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ON THE CLOSEDNESS OF SINGULAR LOCI (*)

By MASAYOSHI NAGATA

The set of singular spots in a model M over a ground ring I in the sense of the writer's paper [5] forms a closed set, as in usual algebraic geometry (see [5, III]).

In the present paper, we shall show that the closedness of singular loci is not true in general in the case of models in the 'non-restricted case' (1).

On the other hand, a similar problem can be discussed from the following point of view: Let A be a Noetherian ring. The set of local rings which are rings of quotients of A is called the affine scheme of A (2), which will be denoted by $S(A)$; here we introduce the Zariski topology on $S(A)$. Then we can ask under what condition the singular locus (*i.e.*, the set of non-regular local rings) of a given affine scheme is closed.

We shall prove in § 1 the following:

THEOREM 1. For a Noetherian ring A, the following three conditions are equivalent to each other:

(1) For any finitely generated ring B over A, the singular locus of the affine scheme $S(B)$ of B is a closed set (of $S(B)$).

(2) For any ring B which contains A as a subring and which is a finite module over A, the singular locus of $S(B)$ is closed.

(3) For any prime ideal $\mathfrak{p}$ of A and for any purely inseparable finite integral extension (3) B of $A/\mathfrak{p}$, the singular locus of $S(B)$ is contained in a proper closed subset of $S(B)$.

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(1) In the definition of models, ground rings are assumed to be Dedekind domains which satisfy 'finiteness condition' (for integral extensions), while those in the non-restricted case are arbitrary Dedekind domains.

(2) The notion of scheme (schémata) was treated in the general case by Grothendieck who will publish a joint paper with J. Dieudonné in a near future ; unfortunately the title of his paper is not decided yet.
We shall remark here that, since an everywhere locally closed subset is closed, our results can be applied to schemes or pre-schemes in the sense of Grothendieck.

(3) An integral extension of an integral domain A is, by definition, an integral domain which is integral over A.

Now let G be the class of Noetherian rings A satisfying the conditions in Theorem 1. We shall prove in § 2 the following:

THEOREM 2. The class G is closed under the following three operations: (i) Formation of rings of quotients. (ii) Homomorphisms. (iii) Finitely generated extensions.

In § 3, we shall prove at first the following.

THEOREM 3. Any complete local ring is in the class G.

Then, using Theorem 3, we shall prove more generally the following:

THEOREM 4. Let A be a (Noetherian) semi-local ring. If, for any prime ideal $\mathfrak{p}$ of A and for any purely inseparable finite integral extension B of $A/\mathfrak{p}$, B is analytically unramified, then A is in the class G.

This Theorem 4 leads to:

THEOREM 5. If a class G' of semi-local rings satisfies the following two conditions, then G' is a subclass of G:

(1) If $A \in G'$ and if $\mathfrak{p}$ is a prime ideal of A, then $A/\mathfrak{p} \in G'$.

(2) If $A \in G'$ is an integral domain and if A' is the derived normal ring of A in a finite purely inseparable extension of the field of quotients of A, then (i) A' is a finite A-module and (ii) for any maximal ideal $\mathfrak{m}'$ of A', $A'_{\mathfrak{m}'} \in G'$, or, instead of (ii), (ii′) $A' \in G'$.

In § 4, we shall give an example of a *regular* local ring which is not in the class G and we derive an example in the case of models in the non-restricted case. In § 5 we shall give an example of Noetherian integral domain A, containing any given field of characteristic zero, which is not in the class G.

§ 1. The proof of Theorem 1.

We shall remark at first the following simple fact:

LEMMA 1. If S is the affine scheme of a Noetherian ring A, then the set of local rings in S which are not integral domains is closed.

The proof is easy by virtue of the primary decomposition of the zero ideal of A.

