Leopold Kronecker's conception of the foundations of mathematics

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Résumé : On réduit habituellement les idées de Kronecker sur les fondements des mathématiques à quelque boutade ou à quelques principes rétrogrades. Ces idées constituent pourtant une doctrine originale et cohérente, justifiée par des convictions épistémologiques. Cette doctrine apparaît dans un article intitulé 'Sur le concept de nombre', paru en 1887 dans le *Journal de Crelle,* et surtout dans le dernier cours professé par Kronecker à Berlin au semestre d'été 1891. Le but de cet article est d'en préciser les principes et les éléments en la comparant aux autres entreprises de fondement, afin d'en souligner l'originalité, et de montrer comment elle a déterminé la pratique mathématique de Kronecker.

Abstract: Kronecker's views on the foundations of mathematics are often reduced to jokes and regarded as an outdated set of ill-assorted ideas. A closer look however shows that they constitute an original and coherent doctrine justified by epistemological convictions. This doctrine appears in the article *On the concept of number*, published in the *Journal of Crelle* (1887) and, especially, in the last course taught by Kronecker, in Berlin in the summer semester of 1891. This article would attempt to state the principles and insights of Kronecker's doctrine by comparing it to other foundation schools of thought so as to underline its originality and show the influence this doctrine exerted on Kronecker's own mathematical work.

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It is accepted today that the turn of the 19^{th} century witnessed three competing schools of thought on the foundations of mathematics, namely, the Frege-Russell logicism, the Hilbertian formalism and the Brouwerean intuitionism. Kronecker's name is not attached to any of the three schools of thought. At best, he appears as a precursor of intuitionism. More often than not, his views on the foundations of mathematics were reduced to the famous sentence[1]: "*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*", or regarded as an outdated set of ill-assorted ideas, borrowed by others, among which Hilbert and Brouwer, to formulate their own tenets. A closer look however shows that Kronecker's ideas on the foundations of mathematics constitute an original and a coherent doctrine that deserves consideration on the same level as the other schools of thought. He was moreover known to have played a dominant role in the development of the mathematics towards the end of the 19^{th} century and was an influential member, along side Kummer and Weierstrass, of the Academy of Berlin from 1860 upto 1891, the year of his death. Indeed, it was Kronecker, Kummer and Weierstrass who together conducted the seminar on mathematics whose success earned for Berlin the reputation of the first center of advanced learning. Several articles due to H. Edwards have analysed in depth[2] the reasons behind the real ostracism to which had been exposed Kronecker's work, particularly his ideas on the foundations of mathematics. My aim here is to show that Kronecker's conception stands as a coherent doctrine justified by epistemological convictions. This doctrine was formulated in the article *On the concept of number*, published in the *Journal of Crelle* (1887) and, especially, in the last course Kronecker himself taught in Berlin during the summer semester of 1891[3]. I will spell out the principles and the insights of Kronecker's doctrine and compare it to the other foundation schools of thought so as to underline both the originality of his contribution and the influence it had exerted on his own mathematical work.

1. Mathematics as a natural science

Kronecker conceived mathematics as a natural science or, as he also put it, as an experimental science[4]. This first thesis was the central

[1] This sentence was stated in a lecture for the *Berliner Naturforscher-Versammlung* (1886) and was quoted by Weber in his obituary [Weber 1893, 15].

[2] For example: [Edwards 1987, 1988, 1989, 1995].

[3] See [Boniface-Schappacher 2002].

[4] See [Kronecker 1891, lecture 4, 17, in Boniface-Schappacher, 2002 p. 232]: "Die Mathematik ist wie eine Naturwissenschaft zu behandeln" und [Kronecker 1891, lec-

point of his epistemology and induced his view on the foundations of his discipline. He was inspired by Kirchhoff the physicist. Following the latter, Kronecker was particularly opposed to "those who want to build our knowledge on imprecise logico-philosophical foundations "[5] and contested that definitions can be considered as elements of the foundations of science.

