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# ORTHOTOPY AND SPHERICAL KNOTS

*By* Barry MAZUR

The classical knot theory analyzes imbeddings of the one-sphere in three-space, and its methods conceivably apply, and generalize, to yield some information concerning $n$-sphere knots in $n+2$ space. Most crucial to the theory is the fact that in this range of dimensions, the complementary space of the knot is a delicate indicator of the equivalence class of the knot. (In the classical situation the fundamental group of the complementary space is enough to determine whether the knot is trivial.)

Deviate, however, from this range of dimensions: $n$-sphere knots in $n+2$ space, and the homotopy type of the complementary space gives absolutely no information. It is independent of the knot class.

Concerning ranges of dimension other than "$n$ in $n+2$" very little is known. For instance: It is unknown whether there are any non-trivial imbeddings of spheres in euclidean space, where the codimension of the sphere is different from 2.

There are, however, certain negative results if the dimension of the ambient euclidean space is sufficiently large with respect to the dimension of the sphere.

There is a theorem of Guggenheim:

Theorem: Any two imbeddings of $K^n$ in $E^r$ are isotopic if $n$ is the dimension of K, and

$$r \geq 2n+2.$$

And then, for the case of spheres, there is refinement, due (independently) to Milnor and Wu (unpublished):

Theorem: If $K^n$ is $S^n$, then the above theorem can be improved to read:

$$r \geq 2n+1.$$

The main theorem of this paper is along the lines of these two theorems. It says that for a broad range of dimensions $(r \geq (3n+5)/2)$ any $n$-sphere knot in $E^r$ (fulfilling a certain requirement of local smoothness) is $*$-trivial. (For a definition and treatment of $*$-triviality, see [**2**]. Briefly, a spherical knot is $*$-trivial if there is a homeomorphism of euclidean space onto itself sending the knot onto the standard imbedding of the sphere, such that the homeomorphism is combinatorial except possibly at one point.)

The paper is divided in two parts, the first being devoted to a study of orthotopy, and general position techniques. The second part uses this theory to prove the main theorem.

I am most thankful to Prof. Milnor who allowed me to see his manuscript.

## § 1. Terminology.

I rely upon [3], for general terminology, and permit myself the following loose usage: Homeomorphism will always mean combinatorial homeomorphism ; a subcomplex $A \subset E^r$ will mean that A is a complex whose imbedding homeomorphism

$$i : A \to E^r$$

is piecewise linear ; the "standard" $k$-sphere, $S^k \subset E^r$ is an image of $S^{r-1} \cap L^{k+1}$ under affine transformation, where $L^{k+1}$ is a $(k+1)$-dimensional subvector space of $E^r$, and $S^{r-1}$ is the unit sphere in $E^r$. The metric I shall place on $E^r$ is :

$$||x|| = \max|x_i| \quad \text{if} \quad x = (x_1, \ldots, x_r), \; x_r \in R.$$

A *homogeneous n-manifold* M will refer to a finite complex which is topologically an $n$-manifold, for which A(M), the group of combinatorial automorphisms of M, is transitive (i.e. usually called a combinatorial $n$-manifold).

By a *regular neighborhood*, A, of N(A), a subcomplex of B, I shall mean the closure of the second regular neighborhood (as defined in page 72 of Eilenberg-Steenrod ; I do not mean what they mean by regular neighborhood).

Let $S \subset E^r$ be any set. Then R(S) is the linear manifold spanned by S:

$$R(S) = (x \in E^r | x = \alpha s_1 + (1-\alpha)s_2, \; \alpha \in R, \; s_1, s_2 \in S).$$

If $K^n \subset L^m$ is an $n$-dimensional complex in an $m$-dimensional complex, then the codimension of K in L is:

$$\operatorname{cod} K = m - n.$$

If X is a metric space (i.e. if $X = E^r$) then $d(A, B)$ is the distance from A to B, where A and B are compact sets. Also, let $p \in E^r$, then $B_\varepsilon(p) = (x \in E^r | d(x, p) \leq \varepsilon)$.

Define $E^r_\pm \subset E^r$ to be

$$E^r_+ = [(x_1, \ldots, x_r) \in E^r | x_r \geq 0]$$
$$E^r_- = [(x_1, \ldots, x_r) \in E^r | x_r \leq 0]$$

and they are called the upper and lower half-planes, respectively.

## § 2. The definition of knot equivalence.

I will say that two subcomplexes $K \subset E^r$, $K' \subset E^r$ are *equivalent* (and I denote this by: $K \sim K'$) if there is a homeomorphism

$$T : E^r \to E^r$$

such that

$$T : K \to K'$$

is a homeomorphism of K onto K′. Thus the question of classification of equivalence classes of imbeddings of K in $E^r$ is the classification of the combinatorial type of the "relative" manifolds ($E^r$, K). Equivalence is just what was called an ambient homeomorphism equivalence in [3]. A fact used most frequently in this paper is an immediate corollary of the main theorem of [3]:

THEOREM 1. If $f_t : K \rightarrow E^r$ is an isotopy between K and K′ then $K \sim K'$.

## § 3. Virtual Dimension.

In proving and applying many of the "general position" lemmas that will be developed (all of which involve consideration of the dimension of complexes), I will use a systematic and obvious alteration of the concept of dimension (virtual dimension) which will never be larger than the usual dimension of K (most often smaller), thereby "strengthening" those general position arguments which depend upon the dimension of K being small.

DEFINITION 1. Let $L, K \subseteq E^r$ be two complexes in $E^r$. I will say that the virtual dimension of K with respect to L is less than or equal to $k$ (in symbols: virt $\dim_L(K) \leq k$) if: There is a $k$-dimensional complex P, and a sequence of regular neighborhoods of P: $M_0 \supset M_1 \supset \ldots$, such that $\bigcap_{i=0}^{\infty} M_i = P$, such that there is a homeomorphism of $E^r$ leaving L fixed which brings K into any $M_i$.

If N is a regular neighborhood of K, and L is $E^r$—N, our notation can be reduced to: virt $\dim_L K =$ virt dim K. Notice:

$$\text{virt } \dim_L K \leq \dim K,$$

and that the following three conditions are equivalent:

(i) virt $\dim_L K = 0$
(ii) virt $\dim_K L = 0$
(iii) K and L are *unlinked.*

The generalization of results stated in terms of dimension to corresponding results stated in terms of virtual dimension, being rather straightforward, I henceforth adopt the policy of proving all results merely for dimension, and leaving the transition to virtual dimension to the reader.

For later application of virtual dimension I point out an obvious lemma:

LEMMA 1. Let U be a regular neighborhood of V. Then

$$\text{virt dim } U \leq \dim V$$

## § 4. The Problems of Local Smoothness.

The most obvious distinction between combinatorial imbeddings and differentiable ones is the possibility of a certain local unsmoothness to occur in the combinatorial situation which has no counterpart in the differentiable. The simplest example of these phenomena is obtained by taking a knotted $S^1 \subset E^3$, and considering $E^3 \subset E^4$ imbedded as a linear hyperplane. Then take a point $P \in E^4$ outside of $E^3$, and draw all line segments from $p$ to points on $S^1 \subset E^3$. The locus, $D^2 \subset E^4$, of these line segments is a combinatorial 2-cell, which is "knotted" in $E^4$. A clear manifestation of its "knottedness" is: If $B = B(p)$ is any small ball drawn about $p$, and $S = \partial B \cap D^2$, then S is homeomorphic with $S^1$, and $S \subset \partial B$ is knotted. Such a phenomenon could not occur if $D^2$ were a differentiable disc imbedded in $E^4$. I should like to rule out the possibility of severe local unsmoothness in the imbeddings which I consider.

