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LADISLAV KVASZ

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Henri Poincaré and the Epistemological Interpretation of the Erlangen Program

Ladislav Kvasz
Faculty of Matematics and Physics
Comenius University - Slovakia

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Ladislav Kvasz

Abstract. Mathematicians in the proofs of their theorems often use phrases as "we can see" or "it is obvious". But who are these we and to whom is it obvious? In my paper I shall follow the historical development of modern geometry, and examine in more detail the form of the subject, from the position of which the theories are formulated. I suggest that we should distinguish three kinds of subject of geometrical theories:
- '*perspective subject*' of the projective geometry
- '*meta-subject*' of the geometry of Beltrami-Klein model
- '*scattered subject*' of the geometry of Erlangean Program
After characterizing these forms of the subject in geometry I would like to examine their role in philosophy and show that the differences among the empiricist, Kant's and Poincaré's philosophy of geometry have their origin in the structure of the epistemic subject.

1. The 'Perspective Subject' of the Projective Geometry

The projective geometry is usually regarded as the starting point of the modern synthetic geometry [Piaget & Garcia 1983]. It is because projective geometry is the first mathematical theory, in which the concept of transformation has the central role, as it has in modern mathematics. This historical evidence, which is generally accepted, lacks a deeper epistemological analysis. It would be interesting to ask, which changes in the epistemological framework of geometry enabled Desargues to make his fundamental discoveries. I would like to show, that it was the introduction of the 'perspective subject' into geometry. But before we turn to Desargues, let us make a digression to Renaissance painting, where the ideas and methods of projective geometry have their prehistory.

If we compare the paintings of the Renaissance painters with the paintings of the preceding period, a radical difference strikes us in immediately. The Gothic paintings lack depth. The figures are placed one beside the other, house beside house, hill beside hill, without any attempt to capture the depth of the space. In handbooks on Gothic painting, we can find the explanation for this. This kind of painting was in accordance with the general aims of the painter. The painter's task was not to paint the world as it appeared to him. He had to paint it as it really was, to paint it as it appeared to God. The distant objects appear to us smaller, but they only appear so, in reality they are not smaller at all. So the painter must not paint them smaller.

A quite different aim of painting was followed by the Renaissance painters. They wanted to paint the world as they saw it, to paint it from a point of view, to paint it in a perspective. They wanted to paint the objects in such a way that the picture would evoke in the spectator the same impression as if he were looking at the real object. So it had to evoke the illusion of depth. Albrecht

Dürer (1471-1528) showed us in one of his drawings (Fig. 1) a method by which to reach this goal.

