition.* — *Let* $f: X\to \mathbf{R}$ *be a topologically nondegenerate function with three critical points of index* 0, m *and* $2m$. *If* X *admits a transverse foliation relative to* f, *then there is a map* $g: \partial D^{2m}\to S^m$ *such that* X *is homeomorphic to the* CW-*complex*

$$S^m \cup_g D^{2m} \; (m=1, 2, 4, 8).$$

Proof. — In Section 3C we defined a deformation J as a composition of deformations J_a, each of which is the identity outside a coordinate system (φ_a, U_a); furthermore, in $U_a(4)$ the deformation J takes place along the n^{th}-coordinate line of $\varphi_a=(\varphi_{a_1}, \ldots, \varphi_{a_n})$. In view of the definitions of transverse foliation we can and will assume that J is so chosen that points are dropped along trajectories.

Now let a_1 be the critical point of index m, and introduce polar coordinates of type m in $U_{a_1}(4)$. Consider $W=\{x\in U_{a_1}(4) \mid r_1(x)\leqslant 2, r_2(x)\leqslant 2, r_1(x).r_2(x)\leqslant 1\}$; then the segments of the trajectories in W are represented by $r_2 \leq 2$, $(\omega_1, \omega_2, r_1, r_2) = \text{constant}$. We define the homeomorphism τ by $x \to \tau(x) = J(x, 1)$, and form the compositions τ^i, $i = 1, 2 \ldots$, following the notation and constructions of Section 4B. Recall that $\tau^\infty(W)$ *is coordinatized by* $\{r_1, \omega_1; r_2, \omega_2 \mid r_2 \leq 2, r_1 r_2 \leq 1, r_1 \leq \infty\}$, and that $\tau^\infty(D^-(2)) = S^- - a_0$ is then represented by $r_2 = 0$.

In order to define g the trajectories will be altered such that S^- consists of end points of trajectories, as follows:

1) We leave the trajectories unchanged in the set of points

$$\left\{\begin{array}{l} r_1 r_2 \geq \dfrac{2}{\pi} \text{ arctg } (r_1) \quad \text{and} \\ r_2 \geq \dfrac{2}{\pi} \text{ for } r_1 = 0. \end{array}\right. \tag{1}$$

2) For the set

$$r_1 r_2 \leq \frac{2}{\pi} \text{ arctg } (r_1) \quad \text{and} \quad r_2 \neq 0 \tag{2}$$

we introduce new trajectories represented by $(\omega_1, \omega_2, r_1) = \text{constant}$: (see next page).

Recall from Section 4B that $X - S^- = \tau^\infty(U_{a_2}(a))$ is homeomorphic to $\mathbf{R}^{2m}$. The trajectories emanating from a_2 either 1) end at a_0, or 2) traverse the set

$$\{x \in X \mid r_2(x) = 2, r_1(x) . r_2(x) < 1\};$$

in that case the trajectory enters the set defined by 2) at a lower point, after which it follows the new trajectory to its end in S^-.

In order to define the map $g : \partial D^{2m} \to S^-$ we first consider $\tau^\infty(U_{a_2}(2)) = \text{Int } D^{2m}$, together with the (new) trajectories emanating from a_2, which cover this open $2m$-disc. The closed disc $D^{2m} \cup \partial D^{2m}$ is defined by closing D^{2m} with one point for each trajectory. Finally g is defined by assigning to the endpoint in $D^{2m} \cup \partial D^{2m}$ of each trajectory the end point in S^- of the same trajectory in $D^{2m} \subset X$. Clearly g is continuous and provides the desired attachment.

D) *Proposition. — Let* X *be the manifold of Proposition* 7C *with given function* f *and transversal foliation.*

There is a homeomorphic imbedding $g : \partial D^m \times D^m \to \partial D^{2m}$ *such that if* $Y = D^{2m} \cup_g (D^m \times D^m)$, *then* X *is homeomorphic to the identification space* $Y/\partial Y$.

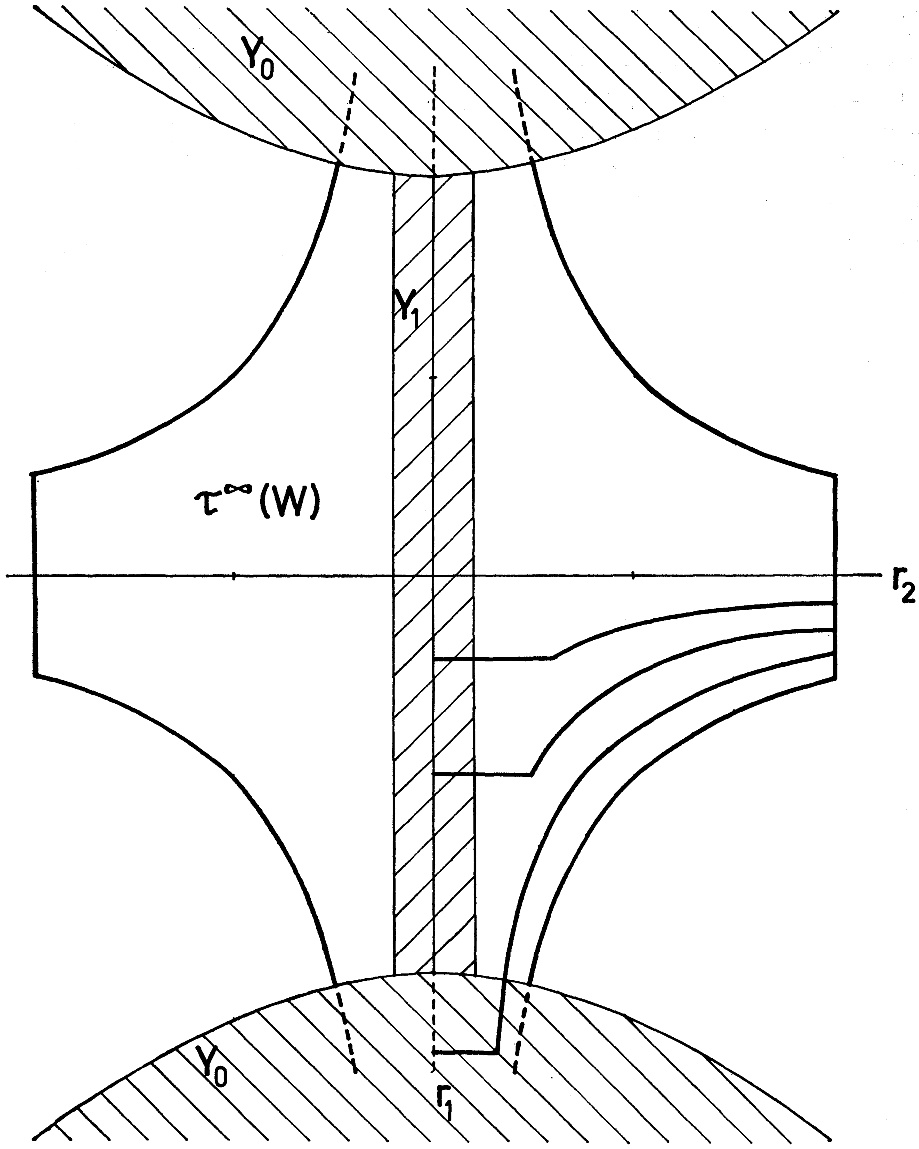
Proof. — Without loss of generality we can suppose that f is represented in polar coordinates in $U_{a_1}(4)$ by

