BARRY MAZUR

The definition of equivalence of combinatorial imbeddings

Publications mathématiques de l'I.H.É.S., tome 3 (1959), p. 5-17
<http://www.numdam.org/item?id=PMIHES_1959__3__5_0>

THE DEFINITION OF EQUIVALENCE OF COMBINATORIAL IMBEDDINGS

By BARRY MAZUR

In treating the intriguing problem of equivalences of imbeddings of complexes K in E^r, a brutish technical question arises: The question of whether or not one can extend an isotopy from K to K′ to an isotopy of the entire ambient space E^r onto itself (*which would bring of course,* K *to* K′).

For if one could not, a multiplicity of possible definitions of knot-equivalence would arise. And what is worse, an isotopy from K to K′ would tell little about the relationship between the respective complementary spaces.

So, the purpose of this paper is to prove just that: Any isotopy of a subcomplex $K \subseteq E^r$ to K′ can be extended to an isotopy of E^r to E^r.

The construction of the extended isotopy is done in two stages. The first stage is to reduce the problem to one of moderately local considerations. The second is to solve the local problem remaining. The technical machinations involved in solving the local problem consists in attempting to restate it as a common extension problem.

§ 1. Terminology Section.

E^r will stand for euclidean r-space complete with metric and linear structure. B^r is to be the closed unit ball in E^r. By finite complex, subcomplex, and subdivision, I mean what is usually meant.

DEFINITION. An open finite complex F is just a pair (K, L), where K and L are finite complexes, $L \subset K$ and every simplex of L is a face of a simplex of K. The *geometric* realization $|F|$ of F will be the space $|K| - |L|$, where $|X|$ is the geometric realization of the complex X.

A *simplicial map* φ of F to F′, where F and F′ are open finite complexes:

$$F = (K, L)$$
$$F' = (K', L')$$

will be a map φ of the simplices of K to the simplices of K′, taking L into L′, and $\partial_i \varphi(\Delta) = \varphi(\partial_i \Delta)$ as long as $\partial_i \Delta \notin L$, where $\Delta \in K$ and ∂_i is the i^{th} face operator. (Thus, $|\varphi|$ will be continuous as a function $|\varphi| : |F| \to |F'|$, but not necessarily on $|K|$.)

REMARK. If L is any subcomplex of a complex K, K—L can be considered as an *open complex*.

I shall constantly confuse a complex with its geometric realization. A closed neighborhood will refer to the closure of an open set. And if $X \subseteq Y$, I define

$$\partial X = CL(X) \cap CL(E^r - X)$$

where $CL(X)$ = the closure of X in Y; I don't refer to Y, in the terminology, because there will never be any ambiguity. If v is a vertex of a complex K, then St v is the complex in K generated by all simplices containing v. If $X \subseteq Y$, $\operatorname{int} X = X - \partial X$. There are three words which I will have recourse to use:

1) *Simplicial map:* $\varphi : K \to K'$. This is given its usual meaning.

2) *Combinatorial map:* $\varphi : K \to K'$. This will mean simplicial with respect to some subdivision of K and K'.

3) *Piece-wise linear map:* $\varphi : K \to E^r$. This will mean linear on each simplex of K, and will be reserved for use only when the range is E^r.

By $1 : M \to M$, I shall mean the identity map of a set onto itself. If $A, B \subseteq E^r$, $J(A, B)$ will stand for the set $\{t\alpha + (1-t)\beta \mid \alpha \in A, \beta \in B\}$, for $0 \leqslant t \leqslant 1$.

FUNCTION SPACES. If K, L are complexes (open or not), let $M(K, L)$ be the set of all simplicial maps of K into L. Then $M(K, L)$ can be given a topology in a natural way. The easiest way to describe this topology is to imbed L in E^N for large N. Then

$$M(K, L) \subseteq M(K, E^N)$$

and $M(K, E^N)$ is given a metric as follows: if $\varphi, \psi \in M(K, E^N)$

$$\delta(\varphi, \psi) = \max_{v \in V(K)} \| \varphi(v) - \psi(v) \|$$

where $V(K)$ is the set of vertices of K. $M(K, L)$ inherits a topology in this way, and it is a simple matter to show that it is independent of the imbedding $L \subseteq E^N$.