Situations such as the above are eliminated by requiring that the imbedding be *locally unknotted* (for the definition ; see [**2**]).

More convenient for the purpose of this paper is a different local smoothness condition:

Definition 2. A subcomplex $K \subset E^r$ is called *homogeneously imbedded* (or just: *homogeneous*) if for any continuous family of homeomorphisms

$$P_t : K \to K$$

such that $P_0$ is the identity, and for any regular neighborhood N of K, there is a homeomorphism

$$P : E^r \to E^r$$

such that $P | E^r - N = 1$ and $P | K = P_1$.

I don't know whether or not the two conditions local unknottedness and homogeneity are the same. That neither restriction is very restrictive may be seen by the following heuristic statement which would lead to unwarranted digression, if I were to attempt to make it precise. Let $\Sigma$ be a combinatorial imbedding of a $k$-sphere in $E^r$ which is a "very close approximation" to S, a differentiable imbedding. Then $\Sigma$ is both homogeneous and locally unknotted.

## § 5. The knot Semi-Groups.

There is a natural additive structure to the set of all equivalence classes of $n$-manifolds combinatorially embedded in $E^r$ (see [**1**] for precise definition), where if $M_0$ and $M_1$ are two knotted $n$-manifolds in $E^r$, $M_0 + M_1$ is essentially obtained by displacing the $M_i$ so that one lies in the upper half-plane and the other in the lower half-plane, then join the $M_i$ by removing an $n$-simplex $\Delta_i$ from each, and attaching a tube, $S^{n-1} \times 1$ such that

$$S^{n-1} \times 0 = \partial \Delta_0 \subseteq M_0$$
$$S^{n-1} \times 1 = \partial \Delta_1 \subseteq M_1.$$

This process is standard, and I call the resulting semi-group of knots $K_n^r$

There are sub-semi-groups that should be singled out:

1) $\Sigma_n^r$: the semi-group of spherical knots;
2) $S_n^r$: the semi-group of locally unknotted spherical knots;
3) $H_n^r$: the semi-group of homogeneous spherical knots (See [2]).

## § 6. General Position and Orthotopy — Part I.

Although our ultimate concern will be with isotopies, we shall have to deal with something not quite as restrictive in search of isotopy.

DEFINITION 3. *A local isotopy* $\varphi_t : K \to E^r$, will be a map $\varphi : I \times K \to E^r$ which is simplicial for a fixed subdivision of K and for each $t$. It is nonsingular on each simplex of K, for each $t$, and piecewise linear in $t$ for fixed $p$, the subdivision of I being independent of $p$.

DEFINITION 4. *An orthomorphism* $\varphi : K \to E^r$ is a simplicial map, nonsingular on each simplex in K, and satisfies the following condition (which assures that self-intersections of K are not too high in dimension):

If $\Delta_1$, $\Delta_2$ are distinct simplices in K such that $\varphi(\text{int } \Delta_1) \cap \varphi(\text{int } \Delta_2)$ is non-empty, then,

$$\text{codim } R(\Delta_\alpha, \Delta_\beta) \leq 1.$$

DEFINITION 5. An *orthotopy* $\varphi_t = K \to E^r$ is (i) an orthomorphism for each $t$ (ii) a local isotopy.

Essential to an analysis of the problem of knotted spheres in euclidean space is the following generalization of a theorem of Guggenheim.

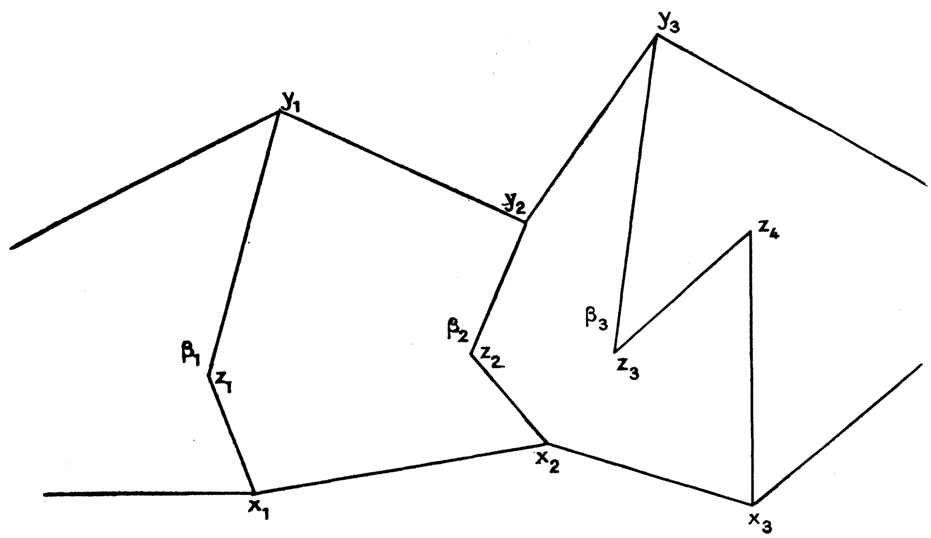


Fig. 1

THEOREM: Let K and K′ be simplicially isomorphic complexes $\psi : K \to K'$ imbedded piece-wise linearly in $E^r$. There is an orthotopy $\psi_t$ between K and K′. More precisely, there is an orthotopy $\psi_t : K \to E^r$ such that $\psi_0 = 1$ and $\psi_1 = \psi$.

PROOF. Draw polygonal arcs $\beta_i$ from the vertices $w_i$ of K to the corresponding vertices $\psi(w_i) = w'_i$ of K'. See Fig. 1.

## § 7. Perturbation into General Position.

Let V be the set of all vertices of the $\beta_j$'s. Let $P_v$ for $v \in V$ stand for the set of all hyperplanes spanned by subsets of vertices in $V - \{v\}$.

Notice that $P_v$ is always a finite union of hyperplanes, hence a closed $(r-1)$-dimensional set.

DEFINITION 6. I shall say: Figure 1 is in general position if $v \notin P_v$ for all $v \in V$. It will be a great simplification if the problem of proving the orthotopy theorem reduces to proving it for the case when Figure 1 is in general position.

This will be so if the following lemma is proven.

LEMMA 2. It is possible to "put" the entire array $K \cup K' \cup (\cup_i \beta_i)$ of Figure 1 in general position by an arbitrarily slight isotopy.

PROCEDURE: Order the vertices of V, $V = (v_1, \ldots, v_q)$. One can find a $v_1^{(1)}$ arbitrarily close to $v_1$, so that $v_1^{(1)} \notin P_{v_1}$. (For $P_{v_1}$ is of codimension one in $E^r$).

