Scientific Intuition of Genii Against Mytho-'Logic' of Cantor's Transfinite 'Paradise'

Alexander A.Zenkin
Computing Center after A.A.Dorodnitsyn of the Russian Academy of Science (Moscou)

Abstract: In the paper, a detailed analysis of some new logical aspects of Cantor's diagonal proof of the uncountability of continuum is presented. For the first time, strict formal, axiomatic, and algorithmic definitions of the notions of potential and actual infinities are presented. It is shown that the actualization of infinite sets and sequences used in Cantor's proof is a *necessary*, but hidden, condition of the proof. The explication of the necessary condition and its factual usage within the framework of Cantor's proof makes Cantor's proof invalid. It's shown that traditional Cantor's proof has a second *necessary*, but hidden as well, condition which is teleological by its nature, i.e., is not mathematical. The explication of the second necessary condition makes Cantor's statement on the uncountability of continuum unprovable from the point of view of classical logic.

One of the most dramatic facts in history of science is connected just with the notion of actual infinity and consists in the following. On the one hand, Aristotle, Berkeley, Locke, Descartes, Spinoza, Gauss, Kant, Cauchy, Kronecker, Hermite, Poincaré, Bair, Borel, Brouwer, Quine, Wittgenstein, Weyl, Luzin, and a lot of other outstanding *creators* of classical logic and classical mathematics, during millenniums, stated categorically and insistently warned about that the actual infinity is a self-contradictory notion and its usage in mathematics is inadmissible (e.g., according to Poincare, "*There is no actual*

Philosophia Scientiæ, 9 Cahier 2, 2005, 145–163.

infinity. - Cantorians forgot that and fell into contradictions. Later generations will regard Mengenlehre (set theory) as a disease from which one has recovered”). On the other hand, a lot of outstanding scientists of the XX c.c., such as Hilbert, Church, Turing, Gödel, etc., and modern axiomatic set theory as a whole ignored this (intuitive though) opinion of genii and accepted Cantor’s “transfinite paradise”, based on the actual infinity. In the paper some logical, philosophical, and psychological aspects of this historical “favorable opposition” are analyzed, and the fact that the scientific intuition of genii has won the historical battle against the mytho-’logic’ of Cantor’s transfinite ‘paradise’ is explained and logically justified.

1 Introduction

Cantor’s set theory is a Trojan Horse of the mathematics-XX: on the one hand, it is a natural, visual, universal language to describe mathematical objects, their properties and relations, originating from Aristotle’s syllogistics and the famous Euler’s “logical circles”, and just therefore this language was accepted by all mathematicians with a natural enthusiasm. However, on the other hand, together with the language, Cantor’s transfinite conceptions and constructions (like the actualization of all infinite sets, a distinguishing of infinite sets by a number of their elements (by their cardinalities), the hierarchy of ordinal and cardinal transfinite numbers, continuum hypothesis, etc.) went into the mathematics-XX. Just the Cantor’s actualization of infinite sets generated a lot of set-theoretical paradoxes and, ultimately, the Third Great Crisis in foundations of mathematics in the beginning of the XX c. The theme itself of the present conference shows that the problem of the actual infinity is not closed and the Third Great Crisis in foundations of mathematics goes on hitherto.

From the earliest times, classical mathematics is, by right, called the “Science on Infinity”, so the understanding of a genuine nature of mathematical infinity has a more closed relation to the foundations of mathematics than, say, a proof of a consistency of *classical* arithmetic, in general, and of the classical Pascal’s mathematical induction, in particular, by means of the meta-mathematical ‘*transfinite* induction’ up to the famous Cantor’s transfinite number:

$$\varepsilon_0 = \omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge\omega^\wedge \ldots$$

By the way, this notation is read in reality not aflat, *i.e.*, “from left to right”, as it is accepted in usual mathematics, but vertically, i.e., “from bottom (the very left 'ω') to top ('$\omega^\wedge \ldots$')" and it realizes a first footstep on the way of the ambitious Cantor's intention to build a transfinite "Stairway up to Heaven" [Cantor 1914], [Cantor 1985], [Katasonov 1999].

Long before Cantor, Aristotle first explicitly distinguished and described two opposite types of infinite — *potential* infinite and actual infinite. The essence of the potential infinity was defined by Aristotle as follows: "*...the infinite exists through one thing being taken after another, what is taken being always finite, but ever other and other*" [Aristotle], [Moor 1993]. Below we shall show that the famous Peano's axiom system from which, according to Poincare, "*almost all mathematics can be deduced*" [Poincaré H. 1983], is a literal and natural formalization of the very essence of the Aristotle's definition of the *potential* infinity (see point 4.).