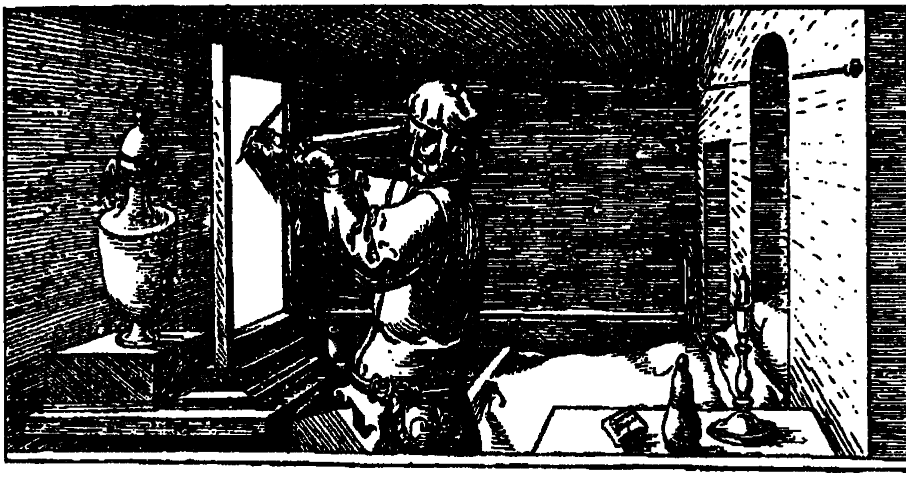


Fig. 1

I shall describe Dürer's procedure in more detail, because it enables me to show what is common and what is different in the perspectivist and projective picturing. Imagine that we want to paint some object so that its picture would evoke in the spectator exactly the same impression as if he was looking onto the original object. Let us take a perfectly transparent foil, fix it onto a frame and put it between our eye and the object we are intending to paint. We are going to dab paint onto the foil, point by point in the following way. We choose some point on the object (let it for instance be brown), mix paint of exactly the same color and dab it on that point of the foil, where the ray of light coming from the brown point of the object into our eye, intersects the foil. If we have mixed the paint well, the dabbing of the paint onto the foil should not be visible. After some time spent by such dotting we create a picture of the object, which evokes exactly the same impression as the object itself.

By a similar procedure the Renaissance painters discovered the principles of perspective. Among other things, they discovered that in order to evoke the illusion of two parallel lines, for instance two opposite sides of a ceiling, they had to draw two convergent lines. They discovered this but did not know why it was so.

The answer to this as well as many other questions was given by projective geometry. Girard Desargues (1593 - 1662), the founder of projective geometry came up with an excellent idea. He *replaced the object with its picture*. So while the painters formulated the problem of perspective as a relation between the picture and reality, Desargues formulated it as a problem of the relation of two pictures.

Suppose that we already have a perfect perspectival picture of an object, for instance of a jug and let us imagine a painter who wants to paint the jug using our dotting procedure. At a moment when he is not paying attention, we can replace the jug by its picture. If the picture is good, the painter should not notice it, and instead of painting a picture of a jug he could start to paint a picture of a picture of the jug. Exactly this was done by Desargues, and it is the starting point of projective geometry.

The advantage brought by Desargues' idea is that, instead of the relation between a three-dimensional object and its two-dimensional picture we have to deal with a relation between two two-dimensional pictures. After this replacement of the object by its picture it is easy to see that our dotting procedure becomes a central projection of one picture onto the other with its center in our eye. I have mentioned all this only to make clear, that the *center of projection represents (in an abstract form) the point of view from which the two pictures make the same impression.*

Before we start to study the central projection of some geometrical objects, we have to clarify what happens with the whole plane on which these objects are drawn. To make the central projection a mapping, Desargues had first of all to supplement both planes with infinitely remote points. After this the line a consists of those points of the plane a which are mapped onto the infinitely remote points of the second plane ß. On the other hand, the line b consists of the images of the infinitely remote points of the plane a. So by supplementing the infinitely remote points to each plane the central projection becomes a one-to-one mapping.

Desargues in this way created a technical tool for studying infinity. The idea is very simple. The central projection projects the infinitely remote points of the plane α onto the line b of the plane ß. So if we wish to investigate, what happens at infinity with some object, we have to draw it on the plane a and project it onto the plane ß. If we draw two parallel lines on the plane α, we shall see that their images on the plane ß intersect at one point of the line b. From this we can conclude that the parallel lines do intersect at infinity, and the point of their intersection is mapped onto that point of the line b where their images intersect (see Fig. 2).

