PHILOSOPHIA SCIENTIÆ

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Philosophia Scientiæ, tome 1, nº 3 (1996), p. 1-15
<http://www.numdam.org/item?id=PHSC_1996__1_3_1_0>

The Axiomatic Method and Ernst Schröder's Algebraic Approach to Logic

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Philosophia Scientiæ, 1 (3), 1996, 1-15.

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Abstract. Poincaré's scepticism towards attempts to found geometry axiomatically, i.e. on self-evident truths which are in no need resp. incapable of proof, can be seen as the symptom of an epistemological crisis of traditional axiomatics. This crisis is illustrated by discussing the various attempts of Ernst Schröder (1841–1902) to found his abstract algebra and his algebra of logic on 'axioms'. In the course of his studies the terminological inexactness brought Schröder to abandon the notion of 'axiom' from his theory. In the last stage of development he founded his algebra and logic of binary relatives on a set of 29 'conventional stipulations'. In Schröder's opinion, however, geometry needed real axioms, contrary to logic and arithmetic. In his architecture of science geometry is more than a mere branch of logic but the most elementary member in the series of physical sciences. Geometrical axioms are thus claimed to be materially true.

Résumé. Il est possible de considérer le scepticisme de Poincaré à l'égard des essais d'un fondement axiomatique de la géométrie (un fondement sur des vérités évidentes qui ne doivent pas — respectivement qui ne peuvent pas — être prouvées) comme un symptôme de la crise épistémologique de l'axiomatique traditionelle. La crise est illustrée par les divers essais de Ernst Schröder (1841–1902) de fonder son algèbre abstraite et son algèbre de la logique sur des "axiomes". L'inexactitude terminologique amène Schröder à abandonner au cours de son étude la notion d'"axiome" dans sa théorie. Au terme de sa réflexion il fonde son algèbre et sa logique de relatives binaires sur un ensemble de 29 "stipulations conventionnelles". Cependant, selon la conception de la géométrie de Schröder, la géométrie a besoin d'axiomes réels, contrairement à la logique et à l'arithmétique. Dans son architecture des sciences la géométrie n'est pas une simple partie de la logique, elle est plutôt la branche la plus élémentaires des sciences physique. Dès lors les axiomes de la géométrie se doivent d'être matériellement vrais.

1. Crisis of Axiomatics

Henri Poincaré's criticism of axiomatics in his *La Science et l'Hypothèse* [Poincaré 1902a] is well-known. He claimed, e.g., that geometrical axioms are neither synthetic judgements *a priori* nor experimental facts. They are stipulations based on conventions. Thus geometrical axioms are definitions in disguise [ibid., II, 3].

It should be stressed that this criticism did not attack modern axiomatics as presented, e.g., by David Hilbert at the turn of the century[1], but rather the traditional axiomatic method as created by Aristotle and applied in Euclid's geometry[2]. Let me illustrate this traditional view on axiomatics with a quote from a source also published close to the turn of the century. Robert Adamson defines

1 For Poincaré's criticism on Hilbert's geometrical axiomatics see his review of Hilbert's „Grundlagen der Geometrie“ [Poincaré 1902b], English translation by Edward V. Huntington [1904], and some passages *in* «Les mathématiques et la logique» [Poincaré 1905], comments which emerge in a general criticism on formalism.

2 For the distinction between traditional ('classical') and modern axiomatics cf., e.g., [van der Waerden 1967].

the notion of 'axiom' in Baldwin's *Dictionary of Philosophy and Psychology* [Adamson 1901] as "a proposition, general in import, and held as standing in no need of, or indeed as incapable of, proof. Axioms are self-evident truths" [ibid., 97]. Adamson hints at the Aristotelian distinction between those axioms which are presupposed in every kind of thinking (*communes animi conceptiones*) and special axioms which belong to specific topics. With respect to mathematical axioms Adamson writes: "In mathematics, the form is commonly restricted to the self-evident propositions on which geometry is based, and those facts of general experience which are so familiar that every one must admit them" [ibid., 98].