Now we shall prove Theorem 1. It is obvious that (1) implies (2). Assume that (2) holds. Let B be as in (3). Then the direct sum $A+B$ satisfies the condition for B in (2) [1]. Hence the validity of (2) implies the closedness of the singular locus of $S(B)$. Since B is an integral domain, the singular locus is a proper subset. Thus (2) implies (3). Assume that (3) holds. Then the proof of closedness of the singular locus

[1] The direct summand A of $A+B$ may or may not be called a subring of $A+B$. But, denoting by Φ the natural homomorphism from A into $A/\mathfrak{p}$, the mapping from A into $A+B$ defined by $a \to (a, \Phi(a))$ gives an imbedding of A into $A+B$ so that A becomes a subring of $A+B$ (under the definition requiring a subring to have the same identity as the original ring).

in the case of models, given in [5, III], can be adapted, and we see the validity of (1); here we shall sketch the adaptation. By Lemma 1, we may assume that B is a finitely generated integral domain over $A/\mathfrak{p}$ with a prime ideal $\mathfrak{p}$ of A. The first step is to show that, for any such B, the singular locus of $S(B)$ is contained in a proper closed subset of $S(B)$. If B is separably generated over $A/\mathfrak{p}$, then by the normalization theorem for separably generated extensions (see [5, II]), we may assume that there is a polynomial ring P over $A/\mathfrak{p}$ such that B is a finite separable integral extension of P. Then by the non-triviality of the discriminant, we see that the singular locus of $S(B)$ is contained in a proper closed subset. If B is not separably generated over $A/\mathfrak{p}$, then denoting by p the characteristic of $A/\mathfrak{p}$, let $A'=A[a_1, \ldots, a_n]$ be such that (i) $B[a_1, \ldots, a_n]$ is separably generated over A' and (ii) $a_i^p \in A[a_1, \ldots, a_{i-1}]$. Then we prove the assertion by induction on n by virtue of the following:

Lemma 2. Let R be a local integral domain and let a be an element such that (i) a is not in the field of quotients of R and (ii) $a^p \in R$, where $p \neq 0$ is the characteristic of R. If $R[a]$ is a regular local ring, then R is regular, too.

For the proof, see [5, II].

Then we prove the following:

Lemma 3. Let A be a Noetherian ring with a prime ideal $\mathfrak{p}$ such that $A_\mathfrak{p}$ is a regular local ring. Let $x_1, \ldots, x_r$ be a regular system of parameters of $A_\mathfrak{p}$ $(x_i \in A)$ and let f be an element of A which is not in $\mathfrak{p}$ but is in every prime divisor of $\Sigma x_i A$ other than $\mathfrak{p}$. If a prime ideal $\mathfrak{q}$ of A contains $\mathfrak{p}$, if $f \notin \mathfrak{q}$ and if $A_\mathfrak{q}/(\mathfrak{p})$ is regular, then $A_\mathfrak{q}$ is regular.

The proof is easy (see [5, III]).

Now we see that the singular locus of $S(B)$ is closed by virtue of the following criterion of closedness:

Lemma 4. A subset F of an affine scheme S is a closed set of S if and only if the following two conditions are satisfied: (1) If $P \in F$, then any specialization of P is in F and (2) if $P' \notin F$ $(P' \in S)$, then $\Phi_P(F)$ is contained in a proper closed subset of $\Phi_P(S)$, where Φ_P denotes the homomorphism defined by P ([5, I, Chap. II, Prop. 6]).

Thus we prove the theorem.

§ 2. The class of rings which satisfy the conditions in Theorem 1.

We shall recall that G is the class of rings satisfying the conditions in Theorem 1. We shall prove now Theorem 2. (i) follows from the following easy fact:

Lemma 5. If A' is a ring of quotients of A, then $S(A')$ is a subspace of $S(A)$.

(ii) follows from the condition (1) in Theorem 1, while (3) implies (iii).

We shall remark here that the condition (3) in Theorem 1 implies the following two propositions:

PROPOSITION 1. A Noetherian ring A is in G if and only if, for any prime ideal $\mathfrak{p}$ of A, $A/\mathfrak{p}$ is in G.

PROPOSITION 2. A Noetherian integral domain A is in G if and only if the following properties are true: (i) for any prime ideal $\mathfrak{p}$ of rank 1, $A/\mathfrak{p}$ is in G and (ii) for any finite purely inseparable integral extension B of A, there exists an element f of A such that $B[1/f]$ is a regular ring (1).

COROLLARY. Any semi-local ring of rank 1 is in G.

§ 3. Semi-local rings.