LEMMA 3. There is an isotopy $\psi_t^{(1)}$ of the array of figure 1 which leaves all vertices other than $v_1$ fixed, and brings $v_1$ to a $v_1^{(1)}$ such that $v_1^{(1)} \notin P_{v_1}$. In fact, $\psi^{(1)}$ is the identity on simplices outside of St $v_1$

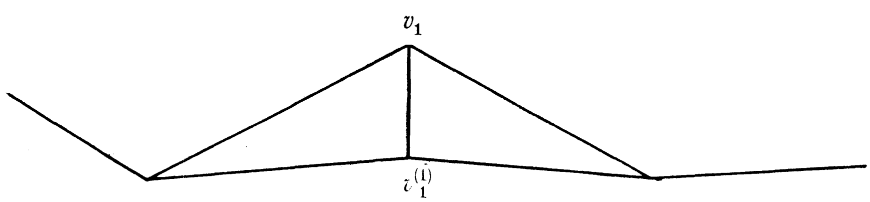


Fig. 2

and brings $\mathrm{St}\, v_1 = J(v_1, \partial \mathrm{St}\, v_1)$ piecewise-linearly to $J(v_1^{(1)}, \partial \mathrm{St}\, v_1)$.

Now we study the new array, as perturbed by $\psi_1$. I will speak of $V^{(1)}$ as the new set of vertices $(V - \{v_1\}) \cup \{v_1^{(1)}\}$, and of $P_v^{(1)}$ as the union of hyperplanes generated by sets of points in $V^{(1)} - \{v\}$.

So, as matters nowst and we have $v_1^{(1)} \notin P^{(1)}_{v_1^{(1)}}$. The next stage in the process is similar. We must find a replacement $v_2^{(2)}$ for $v_2$ so close to $v_2$ that an isotopy $\psi_t^{(2)}$ can be found which leaves all vertices of the array other than $v_2$ fixed and sends $v_2$ linearly to $v_2^{(2)}$ and that $v_2^{(2)} \notin P^{(1)}_{v_2}$. But we need one more thing as well. We need $v_2^{(2)}$ to be taken so close to $v_2$ that the isotopy $\psi_t^{(2)}$ doesn't destroy the fact that $v_1^{(1)} \notin P^{(1)}_{v_1^{(1)}}$, since $P^{(1)}_{v_1^{(1)}}$ changes under the isotopy $\psi_t^{(2)}$. But it is clear that it can be so arranged. Thus we obtain a new array, $P^{(2)}_{v_i^{(2)}}$, and repeat the process.

And so it goes. At the $i^{th}$ stage, it is a question of isotopically perturbing $v_i^{(i-1)}$ to $v_i^{(i)}$ where $v_i^{(i)} \notin P^{(i)}_{v_i^{(i)}}$, and so slightly that one's previous handiwork:

$$v_k^{(i)} \notin P^{(i)}_{v_k^{(i)}} \qquad i > k$$

remains intact. The procedure ends with its final array in general position, proving the lemma.

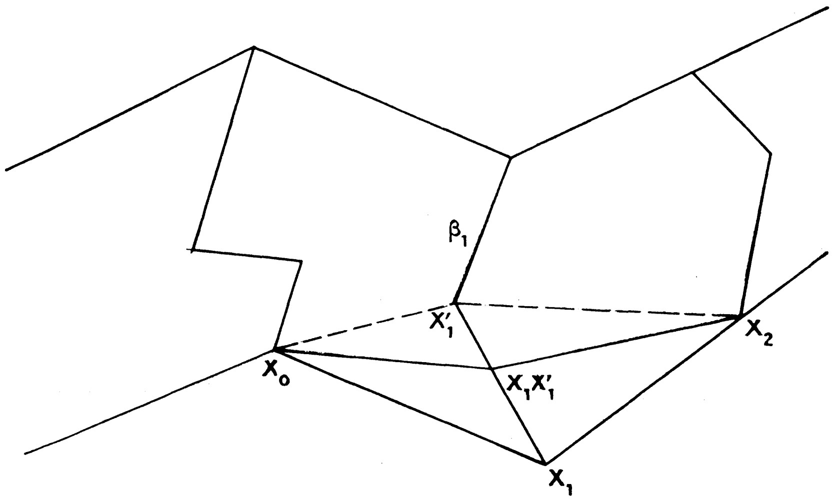


Fig. 3

The orthotopy $\varphi_t$ is obtained, step by step, climbing up the $\beta_j$'s. A typical step would consist in "replacing" one vertex, $x_1$ by the succeeding vertex, $x_1'$ on the arc $\beta_1$. In this manner, the orthotopy $\varphi_t$ will be obtained as the composite of a chain of orthotopies $\psi_t^{(i)}$, $i = 1, \ldots, \nu$, $\psi_t^{(i)}$ will be an orthotopy of the complex $K^{(i-1)}$ to $K^i$, where

$$K^0 = K$$
$$K^\nu = K'$$

and all $K^i$ will have as vertices only those in the array, $K^i$, being obtained from $K^{i-1}$ by chosing one vertex $x \in K^{i-1}$ and replacing the vertex $x$ by its successor $x'$ on the path of the array $\beta_x$, which contains $x$. This can be done, as long as $x$ is not the "last" vertex of $\beta_x$ ; or equivalently as long as $x \notin K'$. ("Successor" means in the direction towards $K'$ along $\beta_x$.) Thus the local isotopy $\psi_t^x$ which sends $x$ to $x'$ may be defined by its action on the vertices of $K^{i-1}$ (and extended piece-wise linearly to $K^{i-1}$):

$$\psi_t^x(v) = v \qquad \text{if } v \in V(K^{i-1})$$
$$\psi_t^x(x) = (1 - t)x + tx' \qquad v \neq x$$

and $K^i$ is, of course, $\psi_1^x(K^{i-1})$. Since the number of vertices of the array is finite this process must terminate, if repeated enough, with a $K^\nu$ such that all vertices of $K^\nu$ are in $K'$, i.e. $K^\nu = K'$. Thus the chain of local isotopies $\psi_t^{(1)}, \ldots, \psi_t^{(\nu)}$ will yield an orthotopy $\varphi_t$ between K and $K'$ if they themselves are orthotopies.

LEMMA 4. The $\psi_t^{(i)}$ are orthotopies.

Let $\psi_t = \psi_t^{(i)}$, dropping the unnecessary superscript. I shall prove:

LEMMA 5. For each $t$, $0 < t \leq 1$, $\psi_t$ is an orthomorphism.

Which clearly implies lemma 4 above, and by induction I assume $\psi_0$ to be an orthomorphism already.