Let me contrast this traditional view with the new vision on axiomatics that can be found in David Hilbert's early philosophy of mathematics, e.g., in his lecture course „Logische Principien des mathematischen Denkens" of the summer term of 1905. In Hilbert's early conception modern structural formalism is, however, only set up[3], since he still starts from specific mathematical fields in which a foundational need has become evident. From the stock of propositions of this field he isolates certain propositions according to some criteria of evidence which are not discussed in detail. Only at this point does the formalistic procedure begin with an examination of these accentuated propositions. The system of these propositions, now called axioms, has to follow certain criteria which are responsible for its truth and completely independent of the process of selection [cf. Hilbert 1905, 8]. The set of axioms has to be complete, i.e., "we will have to demand", Hilbert writes, "that all other facts of the presented field of knowledge follow from the axioms."[4] The axioms have to be independent of one another, i.e., it has to be proved "that none of the axioms could be deduced from other ones by logical inferences". Finally, Hilbert demands a proof of consistency for the axiomatic system, i.e., of the axioms themselves and of all inferences derived from them. With this approach the

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3 It is only set up although Hilbert's famous remark of 1891 "One must be able to say at all times — instead of points, straight lines, and planes — tables, chairs, and beer mugs" [see Reid 1970, 57] can be regarded as a paradigm of modern axiomatics. Hubert C. Kennedy gives prominence to Hilbert's predecessors in this modern view, Moritz Pasch, Gino Fano, and Giuseppe Peano. Nevertheless I doubt his assertion that "the transition from viewing an axiom as 'a self-evident and generally accepted principle' to the modern view took place in the second half of the nineteenth century and can be found in the very brief period from 1882 to 1889" [Kennedy 1972, 133], since still in 1905 Hilbert used both conceptions of axioms; the transition was, thus, not a complete one [cf. Peckhaus 1990].

4 This is of course not yet the formal definition of completeness as maximum consistency as presented, e.g., *in* [Hilbert and Ackermann 1928].

axiomatic method is split into an investigation concerning the axioms themselves and the application of these axioms to deduce or prove theorems.

What I would like to argue for is that Hilbert's suggestions provided a practicable solution for the problems mathematicians had at that time with the traditional notion of 'axiom'[5]. These problems grew into a crisis of traditional axiomatics at the end of the 19th century. Poincaré's scepticism towards geometrical axiomatics is one symptom of this crisis, and in my lecture I would like to hint at another: Ernst Schröder's struggle with axiomatics in his algebra of logic and his universal algebraic programme.

2. Axiomatic Presentation of Schröder's Algebra of Logic

According to Bocheński [1956, 314], Ernst Schröder's (1841–1902) algebra of logic completed the Boolean period in logic. His monumental *Vorlesungen über die Algebra der Logik* [Schröder 1890, 1891, 1895, 1905] formed its climax and, at the same time, its end, despite fruitful impact of this work on modern algebraic logic and model theory. Schröder's logic, especially his calculi of domains and of classes, is usually presented in an axiomatic way. It seems that such axiomatical interpretation could be traced back to Schröder himself, since in the first part of his posthumously published *Abriß der Algebra der Logik* [Schröder 1909-1910], an axiomatic form is chosen.

In this comprehensive presentation of the theories presented in the first two volumes of the *Vorlesungen* it is claimed [Schröder 1909/1966, 666] that logic deals with 'domains' which

> form in respect to the relation to each other the object of a 'theory of domains'. Of what kind these things are will be stipulated by certain general propositions, the so-called 'axioms', which should be valid for all things to be taken into account, and for all domains as meanings of the general symbols. [Schröder 1909/1966, 666]

All attributes of any domain a, b, c, ... are given in a set of seven axioms, two of them split into the dual forms for logical addition and logical multiplication. The basic relation in such domains is the nonsymmetrical 'subsumption' which is binary, reflexive and transitive but besides this, arbitrary. It is designated

5 In the same sense van der Waerden writes: „Durch *Hilberts* Geniestreich waren alle erkenntnistheoretischen Schwierigkeiten, die von jeher mit den geometrischen Grundbegriffen und Axiomen verbunden waren, mit einem Schlage aus der Welt geschafft" [van der Waerden 1967, 3].

by the sign $=\!\!\!\in$ ('sub')[6]. The axiomatic system runs as follows [ibid., 680]:

I	$a =\!\!\!\in a$	identity or tautology axiom,
II	$(a =\!\!\!\in b)(b =\!\!\!\in c) =\!\!\!\in (a =\!\!\!\in c)$	subsumption,
III	$(a =\!\!\!\in b)(b =\!\!\!\in a) = (a = b)$	equality definition,
$\mathrm{IV}_{\times}$	$0 =\!\!\!\in a$	zero postulate,
IV_{+}	$a =\!\!\!\in 1$	one postulate,
V	$1 \not{=\!\!\!\in} 0$	existence postulate,
$\mathrm{VI}_{\times}$	$(x =\!\!\!\in a)(x =\!\!\!\in b) = (x =\!\!\!\in ab)$	definition of logical product,
VI_{+}	$(a =\!\!\!\in y)(b =\!\!\!\in y) = (a + b =\!\!\!\in y)$	definition of logical sum,
$\mathrm{VII}_{\times,+}$	$(a + z)(\bar{a} + z) = z = az \cdot \bar{a}z$	negation or distribution principle.

This axiomatic system has been up to now the base of axiomatic presentations of Schröder's calculus[7]. I only want to hint at the fact that already in 1904 the American postulation theorist Edward Vermilye Huntington had condensed a set of 10 postulates from Schröder's class calculus, using the dyadic relation 'within' ⧀ and discussing independence, completeness, and consistency of these postulates [Huntington 1904, 297].

Both axiomatic systems are doubtless written in the spirit of Hilbert's axiomatics. Nevertheless I wonder whether Schröder himself had anticipated Hilbert's notion of 'axiom' during his lifetime. Maintaining such doubts, I have to face up to the fact that

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6 The distinction is made between a 'primary subsumption', i.e., the incorporation of a class symbol in another, and the "secondary subsumption" which stands for implication. A confusion of the two sorts of subsumption while using them simultaneously is claimed to be impossible in practice. [cf. Schröder 1909/1966, § 22, especially p. 680; cf. also § 11, 667*f.*, and § 84, 716*f.*].

7 It was, e.g., quoted by Randall R. Dipert in his dissertation *Development and Crisis in Late Boolean Logic* [1978, 132–134], and has from there found its way into the recently published *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences* [Grattan-Guinness (Ed.) 1994]. In his *Companion Encyclopedia* contribution on "Algebraic Logic from Boole to Schröder, 1840–1900" Nathan Houser incorrectly writes in the equality definition (axiom III) a negated subsumption sign for the equality sign [Houser 1994, 611].

the system of the *Abriß der Algebra der Logik* was published under Schröder's name, although seven years after his death. Responsible for this publication was the grammar-school professor Karl Eugen Müller, who had been entrusted by the *Deutsche Mathematiker-Vereinigung* after Schröder's death to check Schröder's extensive manuscript *Nachlaß* for material worth publishing, especially material suitable for completion of the unfinished *Vorlesungen*. In his preface to the compendium Müller reports that he has found among the papers "only short sketches, and some remarks to an '*Abriß*' of the algebra of logic, but no realized presentation of any part of that *Abriß*." Müller remarks that he had compiled the *Abriß* "leaning if possible upon the manuscript stuff and the chief work, but — according to Schröder's explicitly stated opinion — regarding recent research" [Müller 1909/1966, 653][8]. I think that it is a reasonable assumption that the axiomatic form is also one of Müller's additions. Further evidence gives Müller's first own contribution to symbolic logic on the foundations of the calculus of domains [Müller 1900]. There he stresses that his presentation differs from Schröder's in regard to the hypothetical axiomatical foundations of the calculus [ibid., 2], and he compiles a set of 9 axioms quite different from that of the later *Abriß* [ibid., 20]. Coming back to Schröder it seems that he stood more than one single step apart from modern axiomatics, because of heuristic reasons, since he regarded logic as representative of a universal theory of algebraic connectives which proceeds basicaly in a combinatorial way.

Examination of Schröder's writings shows that he was inspired by the traditional notion of 'axiom' devoted to Aristotelian ἀξιώματα and Euclidean postulates. Schröder's growing reluctance to found his logical theories on axioms becomes obvious. In the following I intend to show chronologically the changing status of those proposition he called "axioms."