As to the actual infinity, Aristotle postulated the quite well argued Thesis which, in its later, canonized in Middle Ages, transcription, sounds so: "*Infinitum Actu Non Datur*", i.e., "there will not be an actual infinite. — [...] the infinite has a *potential* existence" [Aristotle]. So, according to Aristotle, the actual infinity is impossible not only in Nature, but also in Science, *i.e.*, it means logically that the actual infinity notion is self-contradictory.

However, in the second half of the XIX Century, Georg Cantor "rejected the scientific authority of Aristotle, Leibniz, Gauss, Cauchy, etc." and declared the *contradictory* Thesis: "*all sets are actual*" [Cantor 1914], [Cantor 1985]. All Cantor's transfinite constructions and his paradigmatic statements about existence of different infinities are based on the actual infinity. Starting with Kronecker, a lot of outstanding logicians and mathematicians rejected Cantor's "Study on Transfinitum", and Poincare was one of the most deep and subtle critics of the basis of Cantor's theory. In particular, Poincare flatly stated: "*There is no actual infinity. — Cantorians forgot that and fell into contradictions. Later generations will regard Mengenlehre (set theory) as a disease from which one has recovered*" [Poincaré H. 1983].

And today there is a lot of mathematicians who reject Cantor's idea on actual infinity. For example, the outstanding logician and expert specialized in foundations of mathematics, S. Feferman writes in his recent remarkable book "In the Light of Logic" [Feferman 1998]:

> "[...] there are still a number of thinkers on the subject <on Cantor's transfinite ideas - AZ> who in continuation of Kronecker's attack, object to the panoply of transfinite set theory in mathematics [...] In particular, these opposing

> points of view reject the assumption of the actual infinite (at least in its non-denumerable forms). Following this up, alternative schemes for the foundations of mathematics have been pursued [. . .] in a direct and straightforward way on philosophically acceptable non-Cantorian grounds.
>
> Furthermore, a case can be made that *higher set theory is dispensable in scientifically applicable mathematics*, *i.e.*, in that part of everyday mathematics which finds its applications in the other sciences. Put in other terms: *the actual infinite is not required for the mathematics of the physical world*".

The same opinion is advanced by Ja.Peregrin ("There is not an actual infinity" [Peregrin 1995]), V.F.Turchin ("*For actual infinity we have no place [. . .] in the global cybernetic theory of evolution and in the constructivist foundation of mathematics*" [Turchin 1991]), P.Vopenka ("*the set theory whose energies were directed to the actualization of potential infinity turned out not to be able to eliminate the potentiality. . .*"), and many other modern experts in foundations of science.

Thus, as we can see, the acceptance of the Cantor's actual infinity conception is not unanimous in the modern mathematical community and it demands further clarification.

However that may be, a quite non-trivial question arises: Aristotle and Poincare, and also Leibniz, Berkeley, Locke, Descartes, Spinoza, Gauss, Kant, Cauchy, Kronecker, Hermite, Bair, Borel, Brouwer, Quine, Wittgenstein, Weyl, Luzin, and a lot of other outstanding creators of the classical, *i.e.*, according to Feferman [Feferman 1998], 'really working' today logic and the classical, 'really working' mathematics stated categorically and warned about that the actual infinity is a self-contradictory notion and its usage in mathematics is inadmissible. However modern meta-mathematics and axiomatic set theory have ignored their opinion and accepted Cantor's theory. Why? Because this opinion was based only upon the intuition (of genii though), but Cantor's set theory was based on his famous theorem on the uncountability of real numbers (continuum) [Cantor 1985]. So, just this Cantor's theorem is the only basis and acupuncture point of modern meta-mathematics and axiomatic set theory in the direct sense that if the Cantor's famous diagonal proof of this theorem is wrong then all the Cantor's transfinite 'paradise' of these sciences fall to pieces as a house of cards.

2 Cantor's Theorem on Continuum's Uncountability

However, this Cantor's theorem is a next Trojan Horse. To understand this, consider the *traditional* Cantor's diagonal proof [Cantor 1985], [Hodges 1998], [Alexandrov 1948], [Capiński & Kopp 1999]. Here $N = 1, 2, 3, ...$, X is a set of all real numbers of the interval (0,1) and, for simplicity, we shall use the binary system to represent real numbers. Remark that as it was shown in [Zenkin 2000b, 1997b], the further conclusions hold for any radix > 2.