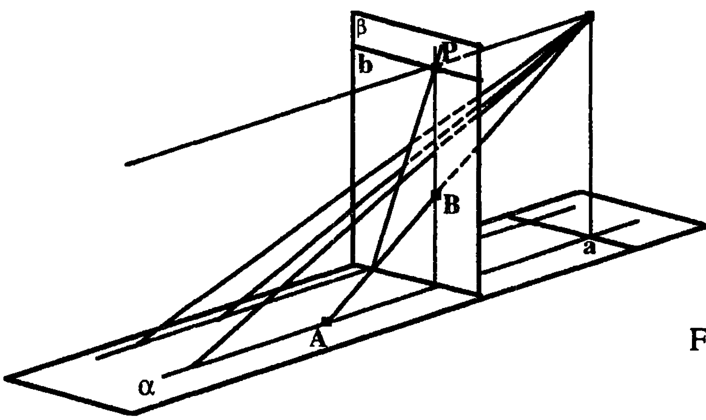
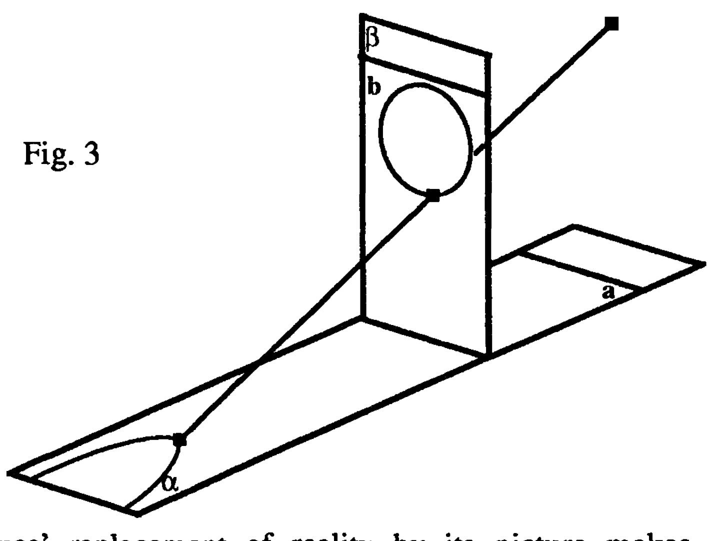


Fig. 2

If we draw a parabola on the plane α, we shall see that the parabola touches the infinitely removed line (see Fig. 3). This is the difference between the parabola and the hyperbola, which intersects the infinitely removed line at two points. So *Desargues found for the first time a way to give to the term infinity a clear, unambiguous and verifiable meaning.*

Fig. 3



Desargues' replacement of reality by its picture makes it possible to study the transformations of the plane on which the objects are placed independently of the objects themselves. We could say, to study the transformations of the empty canvas. We have seen that exactly these rules of transformation of the plane enforce the fact that the images of parallel lines are not parallel. It is not an individual property of the lines themselves but the property of the plane on which these lines are placed. So *projective geometry brings the* which these lines are placed. So *projective geometry brings the background (the plane or the space) into the theory.* Euclidean geometry studied triangles, circles, etc., but these object were, so to speak, situated in the void. In projective geometry the object becomes situated on the plane.

In pictures of projective geometry there is a remarkable point — different from all other points — the center of projection. As was shown above, the center of projection represents in an abstract form the eye of the painter from Dürer's drawing. Beside the center of projection in pictures of the projective geometry there is also a remarkable straight line. It is the line *a*, which is responsible for many of the singularities which occur in the geometrical formations by the projection. The position of the line *a* on the plane a is determined by the center of projection, which represents the eye of the spectator. So it is not difficult to see that the line *a* represents the horizon. We have not met anything similar to this line in Euclidean geometry. The Euclidean plane is absolutely homogeneous, all its lines are equivalent. So *instead of the Euclidean looking from nowhere onto a homogeneous world, the point of view is explicitly incorporated into the theory. It is present in the form of the center of projection and of the horizon which belongs to this center.*

This incorporation of the point of view into the theory made it possible to broaden qualitatively the concept of geometrical transformation. We cannot say that Euclid did not use transformations. In some constructions he uses rotations, translations, etc. But since he did not have the point of view incorporated into his theory, he was able to define only very few transformations. To define a transformation means to specify what changes and what remains unchanged. As the point of view was not a part of his theory, Euclid had to define his transformations in the same way for each point of view. This means that his transformations could not change the form of the geometrical object. Desargues, having explicitly introduced the point of view in his theory, was able to define a qualitatively larger class of transformations. He could define what remains unchanged and what is changed with respect to a unique point. Exactly thus is defined a projective transformation. Two figures are projectively equivalent if there exists a point from which they appear the same.