§ 2. On the definition of knot and knot-equivalence.

Whereas in the classical (one-dimensional) knot theory, a unique and natural notion of equivalence of imbedding more or less immediately presents itself, in general dimensions this is not so, and some choices must be made. For instance, by the title alone I am already committed to the genre of combinatorial as opposed to differentiable. Doubtless it is of no concern, and anyone familiar with the liaison between the two concepts can easily make the appropriate translation to the domain of differentiable imbeddings.

Then there are two possible points of view towards a complex K knotted in W.

I. The knot may be considered as the imbedding:

$$\varphi : K \to W$$

or

II. It may be looked upon as the subcomplex $K' \subseteq W$, $K' = \varphi(K)$, where the precise imbedding $\varphi : K \to K'$ has been lost.

I take the second point of view. (In most cases where I and II differ significantly, I is woefully unnatural. Also, most crucial when K is a sphere is the concept of knot addition, an operation more at home with II than I.) So, by a knotted complex K in W will be meant a subcomplex K′ such that there is a combinatorial homeomorphism

$$\varphi : K \to K' \qquad \text{(i.e. } K \approx K'\text{)}.$$

Finally, three notions come to mind as candidates for the definition of knot-equivalence. To distinguish them, I give them the names:

1) Isotopy equivalence.
2) Ambient isotopy equivalence.
3) Ambient homeomorphism equivalence.

My list calls for a few definitions.

§ 3. The three equivalence relations.

Definition 1. Let K and K′ be combinatorially isomorphic subcomplexes of E^r. A *continuous isotopy* φ_t between K and K′ will be: a sequence of combinatorial homeomorphisms with respect to some fixed subdivision of K:

$$\varphi_t : K \to E^r$$

(alternatively referred to as

$$\varphi : I \times K \to E^r, \qquad \varphi_t(k) = \varphi(t, k)),$$

such that $\varphi_0 = 1$, and $\varphi_1 : K \overset{\approx}{\to} K'$, and φ_t is continuous in t (i.e. φ_t considered as an arc in the function space $M(K, E^r)$). A *combinatorial isotopy* between K and K′ is a continuous isotopy such that if $\varphi_k : I \to E^r$ is defined to be the map $\varphi_k(t) = \varphi_t(k)$ for fixed $k \in K$, φ_k is piece-wise linear with respect to a fixed subdivision S(I), independent of k. I'll call K and K′ continuously isotopic (combinatorially isotopic) if there exists a continuous (or combinatorial) isotopy between them.

A fairly easily obtained result (mentioned to me once by M. Hirsch) simplifies things somewhat: *If* K *and* K′ *are continuously isotopic, they are also combinatorially isotopic.* (I omit the proof.) Therefore, in what follows, I suppress unnecessary adjectives and refer to K and K′ as merely: isotopic.

Definition 2. Let K, K′ be two subcomplexes in E^r. By an *ambient isotopy* between K and K′, I shall mean a sequence of combinatorial homeomorphisms:

$$\varphi_t : E^r \to E^r$$

such that φ_t is (again) continuous in t (in the usual function-space sense of the word). I do *not* require that φ_t be combinatorial for fixed subdivision of E^r independent of t. I do require, however, that $\varphi_t | K$ is an isotopy between K and K'. Finally I require $\varphi_0 = 1$. It is clear, then, that if K and K' are ambiently isotopic, they are isotopic; every ambient isotopy restricts to an isotopy.

DEFINITION 3. If f_t is the isotopy obtained by restricting the ambient isotopy F_t to K, I shall say: F_t covers f_t. And so, the first two notions of knot-equivalence are:

I) $K \underset{\iota}{\sim} K'$ if and only if K, K' are isotopic

II) $K \underset{a}{\sim} K'$ if and only if K, K' are ambiently isotopic.

EQUIVALENCE THEOREM. The two equivalence relations $\underset{\iota}{\sim}$ and $\underset{a}{\sim}$ are the same.

The equivalence theorem will be proved once it is shown that (Extension Theorem): Every isotopy f_t is covered by an ambient isotopy F_t.

The proof of this theorem is the main result of the succeeding chapters.

Lastly, the third equivalence relation: *ambient homeomorphism equivalence.*

DEFINITION 4. $K \underset{h}{\sim} K'$, or K and K' are ambient-homeomorphism equivalent if there is an orientation preserving combinatorial homeomorphism

$$h : E^r \to E^r$$

such that $h : K \approx K'$.