3. Development of Schröder's 'Axiomatics'

3.1. The "One and Only Axiom" in the *Lehrbuch der Arithmetik und Algebra* (1873)

Schröder's first logical considerations can be found in his *Lehrbuch der Arithmetik und Algebra* published in 1873, which treats pure mathematics as the theory of numbers. Natural numbers are introduced by the countability of things, each thing being a unit. A

8 For Müller's biography, his editorial work, and the fate of Schröder's Nachlaß, cf. [Peckhaus 1988].

natural number is defined as a sum of units. Schröder's theory of numbers is based on the "one and only axiom", the "axiom of the inherence of the signs". He demands that this axiom is presupposed in every deductive science and it gives the certainty "that in all our arguments and inferences the signs inhere in our memory — and even more on the paper. [...] Without this principle", he continues,

> which is derived by induction or generalization from a very rich experience, every deduction would indeed be illusory, since every deduction begins when — after having sufficiently clothed the basic features of things into signs — the investigation of the things has made room for the investigation of their signs. [Schröder 1873, 16*f.*]

It should be stressed that axioms of such kind that provide the conditions that make it possible to set up systems of propositions in mathematics or logic at all were not so unusual at that time as the heavy criticism of Frege and Kerry might suggest [cf. Frege 1884, VIII; Kerry 1890, 333-336]. Similar axioms can be found in Dedekind's and Hilbert's early foundational studies[9]. Such axioms are not formal because of their empirical origin.

3.2. "Axiomatics" in the *Operationskreis des Logikkalkuls* (1877)

Without completing his large-scale number theory (four volumes were planned, only one was published) Schröder switched his interests toward logic. In 1877 he published his *Operationskreis des Logikkalkuls*, a concise presentation of the Boolean calculus, using, contrary to Boole, the inclusive interpretation of the logical sum, and stressing duality between conjunction and adjunction. The *Operationskreis* consists of 40 numbered propositions, of which the first 20 concern the direct logical connectives addition and multiplication and the second half their inverses. One definition and two axioms are presupposed in this set of propositions [Schröder 1877, 5]:

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9 Dedekind presupposes such conditions when writing about mental practices: „Es kommt sehr häufig vor, daß verschiedene Dinge *a*, *b*, *c*, ... aus irgendeiner Veranlassung unter einem gemeinsamen Gesichtspunkte aufgefaßt, im Geiste zusammengestellt werden" [1888, § 1], and Hilbert explicitely formulates an "axiom of the existence of an intelligence" which runs as follows: „Ich habe die Fähigkeit, *Dinge* zu denken und sie durch einfache Zeichen (*a*, *b*, ... *X*, *Y*, ...) derart in vollkommen charakteristischer Weise zu bezeichnen, dass ich sie daran stets eindeutig wiedererkennen kann; mein Denken operiert mit diesen Dingen in dieser Bezeichnung in gewisser Weise nach bestimmten Gesetzen[,] und ich bin fähig, diese Gesetze durch Selbstbeobachtung zu erkennen und vollständig zu beschreiben" [Hilbert 1905, 219].

i The definition of the equality of class symbols.

ii Axiom: every class symbol is equal to itself.

iii Axiom: If two class symbols are equal to a third, they are also equal to one another.

Schröder states that 13 of the theory's propositions "have to be stated for the present as formal axioms" [Schröder 1877, 5][10]. In his *Operationskreis* Schröder uses the term 'axiom' for unproved or unprovable theorems of his calculus. Such axioms also cover definitions which introduce schematic signs for, e.g., operations, and define the characteristics of such operations directly or implicitly. Such axiomatic definitions can be connected with postulates that maintain the existence of objects of the calculus. Schröder connects, e.g., the axiomatic definitions of logical sum and logical product to the axiomatic postulate that addition and multiplication of class symbols lead again to class symbols, and that these operations can always be realized. Schröder stresses that the theorems of the algebra of logic are intuitive, that they are directly evident. The assertions called 'axioms' could be regarded as implications, directly given together with the definitions. Therefore they are not empirical, but formal, i.e., derived from evident intuitions [Schröder 1877, 4].