> It was often said that mathematics has to start with definitions and that mathematical propositions have to be deduced from those definitions and from ground postulates. However, definitions per se, are already an impossibility, as Kirchhoff underlined, because each definition uses its own concepts, which in their turn have to be defined, etc.[6]

Instead of definitions, Kirchhoff and Kronecker placed *phenomena* at the center of the foundations of the natural sciences and mathematics. The task of mechanics and of the natural sciences, Kirchhoff stated, is in general to describe phenomena *simply* and *completely* (*einfach und vollständig*). "Now mathematics, Kronecker added, is nothing else than a science of nature, therefore it must also 'describe phenomena simply and completely'. Foundations result therefrom"[7]. According to Kronecker, for the foundations of mathematics, as well as of the natural sciences, phenomena are basic concepts and principles which are *given by experience* and open to modification in the course of the development of the subject matter. This is the first consequence of the view of mathematics as an experimental science.

This first consequence of the Kronecker conception throws light on the import of Kronecker's sentence quoted in the introduction saying that "natural numbers were created by God, everything else is the work of men ". It means that ordinal numbers, that Kronecker assumed to

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ture 6, 28, in Boniface-Schappacher, 2002, 240]: "(...) die Definitionen der Erfahrungswissenschaften, — d.h. der Mathematik und der Naturwissenschaften, ...".

[5]"Ich stelle mich damit denjenigen gegenüber, welche unsere Wissenschaft auf unpräzisen, logisch-philosophischen Fundamenten aufbauen wollen" [Kronecker 1891, lecture 2, 9, in Boniface-Schappacher, 2002, 226]

[6]"Man hat häufig gesagt, die Mathematik müsste mit Definitionen beginnen, und aus ihnen zusammen mit den postulierten Grundsätzen seien die mathematischen Sätze abzuleiten. Nun sind aber Definitionen an sich schon eine Unmöglichkeit, wie Kirchhoff zu sagen pflegte, denn jede Definition braucht ihre Begriffe, welche wieder zu definieren sind u.s.w. [Kronecker 1891, lecture 2, 8-9, in Boniface-Schappacher, 2002, 225].

[7]"Nun ist aber die Mathematik nichts anderes als eine Naturwissenschaft und es kommt also auch bei ihr darauf an, die Erscheinungen "einfach und vollständig zu beschreiben. Die Begründung ergiebt sich dann von selbst" [Kronecker 1891, lecture 2, 9, in Boniface-Schappacher, 2002, 226]

be at the basis of pure mathematics, need not be defined. They have to be considered as *given* and as the only *given* objects of this science. All other objects must be built from them. It is for this reason that Kronecker, in opposition to the other foundation schools of thought in mathematics, as led by Frege, Hilbert or Brouwer, never tried to seek the ultimate foundation of the basic concepts of his discipline outside mathematics itself. It is noteworthy to recall here,

- that Frege wanted to found on logic mathematics, with arithmetic in the first place considered as the science of truth,
- that, after the failure of logicism to found arithmetic, Hilbert tried *at the same time* to base arithmetic and logic on sensitive intuition, more precisely on a set of signs with no meaning, 'on paper' as Brouwer ironically observed,
- and that, according to Brouwer himself, mathematics starts with the 'Primordial Intuition of Time' and hence is founded on the mind of what Brouwer called 'the Subject', 'the Creating Subject' or the 'Idealized Mathematician'.

It is within mathematics itself that Kronecker sought to ground his discipline. Thus, in order to define cardinal numbers, he used two fundamental mathematical notions inherited from Gauss, the notions of equivalence and invariant. For equivalence relation, he considered the one-to-one relation between finite sytems of distinct objects and, for an invariant, he turned to a system of the class, more precisely, to the collection of the first ordinal numbers — or even of the fingers. For instance, three fingers is the characteristic invariant of the class of collections of three objects. This definition is rather close to that of Frege and of Russell. Both used an equivalence relation between collections. Russell defined number as the class of all the collections equivalent to a given collection, Frege as the extension of the concept of "'equinumerous' (*gleichzahlig*) to a given concept". In order to establish his definition, Kronecker instead considered as invariant, a *representative* of the class of the equivalent collections. Such a representative has the avantage of giving a more real basis to the concept of number. It is moreover a way to avoid the explicit or implicit use of the notion of infinite set thought by Kronecker as being too abstract.