Cantor's Theorem (1890). *X is uncountable.*

Poof (by Reductio ad Absurdum method). Assume that X is *countable.* Then there is an enumeration of *all* reals from X. Let

$$x_1, x_2, x_3, \ldots \tag{1}$$

be some arbitrary enumeration of *all* real numbers of X, *i.e.*, a one-to-one correspondence between *all* elements of X and *all* elements of N. Applying his famous diagonal method to enumeration (1), Cantor constructs a *new* (anti-)diagonal real, say, y^*, which, by its construction, differs from every real of enumeration (1), i.e., the new real y^* does not belong to enumeration (1), and, consequently, the given enumeration (1) is *not* an enumeration of *all* real numbers from X. The obtained contradiction, according to the meta-mathematical version of the Reductio ad Absurdum (further - RAA) method, "proves" that the assumption "X is *countable*" is *false.*

3 First Hidden *Necessary* Condition of Cantor's Proof

In the middle of the 20^{th} century, meta-mathematics declared Cantor's set theory as "naïve" [Kleene 1957] and soon the very mention of the term "actual infinity" was banished from all meta-mathematical and set theoretical tractates. The ancient logical, philosophical, and mathematical problem, which during millenniums troubled outstanding minds of humankind, was "solved" according to the principle: "there is no term – there is no problem". So, today we have a situation when Cantor's theorem and its famous diagonal proof are accepted and described in every manual of axiomatic set theory, but with no word as to the "actual infinity". However, it is obvious that if the infinite sequence (1) of Cantor's proof is *potential* then no diagonal method will allow to construct an *individual* mathematical object, *i.e.*, to *complete* the *infinite* binary sequence y^*. Thus, just the *actuality* of the infinite sequence (1) is a *necessary* condition (a Trojan Horse) of Cantor's proof, and therefore the traditional, set-theoretical formulation of Cantor's theorem (above) is simply wrong from the standpoint of classical mathematics, and must be re-written as follows without any contradiction with any logic.

Cantor's Theorem (corrected-1). If X is *actual*, then X is *uncountable*.

This corrected formulation of Cantor's theorem leads to some very non-trivial consequences.

3.1 Relativity of Uncountability

The notorious uncountability of the continuum is interpreted by modern axiomatic set theory as some inner, 'genetic', absolute property of the continuum. However, as the corrected formulation of Cantor's theorem shows, the uncountability is not an absolute property of the continuum, but rather a *conditional* one, valid (if any, see below) only within the framework of the Cantor's paradigm of the actualization of all infinite sets.

3.2 Uncountability as a Matter of Taste

Since so far no Cantorian (Poincaré's term) has proved the actuality of infinite sets, in general, and of the set X, in particular, the actuality of X is an *inauthentic* statement, *i.e.*, from the classical logic point of view, it is a *second assumption* of Cantor's RAA-proof, and we get a Reductio ad Absurdum with … *two assumptions.* But if a deduction, containing two inauthentic premises, has led to a contradiction, it means that *at least one* of the premises is false [Aristotle], [Hodges 1998], [Zenkin 2001]. According to Aristotle's classical logic, this "*at least one*" means that in such a case we have not the only Cantor's conclusion, but the following *three*, equal in logical rights, alternative conclusions:

(i) X is *actual* and X is NOT-*countable* (Cantor).

(ii) X is NOT-actual (*i.e., potential*) and X is *countable* (Aristotle).

(iii) X is NOT-actual (*i.e., potential*) and X is NOT-*countable* (anonymous).

From the classical logic point of view, the Cantor's proof itself is not able to answer the question what an alternative of these three, *including the initial Cantor's statement* (i), is true in reality.

Thus, the famous Cantor's "proof" of the uncountability of the set X of all real numbers *proves nothing* and reduces the sacramental meta-mathematical question as to a differentiation of infinite sets by their cardinalities to a matter of *belief*: if you like the actuality, you can choose the first alternative (i) after Cantor; if one (together with all classical mathematics) trust in Aristotle's Thesis, (s)he can choose the second alternative (ii); somebody who trusts in nothing can privatize the third (so far ownerless) alternative (iii) without any contradiction with any logic.

3.3 The Main Theorem of Meta-Mathematics as Disposable

As is well-known, the famous Pythagoras theorem has been proved by every generation of school children for 2600 years now. And every good pupil gets the same result: $a^2 = b^2 + c^2$. So, in real mathematics any theorem, once proven, retains its validity independently of the number of repetitions of its proof. Apparently, some meta-mathematical theorems do not save such the "peculiarity" of classical mathematics.

Indeed, from the once (supposedly) proven Cantor's theorem (corrected-1), according to the contraposition law of elementary logic which holds even in meta-mathematical logic [Kleene 1957] (remind: from a proven "if A then B" the authentic statement "if NOT-B then NOT-A" is deduced immediately), we obtain the following quite interesting logically and very deep epistemologically conclusion.

