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Beth and Bernays on Intuitionism

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Abstract. What is common to Beth's and Bernays' reflections about intuitionism concerns its philosophical aspects, in particular its basic notion: mental evidence as something irrefutable and fixed forever. At the beginning Bernays, as a collaborator of Hilbert's, considered intuitionism too extremistic since it assigned a foundational role exclusively to (mental) evidence, neglecting the role of abstraction. Later, when he approached Gonseth's epistemology, he even stated that evidence as something fixed forever cannot exist. What is evident can vary when the "horizon of experience" varies: evidences are acquired. On his side Beth at first believed that intuitionism was the only reliable foundational school and used this fact to defend Kant's epistemology. Later he started to doubt the evidence of natural numbers and more and more came to believe that in general evidence is not reliable. In 1950 he enlarged on his reflection about evidence by including it among the postulates of the Aristotelian theory of science, deducibility and reality being the other postulates. All of these are unreliable. As Kant shared the Aristotelian theory of science Beth concluded to the necessity of abandoning Kant's philosophical system. Furthermore, as Kant's and Brouwer's thought had led him to underestimate logic, Beth felt the need to re-evaluate logic and to devote himself to it (and obtained many interesting metatheorical results). Beth and Bernays had direct exchanges of ideas about the notion of evidence. In 1943, when Bernays still believed in a limited philosophical role for evidence, Beth wrote him that, although some evidences in mathematics may exist, it is very difficult to express them in an unexceptionable way, and that language itself contributes to make concepts evident. Later, in 1958, when Bernays did not believe in evidence at all, Beth, starting from a philosophical analysis of evidences, shared with Bernays the idea of acquired evidences. Finally, Beth stressed that in the literature "intuition" is often confused with "evidence". He recognised the presence in mathematics of a "creative intuition", of a "global intuition" and of an "intuition of the infinite". The existence of intuition was also supported by the Löwenheim-Skolem paradox and had as a consequence that reality is not a unique block but is built by various spheres (logic being one of them). The relationship between the spheres was described by Beth with reference to Bernays' notion of complementarity.

Résumé. Ce qui est commun à Beth et Bernays dans leurs réflexions sur l'intuitionnisme concerne les aspects philosophiques, en particulier leur thèse selon laquelle l'évidence mentale est quelque chose d'irréfutable et d'établie pour toujours. Bernays, étant un collaborateur de Hilbert, considère l'intuitionnisme trop extrémiste parce qu'il attribue un rôle fondationel exclusivement à l'évidence (mentale), en oubliant le rôle de l'abstraction. Plus tard, quand il s'approche de l'épistémologie de Gonseth, il se rend compte que l'évidence établie pour toujours n'existe pas. Ce qui est évident peut varier quand l'"horizon d'expérience" varie : les évidences sont acquises. De son côté, Beth croit d'abord que l'intuitionnisme est la seule école fondationnelle valable et utilise cet aspect pour défendre l'épistémologie Kantienne. Plus tard, il va douter de l'évidence des nombres naturels et met en question l'évidence en général. En 1950, il étend ses réflexions sur l'évidence en l'insérant parmi les postulats de la théorie Aristotélicienne des sciences, la déductibilité et la réalité étant les autres postulats. Ils manquent tous de fiabilité. Puisque Kant partageait la théorie Aristotélicienne des sciences, Beth abandonne le système philosophique Kantien. De plus, comme la pensée de Kant et de Brouwer l'ont porté à mésestimer la logique, Beth s'engage à la réévaluer et à devenir lui-même logicien (en obtenant plusieurs résultats métathéoriques intéressants). Beth et Bernays ont des échanges d'idées sur la notion d'évidence. En 1943, quand Bernays croit encore en un rôle philosophique limité de l'évidence, Beth lui écrit que, bien qu'il y a de l'évidence dans les mathématiques, il est très difficile de l'exprimer de manière parfaite, et que le langage lui-même contribue à rendre évidents les concepts. Plus tard, en 1958, quand Bernays ne croit plus à l'évidence, Beth, à partir de son analyse philosophique des évidences, partage avec Bernays l'idée d'évidence acquise. Enfin, Beth remarque que dans la littérature le concept d'"intuition" est souvent confondu avec celui d'"évidence". Il reconnaît la présence dans les mathématiques d'une "intuition créative", d'une "intuition globale" et d'une "intuition de l'infini". L'existence de l'intuition est aussi confirmée par le paradoxe de Löwenheim-Skolem qui implique que la réalité n'est pas un bloc unique mais est formée par plusieurs sphères (la logique étant l'une d'entre elles). La relation entre ces sphères est décrite en utilisant la notion de complémentarité de Bernays.