$$f(x) = -r_1^2(x) + r_2^2(x).$$

This implies $f(a_0) < -16$. Let $Y_0 = \{x \in X \mid f(x) \leq -4\}$.

The trajectories in Y_0 which end at a_0 form a transverse foliation of $Y_0 - a_0$. The direction of the ray at a_0, together with the values of f along the ray, determine polar coordinates (r^*, ω^*), by which Y_0 *can be mapped homeomorphically onto a closed 2m-disc*:

$$Y_0 \rightarrow \{(r^*, \omega^*) \mid r^* \leqslant 1\}, \quad \text{where} \quad r^*(x) = \frac{f(x) - f(a_0)}{-4 - f(a_0)}.$$



Consider now the $2m$-disc Y_1 given by $Y_1 = \{x \in U_{a_1}(4) \mid f(x) \geqslant -4, r_2(x) \leqslant \rho\}$ for some sufficiently small $\rho > 0$; see the figure in C). The (old) trajectories define an attaching map g of Y_1 to Y_0 along

$$Y_{01} = Y_0 \cap Y_1 = \{x \in U_{a_1}(4) \mid f(x) = -4, r_2(x) \leqslant \rho\}.$$

Y_{01} is clearly homeomorphic to $\partial D^m \times D^m$. All trajectories of the transverse foliation meet the boundary ∂Y of $Y = Y_0 \cup_g Y_1$ transversally. That is in particular true along the part $\partial Y_1 \cap \partial Y$, where the trajectories are represented by $(\omega_1, \omega_2, r_1 \cdot r_2) = \text{constant}$.

Let $Y_2 = \overline{X - Y}$. Since each trajectory emanating from a_2 meets ∂Y transversally in exactly one point, we see that these rays together with the values of f define polar coordinates in Y_2. Thus $Y_2 \subset X$ is also a closed $2m$-disc, and $X = Y \cup Y_2$ is homeomorphic to $X/Y_2 = (X - \operatorname{Int} Y_2)/(Y_2 - \operatorname{Int} Y_2) = Y/\partial Y$.

Remark. — It is clear that Int Y is *a neighbourhood of* S^- *in* X. Is it a tubular neighbourhood? In connection with this problem it is natural to ask how the $(m-1)$-sphere $g(\partial D^m \times 0)$ lies in the $(2m-1)$-sphere ∂Y_0; in particular, is it unknotted? We cannot answer that question in general, but in out next sections we conclude "unknotted" in some special cases. This leads to interesting consequences.

8. The differentiable case.

A) *Introduction to the knot problem.*

In this section we suppose X and f are C^∞ and $f : X \to \mathbf{R}$ is a C^∞-nondegenerate function with three critical points a_0, a_1, a_2. We fix a C^∞-Riemannian metric on X, which in some neighbourhood of each critical point, equals the Euclidean metric $ds^2 = \Sigma_1^n d\varphi_i^2$ in preferred coordinates as in Section 1A. As in Proposition 7D we construct

1) the disc Y_0, which is diffeomorphic to D^{2m};

2) the space Y_1, which is diffeomorphic to $D^m \times D^m$;

3) the space Y_2, which is homeomorphic to D^{2m}, and is diffeomorphic except along the $(2m-2)$-manifold $Y_{012} = Y_0 \cap Y_1 \cap Y_2$ ("edge") in the boundary ∂Y_2.

The boundary of the differentiable manifold $Y = Y_0 \cup Y_1$ is not smooth along this same edge. This however is not the main difficulty for the analysis of X, as it is not hard to "round off" these edges. We would like to deduce that Y^+, obtained from Y after suitably rounding off, is an m-ball bundle over S^m, in which case X would have to be diffeomorphic to one of the examples of Section 2.

As all depends on how $Y_1 = D^m \times D^m$ is attached to $Y_0 = D^{2m}$ along $Y_{01} \approx \partial D^m \times D^m$, our first main problem is the analysis of imbeddings such as

$$\begin{array}{rcccc} \varphi : & \partial D^m \times D^m & \to & S^{2m-1} & \subset D^{2m} \\ & \| & & \| & \| \\ & Y_{01} & & \partial Y_0 & Y_0 \end{array}$$

Any such imbedding determines by restriction a unique imbedding

$$\begin{array}{c}\varphi^0 : S^{m-1} = \partial D^m \times 0 \to S^{2m-1} \subset D^{2m} \\ \Big\| \\ Y_{01}^{(0)}\end{array}$$

In the next paragraph we define several kinds of equivalence concerning imbeddings.

B) *Four kinds of knot-classes.*

All maps are assumed C^∞. — For the cases C^0 and C^{comb} analogous definitions hold, however. An imbedding φ of a nested sequence

$$Z = (Z_1 \supset Z_2 \ldots \supset Z_n)$$

of manifolds (spaces or sets) into a second sequence $W = (W_1 \supset W_2 \ldots \supset W_n)$, is a sequence of imbeddings $\varphi_{(i)} : Z_i \to W_i$ such that $\varphi_{(i)}$ is the restriction of $\varphi_{(1)}$ to Z_i :

$$\varphi_{(i)} = \varphi_{(1)} \mid Z_i.$$

An imbedding $\varphi : Z \to W$ which has an inverse (also an imbedding)

$$\varphi^{-1} : W \to Z$$

is called a *diffeomorphism of nested sequences*. Several kinds of equivalence can be introduced in the class of all imbeddings of Z into W.

I) *The isotopic knot class* $K(\varphi; Z; W)$.