This self-foundation of mathematics is in line with the second thesis of Kronecker: the necessity for a discipline not to infringe on another discipline. At the beginning of the century, mathematics and the natural sciences were suffering from the infantile illness of lack of confidence in their own methods. But by contrast, by the end of the century, old age illness whetted their desire to dominate the whole of reality. The result in both cases was to turn to philosophy for basic concepts.

The separation between different scientific areas, which seemed necessary to Kronecker in order to gaurd against seeking for the foundations of one science in another area, must be maintained between different disciplines of the same scientific area. The latter separation must not however be detrimental to the applications of discipline (of mathematics) to other disciplines such as astronomy, physics and statistics. This concerns basic concepts and methods which must necessarily be adapted to the subject matter. This last point accounts for Kronecker's particularly firm position on irrational numbers which he considered to be objects of geometry and not of arithmetic, and which he wanted to see restored to their original discipline.

2. Kronecker's realism

If Kronecker assimilates mathematics to the natural sciences, it is because "its objects are as real as those of its sister sciences"[8]. Kronecker explained this point: "Everyone who speaks about mathematical 'discoveries' (*Entdeckungen*), and not about mathematical 'inventions' (*Erfindungen*) feels it. Since only what already exists can be discovered, and only what the human mind produces can be called 'invention'. This is why mathematicians 'discover' results through methods that they 'invented' for that purpose"[9]. Thus, mathematician's freedom is, according to Kronecker, in the invention of new *methods*, not in the creation of new objects, as did Dedekind and Cantor, for instance. I will come back on this point.

Althought Kronecker compared mathematics to the natural sciences, the reality of mathematical objects is not for him those of natural objects. Kronecker is strictly speaking not an empiricist. His mathematics was not an applied science but a pure one; it was not really a natural science, but must *to be dealt with as* a natural science. Kronecker established in

[8]"...ihre Gegenstände sind ebenso wirklich wie diejenigen ihrer Schwesterwissenschaft" [Kronecker 1891, lecture 4, 18, in Boniface-Schappacher, 2002, 232].

[9]"Dass dem so ist, fühlt ein jeder, der von mathematischen 'Entdeckungen', nicht aber von mathematischen 'Erfindungen' spricht. Denn entdeckt kann doch nur dasjenige werden, was bereits wirklich existiert; was aber der menschliche Geist aus sich hervorbringt, das heisst 'Erfindungen'. Daher 'entdeckt' der Mathematiker die Resultate durch Methoden, welche er zu diesem Behufe 'erfunden' hat" [Kronecker 1891, lecture 4, p. 18, in Boniface-Schappacher, 2002, 232-33].

fact an analogy between mathematics and the natural sciences, both being experimental sciences. But, if mathematical objects were not for him objects of the physical world, they were neither simple productions of our mind. Althought he quoted the famous sentence due to Gauss: "number is a product of our sole mind", Kronecker conceived number and mathematics very differently from Gauss. Indeed, in Gauss' conception one can find the first views of mathematics as being a conceptual science, a 'free creation of the human mind', as Dedekind and Cantor put it. For Kronecker, the status of arithmetic as a pure science is linked to its independence of time and of space:

> Indeed, I consider as special disciplines of our science mechanics which operates with the concept of time, geometry which seeks to discover spatial relations where time does not appear, and so-called pure mathematics, where neither time nor space appear and which I want to designate as 'arithmetic'[10].

Kronecker agrees thus far with Gauss, but to him, the purety of mathematics does not mean that it is an abstract science. We saw above that in Kronecker's opinion, mathematics like the natural sciences, must be founded on experience — mathematical experience in the first place; Thus, for Kronecker, mathematical objects are mathematical phenomena, most often, concrete algebraic expressions, which are neither in nature nor in the human mind. By its realism, Kronecker's conception is very different from Brouwer's to which it is often associated. It is closer to Frege's. Frege, for instance, judged certain mathematical creations illegitimate and the comparison he made between mathematics and geography recalls the one made by Kronecker between mathematics and the natural sciences. "A mathematician, as a geographer, does not do what he wants; both do nothing else but discover what exists, and give it a name"[11]. Nevertheless, the difference between Frege and Kronecker is that Frege's realism grounded in logic whereas that of Kronecker is grounded in mathematics.

