JAN KUBARSKI

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Publications du Département de Mathématiques de Lyon, 1994, fascicule 2
, p. 7-126
<http://www.numdam.org/item?id=PDML_1994___2_7_0>

THE CHARACTERISTIC CLASSES OF FLAT AND OF PARTIALLY FLAT REGULAR LIE ALGEBROIDS OVER FOLIATED MANIFOLDS

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CONTENTS

INTRODUCTION

1) This work belongs to the direction initiated by K.Mackenzie in [20; Chap. III. §§ 2.5.7, Ch. IV], [21] and developed by the author in [17], and concerns the "clean" theory of Lie algebroids. [These works isolate this theory from the common theory of Lie groupoids and Lie algebroids].

Originally, the notion of a Lie algebroid was invented by J.Pradines [28], [29] (1967) in connection with the study of differential groupoids, generalizing the construction of the Lie algebra of a Lie group. Since every principal bundle P determines a Lie groupoid PP^{-1} of Ehresmann [6], therefore - in an indirect manner - determines a Lie algebroid $A(P)$. The construction of this object with the omission of the indirect step of Lie groupoids (with the use of the vector bundle TP/G) was made independently by K.Mackenzie [20] and by the author [15]. In [15] there is also a third manner of constructing a Lie algebroid of a principal bundle $P(M,G)$ as an associated bundle $W^1(P)\times_{G^1_n}(\mathbb{R}^n\times\mathfrak{g})$ with the first-order prolongation of P.

Since 1977 another source of transitive Lie algebroids (discovered by P.Molino [23]) has been known, namely, the theory of transversally complete foliations. On this ground R.Almeida and P.Molino discovered in 1985 [3] (see also [24]) non-integrable transitive Lie algebroids (i.e. ones which do not come from principal bundles), refuting an assertion of J.Pradines concerning the non-existence of such objects [30]. More precisely, they proved that a TC-foliation $\mathcal{F}$ has an integrable Lie algebroid if and only if $\mathcal{F}$ is developable. Since the fact that any TC-foliation with nonclosed leaves on a simply connected manifold is not developable is obvious, therefore its Lie algebroid is not integrable. A more concrete example is the foliation of left cosets of a connected and simply connected Lie group by a nonclosed Lie subgroup. In [16] the author gives a direct definition of the Lie algebroid of such a TC-foliation (without using Molino's theory) and develops the method of a Lie algebroid on this ground.

Differential geometry of the last five years has revealed new objects which determine Lie algebroids: Poisson manifolds (A.Coste, P.Dazord, A.Weinstein [5], 1987) and some complete closed pseudogroups (A.Silva [32], 1988). To sum up, the method of a Lie algebroid in differential geometry has acquired weight.

2) Can the characteristic classes known on the ground of principal bundles [Pontryagin classes, the classes of flat or of partially flat principal bundles] be constructed on the level of Lie algebroids ? - was the problem the author posed some five years ago.

The first result in this direction concerns the Chern-Weil homomorphism of principal bundles. In [15] the author observed that the Chern-Weil homomorphism of principal bundles is an invariant of Lie algebroids of these bundles in the case of connected structure Lie groups [the troubles refer only to the domain of this homomorphism]. The full answer to this question is included in work [17] which is based on

(a) the author's observation that the Chern-Weil homomorphism of a connected principal bundle is an invariant of the Lie algebroid of this bundle [this forced the initiation of the notion of a representation of a principal bundle on a vector bundle and the obtaining of some related results],

(b) the construction of an equivalent of this homomorphism for the class of regular Lie algebroids over foliated manifolds [containing the class of transitive ones] (in [17] the author initiated the theory of connections in nontransitive Lie algebroids),

(c) the discovery of a class of transitive non-integrable Lie algebroids having the nontrivial Chern-Weil homomorphism.

Due to (b) and (c), the technique of characteristic classes can be applied to the investigation of the objects different than principal bundles but possessing Lie algebroids, such as TC-foliations, nonclosed Lie subgroups, Poisson manifolds, some pseudogroups, or vector bundles over foliated manifolds.

As to (c), the author calculated the Chern-Weil homomorphism of the Lie algebroid $A(G;H)$ of the foliation of left cosets of a Lie group G by a nonclosed Lie subgroup H. The superposition

$$h_{A(G;H)}:\bigvee(\bar{\mathfrak{h}}/\mathfrak{h})^{*}\rightarrowtail(\bigvee\bar{\mathfrak{h}}^{*})_{I}\xrightarrow{h_P}H_{dR}(G/\bar{H})$$

serves as this homomorphism, where $h_P:(\bigvee\bar{\mathfrak{h}}^{*})_{I}\longrightarrow H_{dR}(G/\bar{H})$ is the Chern-Weil homomorphism of the $\bar{H}$-principal bundle $P=(G\longrightarrow G/\bar{H})$. Next, it was noticed that the case of a compact and semisimple Lie group G is a case in which $h_{A(G;H)}$ is not trivial (more precisely, $h^{(2)}_{A(G;H)} \neq 0$). Adding the simple connectedness of G, we obtain a non-integrable Lie algebroid.

Some version of Bott's phenomenon on the ground of regular Lie algebroids is the aim of work [18]. There, this Vanishing Theorem is interpreted for TC-foliations, especially, for nonclosed Lie subgroups, and used to the proving of the nonexistence of Lie subalgebras of some types.

3) The present work has 3 parts and concerns the construction of the characteristic homomorphisms for flat and for partially flat regular Lie algebroids. The first part is devoted to the investigation of some properties of regular Lie algebroids over Euclidean spaces, needed in the sequel, such as, for example:

— *Any regular Lie algebroid over the foliated manifold* $(\mathbb{R}^p\times\mathbb{R}^q, T\mathbb{R}^p\times 0)$ *possesses a globally determined flat connection and is trivial in the sense that it is isomorpic to the pullback of an entirely nontransitive Lie algebroid over* $\mathbb{R}^q$ *via the projection* $\mathbb{R}^p\times\mathbb{R}^q \longrightarrow \mathbb{R}^q$.