Corollary 1. If X is *countable*, then X is NOT-actually, *i.e., potentially*, infinite.

Now repeat literally the traditional RAA-proof of the Cantor's theorem once again. Assume that X is a *countable* set. Then the sequence (1) of all real numbers is countable too. According to Corollary 1, the sequence (1) will be a *potentially* infinite one. However, the famous Cantor diagonal method is not applicable to potentially infinite sets. Consequently, the Cantor's theorem becomes ... unprovable.

So, the famous Cantor's theorem, in contrast to all mathematical theorems, is not provable twice and is, thus, a disposable meta-mathematical theorem. It's already something like ... not a meta-, but a para-"mathematics".

Taking into account that the set N of finite natural numbers is *countable by definition*, we deduce from the Corollary 1, applied to the natural *indexes* of reals in the sequence (1), the following quite unexpected *reliable* consequence.

Corollary 2. The *countable* set $N = 1, 2, 3, \ldots$ of finite natural numbers is *potentially* infinite.

This corollary means that Cantor's enumeration (1) is a 1-1-correspondence between the *actually* infinite set X of real numbers and the *potentially* infinite set N of natural numbers. From the classical logic point of view such the correspondence is absurd.

Thus, from the main Cantor's theorem it follows that the Cantor's axiomatic statement — "*all sets are actual*" (above) — which is the only basis for all his transfinite ordinal and cardinal constructions, is wrong, *i.e.*, according to Poincaré (and Weyl), all Cantor's set theory as well as all modern "non-naïve" axiomatic set theories are really "built on a sand" [Poincaré H. 1983].

4 A Strict Definition of the Concepts of Potential and Actual Infinities

As was said above, the Cantor's 'naïve' set theory as well as all modern 'non-naïve' axiomatic set theories are based on the actual infinity concept in the sense that if, for example, a set of real numbers as well as these numbers themselves as infinite binary sequences are potential in Aristotle's sense then the Cantor's theorem on the uncountability of continuum becomes unprovable and therefore any differentiation of infinite sets by their cardinalities as well as transfinite cardinal and ordinal 'numbers' themselves lose any sense and attraction. Every meta-mathematician and every set theorist knows well this obvious truth.

However hitherto there is not a strict definition of the concept of the actual infinity. This fact generated a widespread opinion that the ideas of potential vs. actual infinity are vague, fuzzy, pure intuitive, that the idea of actual infinity itself makes sense only within a platonistic conception of *philosophy* of mathematics, and therefore all such ideas and discussions are at the informal, philosophical level and can't be a part of the axiomatic set theories [Feferman 1998].

It is a quite strange situation, isn't so? Indeed, as is well known, according to Bourbaki, "logically speaking, it is possible to deduce almost all modern mathematics from a unified source — the axiomatic set theory" [Bourbaki 1965]. It means that the axiomatic set theory pretends to formalize almost any mathematical theory of any level of complexity and informality. Nevertheless, the most power formal technique of modern axiomatic set theory is hitherto not able to formalize the basic concept of Cantor's 'mathematics' — the concept of actual infinity! Why? - Maybe it can't? - It's difficult to trust in such impotence as to such simple point. Maybe it intentionally doesn't wish to do that intuitively bewaring of unpredictable and fatal consequences of such formalization? — Let history solve this riddle.

However that may be, I believe that if strict mathematical definitions of the concepts of the potential and actual infinities would be at last formulated, then such the definitions could help to understand better a lot of basic problems of the modern Foundations of Mathematics, Logic, and Philosophy of Infinity.

I offer here for the first time a quite natural version of such the definitions [Zenkin 2002a].

4.1 Definitions of the Concept of *Potential* Infinite (PI)

Aristotle's Definition of PI-Concept (AZ: the insertions in brackets are mine): "*...the infinite exists through one thing [$n+1$] taken after [>] another [n], what is taken being always finite [$n < infty$], but ever other and other [$n \to \infty$]*".

Axiomatic Definition of PI-Concept

A1 There exists a 'thing' '0', since *any* well-ordered *finite* sequence of 'things' has a first 'thing' (we denote this first 'thing', say, as '0');

A2 [if n is a 'thing' (a natural) then $n+1$ is a 'thing' (a natural) as well] & [$n < n+1$].

A3 There are not 'things' (natural numbers) that are different from those defined by (**A1**) & (**A2**).

As is easy to see, it is a strict, *formal, axiomatic, inductive* definition of the common series of the common *finite* natural numbers:

$$1, 2, 3, \ldots, n, \ldots \qquad (*)$$

and the points A1-A3 are the first three, well known axioms of Peano's arithmetic [Kleene 1957]. From the strict, axiomatic Aristotle-Peano's definition of PI, the following fundamental mathematical property of the PI-series ($*$) follows.