[10]"Als die speziellen Disziplinen unserer Wissenschaft betrachte ich nämlich: die Mechanik, welche mit dem Begriffe der Zeit operiert, die Geometrie, welche die von der Zeit freien, räumlichen Verhältnisse untersucht und die von Raum und von Zeit freie, sogenannte reine Mathematik, welche ich als 'Arithmetik' bezeichnen möchte" [Kronecker 1891, lecture 2, p. 10, in Boniface-Schappacher, 2002, 227].

[11][Frege 1884, 108].

3. Nominalist or restrictive consequences of Kronecker's realism

Regarding objects, Kronecker's realism was more restrictive than Frege's. In addition to positive integers, Frege indeed considered as arithmetical objects all numbers obtained by widening the concept of number, such as the negative integers, the fractional numbers, the irrationals, the complex numbers, etc. Frege also accepted Cantor's transfinite numbers because, like the preceding numbers, they could also be defined in a purely logical way. For Kronecker, on the other hand, positive integers were the only numbers to be accepted as basic arithmetical objects, because they were the only numbers to be consistent with the experience of counting. It was then not necessary to create other entities which, moreover, would denature the concept of number. Thus, such unnecessary creations were to be avoided. Such was the nominalist aspect of Kronecker's conception.

Kronecker's realism led him to accept only constructive definitions and proofs of existence:

> The point of view on which I desagree with most mathematicians resides in the basic assertion that mathematics and the natural sciences — which have recently been separated by this name from the remaining sciences, the so-called sciences of the mind (*Geisteswissenschaften*) — must not only be free of contradiction, but must also result from experience and, what is even more essential, must dispose of a criterium by which one can decide, for each particular case, whether the presented concept is to subsume, or not, under the definition. A definition which does not achieve this, can be advocated by philosophers or logicians, but for us mathematicians, it is a bad nominal definition. It is worthless.[12]

In the same vein, Kronecker only accepted proofs of existence as rigorous because of providing a method to exhibit the object. This constructivity requirement explains why Kronecker is considered as a precursor

[12] "Der Standpunkt, welcher mich von vielen andern Mathematikern trennt, gipfelt in dem Grundsatz, dass die Definitionen der Erfahrungswissenschaften, — d.h. der Mathematik und der Naturwissenschaften, welche man neuerdings unter jenem Namen von den übrigen Wissenschaften, den sogen. Geisteswissenschaften trennt, — nicht bloss in sich widerspruchsfrei sein müssen, sondern auch der Erfahrung entnommen sein müssen, und was noch wesenlicher ist, das Kriterium mit sich führen müssen, durch welches man für jeden speziellen Fall entscheiden kann, ob der vorliegende Begriff unter die Definition zu subsumieren ist, oder nicht. Eine Definition, welche dies nicht leistet, mag von Philosophen oder Logikern gepriesen werden, für uns Mathematiker ist sie eine bloße Wortdefinition und ohne jeden Wert." [Kronecker 1891, lecture 6, 28, in Boniface-Schappacher, 2002, 240].

of Brouwer's intuitionism. Indeed, for Brouwer just like Kronecker, to be is to be built. However, the nature of the resulting constructions is not the same for the two mathematicians. As we already underlined, for Kronecker, the results of constructions were concrete algebraic expressions which are visible in the outside spatial world. For Brouwer, on the other hand, they are of mental in nature. Indeed, for him, the Kantian apriority of space must be abandoned for 'the Primordial Intuition of Time' which created mathematical 'Two-ity' and the ordinal numbers as well as the continuum. The requirement of constructivity, which was common to Kronecker and Brouwer, had no place in Frege's conception of mathematics. Hilbert fiercely critized Brouwer and Weyl for what he called in 1922 'a dictatorship of prohibitions *à la* Kronecker'. But he then recognized the need for a secure foundation of mathematics and referred to Kronecker in most of his foundational articles, identifying his 'finitism' with Kronecker's position[13]. Today, the restrictive feature of the constructive tendancies, which would imply a 'mutilation' of mathematics is generally emphasized. I will show in the following section how Kronecker's adherence to the constructivity requirement and to his epistemological position influenced his mathematical practice and the development of his mathematical work.