So the main epistemological innovation of Desargues was the incorporation of the «perspective subject» into geometry. The subject is manifested as the center of projection, in respect to which the transformations are defined, and on the other hand as the horizon belonging to this center, which makes it possible to represent the geometry, she or he needs to know the exact role of the center and horizon. She or he has to know, that they are no ordinary geometrical objects, but they represent the «perspective subject» in the geometrical language.

2. The 'Meta-Subject' of the Geometry of the Beltrami-Klein Model

The consistency of non-Euclidean geometry was proven in 1868, twelve years after Lobatchevsky's death, by Eugenio Beltrami (1835 - 1900), who constructed its first model. A simplified version of this model was suggested by Felix Klein in 1871, and I would like to discuss it briefly.

What is a model? The Beltrami-Klein model is based on a simple picture (Fig. 4), on which a circle with some of its chords is represented. The points of the inside of the circle (the circle without the circular line) represents the whole non-Euclidean plane, and the chords (without their ends) represent its straight lines. In this model all the axioms of Euclidean geometry are satisfied. Through two different points of this plane there passes exactly one straight line, etc. — all axioms with the exception of the axiom of parallels. If we choose some straight line, for instance *a*, and some point which does not lie on this line, for instance *A*, through this point we can thus draw two straight lines (p_1 and p_2) which do not intersect the line *a*.

I think that this picture goes beyond the boundaries of

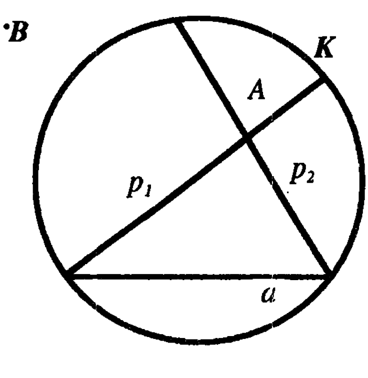


Fig. 4

Desargean representation. Desargues represents the objects on a certain background. Beltrami on the contrary takes a common Euclidean plane, and on this plane he first represents the non-Euclidean background in the form of a circle, which represents the horizon of the non-Euclidean plane, and then on this non-Euclidean background he represents the objects. So the object gets two frames, it becomes situated onto two backgrounds simultaneously. On the one hand, it is situated on the external background of the Euclidean plane, on which it is a chord of the circle. But on the other hand the plane, on which it is a chord of the circle. But on the other hand the same object is situated also on the internal background, where it acts as a straight line of the non-Euclidean plane. This double coordination of the objects made it possible to prove the consistency of non-Euclidean geometry.

If non-Euclidean geometry would be inconsistent, it would contain some theorem about the points and straight lines of the non-Euclidean plane, which could be proven together with its negation. Beltrami's model makes it possible to translate this theorem into the language of Euclidean geometry, which would state the same things, but not about points and lines of the non-Euclidean plane, but about the points and chords of the Euclidean circle *K*. It is not difficult to see, that this translation transforms proofs into proofs, and so we would get a theorem of Euclidean geometry, which also could be proven with its negation. So if non-Euclidean geometry were inconsistent, the Euclidean would be also.

We see, that the model serves as a translators tool, which translates theorems from the internal language of non-Euclidean geometry into the external language of Euclidean geometry.

EXTERNAL LANGUAGE	INTERNAL LANGUAGE
K - circle on the Euclidean plane	-horizon of the non-Euclidean plane
A - point inside of the circle	-point of the plane
B - point outside the circle	-?????
a - chord of the circle	-strait line
p_1 - a chord not intersecting *a*	-a parallel to *a*

Also the pictures are in some sense translatory media. The famous painting of the pipe from Rene Magritte, with the text «*Ceci n'est pas une pipe*» ("This is not a pipe"), shows it quite clearly. On the painting there is no real pipe. The picture is able to represent something, that is not present. So we can understand the picture as a translatory medium between the iconical sign and its meaning, between the picture and the pipe. This the picture has in common with the model. The basic difference is, that in the Desargean representation only one language is present in the picture. The other is absent. This is the theme of Magritte's painting. The pipe is not present. So if we regard the Desargean picture as a translation, it is a translation from iconic signs, which are present in the picture, into meanings, which are not present.