Two *imbeddings* $\varphi_0 : Z \to W$ and $\varphi_1 : Z \to W$ are *isotopic* if there exists a C^∞-map $h_I : W \times I \to W$, such that

1) each h_t is a diffeomorphism of W,

2) h_t is the identity for all t in some neighbourhood in I of the point 0,

3) $h_1 \circ \varphi_0 = \varphi_1$.

This defines an equivalence on differentiable imbeddings of nested sequences, and the equivalence class of $\varphi : Z \to W$ is called the *isotopic knot class of the imbedding* $\varphi : K(\varphi; Z; W)$.

II) *The knot class* $k(\varphi; Z; W)$.

Two imbeddings φ_0 and φ_1 are diffeomorphic if there exists a diffeomorphism

$$f : W \to W$$

such that

4) $f \circ \varphi_0 = \varphi_1$.

This defines an equivalence. The equivalence class of $\varphi : Z \to W$ is called the *knot class* of *the imbedding* $\varphi : k(\varphi; Z; W)$.

III) If we replace condition 3) in I) by

$$h_1 \circ \varphi_0(Z_i) = \varphi_1(Z_i) \quad \text{for} \quad i = 1, \ldots, n$$

a new weaker equivalence is obtained. The equivalence class is the *isotopic knot class of the imbedded nested sequence*: $K(\varphi(Z); W)$.

IV) Analogously, if we replace 4) in II) by

$$f\circ\varphi_0(Z_i)=\varphi_1(Z_i) \quad \text{for} \quad i=1, \ldots, n$$

a new weakest equivalence is obtained, with class: the *knot class of the imbedded nested sequence*: $k(\varphi(Z); W)$.

Lemma. — *In the following scheme the knot class at the initial point of each arrow determines uniquely the knot class at the end of the arrow. In other words: the value of the knot class at the end of the arrow gives at most as much information about* φ *as the value at the initial point.*

$$\begin{tikzcd} & k(\varphi; Z; W) \arrow[dr] & \\ K(\varphi; Z; W) \arrow[ur] \arrow[rr] \arrow[dr] & & k(\varphi(Z); W) \\ & K(\varphi(Z); W) \arrow[ur] & \end{tikzcd}$$

Proof. — This follows immediately from the definitions.

Lemma. — *Let* W *be a* C^∞*-manifold with a smooth boundary* ∂W *and* $\varphi_i : A \to \partial W \subset W$ *two imbeddings for* $i=0,1$.

Then

$$K(\varphi_0; A; \partial W)=K(\varphi_1; A; \partial W) \tag{1}$$

if and only if

$$K(\varphi_0; A; W)=K(\varphi_1; A; W) \tag{2}$$

Hence $K(\varphi; A; \partial W)$ and $K(\varphi; A; W)$ determine each other in case $\varphi(A)\subset\partial W$. Both are also equivalent to $K(\varphi; A\supset A; W\supset \partial W)$ which we will denote by $K(\varphi; A; W\supset\partial W)$.

Proof. — Any diffeomorphism of W onto W preserves the boundary and from this fact follows the if part of the lemma.

Next assume (1). Let $V=\partial W$, and let $h_1 : V\times 1\to V$ be the diffeotopy connecting φ_0 and $\varphi_1=h_1\circ\varphi_0$, as in the definition above. Choose a neighbourhood of V in W which is diffeomorphic to $V\times I$, with $V\times 0$ corresponding to ∂W. Define $h_t(x)=x$ for $t<0$. The diffeotopy h_1 is then restriction to ∂W of the following diffeotopy of W:

$$H_I : W\times I\to W$$
$$H_t(x\times s)=h_{t-s}(x)\times s \quad \text{for} \quad x\times s\in V\times I\subset W$$
$$H_t(w)=w \quad \text{for} \quad w\in W-V\times I.$$

Consequently H_I is a diffeotopy which connects the imbeddings

$$\varphi_i : A\supset A\to W\supset\partial W \quad \text{for } i=0 \text{ and } 1,$$

and the lemma is proved.

We now recall the following generalisation of a theorem of Whitney and Wu:

Theorem of Haefliger [9]. — Let A^p and V^q be C^∞-manifolds which are $(k-1)$-connected and k-connected respectively.

Then

a) Any continuous map of A^p in V^q is homotopic to a differentiable imbedding if $q \geqslant 2p-k+1$ and $2k<p$ (and to a differentiable immersion if $q \geqslant 2p-k$ and $2k<p$, A and V k-connected).

b) Two differentiable imbeddings of A in V which are homotopic as continuous maps, are differentiably isotopic if $q \geqslant 2p-k+2$ and $2k<p+1$.

Remark. — In Section 2B we used part *a)* of this theorem for the case $q=2p=2m$, with the choice $k=1$.

From the theorem we deduce in particular: for $m>2$, any two C^∞-imbeddings φ_i, $i=0,1$, of S^{m-1} in S^{2m-1} are differentiably isotopic. We apply this to obtain the first part of the

Proposition. — *For* $m \neq 2$ *there is exactly one knot class* $K(\varphi; Y^0_{01}; Y_0 \supset \partial Y_0)$, *hence, in view of the lemmas, exactly one knot-class* $k(Y^0_{01}; Y_0 \supset \partial Y_0)$. *If* $m=2$, *the knot class* $K(\varphi; Y^0_{01}; Y_0 \supset \partial Y_0)$ *depends only on the function* f. *The same then holds for* $k(Y^0_{01}; Y_0 \supset \partial Y_0)$, *which we denote by* $k(X, f)$.

To prove the second statement, we first note that any of the knot classes mentioned is independent of the choice of level between $f(a_0)$ and $f(a_1)$, because the 1-parameter group determined by the gradient lines of f define suitable diffeomorphisms of the triples $(Y^0_{01}; Y_0, \partial Y_0)$ associated with any two such levels. Secondly, the knot class is independent of the choice of differentiable Riemannian metrics, for if ds_1^2 and ds_2^2 are any two, then the metric $(1-t)ds_1^2+t\,ds_2^2$ for $t \in I$ determines a diffeotopy of Y^0_{01} in ∂Y_0.

Problem. — It can be established that not every knot class $k(S^1; S^3)$ can arise as above from a function. On the other hand, we know no nontrivial example of a knot which does so arise.

C) *Lemma.* — *Let* $D^m(\rho)=\{(r,\omega) \mid r \leqslant \rho\}$, *and let* $g_i : \partial D^m \times D^m(\rho_0) \to \partial D^{2m} (i=1,2)$ *be two differentiable imbeddings for some* $\rho_0>0$.