The topic of intuitionism represented for both Beth and Bernays an opportunity for developing and specifying reflections on many interesting epistemological themes. Some of them were common to both authors, who eventually arrived at the same conclusions.

1. Bernays and intuitionism

Bernays' interest in intuitionism concerned its epistemological position, i.e. its emphasis on evidence as the only source of certainty and truth.

At the very beginning, when he was a student and collaborator of Hilbert he firmly believed that only a balanced contribution of abstraction and evidence could really be a source of human knowledge. He identified the couple evidence/abstraction with the couple finite/infinite (in mathematics), as he shared Hilbert's project for founding and preserving classical mathematics[1]. With this in mind, he first distinguished between finitistic mathematical notions considered as evidences (on which also intuitionists should agree, as such notions coincided with the mathematics that they accepted), and those mathematical concepts that required some reference to the infinite considered as a form of abstraction (ideal). Then he had the aim of proving the non-contradictoriness of the latter on the basis of the former. He was convinced that such a program was performable. After Goedel's results, he took note both of the fact that it was impossible to prove the non-contradictoriness of the whole of mathematics on the basis of a part of it (the finitistic part) and of the fact that finitistic mathematics was stricter than intuitionistic mathematics. The couple evidence/abstraction could no longer coincide with the couple finite/infinite and the role of this latter was no longer central in mathematics (as something finite could not prove the non-contradictoriness of the infinite). He maintained his idea that both abstraction and evidence are sources of knowledge, by calling the outlook "moderate platonism", according to distinctions presented in "Sur le platonisme dans les mathématiques" [Bernays 1935]. There he had discussed various outlooks on the role of the subject in mathematics: that of extreme platonism - belief in the absolute independence of mathematics on the subject; that of intuitionism - belief in the total dependence of mathematics on the thinking subject; that of moderate platonism - stressing at the same time both the role of the thinking subject and of reality in mathematics. He then tried to clarify which intuitionistic notions went beyond finitistic mathematics and used his answer to support his opinion: those who tried to base all knowledge only

[1] On this subject see also [Franchella 1997b].

on evidence did not manage to limit themselves to this, therefore both evidence and abstraction are needed.

The first concept mentioned by Bernays as going beyond finitistic mathematics was that of natural number. The second concept was the meaning of the universal quantifier as a method for proving, for each of the mathematical entities considered, the property at issue. In both cases the abstraction consists of putting aside the fact that no flesh and blood mathematician could ever actually perform the constructions required for (respectively) producing a high natural number and specifying the method of proof. Bernays had at first (in 1930) discussed this theme by embracing another position: he considered as epistemologically irrelevant the performability of a certain construction by a living mathematician, because he was convinced that finite and infinite are intrinsically different. A large natural number is in any case something finite; the fact that a human being could not ever in his life construct it is only a contingent fact due to contingent physical conditions [Bernays 1930, 39].

After Goedel's results he realised that there are intuitionistic notions that intrinsically are not finitistic, so he reconsidered his former opinion (he now judged as essential those facts that he had earlier seen as contingent) and identified these notions exactly as those referring to operations that could not ever be performed by a living mathematician. I note here that A. Heyting acknowledged Bernays' criticism (together with Griss' criticism of negation) and presented a scale of evidence of intuitionistically questionable notions where he included high natural numbers and the universal quantifier. He began to write a first list of them [Heyting 1949, 306-307], then revised it [Heyting 1958a, 332-337 and 1958b, 103-104], and finally presented a detailed and definitive version [Heyting 1962b, 195] within a scale of degrees of evidence:

> "The highest grade is that of such assertions as 2 + 2 =4. 1002 + 2 = 1004 belongs to a lower grade; we show this not by actual counting, but by a reasoning which shows that in general (n+2) + 2 = n + 4. Such general statements about natural numbers belong to a next grade. They already have the character of an implication (...) This level is formalized in the free variable calculus. I shall not try to arrange the other levels in a linear order; it will suffice to mention some notions which, by their introduction, lower the grade of evidence.
>
> 1) The notion of the order type ω, as it occurs in the definition of constructible ordinals.