On the other hand, Beltrami's representation has both languages, the inner and the outer, present in the same picture. Here the translation is not between the present line of symbols and the absent line of meanings, but between two present lines of symbols. This sofisticated structure of the language can be easily misunderstood. In deed, when Felix Klein formulated his famous formula, which introduced the concept of distance into the model, many of the most important mathematicians, as Weierstrass or Cayley, criticized him very strongly. They said, that Klein's formula is circular, because it defines the concept of distance already using it. What they did not realize was, that these two concepts of distance were in different languages. The *d(X,Y)* is an expression of the internal language, while the expressions */AX/, /XB/, /AY/* and */YB/* are expressions of the external language. That is why no circularity appears.

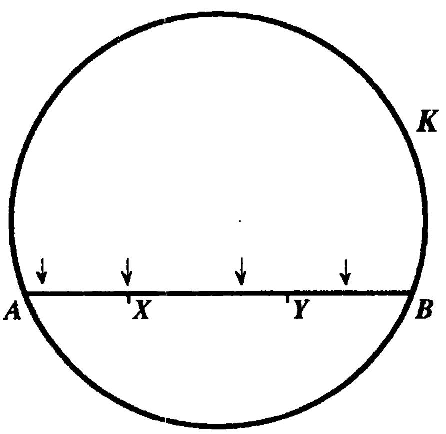


Fig. 5

$$d(X,Y)=\left|\ln\left(\frac{|AX|}{|XB|}:\frac{|AY|}{|YB|}\right)\right|$$

But let us look onto the Klein's formula once more. Its left side belongs to the internal language, its right side belongs to the external language. But where belongs the sign of equation? Neither into the internal language, nor into the external one. It belongs to the metalanguage, to the language of the meta-subject, who coordinates the internal language with the external. So the Klein's formula itself is neither a formula of the internal language nor does it belong to the external language. It is a formula of the meta-language.

We could say, that Desargues has introduced into geometry the viewpoint of the spectator, and Beltrami completed it with the metaviewpoint of the interpreter. From this metaviewpoint *we see* that an inconsistency in the internal language — the language of the non-Euclidean geometry could be translated into a similar inconsistency in the external language — the language of the Euclidean geometry. But this fact of translatibility is neither a fact of the Euclidean geometry, nor a fact of the non-Euclidean one. It is a fact about these geometries, belonging to their metatheory. So we can say, that from the epistemological point of view the main difference between the Desargean geometry an the geometry of the Beltrami-Klein model is the structure of the subject of they language. The language of the Desargean geometry has a perspective subject, which makes it possible to represent an object from different points of view, and so what is infinitely remote from one point can bee studied from another. The language of the Beltrami-Klein model has a meta-subject, from which even theorems about the theory can be formulated and proven.

I am convinced, that the structure of the subjectivity of Kant's theory has many in common with the structure of the language of the Beltrami-Klein's model. Kant in his philosophy distinguishes between the empirical and the transcendental subject. I think, that the relation between them has very much in common with the relation between the internal and the external language of the Beltrami-Klein model. The external language of the Euclidean plane is a necessary prerequisite, which we need, if we want to draw the circle K and construct the internal language of the model. So in the model the relation between the internal and the external languages is asymmetrical. The external language is a prerequisite for the internal one. This can be seen clearly on the Klein's formula for the distance. The distance $d(X, Y)$ in the internal language is defined on the basis of the concept of distance in the external language. So the external language is apriory to the internal. And exactly such relation of apriory exist between the transcendental and the empirical subject.

But beside of these two subjects, I think, in Kant's theory we can find a third one. Let us take Kant's thesis, that the transcendental schemes are apriory conditions of every objective empirical judgment. The structure of this thesis is similar to Klein's formula for distance. We have here an expression of the external (apriory) language, namely the transcendental scheme. Then there are terms of the internal (aposteriory) language, namely the empirical judgments. But to which language belong the copula? The word *are* in this thesis is neither a term of the external, nor a term of the internal language. It belongs to the metalanguage. So in Kant's theory, beside of the transcendental and the empirical subject, there is a third subject, the subject from which the theory is formulated. Kant's Critique of pure reason is written neither in the language of the empirical subject, nor in the language of the transcendental one. It is written in the metalanguage.