If these determine the same differentiable knot class $k(g_i(\partial D^m \times 0); D^{2m} \supset \partial D^{2m}))$ *then for any positive* $\rho<\rho_0$ *there is a diffeomorphism* g *of* D^{2m} *such that*

1) $g_2 = g \circ g_1$

2) *for every* $x \in \partial D^m$ *the restriction of* $g_2^{-1} \circ g \circ g_1$ *to* $x \times D^m(\rho)$ *is an orthogonal map onto* $x \times D^m(\rho)$.

Proof. — By hypothesis there is a diffeomorphism h of D^{2m} such that

$$h \circ g_1 \mid \partial D^m \times 0 = g_2 \mid \partial D^m \times 0.$$

Now $h \circ g_1$ and g_2 define the structure of an orthogonal disc bundle in the tubular neighbourhoods $h \circ g_1(\partial D^m \times D^m(\rho_0))$ and $g_2(\partial D^m \times D^m(\rho_0))$.

Then there exists a diffeotopy h_1 of D^{2m}, which keeps $g_2(\partial M^m \times 0)$ pointwise fixed, for which h_0 is the identity map, whereas for some $0<\rho<\rho_0$, $g_2^{-1} \circ h_1 \circ h \circ g_1$ is an orthogonal bundle map of $\partial D^m \times D^m(\rho)$ (for a proof see Milnor, Theorem (5.2), *Differentiable structures*. Mimeogr. notes Princeton University, Spring 1961). Finally, $g = h_1 \circ h$ is the diffeomorphism of D^{2m} required in the lemma.

Theorem. — *Let* $f : X \to \mathbf{R}$ *be a* C^∞*-nondegenerate function with three critical points. If the knot class* $k(X, f)$ *is trivial, then there is a differentiable* (O_m, D^m)*-bundle* Y^+ *over* S^m *and a diffeomorphism* $\tau : \partial Y^+ \to \partial D^{2m}$ *such that* X *is diffeomorphic to* $Y^+ \cup_\tau D^{2m}$. *We emphasize that* $k(X, f)$ *is trivial except possibly for* $m = 2$.

Proof. — We introduce polar coordinates $(r_1^*, \omega_1^*; r_2^*, \omega_2^*)$ of type m on the $2m$-disc Y_0, with $(r_1^*)^2 + (r_2^*)^2 \leqslant 1$. By Lemma 8C we can assume that $Y_{01} = \{ (\omega_1; r_2, \omega_2) \mid r_2 \leqslant \rho \}$ is represented in Y_0 by the set $\{ (r_1^*, \omega_1^*; r_2^*, \omega_2^*) \mid (r_1^*)^2 + (r_2^*)^2 = 1 \text{ and } r_2^* \leqslant \rho \}$, and that these coordinates, as far as Y_{01} is concerned, are related as follows:

$$(r_1^*, \omega_1^*; r_2^*, \omega_2^*) = \left(\sqrt{1 - r_2^2}, \omega_1; r_2, \tau(\omega_1) \circ \omega_2\right),$$

where $\tau(\omega_1)$ is an orthogonal transformation operating in the $(m-1)$-sphere in which ω_2 varies, and depends only on ω_1.

Y_1 can be coordinatized by $\{ (r_1, \omega_1; r_2; \omega_2) \mid r_1 \leqslant 1, r_2 \leqslant \rho \}$.

We recall that the transversal foliation on Y_0 consists of the halfrays, which end in the origin a_0 and which are orthogonal to the concentric spherical level manifolds of f.

Now we glue Y_1 along Y_{01} to a part $Y_0^\rho \subset Y_0$, which consists roughly of the points

$$\{ (r_1^*, \omega_1^*; r_2^*, \omega_2^*) \mid r_1^{*2} + r_2^{*2} \leqslant 1, r_2^* \leqslant \rho \}$$

but which more precisely has the following properties:

a) $Y_0^\rho \supset Y_{01}$ is diffeomorphic to

$$D^m \times D^m(\rho) \supset \partial D^m \times D^m(\rho)$$

with product space coordinates $(\omega_1^*; r_2^*, \omega_2^*)$ for Y_{01}.

b) The boundary of $Y_0^\rho \cup Y_1$ is smooth. In Y_1 it has an equation of the kind $r_2^* = \gamma(r_1^*)$.

c) This boundary is transversal to the transversal foliation of f. Then $Y^+ = Y_0^\rho \cup Y_1$ is differentiably an (O_m, D^m)-bundle over S^m, which is defined by the gluing transformation $\tau(\omega_1)$ as function of ω_1 (See Steenrod [42, p. 98]). The manifold $X - Y^+$ is foliated by the trajectories which start at a_2. Each of these meets ∂Y^+ transversally in one point.

Then $\overline{X - Y^+}$ is diffeomorphic to D^{2m} and the theorem follows.

9. The various dimensions (C^∞).

A) The case $m = 2$. We have seen in Theorem 6D that in this dimension X (with the standard orientation) has the oriented homotopy type of $P_2(\mathbf{C})$. In the differentiable case we can obtain further precision as follows. In view of the rest of this Section 9A we note that we have no example of any C^∞ manifold X which is C^∞-different from $P_2(\mathbf{C})$.

Definition. — If V and V′ are differentiable n-manifolds, then a differentiable map $\varphi : V \to V'$ is said to be a *tangential homotopy equivalence* if φ can be covered by a bundle map ψ of the principal L_n-bundles of their tangent bundles:

$$\begin{array}{ccc} P(V) & \xrightarrow{\psi} & P(V') \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & V' \end{array}$$

This defines an equivalence relation, and the equivalence class of V is called its *tangential homotopy type*.

With this terminology we can formulate a theorem of Pontrjagin (see Reeb-Wu [25, p. 71]) as follows: *Two oriented differentiable 4-manifolds have the same oriented homotopy type if and only if they have the same oriented tangential homotopy type.*

Proof. — The sufficiency is trivial. To prove the necessity, let $\varphi : V \to V'$ be such an oriented homotopy equivalence.

Then if $\varphi^{-1}P(V')$ denotes the L_4-bundle over V induced from $P(V')$ by φ, then we have the following relations between characteristic classes:

$$\begin{aligned} w_2(\varphi^{-1}P(V')) &= \varphi^* w_2(V') = w_2(V) && \text{(Whitney class)} \\ W_4(\varphi^{-1}P(V')) &= \varphi^* W_4(V') = W_4(V), && \text{(Euler class)} \\ p_1(\varphi^{-1}P(V')) &= \varphi^* p_1(V') = p_1(V) && \text{(Pontrjagin class)} \end{aligned}$$

(The first two lines are consequences of the formulas in Section 5C; the third is a consequence of Hirzebruch's Signature (index) Theorem: $\tau(V) = p_1(V)[V]/3$).