We shall prove at first Theorem 3. By virtue of the condition (3) in Theorem 1, we have only to prove that if B is a complete local integral domain (which is Noetherian), then the singular locus of $S(B)$ is contained in a proper closed subset of $S(B)$. If B contains a field, then the Jacobian criterion (see [6]) can be applied by virtue of Cohen's structure theorem of complete local rings (see [1]), and we see that the singular locus of $S(B)$ is a closed set. If B does not contain any field, then B is of characteristic zero, hence it is a finite separable integral extension of an unramified complete regular local ring, by the structure theorem of Cohen. Therefore we see that the singular locus of $S(B)$ is contained in a proper closed subset of $S(B)$ by a theorem of Cohen on the regularity of rings of quotients of complete regular local rings (2).

In order to prove Theorem 4, we prove the following:

PROPOSITION 3. Let B be a semi-local ring and let $\mathfrak{p}$ be a prime ideal of B. Let B^* be the completion of B and let $\mathfrak{p}^*$ be a minimal prime divisor of $\mathfrak{p}B^*$. Assume that $\mathfrak{p}B^*_{\mathfrak{p}^*} = \mathfrak{p}^*B^*_{\mathfrak{p}^*}$. Then $B_\mathfrak{p}$ is regular if and only if $B^*_{\mathfrak{p}^*}$ is regular.

Proof. By the theorem of transition (see [4]), the length of $B_\mathfrak{p}/\mathfrak{p}^2B_\mathfrak{p}$ is equal to that of $B^*_{\mathfrak{p}^*}/\mathfrak{p}^{*2}B^*_{\mathfrak{p}^*}$ and rank $\mathfrak{p} = \text{rank}\,\mathfrak{p}^*$, which proves our assertion.

In order to prove Theorem 4, it is sufficient to prove the following proposition, by virtue of Theorem 1:

PROPOSITION 4. Let A be a semi-local ring and let A^* be the completion. If the singular locus of $S(A^*)$ is contained in a proper closed subset of $S(A^*)$, then the singular locus of $S(A)$ is contained in a proper closed subset of $S(A)$.

The proof is easy by virtue of Proposition 3.

(1) A ring A is called regular if every local ring which is a ring of quotients of A is a regular local ring.

(2) This theorem of Cohen (see [1]) was generalized by Serre and Auslander-Buchsbaum (see [7]) to arbitrary regular local rings.

COROLLARY 1. If a semi-local ring A has no nilpotent element and if A is analytically unramified, then the singular locus of $S(A)$ is contained in a proper closed subset of $S(A)$.

COROLLARY 2. If every prime ideal of a semi-local ring A is analytically unramified, then the singular locus of $S(A)$ is a closed set of $S(A)$.

In order to prove Theorem 5, we have only to prove, by virtue of Theorem 4, the analytical unramifiedness of members of G', which was proved by Zariski [**8**] (cf. [**3**]) (1).

Zariski's result quoted above (cf. [**3**]) and Corollary 2 to Proposition 4 gives us the following.

PROPOSITION 5. If a class G'' of semi-local rings satisfies the following two conditions, then for any member A of G'', the singular locus of $S(A)$ is a closed set of $S(A)$:

(1) If $A \in G''$, and if $\mathfrak{p}$ is a prime ideal of A, then $A/\mathfrak{p} \in G''$.

(2) If $A \in G''$ is an integral domain and if A' is the derived normal ring of A, then (i) A' is a finite A-module and (ii) for any maximal ideal $\mathfrak{m}'$ of A', $A'_{\mathfrak{m}'} \in G''$, or, instead of (ii), (ii') $A' \in G''$.

We shall give three remarks on the class G' in Theorem 5. We shall denote from now on by G' the largest class of semi-local rings satisfying the conditions in Theorem 5. Then, at first:

PROPOSITION 6. G' is the largest class of semi-local rings satisfying the following two properties:

(1) If $A \in G'$, then every prime ideal $\mathfrak{p}$ of A is analytically unramified and $A/\mathfrak{p} \in G'$.

(2) If $A \in G'$ is a local integral domain, then for any finite purely inseparable integral extension A' of A, A' is in G'.

Proof. We have seen in the proof of Theorem 5 that G' satisfies the above conditions. Conversely, since the analytical unramifiedness of a semi-local integral domain A implies the finiteness of the derived normal ring of A, we prove Proposition 6.

The second remark we want to give here is as follows:

PROPOSITION 7. (1) If B is a semi-local ring which is a ring of quotients of a member A of G', then $B \in G'$.

(2) If $\mathfrak{a}$ is an ideal of $A \in G'$, then $A/\mathfrak{a} \in G'$.