Call $\Delta^t = \psi_t(\Delta)$ for $\Delta$ a simplex in $\widetilde{K} = K^{(i)}$. Assume that $\psi_t$ fails to be an orthomorphism for some $t > 0$. So: int $\Delta_1^t$, int $\Delta_2^t$ intersect, where $\Delta_i^t = \psi_t(\Delta_i)$ and $\Delta_1$, $\Delta_2$ are distinct simplices of K, yet:

$$\text{cod}\,[R(\Delta_1^t,\ \Delta_2^t)] \geq 2$$

Let $x \in \widetilde{K}$ be the unique vertex moved by $\psi_t$, and, by our convention,

$$\psi_t(x) = x_t.$$

I must distinguish between two cases:

I) $\qquad \Delta_1^t \in \text{St}(x_t),\ \Delta_2^t \notin \text{St}(x_t)$

II) $\qquad \Delta_1^t,\ \Delta_2^t \in \text{St}(x_t).$

CASE I: Let $\Delta_2^t = \Delta_2$ be the simplex unmoved by $\psi_t$. The assumption

$$\text{cod}\,[R(\Delta_1^t,\ \Delta_2)] \geq 2$$

gives us

$$\text{cod}\,[R(\Delta_1^t,\ \Delta_2,\ x_1)] \geq 1.$$

I make the notational convention: $\hat{\Delta}^t \subset \Delta^t$ is the face in $\Delta^t$ opposite the vertex $x_t$, for $\Delta_t \subset \text{St}(x_t)$. Thus:

*a)* $\qquad \hat{\Delta}^t \subset \partial\,\text{St}(x_t)$

*b)* $\qquad \hat{\Delta}^t = \hat{\Delta}^0$ for all $0 \leq t \leq 1$.

A useful fact for the arguments that follow is the obvious:

LEMMA 6. Let S be a set, $S \subset E^r$, $x, y \in E^r$ and $a \in R$, $a \neq 1$, then:

$$ax + (1-a)y \in R(S),\ x \in R(S)$$

implies $y \in R(S)$.

A) Assuming (I), then $t \neq 1$.

PROOF: If $t = 1$, then

$$R(\hat{\Delta}_1^t,\ \Delta_2) \ni x_1,$$

for let $\alpha_1 \in$ int $\Delta_1^1, \alpha_2 \in$ int $\Delta_2$ and $\alpha_1 = \alpha_2 = \lambda\xi_1 + (1-\lambda)x_1$, for $\xi_1 \in \hat{\Delta}_1^1$, and $0 < \lambda < 1$. Thus $\alpha_2 = \lambda\xi_1 + (1-\lambda)x_1 \in R(\hat{\Delta}_1^1, \Delta_2)$ and $\xi_1 \in R(\hat{\Delta}_1^1, \Delta_2)$ but, by Lemma 6, one has $x_1 \in R(\hat{\Delta}_1^1, \Delta_2)$; however

$$\text{cod}\,R(\hat{\Delta}_1^1,\ \Delta_2) \geqslant \text{cod}\,R(\Delta_1^1,\ \Delta_2) \geq 2$$

therefore

$$x_1 \in R(\hat{\Delta}_1^1, \Delta_2) \subset P_{x_1},$$

which contradicts general positionality. Therefore $0 < t < 1$.

B) $R(\Delta_1^0, \Delta_2) \subset R(\Delta_1^t, \Delta_2, x_1)$.

To demonstrate this, it suffices to show

$$x_0 \in R(\Delta_1^t, \Delta_2, x_1).$$

But $x_1, x_t \in R(\Delta_1^t, \Delta_2, x_1)$ and since $x_t = tx_1 + (1-t)x_0$ and $t \neq 1$, Lemma 6 again gives

$$x_0 \in R(\Delta_1^t, \Delta_2, x_1).$$

Also, $x_1 \in R(\Delta_1^0, \Delta_2)$: Because if $\alpha_1 \in \operatorname{int} \Delta_1^t, \alpha_2 \in \operatorname{int} \Delta_2, \alpha_1 = \alpha_2$, then $\alpha_2 = \alpha_1 = \lambda\xi_1 + (1-\lambda)x_1$, $\xi_1 \in \Delta_1^0$ and $0 < \lambda < 1$, but

$$\operatorname{cod} R(\Delta_1^0, \Delta_2) \geq \operatorname{cod} R(\Delta_1^t, \Delta_2, x_1)$$
$$\geq \operatorname{cod} R(\Delta_1^t, \Delta_2) - 1 \geq 1.$$

Therefore

$$x_1 \in R(\Delta_1^0, \Delta_2) \subset P_{x_1}$$

again contradicting general positionality.

Case II: Assume again that $\psi_t$ is not an orthomorphism for some $t > 0$.

There are simplices $\Delta_1^t, \Delta_2^t$ such that:

1) $$\alpha^t \in \operatorname{int} \Delta_1^t \cap \operatorname{int} \Delta_2^t$$

2) $$\operatorname{cod} R(\Delta_1^t, \Delta_2^t) \geq 2.$$

A) $$R(\Delta_1^0, \Delta_2^0) \subseteq R(\Delta_1^t, \Delta_2^t, x_0),$$

an evident fact, implying

$$\operatorname{cod} R(\Delta_1^0, \Delta_2^0) \geq \operatorname{cod} R(\Delta_1^t, \Delta_2^t, x_0) \geq 1.$$

B) In fact:

$$\operatorname{cod} R(\Delta_1^0, \Delta_2^0) \geq 2.$$

For, if $\operatorname{cod} R(\Delta_1^0, \Delta_2^0) = 1$,

$$R(\Delta_1^0, \Delta_2^0) = R(\Delta_1^t, \Delta_2^t, x_0)$$

and $x_1 \in R(\Delta_1^t, \Delta_2^t, x_1)$.

Since $x_t, x_0 \in R(\Delta_1^t, \Delta_2^t, x_0)$ and $x_t = (1-t)x_0 + tx_1$, $t \neq 0$, this implies:

$$x_1 \in R(\Delta_1^0, \Delta_2^0) \subset P_{x_1}$$

contradicting general positionality.

Let $\alpha_i^0 \in \operatorname{int} \Delta_i^0$ be the elements for which $\psi_t(\alpha_i) = \alpha^t$.

C) $\alpha_1^0 \neq \alpha_2^0$. For, by (B), $\operatorname{cod} R(\Delta_1^0, \Delta_2^0) \geq 2$, and $\psi_0$ being an orthomorphism, $\operatorname{int} \Delta_1 \cap \operatorname{int} \Delta_2$ is empty. Let $\alpha_i^0 = \delta_i^0 + \lambda_i x$, where

$$\left(\frac{1}{1-\lambda_i}\right) \delta_i^0 \in \hat{\Delta}_i^0.$$

Then:

$$\psi_t(\alpha_i) = \alpha^t = \delta_i^0 + \lambda_i x^t$$
$$= \delta_i^0 + \lambda_i[(1-t)x_0 + tx_1]$$

giving us

D) $$\delta_1^0 - \delta_2^0 = (\lambda_2 - \lambda_1)[(1-t)x_0 + tx_1].$$

Also:

E) $$0 \neq \alpha_1^0 - \alpha_2^0 = (\delta_1 - \delta_2) + (\lambda_1 - \lambda_2)x_0.$$

F) $$\lambda_1 - \lambda_2 \neq 0.$$

If $\lambda_1 = \lambda_2$, one would have, by D),

$$\delta_1^0 = \delta_2^0, \ \alpha_1^0 = \alpha_2^0$$

which would contradict E). So F) follows.