2) The notion of negation, which involves a hypothetical construction which is shown afterwards to be impossible.

3) The theory of quantification. The interpretation of the quantifiers themselves is not problematical, but the use of quantified expressions in logical formulas is.

4) The introduction of infinitely proceeding sequences.

5) The notion of a species."

Eventually, as Bernays pointed out in his 1970 paper, Heyting also recognised the weakness of intuitionism as the other foundational school. He concluded that no foundational school could any longer exist as an absolute foundation of mathematics: mathematical foundational research should change and become a search for capturing the Platonistic, finitistic, and constructivistic aspects of traditional mathematics.

In the meantime, when he moved to Zurich[2] the belief in the joint role of evidence and abstraction made of Bernays an interlocutor of Gonseth. According to Gonseth, every science is a horizon of experience; it does not exhaust all reality but tries to approximate reality by abstraction through concepts that are always open to modifications; this is important, since new problems arise when these concepts are checked against empirical data. Here "empirical" has a wide meaning, including historical and social experience. In every period of time each kind of science has its range of research and of applicability - due to the original problems of applying preceding theorizations. It possesses some basic ideas about its domain (the *veritées préalables*) that in any case only allows for a partial description of reality, because they only designate those aspects of it that are relevant for the topic in question (they are not a picture of the town but simply a map highlighting only the streets and squares of the city). When a new *horizon* is introduced in order to better explain reality, the old *horizon* and the new one can be seen as "complementaries". As Bernays himself recalled in "Über die Ausdehnung des Begriffes der Komplementarität auf die Philosophie" [Bernays 1948b], the concept of complementarity has two meanings, or better nuances, both of them derived from Niels Bohr's physical theory. The first consists in the impossibility for some couples of quantities to be exactly determined both at the same time (such quantities are called complementary). The second refers to comprehensive theories (for instance the corpuscular theory and the wave-theory) that aim to explain the same experiences in two completely different

[2] On this subject see also [Franchella 1997a].

and incompatible ways: such theories are called complementary. In "Überlegungen zu Ferdinand Gonseths Philosophie" [Bernays 1977], Bernays pointed out that complementarity between horizons was of the second type, whereas complementarity of the first type had been used by Gonseth in order to justify his leaving a new concept not entirely specified: a concept cannot be precisely fixed in advance if it is to be used effectively within a theory. Precision and usefulness are complementary properties in the sense that the more defined a concept is, the less serviceable it is.

Within an epistemology of changing horizons evidence can receive only provisional status: what is evident at a certain stage of knowledge may not be regarded as such at a further stage, and new evidences can be acquired. In his cautious acceptance of Gonseth's epistemology, Bernays therefore came to consider the importance of the couple evidence/abstraction as being their relationship, their dialectics. Evidence as something certain, self-evident and forever established has disappeared [see Bernays 1946].

2. Beth and intuitionism

Intuitionism played a role in Beth's scientific career in three main ways. Firstly, as a foundational school; secondly, with reference to its logical system, whose properties had to be studied; thirdly, in its focusing the attention on the concept of evidence.

At the very beginning of his studies Beth saw intuitionism as a foundational school. He lived at that time under what he himself defined as an "exclusively Marburg orientation", which came to him in particular from his supervisor J.C. Franken. This Neokantian education caused him to hold scientific results in high regard, a position which he would never abandon and which induced him to pay attention to new results wherever they came from. On this background he heard of the foundational question in mathematics and became convinced that intuitionism was the only acceptable foundational school. In particular, logicism appeared to him as "out" at that time (during the thirties). As for the other two positions, formalism and intuitionism, he noticed that both appealed to intuition, only intuitionism did it in a more direct way - that is, already in the construction of mathematics itself, while formalism only used intuition at the level of metamathematical analysis. In its turn, this appreciation of intuitionism gave him grounds for defending Kant, as we can see in his 1935 dissertation "Rede en Anschouwing" ("Reason and Intuition") presented at the Faculty of Humanities of Utrecht University. The subject was offered in a competition, proposed by the Faculty on the theme: whether the necessity of space as an *a priori* form of intuition no longer holds, given the possibility of founding geometry on a merely logical basis. This refers to Couturat's criticism of the Kantian theory of spatial intuition in the name of logicism.