4. Operations and methods

For Frege, Russell and Hilbert, founding mathematics resided essentially in putting the mathematical edifice on a solid base. For Kronecker, just as for Brouwer, the question was less that of the *base* than that of the *manner of doing mathematics*. It is foremost a question of methods and concern about operative concepts rather than about objects or the concepts of objects. Kronecker's methods, though constructive, had led to an important development of the subject matter, at least as to the same degree as with less restrictive or less constructive approaches as we will see.

In order to give an idea of these methods, I shall compare the notion of number fields elaborated by Kronecker in his *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* (*Elements of an arithmetic theory of the algebraic magnitudes*), to the one introduced by Dedekind and which is still in use today. Let us first underline that Kronecker did not use the term 'field' (*Körper*) introduced by Dedekind. He rather

[13]He said in 1931: "At roughly the same time [Hilbert was speaking of the year 1888], and, therefore, more than a generation after, Kronecker clearly expressed a conception which he illustrated with numerous examples; this conception today essentially coincides with our finite mode of thinking" [Mancosu 1998, 267].

prefered the term 'domain' (*Bereich*). In a general way, he considered it "appropriated to avoid in the terminology the expressions having strong spatial connotation and to use only general, hardly avoidable expressions, such as the word 'domain' — or to resort to general images"[14]. He used expressions which voided invocation of the idea of spatiality. His aim was to establish a classification of the magnitudes he was dealing with, the criterium of which was algebraic and not spatial. However, the term 'magnitude' for him had no meaning of measure but designated every 'arithmetico-algebraic form'. These forms which constitute the objects of Kronecker's mathematics, will be then classified into kinds and species of a naturalized mathematical science.

Kronecker determined a field from a finite number of non rational fixed magnitudes $R', R'', R''', \ldots$ and considered the field to be the set of all the rational functions of these magnitudes with integral coefficients and called it the 'rationality domain' (*Rationalität-Bereiche*) and denoted it by $(R', R'', R''', \ldots)$. This definition highlights a field as an extension of $\mathbf{Q}$ the field of the rational numbers. It naturally allows new extensions obtained by the method of adjunction introduced by Galois for the resolubility of algebraic equations. Furthermore, Kronecker's definition embraces both transcendental and algebraic extensions and underscores the isomorphism between $\mathbf{K}(R)$ and $\mathbf{K}(X)$ in the case where R is transcendental and X is an independent variable. Kronecker distinguished two types of rationality domains which he called 'natural domains' in the sense that such domains are naturally limited. The first type is the field of rational numbers "which in some way represents the absolute unit of the rationality concept". The second type obtained by taking $R', R'', R''', \ldots$ as independent variables (ie. the fields of rational functions defined on $\mathbf{Q}$). The latter type of rationality domains are arbitrarily delimited on account of the arbitrary choice of elements to include by adjunction. Dedekind did not agree with this arbitrary choice in the construction of the rationality domains. In complete opposition to Kronecker, Dedekind defined a field as a set closed under the four arithmetical operations:

> By field we mean every infinite system of real or complex numbers, so closed and complete that the addition, the substraction, the multiplication and the division of any pair of numbers of the system always produce a

[14]"Eben desshalb (...) halte ich es für angemessen, in der Terminologie die Ausdrücke mit entschieden räumlichem Gepräge zu vermeiden und nur solche, kaum zu umgehende allgemeine Ausdrücke — wie eben jenes Wort 'Bereich' — oder allgemeine Bilder zu gebrauchen (...)" [Kronecker 1881, 250].

> number belonging to the same system. The smallest field is formed by all the rational numbers, the biggest by all numbers[15].

Moreover, Dedekind chose to designate the field by a single letter, generally $\mathbf{K}$, even if this field is an explicit extension of a smaller field (for example of $\mathbf{Q}$ the field of the rational numbers). He reproached Lipschitz for disfiguring (*verunzieren*) the concept of number field by introducing the notation $\mathbf{Q}(\alpha)$ in his account[16].