On the face of it, this last equivalence relation seems weaker than the others — thus:

$$K \underset{a}{\sim} K' \Rightarrow K \underset{h}{\sim} K'.$$

§ 4. The stability of combinatorial imbeddings.

Let the usual metric be placed on the set M of all combinatorial maps

$$\varphi : K \to B^r \subseteq E^r$$

$$\delta(\varphi, \psi) = \max_{v \in V(K)} || \varphi(v) - \psi(v) ||$$

where V(K) is the set of vertices of K. Let $M = N \cup S$, where N is the subset of imbeddings, and $S = M - N$.

LEMMA 1. There is a continuous function ρ on M with the properties:

(i) $\rho(m) \geq 0$, $\rho(m) > 0 \Leftrightarrow m \in N$.

(ii) If φ is an imbedding (i.e. $\varphi \in N$) and $\varphi' \in M$ such that $|| \varphi'(v) - \varphi(v) || < \rho(\varphi)$, for all $v \in V(K)$ then φ' is again an imbedding.

(iii) ρ is the maximal function possessing properties (i), (ii).

PROOF. The proof is immediate once one proves:

LEMMA 2. S is compact. Which is trivial, for M is clearly compact and S closed. Then take $\rho(\varphi)=\delta(\varphi, S)$, and it is again immediate that ρ satisfies the requirements of lemma 1.

§ 5. Demonstration of the Equivalence Theorem.

THEOREM. Any isotopy f_t is covered by an ambient isotopy F_t.

The nature of the proof is to replace f_t by chains of more restricted kinds of isotopies, which reduces the problem to finding ambient isotopies covering these special isotopies. One proceeds to solve the problem for that special class.

DEFINITION 5. A *perturbation isotopy* φ_t is an isotopy such that:

$$||\varphi_1(v)-\varphi_0(v)||<\rho(\varphi_0) \text{ for all } v\in V(K).$$

DEFINITION 6. A *simple isotopy* φ_t is an isotopy such that φ_t is constant on every vertex in V(K), save one, v and the image of v under φ_t is a line segment in E^r.

LEMMA 3. Any isotopy f_t between K and K′ can be replaced by a chain of perturbation isotopies $f_t^{(1)}, \ldots, f_t^{(\nu)}$. That is: $f_t^{(i)}$ is an isotopy between $K^{(i-1)}$ and $K^{(i)}$ where:

$$K^{(0)}=K$$
$$K^{(\nu)}=K'.$$

LEMMA 4. Any perturbation isotopy φ_t can be replaced by a chain of simple isotopies

$$\varphi_t^{(1)}, \ldots, \varphi_t^{(\mu)}$$

which gives us:

LEMMA 5. Any isotopy f_t may be replaced by a chain of simple isotopies:

$$f_t^{(1)}, \ldots, f_t^{(\nu)}$$

Thus our original theorem reduces to the relatively

LOCAL PROBLEM: Given a simple isotopy f_t, find an ambient isotopy F_t covering it.

Clearly the solution of the local problem coupled with Lemma 5 provides a proof of the equivalence theorem.

§ 6. Reduction to perturbation isotopies (Proof of Lemma 3).

Define $\beta(t)=\rho(f_t)$, where ρ is as in Lemma 1. Then β is continuous and positive on I and hence it has a minimum M.

LEMMA 6. One may partition the interval I into

$$0=x_0, x_1, \ldots, x_\nu=1$$

so finely that $||f_{x_i}(v)-f_{x_{i+1}}(v)||<M$ for all $v\in V(K)$ and all $i=0, \ldots, \nu-1$. And therefore we would have

(A) $$||f_{x_i}(v)-f_{x_{i+1}}(v)||<\rho(f_{x_i}).$$

The proof of this comes simply from the continuity of f_t in t. Define

$$f_t^{(i)}=f_{x_i+t(x_{i+1}-x_i)}$$

and the chain of isotopies $f_t^{(1)}, \ldots, f_t^{(\nu)}$ can replace f_t as an isotopy between K and K′.

Also, (A) becomes: $||f_0^{(i)}(v)-f_1^{(i)}(v)||<\rho(f_0^{(i)})$ for all $v\in V(K)$.