Since $H^4(V) = H^4(V') = \mathbf{Z}$, we can apply [25, p. 71] to obtain a bundle isomorphism $\iota : P(V) \to \varphi^{-1}P(V')$. The required covering bundle map is then

$$\psi = \varphi \circ \iota : P(V) \to P(V').$$

Thus we have the

Proposition. — *Let* X *be a differentiable 4-manifold admitting a differentiably nondegenerate function with three critical points. Then* X *with its standard orientation has the oriented tangential homotopy type of* $P_2(\mathbf{C})$.

Remark. — In Section 9B below we give examples of differentiable manifolds with the same oriented homotopy type and different oriented tangential homotopy types (having different Pontrjagin classes) in dimension 8 and 16.

Corollary. — X *admits an almost complex structure, and with it* X *belongs to the complex cobordism class of* $P_2(\mathbf{C})$.

Proof. — Let $\psi : P(X) \to P(P_2(\mathbf{C}))$ be any L_4-bundle map covering an oriented homotopy equivalence $\varphi : X \to P_2(\mathbf{C})$. Each reduction of $P(P_2(\mathbf{C}))$ to the unitary group U_2 (i.e., each almost complex structure on $P_2(\mathbf{C})$) determines through ψ a definite U_2-reduction of $P(X)$.

Suppose we fix any almost complex structure on X, and let $c_r(X)$ denote its Chern classes. Then $c_2(X)[X]=\chi(X)=3$, and using the general relation $p_1(X)=c_1(X)^2-2c_2(X)$, we have

$$c_1(X)^2[X]=p_1(X)[X]+2c_2(X)[X]=3\tau(X)+2\chi(X)=9.$$

Therefore the Chern numbers depend only on the oriented homotopy type of X. But according to Milnor, the Chern numbers characterize the complex cobordism class of an almost complex manifold.

Remark. — If it were known that X admits an *integrable* almost complex structure, then X would admit a complex projective-algebraic structure (Kodaira) and with it X is biholomorphic with $P_2(\mathbf{C})$ (Hirzebruch-Kodaira [13], Van der Ven-Remmert [26]).

Theorem. — *Let* X *be a* C^∞*-4-manifold admitting a* C^∞*-nondegenerate function f with three critical points, for which the knot class* $k(X,f)$ *is trivial. Then* X *is combinatorially equivalent to* $P_2(\mathbf{C})$. *Furthermore* X *is diffeomorphic to* $P_2(\mathbf{C})\#S^{4,\sigma}$ *where* $S^{4,\sigma}$ *is a topological 4-sphere with some differentiable structure* σ, *and* $\#$ *denotes the connected sum of differentiable manifolds.*

Proof. — By Theorem 8C we know that X has the form $Y^+\cup_\zeta D^4$, where Y^+ is a (O_2, D^2)-bundle over S^2, whose boundary ∂Y^+ is diffeomorphic to S^3. By Steenrod [42, p. 99] we know that Y^+ is the associated disc bundle of the Hopf bundle (with Chern class c_1 given by $c_1[S^2]=+1$). Thus from the combinatorial point of view X is the unique C^{comb}-manifold obtained from Y^+ by attaching a cone to its boundary.

From the C^∞-point of view Y^+ is diffeomorphic to a tubular neighbourhood o a $P_1(\mathbf{C})$ in $P_2(\mathbf{C})$. Then Y^+ is also diffeomorphic to the closed complement in $P_2(\mathbf{C})$ of a C^∞-4-disc D^4 with smooth boundary ∂D^4 in $P_2(\mathbf{C})$. $P_2(\mathbf{C})$ then can be obtained from Y^+ by attaching D^4 by some specific diffeomorphism $\zeta_0:\partial D^4\to\partial Y^+$. X on the other hand is obtained from Y^+ by attaching D^4 by some diffeomorphism $\zeta:\partial D^4\to\partial Y^+$. If the diffeomorphism $\zeta^{-1}\zeta_0:\partial D^4\to\partial D^4$ can be extended over D^4 then X is diffeomorphic to $P_2(\mathbf{C})$. If not, then the attachment $\zeta^{-1}\zeta_0:\partial D^4\to\partial D^4$ defines an unusual C^∞-structure $S^{4,\sigma}$ on the 4-sphere $D^4\cup_{\zeta^{-1}\zeta_0}D^4$. In that case $X=Y^+\cup_\zeta D^4$ is diffeomorphic with $P_2(\mathbf{C})\#S^{4,\sigma}$.

Problem. — It is not known whether there exists any unusual C^∞-structure on S^4. Even if $S^{4,\sigma}$ is C^∞-unusual, it is not known whether $P_2(\mathbf{C})\#S^{4,\sigma}$ and $P_2(\mathbf{C})$ are then necessarily non-diffeomorphic.

B) *The case* $m=4$.

Theorem. — *Let* X *be a* C^∞*-8-manifold admitting a* C^∞*-nondegenerate function with three critical points. Then* X *is diffeomorphic to one of the manifolds* $X_h^{8,\zeta}$ *of Section* 2C, *with*

$$h(h-1)/56\equiv 0 \bmod 1. \tag{1}$$

Thus $h=0,1$, 8 *or* 49 *plus an integral multiple of* 56. *Conversely, each such h and diffeomorphism* ζ *defines a manifold* X *satisfying our hypothesis.*

Finally if X_h^{8,ζ_0} *is such a manifold then any other manifold with the same h is diffeomorphic with:*

$$X_h^{8,\zeta}=X_h^{8,\zeta_0}\#S^{8,\sigma}$$

where $S^{8,\sigma}$ *is a topological* 8-*sphere with differentiable structure* σ. *According to Kervaire-Milnor* [16] *there are only two cases, say* $\sigma = 0$ *(usual) or* $\sigma = 1$ *(unusual).*

Proof. — In the construction of Section 8C we found that Y^+ is the C^∞-total space of an (SO_4, D^4)-bundle with base space a topological 4-sphere with the usual C^∞-structure, represented by the zero crosssection of the bundle. That bundle is associated with some principal bundle $\xi_{h,1-h}$ in the notation of Section 2B. Thus Y^+ is diffeomorphic to the C^∞-manifold Y^8_h.

But ∂Y^+ is diffeomorphic to $S^{7,0} = \partial D^8$ the smooth boundary of a C^∞-8-disc, that is a 7-sphere with the usual C^∞-structure. According to our classification of C^∞-structures on S^7 [7, § 6] we conclude that h must satisfy the congruence (1).