Beth began by describing both Kant's theory (in a standard way) and Couturat's criticism of it. Then he proved that there is no substantial difference between geometry and the rest of mathematics. This meant that the question whether in geometry proofs have to be developed in a purely logical way must be transformed into the question whether *in general* mathematical proofs have to be developed in a purely logical way. Here Beth suggested to consider the observations that came from the various foundational schools. Intuitionism led to the statement that mathematical judgements are synthetic, in the sense that they cannot be formed without a previous process of "verification" in (mental) reality. Beth's dissertation therefore concludes in a Kantian style. Mathematical judgements are synthetic a priori, in the sense that they are built after verification and that they are valid forever. This is possible thanks to the cooperation of intuition and reason: intuition builds mathematical objects according to the objective laws of reason.

The influence of intuitionism on this first publication of Beth's can also be recognised in the only point on which he deviates from Kant's opinion. While Kant considered mathematics - like all theoretic sciences - conceivable only as a foundation for empirical sciences, he himself is convinced of the possibility of a purely theoretical construction of mathematics; this is clearly a point he has in common with Brouwer.

Later, Beth began to have some doubts about intuitionism. We find a trace of them in his first criticism of Kant, expressed in his 1938 article "Getalbegrip en tijdsaanschouwing" ("The concept of number and the intuition of time"). In the meantime, he had broadened his horizons by attending conferences and exchanging letters with Fraenkel, Barzin, Errera, Feys, Bernays, Church, Scholz and Tarski. Although he agreed that Brouwer's concept of the intuition of time as a basis for mathematics is acceptable, he regarded the Kantian opposition a priori/a posteriori as unintelligible, since though everyone is convinced of the necessity and generality of the intuitive theory of natural numbers, nobody can give a foundation for this conviction. This is in direct contrast to intuitionism (which bases the theory of numbers on the intuition of time).

In his 1940 booklet *Inleiding tot de wijsbegeerte* (*Introduction to Philosophy*) Beth disavowed any claim to evidence as to the Euclidean axioms and emphasized that none of the foundational schools had the right to consider itself as an absolute basis for mathematics, in particular because "evidence" was not something to be trusted. Consequently Beth also had to change his view about Kant, since, having accepted intuitionism, he had accepted Kant's ideas too.

In 1950, Beth enlarged on his analysis of the failure of foundational schools by deepening his reflections on "theory of science" in general. He identified a theory of science that can be traced back to Aristotle and which is based on three main postulates: reality, deducibility and evidence. This theory remained unchallenged till 1600, when one began to distinguish between rational sciences and empirical sciences. In 1720 the Dutchman Nieuwentyt tried to unify rationalism and empiricism in a non-Aristotelian way: pure mathematics satifies the conditions of deductiveness and evidence; applied mathematics satifies the condition of reality.

At the very beginning of his work, Kant had shared Nieuwentyt's position: that is why he distinguished between synthetic and analytical judgements. After his discovery of Hume, however, Kant subjected mathematical synthetic judgements to a condition of reality (mathematical definitions have to correspond to our space-time intuition). These synthetic a priori judgements represent a return to the Aristotelian theory of any science, where all three conditions are satisfied. In the light of this historical reconstruction Beth drew the conclusion that modern science no longer sees the above theory of science as suitable: physical theories are not evident; quantum logic is not deductive; foundational schools have abandoned the demands of evidence and deducibility.