CONCLUSION. Each $f_t^{(i)}$ is a perturbation isotopy.

§ 7. Reduction to simple isotopies (Proof of Lemma 4).

Let, then, f_t be a perturbation isotopy between K and K′. Order the vertices $\{v_1, \ldots, v_j, \ldots, v_n\}\in V(K)$. I'm going to define a chain of complexes

$$K=K_0,\ K_1, \ldots, K_n=K'$$

and a chain of simple isotopies

$$f_t^{(i)} \qquad i=1, \ldots, n$$

such that $f_t^{(i)}$ is an isotopy between K_{i-1} and K_i. Define $K^{(i)}$ to be the image of K under the piece-wise linear map $\mu^{(i)}$ which acts in this way on vertices:

$$\mu^{(i)}(v_j)=\begin{cases} v_j & \text{if } j>i \\ v'_j & \text{if } j\leqslant i \end{cases}$$

Let $f_t^{(i)}$ be the simple isotopy which acts as follows on the vertices:

$$\begin{aligned} &\{v'_1, \ldots, v'_{i-1}, v_i, \ldots, v_\nu\}=V(K^{(i)}) \\ &f_t^{(i)}(v'_j)=v'_j && j<i \\ &f_t^{(i)}(v_j)=v_j && j>i \\ &f_t^{(i)}(v_i)=(1-t)v_i+tv'_i. \end{aligned}$$

In order to show that:

LEMMA 7. $f_t^{(i)}$ is an isotopy between $K^{(i-1)}$ and $K^{(i)}$,

it remains to show :

LEMMA 8. $f_t^{(i)}$ is a combinatorial homeomorphism for each t.

PROOF. Let $f_0: K\rightarrow E^r$ be the simplicial imbedding which is the injection of K in E^r. Make $\sigma_t^{(i)}: K\rightarrow E^r$ as follows :

$$\sigma_t^{(i)}: v_j\rightarrow\begin{cases} v'_j & j<i \\ (1-t)v_j+tv'_j & j=i \\ v_j & j>i \end{cases}$$

Then $f_t^{(i)}\mu^{(i)}=\sigma_t^{(i)}$.

And so $f_t^{(i)}$ will be a combinatorial homeomorphism if $\sigma_t^{(i)}$ is.

But
$$\delta(\sigma_t^{(i)}, f_0) \leqslant \max_j ||v_j' - v_j||$$
$$\delta(\sigma_t^{(i)}, f_0) \leqslant \max_{v \in \mathrm{V(K)}} ||f_1(v) - f_0(v)||$$
$$\leqslant \rho(f_0).$$

The last inequality occurs since f_t is simple. Thus, by definition of ρ, $\sigma_t^{(i)}$ is a combinatorial homeomorphism, which proves the lemma.

§ 8. Solution of the Local Problem.

PROBLEM. If f_t is a simple isotopy between K and K′, find an ambient isotopy F_t covering f_t.

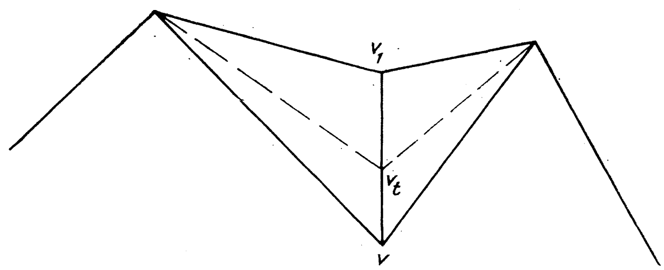

Fig. 1

Terminology: Let v be the vertex of K that f_t moves, $f_t(v) = v_t$. Call V the directed line generated by the vector $\vec{vv_1}$. Call V_x the unique line through $x \in \mathrm{E}^m$ parallel to V and $h : \mathrm{E}^m \to \mathrm{V}$ the map obtained by projecting E^m to V. Call H the hyperplane consisting of the « zeros » of h.

$$\begin{array}{l} \mathrm{E}^m \xrightarrow{h} \mathrm{V} \\ \downarrow \pi \\ \mathrm{H} \end{array}$$

Now, since [fig. 1] f_t is the identity except on St v, we would like to enclose St v in some region Ω so that we may define F_t to be the identity in the complement of Ω, and so that Ω would lend itself nicely to the construction of F_t on it.