3.3. Criticism of Axiomatics in the *Vorlesungen über die Algebra der Logik* (1890–1895)

The terminological inexactness in his early writings inclined Schröder in the course of time to abandon the notion of 'axiom' from his theory. In the first volume of his *Vorlesungen* Schröder distinguishes the following types of sentences: Definitions, i.e., explanations of terms, postulates, principles, or axioms (only 'principles' is italicized) and theorems [Schröder 1890, 165*f.*]. Schröder remarks that it is usually demanded that logic has to explain what definitions, postulates, axioms, and theorems are, but here (which means in the complete *Vorlesungen*) he will abstain from doing so. We have therefore to check Schröder's use of these sorts of propositions in order to find out their status. *Theorems* are propositions derived from definitions and principles. *Principles* give the features of logical symbols which cannot be derived from other propositions of the calculus. In the 'identical calculus', i.e., Schröder's version of Boole's

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10 Randall R. Dipert wrote [1978, 87]: "The *Operationskreis* was one of the first serious attempts to axiomatize the Boolean calculus", an opinion which is questioned in the following.

algebra of logic, there are only three principles: identity, which is given by the reflexivity of the subsumption relation, the inference of subsumption, which asserts the transitivity of the subsumption relation, and the principle $III_{\times}$: If $bc = 0$, then $a(b + c) \mathrel{=\!\!\!\!\in} ab + bc$ [ibid., p. 293]. The last is a consequence of Schröder's proof that the second subsumption of the law of distributivity is independent from the propositions of the identical calculus without negation. *Definitions* introduce atomic expressions of the calculus, such as equality defined as antisymmetry of the subsumption relation [ibid., Def. (1) p. 184], 'identical zero' ('nothing') and 'identical one' ('all') [Def. $(2_{\times})$ p. 184], 'identical multiplication' (conjunction) and 'identical addition' (adjunction) [Def. $(3_{\times,+})$ p. 196] with modified versions [Def. $(4_{\times,+})$ p. 202; Def. $(5_{\times,+})$ p. 205], and negation [Def. (6) p. 302].

Contrary to the types of propositions just mentioned *postulates* were not formalized. They are responsible for the connections between the formal system and perception in applications of logic. Schröder writes:

> As soon as we want to give a meaning to those symbols included in our 'domains' [i.e., manifolds of indetermined elements], i.e., claim that there are real domains accessible to perception which have the respective attributes, *we add* to our definitions certain *postulates*, i.e., we assert that demands for giving evidence of the existence of some domains can generally be accomplished although in this respect we can only refer to perception. [ibid., 212]

Although Schröder spoke in the beginning of "*principles* or *axioms*", he never used the notion of "axiom" throughout the pages of the *Vorlesungen*. That this was not without reason becomes clear from the third volume. In this volume, published in 1895 and devoted to the "algebra and logic of [binary] relatives", he again changed his terminology, skipping all types of propositions mentioned above, and founding his theory on 29 'conventional stipulations' which can also form, as he claimed, the foundation of the complete logic [Schröder 1895, 16]. Later Schröder stresses, referring to Charles Sanders Peirce's early paper "Description of a Notation for the Logic of Relatives" [Peirce 1870]:

> Apart from the fundamental conventions compiled in § 3 [Schröder adds in a footnote: "to be rigorous it should be inserted after 'conventions': and the few so-called principles of general logic, which can be regarded (generally but not formally) as being contained in these conventions"], we indeed do not need a further 'principle' in the theory. And if the question is raised about the axiomatical foundations of our discipline of the algebra and logic of relatives, I can agree to Mr. Peirce [...]. The foundations are of the same rank, they are nothing else, as the known 'principles' of general logic. Contrary to geometry, logic and arithmetic do not need any real 'axioms'. [Schröder 1895, 67]

Schröder agrees to the possibility of a formal conception of geometry. In such cases, however, the geometrical axioms were only assertions of mere assumptions. The question about fulfillment and validity of geometrical propositions in some domains of thinking is then not an object of research. These geometrical propositions can only claim 'relative truth' under the condition that the presuppositions become true. Schröder judges:

> Commonly, and in my opinion justly, this is not done. The geometrical axioms are on the contrary taught, presented and accepted with the claim of real validity, truth, be it for our subjective space perception, be it for reality which is thought to be objectively fundamental for space perception. [ibid., 67]

In this conception advocated by Schröder axioms were not analytical judgments, which say nothing. They were *psychologically* essential for thinking because of the nature of our space perception, but they were not *logically* essential for thinking. "Geometry is more than a mere branch of logic; it is the most elementary member in the great series of physical sciences. Arithmetic is a contrary case" [ibid.]. Concerning his own practice in using the notion of 'axiom' Schröder writes, quoting Peirce:

> Already in vol. 1 I have therefore abstained from using the name 'axiom' for the 'principles' necessary in that course. And these 'principles' were only definitions in disguise — "are mere substitutes for definitions of the universal logical relations". As far as the general logical relations can be defined — says *Peirce* with justice — it is possible to get by without any 'principles' in logic (all axioms may be dispensed with). [ibid., 68]

In a certain sense Schröder comes back to his early conception of the notion of 'axiom' propagated in his *Lehrbuch der Arithmetik und Algebra*. *Axioms* are formulated with the claim to be valid in reality. They therefore concern not only the formal structure but bind this structure to the human capacity to gain knowledge and to the world of experiences. In the programme of an algebra and logic of relatives they are relevant for applications of formal structures to possible representatives of theses structures. They restrict the possible translations of schematic symbols and formal operations. For Schröder geometry is formulated in that way, but he does not deny the possibility of 'formal' (not empirical, not Euclidean) geometries. Their foundations were, however, not axiomatical but based on definitions or conventions[11].

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11 Schröder's use of the notion of 'axiom' is thus in accordance with the position which van der Waerden called 'classical axiomatics' and which is characterized

In its last state of development algebra of logic presupposes a general logic which states the general laws dominating every formal system. If they can be formalized these principles can be included into the system of stipulated conventions, but then they loose their status as principles. The principles of general logic are usually the principle of identity, the principle of (excluded) contradiction and the principle of sufficient reason, Leibniz's necessary truths. They are not axioms, thus do not concern the relation between the thinking subject and the real world, but provide the conditions for every activity of thought. The notion of 'axiom' is removed from mathematical terminology and placed at the intersection between the philosophy of mind and the philosophy of science.

4. Absolute Algebra

Let me finally add a few words on the heuristic behind Schröder's logical conception. The method Schröder used in his logical writings follows his programme to create an "absolute algebra", i.e., a general theory of connectives which had been sketched already in the *Lehrbuch der Arithmetik und Algebra* and in a first step elaborated in the school programme pamphlet *Über die formalen Elemente der absoluten Algebra* published in 1874. In the *Lehrbuch* Schröder formulates a four-step programme of a formal, in the end 'absolute', algebra [Schröder 1873, 293*f.*]:

i Formal algebra compiles all assumptions that can serve for defining connectives for numbers of a 'number field'. Such general 'Numbers' are objects which constitute a manifold ('number field') and which are not determined in any way. Examples for such numbers are "proper names, concepts, propositions, algorithms, numbers [of pure mathematics], symbols for dimensions and operations, points and systems of points, or any geometrical objects, quantities of substances, etc".

ii Formal algebra compiles for every premiss or combination of premisses the complete set of inferences ('separation' of the manifold of resulting formulas).

iii Formal algebra investigates which particular number fields can be constructed by the defined operations.

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by two criteria: „1. Die Gegenstände, auf die sich die klassischen Axiome beziehen, sind von vornherein bestimmt und bekannt. [...] 2. Das zweite Merkmal der klassischen Axiomatik ist, dass derjenige, der die Axiome aufstellt, sie für wahr hält" [van der Waerden 1967, 1*f.*].

iv Formal algebra has finally to decide "what geometrical, physical, or generally reasonable meaning these numbers and operations can have, which real substratum they can be given" [ibid., 294].

In the context of the superior programme of an absolute algebra, logic appears to be an algebraic calculus, i.e., in Schröder's terminology a set of formulas following from an *Operationskreis*, i.e., those fundamental relations between 'general numbers' using more than one connective with their respective inverses. In this model-theoretic approach to logic the general numbers are interpreted by domains, classes, propositions and later relatives, and the connectives are logical addition and logical multiplication (later additionally relative addition and relative multiplication). The architecture of Schröder's *Vorlesungen* follows the four step programme, and it was by this procedure that he found a model which followed the basic assumptions of the identical calculus without negation, but in which one of the two directions of the distributivity law did not hold. Huntington is right in his methodological criticism that in Schröder's algebra of logic the independence of only one single principle is shown, and this not even completely because without regarding negation. This criticism, however, veils one of the most eminent achievements of Schröder's calculus: the formulation of the first example of nondistributive lattices [cf. Curry 1977, 160][12].

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12 For comments on earlier version of the paper I would like to thank Christian Thiel (Erlangen), Rudolf Kötter (Erlangen), Geraldine Brady (Chicago), and Roman Murawski (Poznań).

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