Conversely, we have shown [7, Theorem 6] that each value of h satisfying (1) determines a bundle $Y^8_h \to S^4$ with ∂Y^8_h diffeomorphic to S^7. If we take any diffeomorphism $\zeta : \partial D^8 \to \partial Y^8_h$ we can form $X^{8,\zeta}_h = Y^8_h \cup_\zeta D^8$. By Proposition 2D we know that there is a nondegenerate differentiable function with three critical points. The last part of the theorem follows from an argument as at the end of the previous Section 9A.

Problem. — Are X^{8,ζ_0}_h and $X^{8,\zeta_0}_h \# S^{8,1}$ diffeomorphic for some values h?

Proposition. — *If* X *is diffeomorphic to* $X^{8,\zeta}_h$, *then its Pontrjagin numbers are*

$$p_1(X)^2[X] = 2^2(2h-1)^2$$
$$p_2(X)[X] = [45 + 2^2(2h-1)^2]/7 \qquad (2)$$

Here we have assumed that X has the orientation described in Section 5B.

Proof. — It follows from the construction in § 2C that $\xi_{h,1-h}$ is the normal bundle of S^4 in $X^{8,\zeta}_h$.

Therefore, if $i : S^4 \to X^{8,\zeta}_h$ denotes the inclusion map, then

$$i^* p_1(X^{8,\zeta}_h) = p_1(S^4) + p_1(\xi_{h,1-h})$$

by Whitney duality (using the fact that $X^{8,\zeta}_h$ has no 2-torsion; see Hirzebruch [11, p. 68]). But $p_1(S^4) = 0$, and i^* is an isomorphism by Theorem 5B. Relation (2) of Section 2 states that $p_1(\xi_{h,1-h})[S^4] = \pm 2(2h-1)$, which proves the first relation in (2). The second follows from Hirzebruch's Signature Theorem [11, p. 85]:

$$\tau(X) = 1 = [7p_2(X) - p_1(X)^2]/45.$$

The classification of homotopy types in Theorem 6D together with the above proposition and relation (2), yields the.

Corollary. — *The Pontrjagin numbers of a closed* 3-*connected differentiable* 8-*manifold are not homotopy type invariants.*

We have been informed by Milnor that he also has an example of a homotopy type in which closed manifolds with different Pontrjagin numbers occur. That the Pontrjagin *classes* are not homotopy type invariants has been known for some time; see Dold [4].

In contrast to Corollary 5C dealing with (non-oriented) cobordism classes we cite the

Corollary. — The differentiable 8*-manifolds which admit a differentiably nondegenerate function with three critical points, with the natural orientation of Section* 5B, *lie in infinitely many oriented cobordism classes, characterized by* $h(h-1)$.

Proof. — From Theorem 8E we see that for any integer h satisfying (1), there is associated a manifold $X_h^{8,\zeta}$ of described sort, with Pontrjagin numbers (2). But from Thom [36, Th. IV. 2] we see that these manifolds belong to infinitely many distinct oriented cobordism classes. In fact, since all the $X_h^{8,\zeta}$ have the same Stiefel-Whitney numbers by Proposition 5C, we infer from the work of Milnor and Wall [38, p. 293], that two such manifolds are cobordant if and only if they have the same Pontrjagin numbers.

Remark. — The $\hat{A}$ — genus of the differentiable model $X_h^{8,\zeta}$ (see Borel-Hirzebruch [3]) is known to be an integer, and is

$$2^{-8}A(X_h^{8,\zeta}) = \hat{A}(X_h^{8,\zeta}) = (-4p_2 + 7p_1^2)/2^7\cdot 3^2\cdot 5 = \frac{h(h-1)}{56}. \tag{3}$$

In particular $X_8^{8,\zeta}$ is a differentiable manifold with $\hat{A}(X_8^{8,\zeta}) = 1$. Thus in problem 7 of Hirzebruch [12] the greatest integer $b(k)$ such that for all manifolds M^{4k} with vanishing second Stiefel-Whitney class the A-genus $A(M^{4k})$ is divisible by $2^{b(k)}$, is for $k=2$ equal to $b(2)=8$.

Remark. — Let X denote any manifold as in Theorem 9B. If ΩX denotes the loop space of X based at any point, then its Pontrjagin ring $H_*(\Omega X; \mathbf{Z})$ and its cohomology ring $H^*(\Omega X; Z)$ are both those of $\Omega P_2(\mathbf{K})$. For any such differentiable X we have seen in Theorem 4A that (S^-, a_2) are homotopy complements of type (m, n) in the terminology of Eells [5]; our remark then follows from Theorem 7C of that work. In particular, *the cohomology rings* $H^*(X)$ *and* $H^*(\Omega X)$ *together with* $H_*(\Omega X)$ *are not enough to determine the homotopy type of a closed* 3*-connected* 8*-manifold.*

C) *The case* $m=8$. The following results are proved analogously to those of § 9B.

Theorem. — Let X *be a differentiable* 16*-manifold admitting a differentiably nondegenerate function with three critical points. Then* X *is diffeomorphic to one to the manifolds* $X_h^{16,\zeta}$ *of Section* 2C, *with*

$$\frac{h(h-1)}{16256} \equiv 0 \; mod.\; 1. \tag{4}$$

Thus $h = 0, 1, 128, 16129$ *plus an integral multiple of* 16256.

If X_h^{16,ζ_0} *is such a manifold, then the other such manifolds with the same h are the manifolds which are diffeomorphic with* $X_h^{16,\zeta} = X_h^{16,\zeta_0} \# S^{16,\sigma}$, *where* $S^{16,\sigma}$ *is a topological* 16*-sphere with some* C^∞*-structure* σ. The possible C^∞-structures on S^{16} are not known.

Remark. — We do not know whether for $m=8$ and each h satisfying 4) there is a manifold satisfying our hypotheses. The trouble is as follows: Given a differentiable (SO_8, D^8)-bundle $Y_h^{16} \to S^8$ with ∂Y_h^{16} homeomorphic to S^{15}, the condition

$$\mu(\partial Y_h^{16}) = \frac{h(h-1)}{16256} \equiv 0 \bmod. 1$$

on our invariant μ [7, § 9] is a *necessary* condition that ∂Y_h^{16} be diffeomorphic to S^{15}. It is not known to be sufficient. The extent to which we can obtain a converse to Theorem 9C depends on the solution of the

Problem. — For what value of h is ∂Y_h^{16} the boundary of a parallelizable differentiable manifold? See [7] and Kervaire-Milnor [16] for further details.