§ 9. The region Ω.

There are four properties I shall require:

(i) Ω is a closed neighborhood of int St v, and a finite subcomplex of E^m.
(ii) $\Omega \cap \mathrm{K} = \mathrm{St}\, v$.
(iii) If $x \in \Omega$, $\mathrm{V}_x \cap \Omega$ consists of a single interval.
(iv) If $x \in \mathrm{St}\, v - \partial \mathrm{St}\, v$, $x \subset \mathrm{int}\, \mathrm{V}_x \cap \Omega$.
If $x \in \partial \mathrm{St}\, v$, $\mathrm{V}_x \cap \Omega = \{x\}$.

Call $\pi(\Omega)=\Omega^*$ and $\widetilde{\Omega}=\Omega^*\times I$. Let $I_x=V_x\cap\Omega$ for $x\in\Omega$, and $I_{\omega^*}=I_\omega$ for $\pi(\omega)=\omega^*\in\Omega^*$.

If I_0 is a line segment in E^r, let $M(I_0)$ be the simplicial complex of all simplicial homeomorphisms of I_0 leaving endpoints fixed. There is a chosen element in $M(I_0)$ (denoted 1), namely the identity homeomorphism.

There is a natural map

$$\eta : M(I)\rightarrow M(I_0)$$

which is a homeomorphism if I_0 is of positive length. If I_ω is as above, with $\omega\in\Omega^*$, let's denote this natural map by η_ω

$$\eta_\omega : M(I)\rightarrow M(I_\omega).$$

Define $\tau:\widetilde{\Omega}\rightarrow\Omega$ as follows:

$$\tau(\omega, t)=\eta_\omega(t).$$

Then I require that τ be a simplicial map :

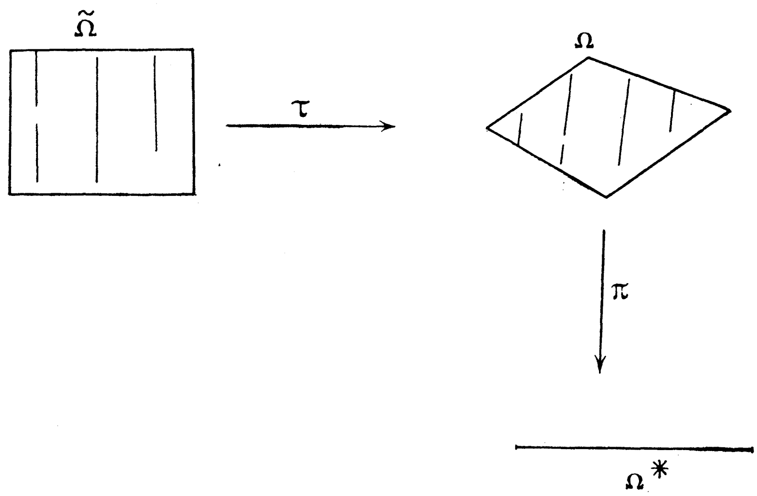

Fig. 2

As a consequence of the axioms, we may deduce these properties of τ :

a) $\tau|\widetilde{\Omega}-(\partial\Omega^*\times I)$ is a homeomorphism.

b) $\tau(\omega^*, t)$ is the unique point $\omega\in\Omega$ lying over ω^*, if $\omega^*\in\partial\Omega^*$.

c) Let $a(x)$, $b(x)$ be functions $a, b:\Omega\rightarrow V$ which assign to each $x\in\Omega$ the upper and lower endpoints (respectively) of the interval $V_x\cap\Omega$. Then a and b are combinatorial functions on Ω.

§ 10. Construction of Ω.

Let O be a finite subcomplex of E^m and a small closed neighborhood of $\overrightarrow{vv_1}$ such that the corresponding axioms (iii) and (iv) hold for it. Moreover, find it small enough so that

$$\Omega_0=J(O,\partial\,\mathrm{St}\,v)$$

has the property that

$$\Omega_0 \cap K = \mathrm{St}\, v.$$

(That such an O can be found is an immediate consequence of the stability lemma.)

LEMMA 9. Any such Ω_0 satisfies the four properties above.
(The proof is mere verification.) So, fix some such Ω.