If this account of Kant is true, so it can be used as an argument in favor of the neo-Kantian position, that even if Kant himself regarded the Euclidean geometry as the only possible, it is easy to bring into accordance Kant's philosophy with the existence of non-

Euclidean geometries. We can do it because the structure of the subject used by Kant is the same as the structure of the one used by Beltrami.

3. The 'Scattered Subject' of the Geometry of the Erlangen Program

At the end of my paper, I would like to show a connection between Poincare's philosophy of geometry and the structure of the subject of the Erlangen Program. I have spent too much time on the first two examples, but I wanted to make the concept of the subject of a geometrical theory as clear as possible. So for the last, perhaps most interesting, but surely most complicated case I can give only the a short account.

In his famous Erlangen program Felix Klein formulated the idea to unify the different geometries of his time, as Euclidean, Lobachevski's, Riemann's, projective, affinne and many others, using the concept of group of transformations. His success is a historical fact, but from an epistemological point of view we have to ask, why even the concept of group was so effective in the unification of geometries.

The answer to this question is similar to the answer to the question we asked at the beginning of the paper about Desargues. In both cases, by Desargues as by Klein, the qualitative progress in geometry is due to changes of the epistemic subject. In the same way as Desargues introduced the 'perspective subject' to geometry, Klein introduced a new structure of epistemic subject, which I would like to call the scattered subject. The reconstruction of the structure of this subject can be found in the *Science and Hypothesis* of Henry Poincaré [Poincaré 1902]. In chapter 4 of his book, Poincaré investigates the relation between the geometrical space and the spaces of our sensory perceptions. He shows, that the space of our visual perceptions is neither homogeneous, nor isotropic and three dimensional. That means that we could not derive the concept of geometrical space from our visual perceptions alone.

Poincaré asserts, that we have derived them from the relations, in which the changes of different kinds of perceptions, visual, tactile and motorical, follow each other. The most important among these relations is the relation of compensation. We can compensate some changes in the visual field by motion. And the structure of these compensations is a structure of group. Thus Poincaré showed, that the visual space of *what we are looking at*, has not the structure of the geometrical space. But let us go further an take a more radical interpretation of Poincaré's words. The structure of the geometrical space has not the space what we are looking *at*, but the space, *from* *where* we are looking. That means, that *the Euclidean geometry is not about points which we see, but about points, from where we are looking*. So every point of the space is a point of view, and between these points acts the group of compensations. So in contrast to the projective geometry, where the subject was, so to say, concentrated into one point, the center of projection, here it is in every point. That is why I suggest to call him the 'scattered subject'.

So the answer to our question, why was Klein able to unify the different geometries, is be the following. He succeeded, because what he did was not taking a concept from algebra and using it in geometry. The structure of group is in geometry present. It forms, as Poincaré has shown, the fundamental structure of our concept of space. So Klein did not bring anything from algebra to geometry. He just made explicit, what in geometry is implicitly present from the beginning, namely the structure of compensations.

I would like to end my paper with a comparison of the philosophy of geometry of the empiricist, of Kant and of Poincaré. I am convinced that the differences among them are exactly the same as the differences between the projective geometry, the Beltrami-Klein model and the Erlangen programm. These differences are due to the differences in the structure of subject, on which these philosophies are based.

The empiricist use the «perspectivist subject» and so the main question for them is the question of the relation of the «picture and its model», that means the question of adequacy of the mental representations of reality. Kant took the metasubject as the basis for his theory. So the main question is not the question of the adequacy, but the question of the apriory conditions of geometry. Poincaré made the next step by taking the scattered subject of the Erlangen programm as the basis for his epistemological considerations. That made it possible for him to give a reconstruction of Kantian apriory as invariants of transformation groups[1].

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