G) $$\frac{\delta_1^0 - \delta_2^0}{\lambda_2 - \lambda_1} \in R(\Delta_1, \Delta_2)$$

for

$$\frac{\delta_i^0}{1-\lambda_i} \in \hat{\Delta}_i^0 \subset R(\Delta_1, \Delta_2).$$

H) $x_1 \in R(\Delta_1, \Delta_2)$, for D) and G) yield

$$\frac{\delta_1^0 - \delta_2^0}{\lambda_2 - \lambda_1} = (1-t)x_0 + tx_1 \in R(\Delta_1, \Delta_2).$$

clearly $x_0 \in R(\Delta_1, \Delta_2)$, and by the induction assumption, $t \neq 0$; H) follows by the application of Lemma 6. But H) contradicts general positionality, since

$$x_1 \in R(\Delta_1, \Delta_2) \subset P_{x_1}.$$

So the orthotopy theorem is proved. With just a bit more care in the proof of the theorem, we could have proved this slightly strengthened version which will be needed later.

Theorem (Extension). Let $F_0$, $F_1$ be imbeddings (or merely orthomorphisms, for that matter) of K in $E^r$. Let $L \subset K$ be a subcomplex and

$$f_t : L \to E^r$$

an orthotopy such that

$$f_0 = F_0 | L, f_1 = F_1 | L.$$

Then there is an orthotopy $F_t$ between $F_0$ and $F_1$ such that

$$F_t | L = f_t.$$

## § 8. The Singularity Locus.

Definition 7. The pre-locus V of an orthomorphism $f: K \to E^r$ is the set of multiple points of $f$ in K. That is,

$$V = \{k \in K \mid \exists k' \neq k, f(k') = f(k)\}.$$

Clearly V is a subcomplex of K. The *locus* L is the image of the pre-locus in $E^r$,

$$L = f(V).$$

The *pre-locus* (and *locus*) of an orthotopy $f_t$, $0 \leq t \leq 1$, is the union of all pre-loci $V_t$ (loci) of the orthomorphisms $f_t$ for each $t$, $0 \leq t \leq 1$:

$$V = \bigcup_{t \in I} V_t.$$

And again, V is a subcomplex of K.

LEMMA 7. Let $f: K^n \to E^r$, where $K^n$ is an $n$-complex, be an orthomorphism, and V its singularity pre-locus. Then

$$\dim V \leq 2n - r + 1.$$

If $f_t: K^n \to E^r$ is an orthotopy, and W its pre-locus, then

$$\dim W \leq 2n - r + 2.$$

PROOF: Let $p \in V$. Then $p \in \Delta_1$, $p \in \Delta_2$, where $\Delta_i$ are the images of distinct simplices of K under $f$.

$$\dim \Delta_1 \cap \Delta_2 \leq \dim R(\Delta_1) \cap R(\Delta_2) = \dim R(\Delta_1) + \dim R(\Delta_2) - \dim R(\Delta_1, \Delta_2).$$

But if $\Delta_1$, $\Delta_2$ have an interior intersection at all, $\dim R(\Delta_1, \Delta_2) \geq r - 1$. So

$$\dim \Delta_1 \cap \Delta_2 \leq \dim R(\Delta_1) + \dim R(\Delta_2) - (r - 1)$$

and since $\dim R(\Delta_i) \leq n$

$$\dim \Delta_1 \cap \Delta_2 \leq 2n - r + 1.$$

COROLLARY 1 (Guggenheim). Any two imbeddings $\varphi_0$, $\varphi_1: K^n \to E^r$ are isotopic if $r \geq 2n + 2$.

For by the orthotopy theorem, there is an orthotopy $\varphi_t$ between $\varphi_0$ and $\varphi_1$. And by the above the dimension of the singularity locus of each orthomorphism $\varphi_t$ is $-1$, or the singularity locus of each $\varphi_t$ is empty. Therefore, the orthotopy is an isotopy.

COROLLARY 2. Let L′, L, N be subcomplexes of $E^r$ and $\varphi: L \to L'$ an isomorphism leaving $L \cap N$ fixed. Then there is an orthotopy $\varphi_t$ from L to L′, leaving $L \cap N$ fixed ($\varphi_1 = \varphi$), such that if $\varphi_t(\Delta)$ and $\Delta'$ have a non-empty interior intersection, for $\Delta \in L - L \cap N$, and $\Delta' \in N$, then $\operatorname{cod} R(\varphi_t(\Delta), \Delta') \geq 1$.

PROOF: Apply the orthotopy theorem with $K = L \cup N$, $K' = L' \cup N$.

COROLLARY 3. In the situation of the above corollary, if

$$r \geq \dim L + \dim N + 2$$

the orthotopy $\varphi_t$ can be chosen to be an isotopy of the complex $L \cup N$. Thus $\varphi_t(l) \in N$ implies $l \in N$ for $l \in L$.

This corollary is interpreted as saying that L and N are unlinkable in $E^r$ if

$$r \geq \dim L + \dim N + 2.$$

## § 9. Some Necessary Facts Concerning General Positionality.

1) *Stability of Orthotopy.*

LEMMA 8. Let $\varphi_t : K \to E^r$ be an orthotopy; then there is a number $\rho(\varphi_t) > 0$ such that if $\varphi'_t : K \to E^r$ is a continuous family of simplicial maps, such that

$$||\varphi_t(v) - \varphi'_t(v)|| < \rho(\varphi_t)$$

for all $t$, and all vertices $v \in V(K)$, then $\varphi'_t$ is again an orthotopy.

The proof is a rewording of the proof of Lemma 1 of [3]. I omit it.

2) Even stronger than an orthomorphism is a map $f : K \to E^r$ such that: 1) $f$ is a simplicial map non-singular on each simplex of K; 2) if int $\Delta_1$ and int $\Delta_2$ intersect, for $\Delta_1$, $\Delta_2$ distinct simplices of K, then $R(\Delta_1, \Delta_2) = E^r$. Just to give such an $f$ a name, I call it *maximally transverse.*

LEMMA 9. If $f : K \to E^r$ is a simplicial map, it is approximable arbitrarily closely by a maximally transverse map

$$f' : K \to E^r.$$

Moreover, if $L \subset K$ and $f|L$ is already maximally transverse, one can have $f'|L = f|L$.

The method of proof has been displayed sufficiently often that I omit the precise proof (or statement) of this lemma.

COROLLARY 4. Any map $f : K^n \to E^r$ for $r \geq 2n + 1$ may be approximated arbitrarily closely by an imbedding $f' : K \to E^r$.

COROLLARY 5. Let $A, B \subset C$ be subcomplexes and virt $\dim_B A +$ virt $\dim_A B + 1 \leq r$. Let

$$\varphi_1 : A \to E^r$$
$$\varphi_2 : B \to E^r$$

be simplicial maps such that $\varphi_1 | A \cap B = \varphi_2 | A \cap B$. Then there is a simplicial map $\varphi$ : $A \cup B \to E^r$ such that $\varphi | A = \varphi_1$, $\varphi | B$ approximates $\varphi_2$, and $\varphi(B - B \cap A)$ is disjoint from $\varphi(A)$. If the $\varphi_i$ were homeomorphisms then so will $\varphi$ be.