§ 11. The problem remaining.

We must define $F : I \times E^m \to E^m$ covering $f : I \times K \to E^m$. Since f_t is the identity in the complement of Ω, we may choose F_t to be the identity on $E^m - \Omega$ (i.e. define) :

$$F : I \times (E^m - \Omega) \to E^m - \Omega$$

as

$$F : (t, x) \to x.$$

And it remains to define the isotopy

$$F : I \times \Omega \to \Omega$$

such that

a) $F_t | \partial\Omega = 1$.
b) F_t covers f_t ; i.e. $F_t(x) = f_t(x) = x_t$, if $x \in \mathrm{St}\, v$.

§ 12. V-homeomorphisms and V-isotopies.

DEFINITION 7. A *full subcomplex* $N \subseteq \Omega$, is one such that if $n \in N$, $I_n \subseteq N$ (i.e. $N = \pi^{-1}\pi N$).

DEFINITION 8. A (combinatorial) V-*homeomorphism* φ of a full subcomplex $N \subseteq \Omega$ onto itself is one such that:

a) φ leaves the endpoints of I_n fixed (for all $n \in N$).
b) φ satisfies the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & N \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \Omega^* & \end{array}$$

(Equivalently, $\varphi(I_x) \subseteq I_x$ for all $x \in N$.) A V-*isotopy* is an isotopy which is a V-homeomorphism at each stage.

LEMMA 10. Any V-isotopy $\varphi_t : \Omega \to \Omega$ must leave $\partial\Omega$ fixed.

PROOF. If φ is a V-homeomorphism $\varphi : \Omega \to \Omega$, then φ leaves ∂I_x fixed, and $\partial\Omega = \bigcup_{x \in \Omega} \partial I_x$. Define $H_v(N)$ to be the set of all combinatorial V-homeomorphisms of N, given the topology it inherits as a subset of M(N, N). There is a chosen element in $H_v(N)$ which I shall denote by 1. It is the identity V-homeomorphism. Let $I_v(N)$ be the topological space consisting of all V-isotopies of N, i.e. all paths in $H_v(N)$ beginning at 1. (By a *path* in $H_v(N)$, I shall always mean one which begins at 1.)

If N and N′ are full subcomplexes $N \subseteq N'$, there are natural restriction maps

$$\rho : I_v(N') \to I_v(N)$$
$$\rho : H_v(N') \to H_v(N)$$

defined in the obvious manner.

§ 13. A V-isotopy F′ covering *f*.

Let N be the full subcomplex of Ω 'generated' by St *v*

$$N = \bigcup_{x \in St\, v} I_x.$$

Define the V-isotopy $F'_t : N \to N$ as follows: for every I_x, let $a(x)$, $b(x)$ be its endpoints, as before.

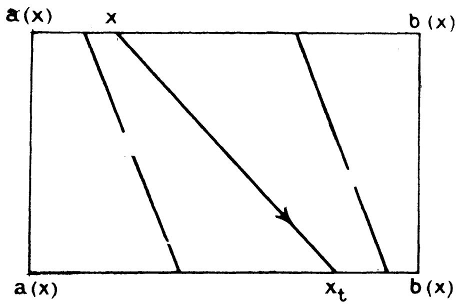

Fig. 3

Define F'_t on I_x by

$$F'_t : a(x) \to a(x)$$
$$F'_t : b(x) \to b(x)$$
$$F'_t : x \to f_t(x) = x_t$$

and extend to I_x by linearity (on each of the subintervals $[a(x), x]$ and $[x, b(x)]$.) F'_t is combinatorial since a and b are combinatorial functions and therefore a V-isotopy. Lastly, $F'_t | St\, v = f_t$, so F'_t covers f_t.

Lemma 10 implies that any V-isotopy F_t which extends F'_t must satisfy *a)*, *b)* of paragraph 12, and hence would yield the ambient isotopy sought for. After these remarks, it is clear that a solution to the local problem is an immediate corollary to:

(Extension Lemma): Let N be a full subcomplex of Ω. Then any V-isotopy of N extends to a V-isotopy of Ω ; or: $\rho : I_v(\Omega) \to I_v(N)$ is onto.

I devote myself, therefore, to a proof of the above.