We shall return in the final paragraph to the notion of evidence. Here we only note that having realised the incompatibility between Aristotle's theory of science and modern sciences, Beth realised that Kant himself had to be abandoned and that there was a need for a new philosophical system. He also felt the need for specifying the general course that the new system should follow, according to him: it should be "rational", i.e. it should be not irrational. He recognised, however, many irrational tendencies in the XXth century. An aspect of these tendencies is the underestimation of logic as having no role in mathematics, and he concentrated on this aspect. As he admitted in his memoirs [Beth 1964, 119], to begin with he had kept away from logic because Kantianism, intuitionism and the movement of "Significs" had all influenced him to do so. Later he learned to appreciate logic through Tarski's semantics (Tarski had lectured in the Netherlands), and he therefore decided to develop this aspect of the battle for reason. The first step was to find the historical grounds for this underestimation. In *La crise de la raison et la logique,* he explained [Beth 1957, 7] that this underestimation had its roots in Descartes and Kant and found corroboration in the case of Brouwer, since all three of them believed that mathematics developed from an intuition of the object.

In his endeavours to re-evaluate logic Beth directly contributed to intuitionism by proving the completeness of a logical intuitionistic system. This was contrary to the intuitionistic viewpoint that had been clearly expressed by Arend Heyting [for instance in 1956, 102], that no formal system can be proved to adequately represent an intuitionistic theory. For, according to intuitionism mathematics is a languageless mental activity, creative and therefore free from laws, that uses methods that cannot be established once for ever. Logic registers the regularities present in the expression of mathematics. Therefore from the intuitionistic viewpoint there is no point in looking for the completeness of logical systems.

Beth did not start from an intuitionistic viewpoint (his metatheory was classical) and his search for completeness had a philosophical reason: to show that logical systems do not impose anything on bodies of natural evidence but that, on the contrary, they extract something which is contained by them: "We can adopt the methods of modern logic without somehow adopting a dogmatic attitude: we are not compelled to impose upon a given body of intuitive arguments a logical structure which is foreign to it, we are able to make explicit any logical structure which it may implicitly contain". [Beth 1956, 381-382]

Summing up, his metalogical results in intuitionism contributed to the larger subject of the "naturalness of logic".

3. Common themes in Beth and Bernays

In re-evaluating logic Beth did not forget the importance of "intuition". This term is frequently confused in the literature with that of evidence (with the meaning of self-evidence). If we keep the two terms (and their meanings) separate we see that Beth very early had abandoned his trust in evidence and consequently his trust in the Aristotelian theory of science. He began by doubting the evidence of natural numbers (1938), then, the evidence of Euclidean geometry and finally evidence in itself (1940). We find traces of an exchange of thoughts between Beth and Bernays on this subject in their correspondence (letters from Bernays dated 31.10.1942 and from Beth dated 14.04.1943). Bernays had at that time affirmed that there was a possibility of some verification of the Euclidean axioms through experience, since the intrinsic limit in distinctness and breadth of evidence was compensated for by a "qualitative, gestaltliche Einstellung":

> "Auch erkenne ich durchaus die Begrenztheit unseres raumanschaulichen Vorstellungsvermögens an, wie sie sowohl in Bezug auf die Schärfe der Unterscheidung wie auch auf den Umfang des Grössenbereiches der anschaulich vorstellbaren Gegenstände besteht. Diese Begrenztheit wird jedoch, wie es mir scheint, in gewisser Hinsicht kompensiert durch eine qualitative, gestaltliche Einstellung, welche ein Moment des Intentionalen in sich schliesst".

Beth observed that he believed there were evidences in mathematics and even as regards reality ("realen Sachverhalten"), only it is always difficult "solche Evidenze einwandfrei und nachprüfbar festzustellen", because "jeder Versuch einer Formulierung riskiert, dass neben der evidenten, gegenständlich bestimmten, Elementen doch wieder formal bestimmte Elemente eingeführt werden". In addition, language, too, contributes to the evidence of concepts: "eine bessere Formulierung bedingt eine Steigerung der Evidenz".

In 1958, after Bernays had lost his trust in evidence as something sure and forever valid, Beth delved into some famous "evidences" from a historical viewpoint. He ended by agreeing with Bernays' concept of "acquired evidences": self-evidence is an unreliable concept; some self-evidences (like those of the Euclidean axioms) are not considered as such any longer, while new ones (like Descartes' *Cogito*) have come into being, as Paul Bernays (whom he explicitly mentions) had already affirmed. [Beth 1958, 233 and also 1961, 125].