Define the map $\varphi' : A \cup B \to E^r$ to be the composite of $\varphi_1$ on A and $\varphi_2$ on B. Then $\varphi' | A$ is already maximally transverse. Approximate $\varphi'$ by $\varphi$, a maximally transverse map $\varphi : A \cup B \to E^r$, such that $\varphi | A = \varphi' | A$, and so close to $\varphi'$ so that $\varphi | B$ is still an imbedding of B (applying the Stability Lemma for imbeddings, Lemma 1 of [3]). $\varphi$ will be an imbedding of $A \cup B$ if

$$\dim A + \dim B + 1 \leq r$$

and, after a simple modification (which I omit) it would be an imbedding even if

$$\text{virt} \dim_B A + \text{virt} \dim_A B + 1 \leq r.$$

## § 10. Part II : The Main Theorem.

I shall use the tools developed in Part I to prove the following theorem: Let S be a $k$-sphere knot in $E^r$ which is homogeneous. Then if

$$r \geq \frac{3k+5}{2}$$

S is invertible. Or, in terms of the semi-groups of [3] in the same range of dimensions, as above

$$H_k^r = G_k^r$$

Coupled with the main theorem in [1], one has: In the same range of dimensions, all homogeneous knots are $*$-trivial. Indeed, with no further complication, let $f_t$ be an orthotopy between two manifolds K and K′ in $E^r$ satisfying assumption (o):

(o) 1) $r \geq \frac{3k+5}{2}$.

2) K is homogeneous.

3) The singularity locus $V \subset K$ of $f_t$ can be brought into a $k$-cell $\Delta^k \subseteq K$ by a continuous family of homeomorphisms $h_t : K \to K$ such that $h_0 = 1$, and $h_1 : V \to \Delta^k$. Clearly condition 3) follows if K is a sphere.

THEOREM 2. If $f_t$ is an orthotopy between K and K′ satisfying condition (o), then :

$$K' = K + S$$

where S is a spherical knot.

The paragraph titles together with the accompanying diagrams provide a rough outline of the method of proof of the theorem.

## § 11. Isolation of the Singularity Locus.

Let $f_t$ be an orthotopy of K′ to K with singularity pre-locus $V \subset K$. Thus

$$f_t : K \to E^r,$$

$f_1 : K \to K \subset E^r$ is the natural injection, and $f_0 : K \to K' \subset E^r$. I should like to find a neighborhood $U_1$ of $f(I \times V) \subset E^r$ for which there exists a regular neighborhood N of V in K, such that

$$f_t : \partial N \to \partial U$$
$$f_t : N \to U.$$

Then U would serve to isolate that part of the orthotopy which had singularities. This would allow us to redefine $f_t$ on U so that the newly-defined $f_t^*$ would have no singularity on U. The resulting imbedding $K^* = f_1^*(K)$ would be equivalent to K′ and "differ from" K merely in U, a set of low virtual dimension,

$$\text{virt dim } U \leq \dim (V \times I).$$

The proof of the main theorem would then follow fairly easily.

## § 12. Regularizing the Orthotopy.

In order to carry out this program one must first replace the orthotopy $f$ by a close approximation $f'$, which has the property that $f'(\mathrm{I}\times\mathrm{N})$ is disjoint from $f'(\mathrm{I}\times(\mathrm{K}-\mathrm{N}))$ for N some regular neighborhood of the singularity prelocus V.

LEMMA 10. There is an orthotopy $f':\mathrm{I}\times\mathrm{K}\rightarrow\mathrm{E}^r$ arbitrarily close to $f$ which still has pre-locus V, and has the property that: $f'(\mathrm{I}\times\mathrm{V})$ is disjoint from $f'(\mathrm{I}\times(\mathrm{K}-\mathrm{V}))$.

PROOF: Apply corollary 5 of section 9 where $\mathrm{A}=f'(\mathrm{I}\times\mathrm{N})$, $\mathrm{B}=f'(\mathrm{I}\times\mathrm{K}-\mathrm{N}))$ in the notation of the corollary. One must check that

$$\text{virt dim}_{\mathrm{B}} f'(\mathrm{I}\times\mathrm{N})+\text{virt dim B}+1\leq r$$

Or:

$$2k-r+3+k+1\leq r.$$

But

$$\frac{3k+4}{2}\leq r$$

which proves the lemma.

1) *Isolation Lemma:* There is:

(1) A one-parameter family $\mathrm{U}_s$, $0\leq s\leq 1$, of closed neighborhoods of $f(\mathrm{V}\times\mathrm{I})$ in $\mathrm{E}^r$, and a continuous family of simplicial homomorphisms $g_s:\mathrm{U}_i\rightarrow\mathrm{U}_s$.

2) A one-parameter family $\mathrm{N}_s$, $0\leq s\leq 1$ of closed neighborhoods of V in K′, and a continuous family $p_s:\mathrm{N}_1\rightarrow\mathrm{N}_s$, of simplicial homeomorphisms

— such that:

3) The map $g:\partial\mathrm{U}_1\times\mathrm{I}\rightarrow\mathrm{E}^r$ is a homeomorphism, where $g$ is

$$g(u,t)=q_t(u),\quad u\in\partial\mathrm{U}_1.$$

4) The map $g:\partial\mathrm{N}_1\times\mathrm{I}\rightarrow\mathrm{K}$ is a homeomorphism, where $g$ is

$$g(n,t)=p_t(n)\quad n\in\partial\mathrm{N}_1.$$

5)
$$f_t:\mathrm{N}_s\rightarrow\mathrm{U}_s$$
$$f_t:\partial\mathrm{N}_s\rightarrow\partial\mathrm{U}_s$$

and

6) The following diagram is commutative:

$$\begin{array}{ccccc}
\partial\mathrm{N}_1\times\mathrm{I} & \xrightarrow{g} & \mathrm{N}_1-\mathrm{int}\ \mathrm{N}_0 & \subset & \mathrm{K}\\
\downarrow{\scriptstyle \mathrm{F}_t} & & \downarrow{\scriptstyle f_t} & & \downarrow{\scriptstyle f_t}\\
\partial\mathrm{U}_1\times\mathrm{I} & \xrightarrow[g]{} & \mathrm{U}_1-\mathrm{int}\ \mathrm{U}_0 & \subset & \mathrm{E}^r
\end{array}$$

where

$$\mathrm{F}_t(n,t)=(f_t(n),t)\quad n\in\partial\mathrm{N}_1$$

7) $\partial U_s$ is a homogeneous manifold combinatorially imbedded in $E^r$, $0 \leq s \leq 1$.

8) $$U_1 \cap f(I \times K) = f(I \times N_1).$$

PROOF: It is standard that one can choose a one-parameter family of regular closed neighborhoods of V in K′ with the properties that:

1) There is a continuous family $p_s : N_1 \to N_s$ of combinatorial homeomorphisms.

2) The map $g : \partial N_1 \times I \to K'$ is a simplicial homeomorphism, where $g$ is $g(n, t) = p_t(n)$ $n \in \partial N_1$.