§ 14. The spaces $X_v(L)$.

Let $EM(I_0)$ be the set of all polygonal paths in $M(I_0)$ beginning at 1. $EM(I_0)$ is endowed with the structure of a simplicial complex in a natural way. (This is standard. It is an infinite simplicial complex given the weak topology.)

LEMMA 11. $EM(I_0)$ is contractible.

And we have the following lemma:

LEMMA 12. $M(I_\omega)$ is a single point 1, if and only if $\omega \in \partial\Omega^*$.

Let $L \subseteq \Omega^*$ be a subcomplex.

$$\begin{array}{ccc} & M(I) \times \Omega^* & \\ & \nearrow^{K} \quad \searrow^{P} & \\ L - L \cap \partial\Omega^* & \longrightarrow & \Omega^* \end{array}$$

Let $X_v(L)$ be the set of all combinatorial cross-sections K over the finite open complex

$$L - L \cap \partial\Omega^*.$$

$X_v(L)$ is given the topology it inherits as a subset of $M(L - L\cap\partial\Omega^*, M(I) \times \Omega^*)$. There is a chosen cross-section in any $X_v(L)$, which consists of the identity function $1: I_l \to I_l$ for any $l \in L$. This I call the identity cross-section (1_L). Since $X_v(L)$ is a topological space, I may speak of paths on it. And again: by a path in $X_v(L)$ I'll always mean one beginning at 1_L.

§ 15. The Cross-Section Extension Lemma.

I state it in a particularly useful way:

Any path of cross-sections K_t in $X_v(L)$ for $L \subseteq \Omega^*$ may be extended to a path of cross-sections $\overline{K}_t$ in $X_v(\Omega^*)$. (Equivalently: there is a path $\overline{K}_t$ in $X_v(\Omega^*)$ such that $\overline{K}_t | L = K_t$, for each t.)

PROOF. Let $L \subseteq \Omega^*$, and $EM(I)$ as before. Thus, paths of cross-sections K_t in $X_v(L)$ correspond simply to cross-sections in the bundle:

$$\begin{array}{ccc} & EM(I) \times \Omega^* & \\ & {}^{K}\nearrow \quad \searrow^{P} & \\ L - L \cap \partial\Omega^* & \longrightarrow & \Omega^* \end{array}$$

Therefore the question of extension of paths in $X_v(L)$ to $X_v(\Omega^*)$ is a question of extension of K to $\overline{K}$:

$$\begin{array}{ccccc} & & EM(I) \times \Omega^* & & \\ & {}^{K}\nearrow & \uparrow^{\overline{K}} & \searrow^{P} & \\ L - L \cap \partial\Omega^* & \to & \Omega^* - \partial\Omega^* & \to & \Omega^* \end{array}$$

But contractibility of $EM(I)$ gives all. (A standard reference to such lemmas on the extension of cross-sections in fibre bundles is: Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press.)

§ 16. The Liaison.

PROPOSITION. If $N \subseteq \Omega$ is a full subcomplex, and $N^* \subseteq \Omega^*$ its image under π, then there is a homeomorphism $\lambda : X_v(L^*) \approx H_v(L)$ such that $\lambda(1_{L^*}) = 1_L$, and

$$\eta_\omega[\xi(\omega)] = \lambda(\xi) \,|\, I_\omega$$

for ξ a cross-section in $X_v(L^*)$.

PROOF.

1) To any cross-section $K \in X_v(L^*)$ I may associate a combinatorial V-homeomorphism $\overline{K} : (L^* - L^* \cap \partial\Omega^*) \times I \to (L^* - L^* \cap \partial\Omega^*) \times I$ where $P = L^* - L^* \cap \partial\Omega^*$ is considered as a subset of $\widetilde{\Omega}$. Define $\overline{K}(l \times I) = K(l)(I)$, where $K(l) \in M(I)$ is considered as a function of I. If K is a combinatorial cross-section with respect to the simplicial structure on $M(I) \times \Omega^*$, then $\overline{K}$ is a combinatorial V-homeomorphism.