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(1) The result of Zariski, which we are using, is as follows:

Let R be a normal local ring. If there exists a non-unit a of R such that every prime divisor of aR is analytically unramified, then R is analytically unramified.

Observe that the above result can be generalized trivially to a normal semi-local ring R, provided that aR is contained in every maximal ideal of R.

Observe also the following well known fact which can be proved easily:

Let R be a semi-local integral domain. If there exists a finite integral extension of R which is analytically unramified, then R is analytically unramified.

(3) If a semi-local ring B is of finitely generated type ([1]) over a member A of G', then $B \in G'$ ([2]).

Proof. (1) and (2) are obvious by the definition of G'. As for (3), we can easily reduce it to the case where B is a local integral domain. By virtue of (1), we may assume that A is a local integral domain dominated by B. By induction on the number of generators of a ring, of which B is a ring of quotients, we may assume that $B = A[x]_{\mathfrak{p}}$ with a prime ideal $\mathfrak{p}$ of $A[x]$ $(x \in B)$. By virtue of (1), we may assume that $\mathfrak{p}$ is a maximal ideal of $A[x]$. Since G' is closed under purely inseparable finite integral extensions of members which are integral domains, we see easily that (1) in order to prove Proposition 7, it is sufficient to prove the analytical unramifiedness of B (using an induction argument on rank A) and (2) if $A \in G'$ is an integral domain, then any finite integral extension of A is in G'. Now we shall prove the analytical unramifiedness of B by induction on rank A. Since the derived normal ring of A' is finite over A, we may assume that A is normal. If x is transcendental over A, then we see that the analytical unramifiedness of B immediately follows from the result of Zariski quoted above (see foot-note (1) in the preceding page). Therefore we assume that x is algebraic over A. By virtue of the remark (2) stated above, we may assume that x is in the field of quotients of A. Let $f(X)$ be the irreducible monic polynomial over the residue class field $A/\mathfrak{m}$, $\mathfrak{m}$ being the maximal ideal of A, such that $f(x) \in \mathfrak{p}/(\mathfrak{m})$. Then, again by virtue of the remark (2) above, we may assume that $f(X)$ is a separable polynomial. Therefore, we may assume that $x \in \mathfrak{p}$. Now we shall use the following version of the result of Zariski quoted above:

LEMMA 6. Let R be a local integral domain. If there exists a non-unit x in R such that (i) xR has no imbedded prime divisors and (ii) for any prime divisor $\mathfrak{q}$ of xR, $\mathfrak{q}$ is analytically unramified and $R_{\mathfrak{q}}$ is a discrete valuation ring, then R is analytically unramified ([3]).

The proof of Lemma 6 was really given by Zariski [**8**].

By virtue of Lemma 6, we have only to show the element x satisfies the conditions in Lemma 6 applied to B. Since A is normal, the relations of x over A are generated by linear relations $ax - b = 0$, the set $\mathfrak{b}$ of b's forms a purely rank 1 ideal and $A[x]/(x) = A/\mathfrak{b}$, which proves the validity of (i). Let $\mathfrak{q}$ be any prime divisor of xB. Then $\mathfrak{q} \cap A$ is a prime divisor of $\mathfrak{b}$ and $x \in A_{(\mathfrak{q} \cap A)}$, hence $B_{\mathfrak{q}} = A_{(\mathfrak{q} \cap A)}$, which is a discrete valuation ring. Analytical unramifiedness of $\mathfrak{q}$ follows from the induction assumption, and the validity of (ii) is proved. Thus Proposition 7 is proved.

([1]) We say that a ring B is of finitely generated type over another ring A if B is a ring of quotients of a finitely generated ring over A.

([2]) Observe furthermore that every complete semi-local ring is in G'.

([3]) This can be generalized to semi-local integral domains, provided that x is in every maximal ideal of R. The generalization of this (in the semi-local case) to non-integral domains is trivially easy.

As a corollary of Proposition 7, we have another remark on G' as follows:

PROPOSITION 8. If an integral domain B is of finitely generated type over a member A of G', then the derived normal ring of B is a finite B-module.

Proof. It is sufficient to prove the case where B is finitely generated over A. Proposition 7 shows that if $\mathfrak{p}$ is a prime ideal of B, then the derived normal ring of $B_{\mathfrak{p}}$ is a finite $B_{\mathfrak{p}}$-module. Therefore the proof of the finiteness of the derived normal ring of an affine ring over a ground ring, given in [5, I], can be applied and we prove the assertion.