Moreover, after Lemma 10, I assume $f$ to be such that $f(I \times N_1)$ is disjoint from $f(I \times (K - N_1))$. It then follows that $f(I \times N_s)$ is disjoint from $f(I \times (K - N_s))$. Now let $M_s = f(I \times N_s) \subset E^r$, and choose a combinatorial (continuous), monotonic increasing function $\varepsilon(s) > 0$, so small that $R_{\varepsilon(s)}(V)$ is disjoint from $\partial N_s$.

## § 13. Explicit Description of $U_s$.

Let

$$d(p) = d(p, f(V \times I))$$
$$d_s(p) = d(p, f[(K - \text{int } N_s) \times I])$$

for $p \in E^r$.
Define

$$U_s(p) = B_{\lambda_s(p)}(p)$$

where

$$\lambda_s(p) = \min\left[\left(\frac{d_s(p)}{d(p) + d_s(p)}\right) . \varepsilon(s), \ d_s(p)\right]$$

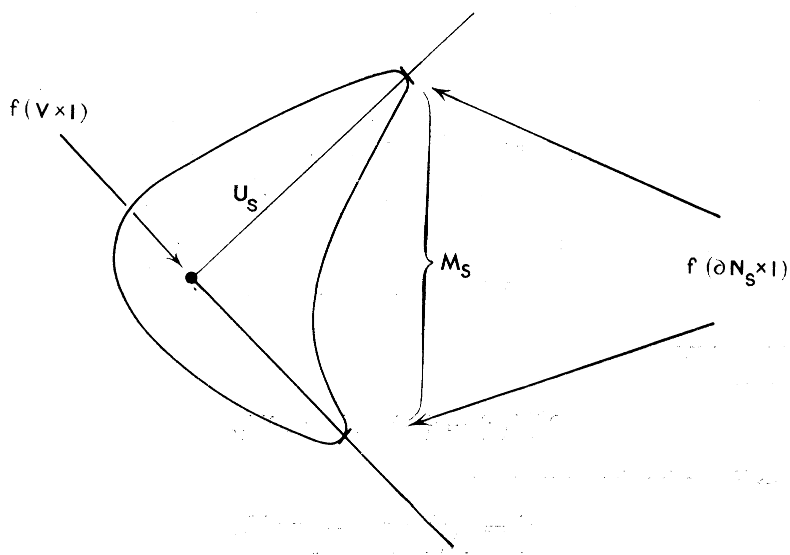


Fig. 4

and

$$U_s = \bigcup_{p \in M_s} U_s(p).$$

## § 14. Pictorial description of $U_s$.

I represent, in figure 4, $M_s$ by the V-shaped arc ; the vertex represents $f(V \times I)$ and the endpoints represent $f(\partial N_s \times I)$. Then $U_s$ is obtained simply by "thickening" every point on $f(\text{int } N_s \times I)$ a very little bit, the amount of thickening decreasing to zero as one approaches $f(\partial N_s \times I)$. The closure of this is $U_s$.

The proof that $U_s$, so defined, actually satisfies properties (1) thru (8) is straightforward ; I omit it therefore.

## § 15. $K^*$: The Modification of K.

LEMMA 11. $f_t | \partial N_1$ is an isotopy of $\partial N_1$ in $\partial U_1$.

PROOF: For $f_t$ is an orthotopy and the singularity pre-locus V is disjoint from $\partial N_1$.

Thus let $F_t : \partial U_1 \to \partial U_1$ be an ambient homeomorphism covering $f_t$, applying Theorem 2 of [3].

LEMMA 12. There is a continuous family of homeomorphisms $G_t : \partial U_1 \times I \to \partial U_1 \times 1$ such that

$$G_t(u, 1) = F_t(u)$$
$$G_t(u, 0) = u$$

for $u \in U_1$.

PROOF: Define $G_t(u, s) = F_{st}(u)$ for $u \in U_1$, $0 \leq t, s \leq 1$ ; now define a homeomorphism

$$G_t^{(1)} : U_1 \to U_1$$

by

$$G_t^{(1)}(u) = u, \quad \text{if} \quad u \in U_0$$
$$G_t^{(1)}(u) = g G_t g^{-1}(u), \quad \text{if} \quad u \in U_1 - U_0$$

where $g$ is the homeomorphism

$$g : \partial U_1 \times I \to U_1 - \text{int } U_0$$

of the isolation lemma.

Notice that:

$$G_1^{(1)} f_0(n) = f_1(n) \quad \text{if} \quad n \in \partial N_1.$$

Define $h : K' \to E^r$ to be the composite

$$h(x) = f_1(x), \; x \in K' - \text{int } N_1,$$
$$h(x) = G_1^{(1)} f_0(x), \; x \in N_1.$$

The two definitions agree on $\partial N_1$, and $h$ is actually a homeomorphism since $f_1(K'—N_1)$ is disjoint from $U_1$. The image:

$$K^* = h(K')$$

is the necessary modification.

Lemma 13. $K^* \sim K'$.

Proof: One must obtain a homeomorphism $H : E^r \to E^r$ carrying $K'$ to $h(K')$. $G_1^{(1)}$ does this on $U_1$. In the bounded manifold $M = E^r—\text{int } U_1$, $f_t$ is an isotopy of $K'—\text{int } N_1 = L$ with the property that $f_t(\partial L) \subseteq \partial M$, and $f_t|\partial L$ is covered by an ambient isotopy $G_t^{(1)}|\partial M = \partial U_1$. Using Theorem 2 of [3] again, one can find an ambient isotopy $H_t$ covering both $G_t^{(1)}|\partial M$ and $f_t|\partial L$. Then the homeomorphism

$$H(x) = H_1(x) \qquad x \in E^r—\text{int } U_1$$
$$H(x) = G_1^{(1)}(x) \qquad x \in U_1$$

sends $K'$ to $K^*$, establishing their equivalence.

## § 16. Summarizing the Relevant Properties.

1) $K^* \sim K'$ ;
2) $K \cap U_1 = f_1(N_1)$ ;
3) $K^* \subset K \cup U_1$ ;
4) $K \cap E_-^r$ consists of a $k$-cell, $E_-^k$, imbedded as the standardly imbedded lower hemisphere of $S^k$ ;
5) virt $\dim_k U_1 \leq \dim(V \times I)$.

## § 17. Bringing $U \cap K$ into the Lower Half-Plane.

Lemma 14. There is a homeomorphism

$$P : E^r \to E^r$$

which has the properties:

1) $P : K \to K$ ;
2) $P : K \cap U \to E_-^r$, the lower half-plane.

For since $V \subset K$, the singularity locus, is assumed contractible to $E_-^k \subset K$ ((3) of condition (0)), there is a continuous family $p_t : K \to K$, such that $p_0 = 1$ and $p_1 : V \to E_-^k$. Since N is a regular neighborhood of V, a continuous family $p_t$ can be found which also has the property:

$$p_1 : N \to E_-^k.$$

By homogeneity of K, $p_1$ can be extended to a homeomorphism $P : E^r \to E^r$.