2) To any combinatorial V-homeomorphism $\varphi : P \to P$, $P \subseteq \widetilde{\Omega} - I \times \partial\Omega^*$ one may associate a combinatorial V-homeomorphism φ' making the diagram below commutative:

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \tau(P) & \xrightarrow[\varphi']{} & \tau(P) \end{array}$$

(since τ is a homeomorphism on $\widetilde{\Omega} - I \times \partial\Omega^*$). Moreover, φ' is extendable to a V-homeomorphism φ''

$$\varphi'' : \tau(P) \cup \pi^{-1}(\partial\Omega^*) \to \tau(P) \cup \pi^{-1}(\partial\Omega^*)$$

(in fact, since I_ω is a single point for $\omega \in \partial\Omega^*$, a V-homeomorphism *must* be the identity map on $\pi^{-1}(\partial\Omega^*)$) and so we must define

$$\varphi'' | \pi^{-1}(\partial\Omega^*) = 1.$$

It is evident that φ'' so defined is a V-homeomorphism on the larger set.

3) Combining 1) and 2), one obtains a map

$$\lambda : X_v(L^*) \to H_v(L)$$
$$\lambda(K) = (\overline{K})'' | L$$

(Notice : $\overline{K}$ is defined on $(L^* - L^* \cap \partial\Omega^*) \times I$ and $\overline{K}'$ on $L - L \cap \pi^{-1}\partial\Omega^*$, and finally $\overline{K}''$ on $(L - L \cap \pi^{-1}\partial\Omega^*) \cup \pi^{-1}\partial\Omega^* = L \cup \pi^{-1}\partial\Omega^*$. So we must restrict $\overline{K}''$ back to L.)

It is straightforward that λ is a homeomorphism. (The construction of λ^{-1} is immediate.)

§ 17. **Conclusion.**

The proof of the extension lemma now follows easily:

LEMMA 12. Any path F_t in $H_v(N)$ extends to a path $\widetilde{f_t}$ in $H_v(\Omega)$ for $N \subseteq \Omega$, a full subcomplex.

PROOF. By the previous paragraph, $\lambda^{-1}(F_t)$ is a path in $X_v(N)$, and by the cross-section extension lemma, it extends to a path $\widehat{\lambda^{-1}F_t}$ in $X_v(\Omega^*)$. Finally, it is evident that

$$\widetilde{f_t} = \lambda(\widehat{\lambda^{-1}F_t}) \in I_v(\Omega)$$

is an extension of F_t.

And so, the main theorem follows.

§ 18. A Strengthening of The Main Theorem.

To avoid any undue complication in the proof of the main theorem, I stated it in its simplest, and therefore least useful, form. For later applications, I will need a strengthened statement of the theorem, which 'globalizes' the range space.

By a homogeneous bounded n-manifold I shall mean a finite complex, which is, topologically, a bounded n-manifold, and which has a group of combinatorial automorphisms which is transitive on interior points, and on each connected component of the boundary.

The strengthened theorem will involve replacing E^r by a general homogeneous manifold, W. I must stipulate, therefore, what I mean by an isotopy $f_t : K \to W$.

DEFINITION. Let K and W be finite complexes with particular triangulations. Then $\Psi : K \to W$ is a piecewise linear map if simplices of K are mapped linearly into simplices of W.

DEFINITION. An *Isotopy* $f_t : K \to W$ is a continuous family of homeomorphisms of K into W, piecewise linear for a fixed triangulation of K and W (independent, of course, of t), where $0 \leq t \leq 1$.

DEFINITION. An ambient isotopy $F_t : W \to W$ is a continuous family of combinatorial automorphisms of W such that $F_0 = 1$, and $F_t | K = f_t$ is an isotopy of K in W. F_t is then said to be an ambient isotopy covering f_t.

THE STRENGTHENED VERSION. Let M and W be bounded homogeneous manifolds (possibly not of the same dimensions), and let M′, W′ be the boundaries of M and W, respectively. Let f_t be an isotopy of M through W. Thus, M is to be regarded as a submanifold of W, and f_0 is the inclusion map.

Let us assume that $f'_t = f_t | M'$ is an isotopy of M′ through W′. Finally, let F'_t be an ambient isotopy of W′ covering f'_t.

Then, there is an ambient isotopy F_t covering f_t such that $F_t | M' = F'_t$.

I omit a proof of this elaboration. Such a proof may be obtained by merely restating the arguments of the main theorem in this more general setting. It would involve complications only in terminology.

Reçu le 16 *novembre* 1959.