Compared to Dedekind's definition, Kronecker's definition emphasizes the dynamic and operative features. Indeed, Dedekind's definition lays emphasis on the stability of the set for the four arithmetical operations and therefore on the closure of the set. He stated that he chose the name 'Körper' because of its use in many sciences, referring to things that are 'complete, entire and closed'. In constrast Kronecker's definition focuses on the notion of adjunction of elements providing an unlimited potentiality, since a domain can be extended at every moment by progressive adjunction of magnitudes outside it. The differences in the definitions and the notations adopted by the two mathematicians on the concept of field reveal differences in their respective epistemological positions. According to Dedekind, a field is a mathematical object and, hence, it has to be given by its characteristic properties and not by a representation, always open to arbitrariness or to particularity. To Kronecker, on the contrary, a field is more an operative concept than an object. It must be given concretely by a representation, by a symbol or by a sign, with a specification of the operation to perform (here the adjunction of an element). Thus, for Dedekind the goal is that of conceptual simplicity, while for Kronecker, it is that of operative effectivity.

Conclusion

The foundations of mathematics, as of every science, has two components: the foundation of the basic objects and the determination of the allowed operations and methods. As the remarks ending the second lecture of his 1891 course testify, Kronecker was rather sceptical about the first of the two components:

> Nobody will contest that the greatest mathematical results of the preceding century and of the beginning of the current one are due to Euler, Lagrange, Laplace, Cauchy and Gauss. The erroneous results of these men's works

[15] [Dedekind 1871, *Werke*, 224].
[16] [Dedekind 1876, Werke, 469].

> must be put under a microscope, in spite of the supposed shortcomings of the foundations. I would like to highlight this noteworthy phenomena by a comparison. You all know the guessing game in which everyone at gathering was asked to choose any number he preferred. Everyone was then asked to perform the same sequence of numerical operations on his number. At the end and to the general surprise of all, every participant got the same result. Mathematicians know that any chosen number can be eliminated [...] Explanations of the basic concepts of mathematics are similar; important results turn out to be completely independent of them[17].

Kronecker thus judged it rather uninteresting the foundation of basic concepts. For him, basic concepts must be constructive and deduced from experience. He paid a greater attention to methods and the determination of allowed operations. His epistemological position imposes restrictive demands. This has generally been underlined but extremely caricatured. The result is that Kronecker has rarely been recognized for the fecondity of his approach. Poincaré, for example, observed that Kronecker could only make discoveries in mathematics, "forgetting that he was a philosopher and by leaving his own principles which were in advance doomed to sterility"[18]. These principles however appear as a powerful motive behind Kronecker's mathematics, provided they are well understood. Kronecker himself considered them as a source of freedom and he identified innovation in mathematics with the invention of methods as attests his own mathematical work, an approach which continues to nourish mathematical research today.

[17]"Und doch wird niemand leugnen, daß die größten mathematischen Errungenschaften im vorigen Jahrhundert und in der Wende zu dem unsrigen einem Euler, Lagrange, Laplace, Cauchy, Gauss zu verdanken sind. Die falschen Resultate in den Werken dieser Männer sind trotz der vermeintlichen Mangelhaftigkeit der Begründung des Fundaments mit der Lupe zu suchen. Ich möchte Ihnen diese merkwürdige Erscheinung wieder durch einen Vergleich verständlich machen. Sie kennen alle das Rätselspiel, wonach jemand sämtlichen Personen einer Gesellschaft vorschreibt, jede für sich solle sich eine bestimmte Zahl denken. Alsdann läßt er mit derselben eine Anzahl Rechenoperationen ausführen und bringt es schließlich dahin, daß er von sämtlichen Beteiligten zu aller Erstaunen das gleiche Resultat verkündigen lassen kann. Die Mathematiker wissen, daß sich die von jeder Person gedachte beliebige Zahl eliminiert hat. — Das sind die Erklärungsarten der mathematischen Grundbegriffe; die schließlichen Resultate sind von diesen vollständig unabhängig". [Kronecker 1891, lecture 2, 11-12, in Boniface-Schappacher, 2002, 228].

[18][Poincaré 1898, 17].

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