Proposition. — If X is diffeomorphic to $X_h^{16,\zeta}$, then its Pontrjagin numbers are zero except for

$$p_2(X)^2[X] = 6^2(2h-1)^2$$
$$p_4(X)\,[X] = \frac{3^4\cdot 5^2\cdot 7 + 19\cdot 6^2\cdot(2h-1)^2}{381}. \tag{5}$$

Remark. — According to Corollary 4 of Atiyah-Hirzebruch [2], the modulo 48 reduction of the Pontrjagin class $p_1(X_h^{8,3}) = \pm 2(2h-1)$ is a homotopy type invariant. Because of Theorem 6D *we can use that reduction to distinguish the oriented homotopy types of our differentiable 8-manifolds.*

It is remarkable that this yields exactly the restriction to C^∞-manifolds of the classification of homotopy types of the C^{omb}-manifolds X_h^8, as given in Theorem 6D. For our manifolds $X_h^{8,\zeta}$ the Atiyah-Hirzebruch invariant therefore gives the complete homotopy type classification. For C^∞-16-manifolds a corresponding statement is not known.

10. Combinatorial manifolds without differentiable structure.

A) *Proposition. — Let X_h^{2m} be the combinatorial manifold of Section 2C; $m=4$ or 8. If X_h^{2m} admits a differentiable structure compatible with its combinatorial structure, then*

$$h(h-1)/56 \equiv 0 \ \mathit{mod.}\ 1 \ \mathit{if}\ m=4,$$
$$h(h-1)/16256 \equiv 0 \ \mathit{mod.}\ 1 \ \mathit{if}\ m=8.$$

By Theorem 9B the converse is true for $m=4$, and is unknown for $m=8$.

Proof. — Any such X_h^{2m} has second Stiefel-Whitney class $w_2(X_h^{2m})=0$, whence by a theorem of Hirzebruch-Borel [3] its $\hat{A}$-genus is an integer. But $\hat{A}(X_h^8) = h(h-1)/56$ as in (3) of Section 9, and $\hat{A}(X_h^{16}) = (-192p_4 + 208p_2^2)/2^{15}\cdot 3^4\cdot 5^2\cdot 7 = h(h-1)/16256$. The proposition follows.

Remark. — Thom [37] used the example X_h^8 to show the existence of a combinatorial manifold with no *compatible* differentiable structure. His proof was based on the fact that the combinatorial Pontrjagin number $p_2(X_h^8)\,[X_h^8]$ is not an integer for certain values of h (e.g., $h=2$). Note by (2) of § 9 that X_7^8 has integral Pontrjagin numbers $p_1^2 = 676$ and $p_2 = 103$ and yet no C^{omb}-compatible differentiable structure.

Remark. — Since Thom's rational Pontrjagin classes of combinatorial manifolds are combinatorial invariants, we see that *there are infinitely many combinatorially inequivalent* *manifolds among the* X_h^{2m} *in each homotopy type*; $m=4$ *or* 8. We have the following alternatives:

Either. — 1) Certain two C^{comb}-different manifolds $X_{h_0}^{2m}$ and $X_{h_1}^{2m}$ are homeomorphic, whence we can conclude that

a) The combinatorial Pontrjagin classes are not topological invariants; and

b) The Hauptvermutung for manifolds, saying that homeomorphic C^{comb}-manifolds are C^{comb}-equivalent, is false; *or*

2) No two such manifolds are homeomorphic, whence our combinatorial classification and the homeomorphism classification coincide. In that case we have new examples of non-homeomorphic manifolds of the same homotopy type.

B) *Theorem.* — *Let* X_h^{2m} *be the combinatorial manifold of Section* 2B. *If* X_h^{2m} *admits any differentiable structure compatible with its topology, then*

$$h \equiv 4j \text{ or } 4j+1 \text{ mod. } 12 \quad \text{for } j \in \mathbf{Z} \ (m=4)$$
$$h \equiv 8j \text{ or } 8j+1 \text{ mod. } 120 \quad \text{for } j \in \mathbf{Z} \ (m=8).$$

For example, X_2^8 *does not admit any differentiable structure.*

Proof. — Given a differentiable structure on X_h^{2m}, we apply a theorem of Smale [31, Theorem D] to show that there is a differentiably nondegenerate function with precisely three critical points. By Theorems 9B and 9C we find that X_h^{2m} is diffeomorphic to some $X_{h'}^{2m,\zeta}$ with $h'(h'-1)\equiv 0(56)$ if $m=4$, and $\equiv 0$ (16256) if $m=8$.

Thus for $m=4$ we have $h'\equiv 0, 1, 8, 49$ mod. 56, whence $h'\equiv 4j$ or $4j+1$ mod. 12 for every $j\in\mathbf{Z}$.

But by Corollary 6D we have $h-h'\equiv 0$ mod. 12 or $h+h'\equiv 1$ mod. 12, and the theorem follows for $m=4$. The case $m=8$ is similar.

Example. — X_4^8 does not admit a C^{comb}-compatible C^∞-structure. We do not know whether it admits a C^0-compatible C^∞-structure.

Corollary. — *Three of the six homotopy types of combinatorial manifolds* X_h^8 *contain differentiable representatives; the other three do not. Forty five of the sixty homotopy types of combinatorial manifolds* X_h^{16} *do not contain a differentiable representative.*

Remark. — The first examples of closed combinatorial manifolds having the homotopy type of no differentiable manifold were given by Kervaire [15] for dimension 10 and by Smale [31] for dimension 12. See also Wall [39]. In view of recent work on the structure of the group Γ^n (see Smale [32] and Kervaire-Milnor [16]), it seems quite possible that every combinatorial n-manifold with $n<8$ does admit a differentiable structure. In that case our 8-manifolds would be non-smoothable examples of the lowest possible dimension.

C) *Theorem.* — *The following statements concerning any two of our* C^{comb}-*manifolds* X_h^{2m}, $h=h_0$ *and* $h=h_1$ ($m=4$ *or* 8) *are equivalent:*

1) *They are* C^{comb}-*equivalent.*

2) *They have the same Pontrjagin classes.*

3) $h_1=h_0$ *or* $h_1=1-h_0$.

Proof. — We represent X_h^{2m} as $Y_h^{2m} \cup C(\partial Y_h^{2m})$ where $C(\partial Y_h^{2m})$ is the cone over the boundary of the C^∞-total space Y_h^{2m} of an (SO_m, D^m)-bundle over $S^{m,0}$. See Section 2. This representation includes a unique combinatorial structure on X.