Main Theorem. There does not exist a last element in the series ($*$). PROOF. Assume that n^* is a last element of ($*$). Since n^* is natural then $n^* + 1$ is natural too and $n^* + 1 > n^*$. So, n^* is not a last element of the series ($*$). Contradiction. Q.E.D.

So, the main theorem is a sufficient basis in order to state that the infinity of the series ($*$) is potential, and vice versa.

Algorithmic Definition of PI-Concept

Finally, we can give the following *algorithmic* definition of the PI-notion [Zenkin 2002a].

It is obvious that the process of the generating of the series ($*$) can be presented by the following computer program (or, if you please, by the following simplest Turing machine):

```
BEGIN
 INTEGER i; LABEL L;
      i:=0;
 L:   i:= i+1;                                        (TM)
      PRINT(i);
      GOTO L
END
```

The Turing machine (TM) is simply printing the sequence 1, 2, 3, ... and so on ad infinitum. The process itself realized by the program possesses the following two obvious properties.

- **P1.** The step-by-step PI-process (TM) of the construction of series ($*$) *never arrives at* its 'STOP' ('HALTING') state. Consequently,
- **P2.** The PI-process (TM) can produce no *individual* mathematical object as its *final result.*

We have thus established the following unique historical fact: contrary to the deep-rooted and widespread opinion in modern set theory that the concept of potential infinity is a vague, informal, intuitive one which therefore can't be used within the framework of modern axiomatic set-theoretical systems, this concept was in reality absolutely strictly defined as far back as IV B.C. by Aristotle though in a fully adequate, but verbal form.

4.2 Definitions of the Concept of *Actual* infinite (AI)

Cantor's Definition of the AI-Concept

It might seem that since we have a strict PI-definition and since AI-notion is contrary to PI-notion, then one might get a strict AI-definition by means of a simple logical negation of the PI-definition. However, Cantor's AI-definition is not a trivial logical negation of Aristotle's definition of PI-notion. Indeed, Cantor writes [Cantor 1914] (almost verbatim): "it is well known that a number of *finite* natural numbers in the series (∗) is *infinite*, and therefore there is not a last number in (∗), <*i.e.*, the series (∗) is *potential* - AZ>; however contradictory it might seem <it's really contradictory very much, as Cantor himself was well understanding! - AZ>, there is in fact no absurdity <"*The essence of pure mathematics is its freedom*"! - So any fantasies are admissible! - AZ> in that to *denote* the *series* (∗) as a whole with a *name* (or *symbol*), say, 'ω', to *call the name* 'omega' an *integer* and then to go on further a 'count':

'omega', 'omega'+1, 'omega'+2, 'omega'+3, ...

and, one can add, to make that in a complete conformity with ... the Aristotle-Peano's axiom: "if a 'thing' is <*called* - AZ> integer then the 'thing'+1 is integer too *for any* 'thing' entirely independently on a real nature of the 'thing' and what we think of the 'thing' (*i.e.*, independently on whether the 'thing' is formal or informal, variable or constant, potential or actual, etc.).

In a word, Cantor accepts Aristotle-Peano's axioms A1-A3 and the main Theorem-1 for PI-objects (otherwise even axiomatic set theorists would be forced to repudiate his AI-definition of the series (∗) and all his 'theory' of transfinite ordinals as having no relation to mathematics), but rejects the properties P1 and P2 of the *algorithmic* definition of PI-notion as follows.

P2*. The PI-process (TM) *produces (as its final result)* a series (∗) as an *individuall* 'mathematical' object - as the famous Cantor's minimal transfinite ordinal type 'omega' that is a "completed, not changeable, but definite and invariable in all its parts" entity [Cantor 1914]. And *therefore*

P1*. The step-by-step PI-process (TM) of the construction of series (∗) *arrives at* its 'STOP' state (but has not a last (*halting*) *step*!).

REMARK 1. The point P1* is a necessary feature of the actual infinity definition, since otherwise the process (TM) must continue ever, e.g., in the time when Cantor claims the PI-series (∗) an integer 'omega' and constructs his known 'transfinite stairs up to a heaven' [KATASONOV 1999]. That is if P1* does not hold then Cantor's 'omega' can't be a "completed, not changeable, but definite and invariable in all its parts" *individual* mathematical object.

However that may be, the main argument of modern axiomatic set theorists against a formal explication of the AI-notion becomes unsound and now there are not sensible, at least, logical objections against the explicit inclusion of Cantor's axiom into any modern axiomatic set theory.