On the other hand, Beth left some space open to "intuition" as something creative. He felt obliged to leave space for intuition as a consequence of the Löwenheim-Skolem paradox. This comes from the Löwenheim-Skolem theorem, which states that "a denumerably infinite system of axioms, which is formalised by means of elementary logic and consistent with regard to deduction procedures based upon this logical system, can be realised by means of a finite or a denumerably infinite model" [Beth 1959, 488]. Now the axiom systems for classical set theory that have been established by Skolem, Fraenkel and von Neumann satisfy the requirements of the said theorem, since they can be formalised by means of a denumerably infinite set of expressions of elementary logic. They therefore have a denumerably infinite model (a finite model obviously being excluded). The paradox is that these axioms "have been established in order to create a reliable basis for operating with sets of a higher cardinal number than the denumerably infinite" [Beth1959, 489]. Beth reflected for a long time on the Löwenheim-Skolem theorem, gave a topological proof of it [Beth 1951], and proposed to interpret this paradox as an indication of the conclusion that deductive theories cannot, in general, provide an adequate description of mathematical structures: therefore, it seems likely that our knowledge of such structures has, at least partly, an intuitive, an immediate character. In his 1957 booklet, *La crise de la raison et la logique*, he had mentioned two kinds of intuition that are present in mathematics: the *intuition globale* that allows us, for instance, to realize the appearance of the same formula in different places within a given derivation and the *intuition de l'infini* that appears by thinking of a non-closed semantic tableau. Beth did not give any further details about these two intuitions except that intuition does have a place in mathematical discovery [Beth 1957, 29]. In a letter (dated 9/6/1951) from Beth to J. Piaget, he spoke of an *idée-clef* that must intervene when solving mathematical problems:

> "Résoudre un problème mathématique c'est en général dessiner un nouveau type d'action moyennant une nouvelle coordination des types d'action déja disponibles. Il s'agit alors de trouver l'idée-clef permettant d'effectuer cette nouvelle coordination. Dans cette recherche de l'idée-clef, la pensée est dirigée, canalisée, par des 'forces mentales' qui la poussent et la retiennent; parmi ces forces mentales il y a: les connaissances mathématiques dont le chercheur dispose, un certain fonds de méthodes de solution, certaines images intuitives d'usage personnel, d'ordinaire très vagues et très variables, la conscience du problème, les conditions imposées par la logique. En général, ces dernières n'interviennent que tardivement, l'idée-clef étant trouvée et éprouvée, au moment qu'il s'agit de formuler une démonstration en règle. Cela n'implique pas, comme pensent beaucoup de mathématiciens que la logique est stérile au point de vue mathématique; en effet, ce n'est que la démonstration en règle qui permette de juger de la portée de l'idée-clef. Parfois l'analyse logique montre que l'idée-clef ne revient qu'à une application plus ou moins ingénieuse d'une méthode déja connue; en d'autres cas, l'idée-clef se révèle capable d'applications fort variées et la solution du problème originale n'est donc que l'introduction à un développment nouveau en mathématiques".

Notwithstanding the indefiniteness of the concept "intuition" embedded in it, this conclusion that pure logic does not exist immediately led to conclusions about reality in general. Beth hypothesized [Beth 1959, 644-645] that there were various "spheres of reality" (called also "zones" or "aspects" of reality), of which he listed: physical reality, social reality, subjective reality and logical reality. Each of them is autonomous: all their attempts to limit each other, fail. An example was the failure of foundational schools, which wanted to give a basis to mathematics by imposing on it certain restrictions ("to be logical, use finitistic methods and be constructive"). Still, the spheres are not entirely unconnected. The proof is given exactly by the fact that pure logic cannot exist and also by the fact that human knowledge penetrates *various* zones of reality.

Beth also tried to specify somehow the relationship between the spheres and here we find a final reference to Bernays. As we said above in Section 1, Bernays, in "Über die Ausdehnung des Begriffes der Komplementarität auf die Philosophie" [1948b], had presented two meanings of the concept of complementarity. The second meaning (that referred to comprehensive theories, like the corpuscular theory and the wave theory that aim to explain the same experiences in two completely different ways, incompatible with each other) was what Beth had in mind. Beth wrote [1959, 645]:

> " It seems reasonable however tentatively to consider the various spheres of reality as *complementary aspects* of one and the same substratum, in the same sense as we have recently been taught to speak of complementary aspects of physical reality; (cf. P. Bernays)."

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