It is known (Milnor [19] and Shimada [29]) that such bundles ξ are classified by their Euler number $W_m(\xi)$ $[S^m]$, which in the present case is 1, and their Pontrjagin number

$$\begin{aligned} p_{m/4}(\xi)\ [S^m] &= \pm 2(2h-1) \text{ for } m=4 \\ &\quad \pm 6(2h-1) \text{ for } m=8. \end{aligned}$$

But as in Proposition 9B we know that $p_{m/4}(\xi)$ corresponds to $p_{m/4}(X)$ under the imbedding $i : S^m \to X$, where $p_{m/4}(X)$ is in view of Thom's theory [37] on Pontrjagin classes for C^{comb}-manifolds, an invariant of the combinatorial structure of X.

Now if (1) holds then consequently

$$(2h_1-1) = \pm(2h_0-1)$$

which implies (2) and (3). On the other hand (2) is clearly equivalent to (3), and if (3) holds and $h_1 = 1 - h_0$ (the other case is trivial) then (1) holds in view of the explicit combinatorial homeomorphism given in Lemma 6B.

REFERENCES

[1] J. F. Adams, On the non-existence of elements of Hopf invariant one, *Annals of Math.*, *72* (1960), 20-104.

[2] M. F. Atiyah and F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds, *Bull. A.M.S.*, *65* (1959), 276-281.

[3] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces, III, *Am. J. Math.*, *82* (1960), 491-504.

[4] A. Dold, Über fasernweise Homotopieäquivalenz von Faserräume, *Math. Z.*, *62* (1955), 111-136.

[5] J. Eells, Alexander-Pontrjagin duality in function spaces, Proc. Symp. in Pure Math., *A.M.S.*, vol. 3 (1961), 109-129.

[6] J. Eells and N. H. Kuiper, Closed manifolds which admit nondegenerate functions with three critical points, Proc. Amsterdam, *Indagationes Math.*, *23* (1961), 411-417.

[7] J. Eells and N. H. Kuiper, An invariant for certain smooth manifolds, *Annali di Math.* (1962).

[8] H. Freudenthal, Zur ebenen Oktavengeometrie, Proc. Amsterdam A, 56 = *Indag. Math.*, *15* (1953), 195-200.

[9] A. Haefliger, Differentiable imbeddings, *Bull. A.M.S.*, *67* (1961), 109-114.

[10] P. Hilton, *Homotopy theory and duality*, II, Notes Cornell Univ., 1959. (As this is not available any more, compare Hilton-Wylie, *Algebraic topology*.)

[11] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Springer 1956.

[12] F. Hirzebruch, Some problems on real and complex manifolds, *Annals of Math.*, *60* (1954), 213-236.

[13] F. Hirzebruch-Kodaira, On the complex projective spaces, *Journ. Math. Pur. Appl.*, *36* (1957), p. 201-216.

[14] I. M. James, Multiplications on spheres II, *Trans. A.M.S.*, *84* (1957), 545-558.

[15] M. Kervaire, A manifold which does not admit any differentiable structure, *Comm. Math. Helv.*, *34* (1961), 257-270.

[16] M. Kervaire and J. Milnor, *Groups of homotopy spheres*, I, II (in preparation).

[17] N. H. Kuiper, A continuous function with two critical points, *Bull. A.M.S.*, *67* (1961), 281-285.

[18] B. Mazur, *The definition of equivalence of combinatorial imbeddings*, Public. de l'Institut des Hautes Études scientifiques, *3* (1959), 97-119.

[19] J. Milnor, On manifolds homeomorphic to the 7-sphere, *Annals of Math.*, *64* (1956), 399-405.

[20] J. Milnor, *On the relation between differentiable manifolds and combinatorial manifolds*, Notes, Princeton Univ., 1956.

[21] J. Milnor, *On simply connected 4-manifolds*, Symp. Intern. de Top. Alg., Mexico, 1958, 122-128.

[22] J. Milnor, Some consequences of a theorem of Bott, *Annals of Math.*, *68* (1958), 444-449.
[23] M. Morse, The calculus of variations in the large, *A.M.S. Coll. Publ.*, *18*, 1934.
[24] M. Morse, Topologically non-degenerate functions on a compact *n*-manifolds, *J. d'Analyse Math.*, VII (1959), 189-208.
[25] G. Reeb, Sur certaines propriétés topologiques des variétés feuilletées, *Act. sci. ind.*, *1183* (1952), 91-154.
[26] R. Remmert-T. v. d. Ven, Zwei Sätze über die Komplex projektive Ebene, *Nieuw Archief voor Wiskunde*, VIII (1960), 147-157.
[27] H. Seifert und W. Threlfall, *Lehrbuch der Topologie*, Leipzig, 1934.
[28] H. Seifert und W. Threlfall, *Variationsrechnung im Grossen*, Teubner, 1934.
[29] N. Shimada, Differentiable structures on the 15-spheres and Pontrjagin classes of certain manifolds, *Nagoya Math. J.*, *12* (1957), 59-69.
[30] S. Smale, On gradient dynamical systems, *Annals of Math.* (1961).
[31] S. Smale, The generalized Poincaré conjecture in higher dimensions, *Bull. A.M.S.*, *66* (1960), 373-375.
[32] S. Smale, Generalized Poincaré's conjecture in dimensions greater than four, *Annals of Math.*, *74* (1961), 391-406.
[33] J. Stallings, The topology of high-dimensional piecewise linear manifolds, *Annals of Math.* (in preparation).
[35] R. Thom, Espaces fibrés en sphères et carrés de Steenrod, *Ann. Sci. Écol. norm. sup.*, *69* (1952), 109-182.
[36] R. Thom, Quelques propriétés globales des variétés différentiables, *Comm. Math. Helv.*, *28* (1954), 17-86.
[37] R. Thom, *Les classes caractéristiques de Pontrjagin des variétés triangulées*, Symp. Inter de Top. Alg., Mexico, 1958, 54-67.
[38] C. T. C. Wall, Determination of the cobordism ring, *Annals of Math.*, *72* (1960), 292-311.
[39] C. T. C. Wall, Classification of $(n-1)$-connected $2n$-manifolds, *Annals of Math.*, *75* (1962), 163-189.
[40] J. H. C. Whitehead, On C^1-complexes, *Annals of Math.*, *41* (1940), 809-824.
[41] J. H. C. Whitehead, Manifolds with transverse fields in euclidean space, *Annals of Math.*, *73* (1961), 154-213.
[42] N. E. Steenrod, *Topology of Fibre Bundles*, Princeton, 1951.
[43] S. Smale, *A Diffeomorphism Criterion for Manifolds* (in preparation).

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