REMARK 2. The original Cantor's definition of the actual infinity of the PI-series (*) is used above because today there is not a better definition of the AI notion in all modern 'non-naive' axiomatic set theories. So, a true sense of Cantor's algorithmic AI-definition above is as follows: of course, the *infinite* series (∗) is potential, but "however contradictory it might seem, there is in fact no absurdit" in order to claim it completed and to accept that the *potentially infinite* series (∗) is *actual*. Possibly, for somebody such Cantor's AI-definition may seem to be inconsistent, but, unfortunately, classical logic is powerless before such 'argumentation' and is not able to prove strictly this inconsistency, since Cantor 'transforms' this evident inconsistency into an *axiomatic definition* of the AI-concept.

REMARK 3. As was shown above, Cantor accepts the Aristotle-Peano's axioms A1-A3 and the Main Theorem, *i.e.*, in reality he accepts the formal definition of *potential* infinity, but rejects only algorithmic consequences of the definition. Just this, according to Wittgenstein (see [Wittgenstein 1956], [Hodges 1998]), "skilful trick" masks the contradictory nature of Cantor's definition of actual infinity.

REMARK 4. Our formalization of AI-concept above was based on Cantor's definition relating to the *discrete* series (∗). However there is another Cantor definition of AI concerning the *continuum*. Here is this definition (almost verbatim). The notion of a continuous function is a basic object of mathematical analysis. However before to speak about mathematical properties of a continuous function given on a segment (domain), say, $X = [0, 1]$, the domain itself must contain *all* its elements (points or real numbers), *i.e.*, the set X must be a "completed ... and invariable in all its parts" entity, *i.e.*, the infinity of the set X of reals must be *actual*.

On the face of it, the definition really seems to be a clear, visual, and convincing one: everybody is allegedly seeing the points, 0 and 1, drawn, say, by a chalk on a blackboard, and is intuitively sure that every point between 0 and 1 can be "seen" in the similar way. However, in reality the problem on whether the set X is actual is reduced to the new/old question what the real number is [Gowers 2000, Zenkin 2003]: in the most common case, a real number is an *infinite* sequence of 0s and 1s and therefore the set X may be actual iff actual is every infinite sequence, presenting a real number from X. The last means that actual must be a sequence of natural *indexes* $\{1, 2, 3, \ldots\}$ of the digits of the sequence. Thus the question on the *actuality* of the *continuum* X is ultimately reduced to the initial question on whether one accepts/rejects the actuality of the *discrete* series $(*)$.

5 Second Hidden *Necessary* Condition of Cantor's Proof

The known meta-mathematical logician W.Hodges analyzes very professionally and criticizes hard a lot of objections against "Cantor's diagonal argument" in his famous paper [Hodges 1998]. In particular, he writes:

> "It was surprising how many of our authors failed to realise that *to attack an argument, you must find something wrong in it* <the italic type here and further is of mine - AZ>. [. . .] The commonest manifestation <of the objections against Cantor's proof - AZ> was to claim that Cantor had chosen the *wrong enumeration of the positive integers*. His argument only works because the positive integers are listed in such a way that each integer has just finitely many predecessors. If he had re-ordered them so that some of them come after infinitely many others, then he would have been able to use these late comers to enumerate some more reals, for example, <the non-indexed Cantor's anti-diagonal real y^* - AZ> [. . .] The existence of a different argument that fails to reach Cantor's conclusion tells us nothing about Cantor's argument."

These Hodges' statements contain at least *three* logical errors: 1) the "different argument" does not simply "fail to reach Cantor's conclusion", but it *disproves* the "Cantor's conclusion"; the last, from the logic point of view, tell us quite much about a doubtful legitimacy of 'Cantor's diagonal argument'; 2) Cantor's diagonal rule (here: "if a diagonal digit is 0 then the corresponding digit of y^* is 1, and vice versa") is applied to real numbers themselves in the sequence (1) but not to their indexes, *i.e.*, any re-indexing which does not change the number and order of reals in (1) does not change the Cantor's anti-diagonal real y^*, *i.e.*, such the re-indexing is admissible from the point of view of Cantor's diagonal algorithm; and 3) if Hodges (after Cantor) likes some ("good") enumerations and does not like other ("bad") ones he as a symbolic logician ought to formulate a legible *logical* criterion to tell the difference between these "good", very desired (by Cantorians) indexings of reals in (1) and "bad", strictly forbidden ones.

Unfortunately, Hodges did not do that. But if to translate his pure intuitive and quite doubtful meta-mathematical "veto" on the "bad" enumerations that "fail to reach Cantor's conclusion" into the common mathematical language, we shall have the following *second* tacit *necessary* condition of Cantor's proof [Zenkin 2003, 2004].