Remark. The proof quoted just above proves the following fact:

Let B be a finitely generated integral domain over a Noetherian integral domain A. If the derived normal ring of A in any finite purely inseparable extension of the field of quotients of A is a finite A-module and if, for every prime ideal $\mathfrak{q}$ of B, the derived normal ring of $B_{\mathfrak{q}}$ is a finite $B_{\mathfrak{q}}$-module, then the derived normal ring of B is a finite B-module (hence the same is true for any ring of quotients of B).

§ 4. The first example.

Let k_0 be a perfect field of characteristic $p \neq 0$ and let $v_1, \ldots, v_n, \ldots$ be infinitely many algebraically independent elements over k_0. Set $k = k_0(v_1, \ldots, v_n, \ldots)$. Let $x_1, \ldots, x_r\ (r \geqslant 2)$ be analytically independent elements over k. Set $A = k^p\{x_1, \ldots, x_r\}[k]$. Then A is a regular local ring (see [2]). Let $p_1, \ldots, p_n, \ldots$ be infinitely many prime elements of A such that $p_iA \neq p_jA$ if $i \neq j$. For each natural number n, we set $q_n = p_1 \ldots p_n$. Set $c = \Sigma v_i q_i$. We shall show that the singular locus of $S(A[c])$ cannot be contained in any proper closed subset of $S(A[c])$. In order to prove it, it is sufficient to show that if a prime ideal $\mathfrak{p}$ of rank 1 in $B = A[c]$ contains one of p_i, then $B_{\mathfrak{p}}$ is not normal, which can be seen as follows: Set $c_n = (c - \Sigma_1^{n-1} v_i q_i)/q_{n-1}$. Then $B_{\mathfrak{p}} = A_{(p_i)}[c_i]$, which does not contain c_{i+1}. The derived normal ring of B contains all of the c_n, hence $B_{\mathfrak{p}}$ is not normal.

We shall now consider the models in the non-restricted case. Consider the case where $r = 2$ in the above example and set $I = A[1/x_1]$. Then I is a Dedekind domain and $B[1/x_1] = I[c]$ is an affine ring over I. The above proof shows that the singular locus of the affine model of $I[c]$ is not closed.

§ 5. The second example.

Let Z_0 be a principal ideal integral domain which contains infinitely many prime ideals such that, for any prime ideal $\mathfrak{p}$ of Z_0, $Z_0/\mathfrak{p}$ is not of characteristic 2: For instance, let Z_0 be $k[x]$ with an arbitrary field of characteristic zero and a transcendental element x over k. We take infinitely many prime elements $p_1, \ldots, p_n, \ldots$ $(p_i Z_0 \neq p_j Z_0$ if $i \neq j)$. Starting from Z_0, we construct a infinite sequence of successive extensions $Z_0, Z_1, \ldots, Z_n, \ldots$ as follows:

When Z_{i-1} is defined, let A_i be the ring $Z_{i-1}[\sqrt{p_i^3}]$. Z_i is a ring of quotients of A_i

such that for each prime ideal $\mathfrak{q}$ of Z_{i-1}, there exists one and only one prime ideal of Z_i which lies over $\mathfrak{q}$.

The existence of Z_i can be proved inductively, together with the following properties: For each prime ideal pZ_0 of Z_0, there exists one and only one prime ideal, say $\mathfrak{p}$ of Z_i and (i) if $pZ_0=p_jZ_0$ with $j\leqslant i$, then $(Z_i)_{\mathfrak{p}}$ is not a valuation ring and $\mathfrak{p}$ is generated by p_j and $\sqrt{p_j^3}$; (ii) if $pZ_0\neq p_jZ_0$ for any $j\leqslant i$, then $\mathfrak{p}$ is generated by p.

Let A be the union of all the Z_n. Then:

(i) *For any* non-zero element a of A, there exist only a finite number of prime ideals containing a. (For a is in some Z_i.)

(ii) If $\mathfrak{p}$ is a prime ideal of A, then $A_{\mathfrak{p}}$ is Noetherian.

(iii) $A_{\mathfrak{p}}$ is not normal if and only if $\mathfrak{p}$ contains some of the p_i.

The properties (i) and (ii) implies that A is Noetherian, hence property (iii) shows that the singular locus of $S(A)$ is not closed.

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