Let

$$P(K^*) = K_2^*$$

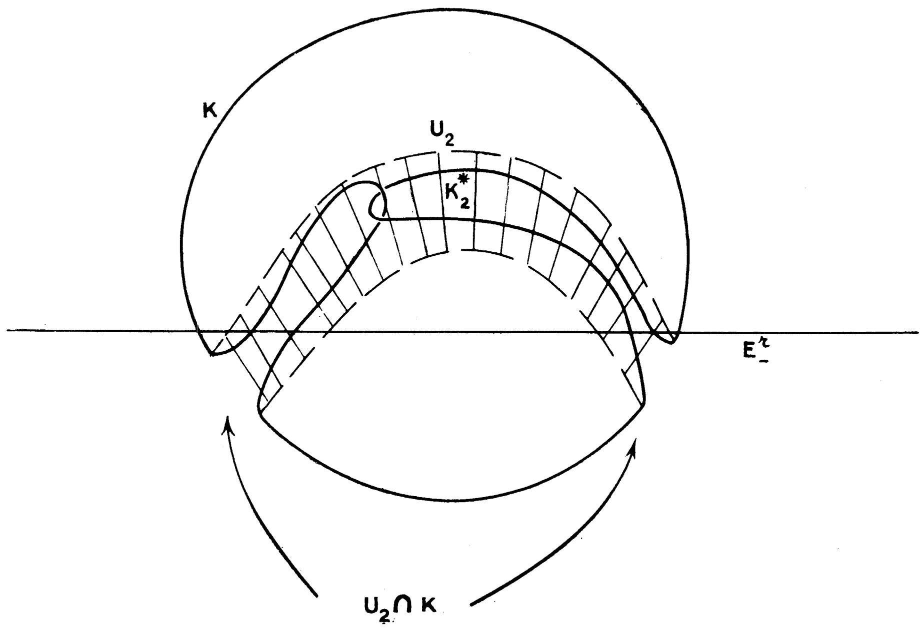


Fig. 5

and

$$P(K) = K$$
$$P(U_1) = U_2.$$

Then one has:

1) $U_2 \cap K_2'$ is in the lower half-plane ;
2) $K_2^* \sim K_2'$ ;
3) $K_2^* \subset K \cup U_2$ ;
4) virt $\dim_K(U_2) \leq \dim(V \times I)$.

## § 18. Bringing $U_2$ into the Lower Half-Plane.

Lemma 15. There is a homeomorphism

$$f_1 : U_2 \to E_-^r.$$

leaving $U_2 \cap K$ fixed.

Proof: Obvious.

Lemma 16. There is a homeomorphism $f_2 : U_2 \to E_-^r$ such that

1) $U_2 \cap K$ is left fixed ;
2) $f_2(U_2 - U_2 \cap K) \subset E^r - K \cap E_-^r$.

Proof: It is a simple application of Corollary 5, after one checks that

$$\text{virt } \dim_K U_2 + \text{virt } \dim K + 1 \leq r$$

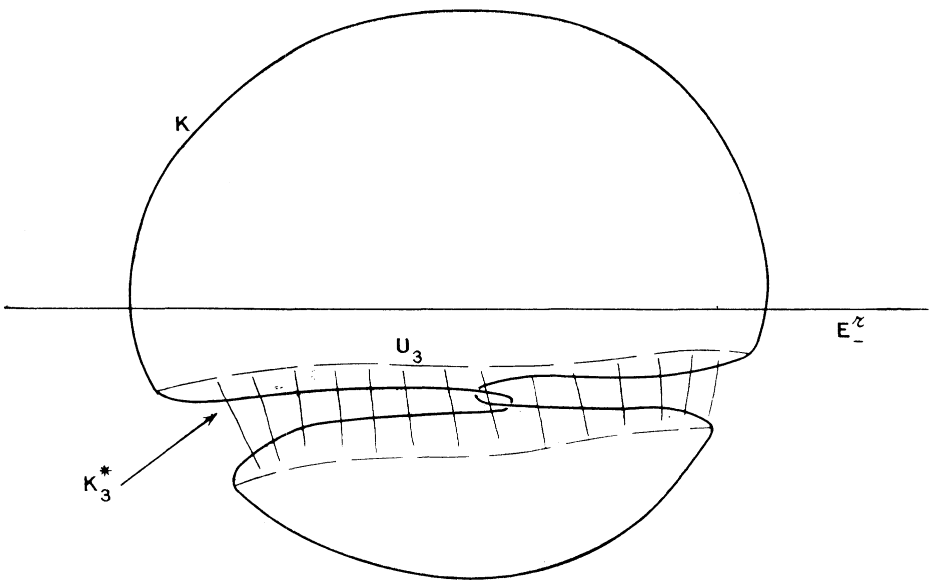


Fig. 6

or

$$2k - r + 3 + k + 1 \leq r$$

But this is the case, for

$$\frac{3k+4}{2} \leq r.$$

LEMMA 17. There is an isotopy $k_t : E^r \to E^r$ such that

1) $k_t | K_2 = 1$ ;

2) $k_1 | U_2 = f_2$ (thus $k_1(U_2) \subset E^r_-$).

PROOF: Apply Corollary 3 with $U_2 = L$, $f_2(U_2) = L'$, $K = N$ and observe that

$$\text{virt dim}_K\, U_2 + \text{virt dim}\, K + 2 \leq r$$

or

$$2k - r + 3 + k + 2 \leq r$$

for

$$\frac{3k+5}{2} \leq r.$$

Call:

$$k_1(K^*_2) = K^*_3,\; k_1(U_2) = U_3$$

and: 1) $K^*_3 \sim K^*_2 \sim K'$

2) $K^*_3 \subset K \cup U_3$

3) $U_3 \subset E^r_-$;

therefore:

$$K^*_3 \cap E^r_+ = K \cap E^r_+.$$

### § 19. The Decomposition : $K \sim K' + S$.

LEMMA 18. $K_3^* = K + S$ where S is a spherical knot.

For define: $S = (K_3^* \cap E_-^r) \cup E_+^k$, where $E_+^k$ is the standardly imbedded upper hemisphere of the standard $k$-sphere in $E^r$, $E_+^k \subset E_+^r$.

LEMMA 19. $E_-^r \cap K_3^*$ is a $k$-cell, and $E^{r-1} \cap K_3^*$ is the standard $k-1$ sphere in $E^{r-1}$, where $E^{r-1}$ is the hyperplane $E_-^r \cap E_+^r$.

PROOF: Obvious from the construction of $K_3^*$:

$$E_-^r \cap K_3^* \subset E_-^k \cup U_3$$

(Because $K_3^* \subset K \cup U_3$, and $K \cap E_-^r = E_-^k$.)

Therefore the boundaries match:

$$\partial(E_-^r \cap K_3^*) = \partial E_+^k$$

and S is actually a sphere.

LEMMA 20. $K + S = K_3^*$.

This is obvious. ($S \cap E_+^r$ is the standard $k$-cell, $E_+^k$, $K \cap E_-^r$ is the standard $k$-cell $E_-^k$. Therefore their sum consists simply of:

$$(K - \text{int } E_-^k) \cup (S - \text{int } E_+^k) = X.$$

But this X is just $K_3^*$.

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