Hidden Cantor-Hodges' Postulate. From the assumption of Cantor's RAA-proof that "X is *countable*", it follows that X is equivalent to *any countable* set, *i.e.*, there is a one-to-one correspondence between X and any other countable set, e.g., any proper infinite subset of N. However, *within the framework of Cantor's diagonal proof*, only such one-to-one correspondences, or indexings of reals in (1), are admissible which utilize *all* elements of $N = \{1, 2, 3, \ldots\}$. Any other indexings which utilize *not all* elements of the set N are forbidden categorically.

I would like to underline here that the hidden Cantor-Hodges' postulate is just the second *necessary* condition of Cantor's diagonal argument since only 'good' indexings of reals in (1) allow "to reach Cantor's conclusion", but any "bad" indexings, according to W.Hodges, "fail to reach Cantor's conclusion", *i.e.*, make Cantor's theorem unprovable.

Thus, the finally completed and explicit formulation of the statement which Cantor tries to prove is in reality as follows [Zenkin 2004, 2003, 2001, 2000b, 1999]:

Cantor's Theorem (corrected-2). The set X is *uncountable iff*: (a) the set X is *actually* infinite; (b) the hidden Cantor-Hodges' postulate holds.

As it was shown above, the first necessary condition (a) of Cantor's theorem makes it invalid. As regards the second necessary condition (b), within the framework of the diagonal proof, the hidden Cantor-Hodges' postulate does not follow from the Cantor's RAA-assumption that "X is *countable*", roughly violates the transitivity law of the equivalence relation for (any) countable sets, and is simply a *teleological* one: only such indexings are admissible which allow to reach a *desired* result, all other indexings which use *not all* elements of N and therefore "fail to reach Cantor's conclusion" are inadmissible. As is well known *any teleological* arguments have no relations to classical logic and classical mathematics.

6 Why Kronecker, Poincaré and other Mathematicians Could not Disprove the Ten Strings of Cantor's Diagonal Proof *Mathematically*? [Zenkin 2001]

Now we have a unique possibility to solve, for the first time, a very nontrivial historical (and *psychological*) riddle, viz. to answer the question why the greatest mathematicians of the XIX-XX centuries, such as Kronecker, Poincare, Brouwer, and others, who categorically rejected Cantor's "Study on Transfinitum", could not prove *mathematically* the logical failure of the Cantor's basic theorem as it is usually accepted in mathematics?

The answer consists in the following. As is known, the "dubious" mathematical theorem can be disproved, at least, by the two ways: 1) if it's possible to detect an error in the proof itself (as, *e.g.*, W.Hodges demands that), or 2) if it's possible to prove that *even one necessary* condition of the proof is logically inconsistent (false, contradictory or simply absurd).

In this connection, the meta-mathematical "sarcasm" of the symbolic logician W.Hodges — as to "It was surprising how many of our authors failed to realize that *to attack an argument, you must find something wrong in it.*" — looks quite strange, since every really working mathematician knows well that in mathematics any proof is absurd not only when it contains "something wrong", but also when even one *necessary* condition of the "proof" is absurd.

However that may be, the Cantor's diagonal procedure itself, apparently, does not contain direct formal errors, — otherwise, I believe, the really working mathematicians would detect such errors long ago, — that is why the first, common way to disprove Cantor's theorem turned out not too effective. Nevertheless, the second way remains - to prove, that "even one necessary condition of the proof is absurd". And here we come across the unique situation in the all millennial history of mathematics, namely, the traditional meta-mathematical formulation of the famous Cantor's theorem on the continuum uncountability contained never the necessary conditions of its own proof in an explicit form. But it is obvious, that it is impossible to refute or to prove an absurdity of what did not exist at all! Just therefore neither Kronecker and Poincare, nor any other Cantor's opponents had ever a "physical" possibility to give a common mathematical refutation of his main theorem: no mathematician, even most ingenious one, simply could ever take into his head a thought itself that it's possible during hundred years, according to W.Hodges, "to teach other people" how to formulate and prove the, ostensibly, mathematical Cantor's theorem [Hodges 1998], omitting ... the necessary conditions of its proof. From the point of view of really working mathematics and Aristotle's logic, it is a scandalous, pathological nonsense in history of mathematics (Brouwer), and, according to V.I.Arnold, the nonsense with "very harmful and grave scientific, pedagogical, and social consequences" [Arnold 1999], [Zenkin 2000a].

7 Conclusions

1. The traditional Cantor's proof of the uncountability of real numbers contains two hidden necessary conditions. The explication of these two conditions makes Cantor's proof invalid.

2. The great intuitive insight of Aristotle, Poincare, and other outstanding logicians and mathematicians as to "Infinitum Actu Non Datur" is rigorously proved.

3. It is obvious that the logical failure of Cantor's theorem on the uncountability of continuum changes essentially traditional logical and methodological paradigms of mathematics-XX and philosophy of infinity, and opens a way to solve main problems connected with the I, II, and III Great Crises in foundations of mathematics. [Kleene 1957], [Zenkin 2004-1999, 1997].

References

Alexandrov, P.S. 1948 *Introduction to Common Theory of sets and functions*, Moscow-Leningrad: Gostehizdat.

Aristotle
Physics, Book III.

Arnold, V. I.

1999 Anti-Scientific Revolution and Mathematics, *Vestnik RAS*, 1999, No. 6, 553–558.

Bourbaki, N.

1965 *Set Theory* Moscow : "MIR", 1965

Cantor, Georg

1914 *Foundations Of Common Study About Manifolds*, A.V.Vasil'jev ed. New ideas in Mathematics, 6, Sankt-Petersburg.

1985 Proceedings in Set Theory, Moscow: NAUKA, 1985

Capiński, M., and Kopp, E.

1999 *Measure, Integral and Probability*, London: Springer, 1999.

Feferman, S.

1998 *In the Light of Logic*, Logic and Computation in Philosophy series, Oxford: Oxford University Press, 1998.

Gowers, T.

2000 What is so wrong with thinking of real numbers as infinite decimals? http://www.dpmms.cam.ac.uk/w̃tg10/decimals.html

Hodges, W.

1998 An Editor Recalls Some Hopeless Papers, *The Bulletin of Symbolic Logic*, 4 (1), 1–17 1998.

Katasonov, V.N.

1999 *The Struggled One Against Infinite. Philosopical and Religious Aspects of Cantor's Set Theory Genesis*, Moscow: Martis, 1999.

Kleene, S.

1957 *Introduction to metamathematics*, Moscow: MIR, 1957.

Moor, A.W.

1993 *The infinity. - The Problems of Philosophy. Their Past and Present*, Ted Honderich ed., London - New-York: 1993.

Peregrin, Ja.

1995 "Structure and Meaning" at:

http://www.cuni.cz/p̃eregrin/HTMLTxt/str&mea.htm

Poincaré, H.

1983 *On Science*, Moscow: Science, 1983.

TURCHIN, V. F.

1991 "Infinity", at: http://pespmc1.vub.ac.be/infinity.html

WITTGENSTEIN, L.

1956 *Remarks on the foundations of mathematics*, Oxford: Blackwell, 1956.

ZENKIN, A.A.

2004 On Logic of "plausible" meta-mathematical misunderstandings, (All-Russian Conference "Scientific Session of MEPI-2004".) Moscow Engineering & Physical Institute, *Proceedings*, 2004, 182-183.

2003 A priori logical assertions with a zero ontology, in *Mathematics and Experience*, A. G. Barabashev, (ed.), Moscow: Moscow State University Publishing House, 2003, 423–434.

2002a As to strict definitions of potential and actual infinities, FOM-archive at:

http://www.cs.nyu.edu/pipermail/fom/2002-December/006072.html

2002b Gödel's numbering of multi-modal texts, *The Bulletin of Symbolic Logic*, 8, (1), 180, 2002.

2001 Infinitum Actu Non Datur, *Voprosy Filosofii (Problems of Philosophy)*, 9, 2001, 157–169.

2000a Scientific Counter-Revolution in Mathematics, Moscow: NG-SCIENCE, Supplement to the Independent Newspaper: *Nezavisimaya Gazeta*, 19 July 2000, 13.

2000b Georg Cantor's Mistake, *Voprosy Filosofii (Problems of Philosophy)*, 2, 2000, 165–168. See also:

http://www.ccas.ru/alexzen/papers/The_Cantor_Paradise.htm,

http://www.hist-analytic.org

and a discussion about Cantor's Diagonal Proof at:

http://www.philosophy.ru/library/math/cantor.htm (in Russian).

1999 Cognitive (Semantic) Visualization of The Continuum Problem and Mirror Symmetric Proofs in the Transfinite Numbers Theory, *Visual Mathematics*, 1 (2).

http://members.tripod.com/vismath1/zen/index.html.

1997a Cognitive Visualization of the Continuum Problem and of the Hyper-Real Numbers Theory, Intern. Conference "Analyse et Logique", UMH, Mons, Belgia, 1997. Abstracts, 93–94.

1997b “Cognitive Visualization of Some Transfinite Objects of Cantor Set Theory”, in *Infinity in Mathematics: Philosophical and Historical Aspects*, A. G. Barabashev, ed., Moscow: Janus-K, 1997, 77–91, 92–96, 184–189, 221–224.