Next, the invariant cross-sections with respect to a representation of the trivial transitive Lie algebroid $T\mathbb{R}^p\times\mathfrak{g}$ and of a regular Lie algebroid over the foliated manifold $(\mathbb{R}\times M, T\mathbb{R}\times E)$ are studied. The results obtained here are used further, for example, in the proofs of the homotopic invariance of the characteristic homomorphisms with respect to subalgebroids. These results are elementary but with the use of a theorem about some system of partial differential equations with parameters (given here together with the proof). Some of them are known from works of R.Almeida and P.Molino [3] or K.Mackenzie [20] (but with other proofs, more sketchy or less algebraic).

The second part is devoted to the characteristic homomorphism of a flat regular Lie algebroid. This part has 7 chapters. In Chap. 1 the author introduces the theory of cohomology with coefficients for arbitrary Lie algebroids, defining three operators ι_ξ, θ^T_ξ, d^T and proving their fundamental properties [given in K.Mackenzie [20] with the proof "standard"]. The characteristic homomorphism of a flat regular Lie algebroid equipped with some subalgebroid is constructed in Chap. 3. Chaps. 4 and 5 concern its properties: the functoriality and the dependence on a subalgebroid. In 5 we introduce the notion of a homotopy between Lie subalgebroids (Def. 5.2) and prove the equivalence of the characteristic homomorphisms for homotopic Lie subalgebroids. We add that [Prop. 5.5.3] two homotopic H-reductions P_t, $t=0,1$, of a principal bundle $P(M,G)$ determine homotopic subalgebroids, and that the converse thorem is not true unless P_t and G are connected. The homomorphism constructed agrees with a suitable one for a flat principal bundle with a given reduction if the flat regular Lie algebroid comes from such a bundle. According to the above, these homomorphisms are equivalent not only for two homotopic reductions but also, more, for two reductions having homotopic Lie subalgebroids. In Sec. 6.2 it is pointed out that the so-called foliated bundle (P,P',ω) where P' is a reduction of P and ω is a connection in P flat over an involutive distribution F gives a flat regular Lie algebroid $(A(P),A(P'),\lambda|F)$ over the foliated manifold (M,F) and then, the characteristic homomorphism

$$\Delta_{\#}^{F}: H(\mathbf{g}; A(P')^{F}) \longrightarrow H_{F}(M)$$

(having the values in the tangential cohomology algebra $H_F(M)$ of (M,F)). The "tangential characteristic classes" of (P,P',ω) - the cohomology classes from the image of $\Delta_{\#}^{F}$ - measure the independence of ω and P', i.e. they do exactly the same as the exotic characteristic classes.

An interpretation of the homomorphism introduced, on the ground of TC-foliations, especially, for nonclosed Lie subgroups, is given in Chap. 7. There are obtained some examples on the ground of nonclosed Lie subgroups (in transitive and in non-transitive cases) having nontrivial the characteristic homomorphism.

Part III concerns the characteristic homomorphism of partially flat regular Lie algebroids, generalizing this notion from the theory of Kamber-Tondeur [10]. Here, some idea of G. Andrzejczak (unpublished) of a change of variables in the Weil algebra [offering facilities for the operating on it] is used in the construction of the Weil algebra for a bundle of Lie algebras.

PART I

LOCAL PROPERTIES OF REGULAR LIE ALGEBROIDS OVER FOLIATED MANIFOLDS

1. TRIVIAL REGULAR LIE ALGEBROIDS

We assume that in our work all the manifolds considered, are of the C^∞-class and Hausdorff, and that the manifolds M, M',... over which we have Lie algebroids are, in addition, connected. By $\Omega^0(M)$ we denote the ring of C^∞ functions on a manifold M, by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M, and by $Sec\,A$ the $\Omega^0(M)$-module of all C^∞ global cross-sections of a given vector bundle A (over M).

We recall [17] that by a *regular Lie algebroid over a foliated manifold* (M,E) (E is a constant dimensional C^∞ involutive distribution on M) we mean a system $A=(A,[\![\cdot,\cdot]\!],\gamma)$ consisting of (a) a vector bundle A over M for which there is defined an $\mathbb{R}$-Lie algebra structure $[\![\cdot,\cdot]\!]$ in the space $Sec\,A$ of global C^∞ cross-sections, (b) a homomorphism of vector bundles $\gamma:A\longrightarrow TM$ (called an *anchor*) such that $Im\,\gamma=E$, $Sec\,\gamma:Sec\,A\longrightarrow\mathfrak{X}(M)$ is a homomorphism of Lie algebras and the following equality $[\![\xi,f\cdot\eta]\!]=f\cdot[\![\xi,\eta]\!]+(\gamma\circ\xi)(f)\cdot\eta$, $f\in\Omega^0(M)$, $\xi,\eta\in Sec\,A$, holds.

If $E=0$, then A is called [20] *completely intransitive.* It is simply a bundle of Lie algebras (Lie algebras $A_{|x}$ and $A_{|y}$, $x,y\in M$, need not be isomorphic, although the bracket $[\xi,\eta]$ of C^∞ cross-sections of A - defined point by point: $[\xi,\eta]_x=[\xi_x,\eta_x]$ - is C^∞, too).

One of the most important constructions of the building of a new regular Lie algebroid is the *inverse-image* $f^{\wedge}A$ by a homomorphism of foliated manifolds $f:(M',E')\longrightarrow(M,E)$ [17]:

$$f^{\wedge}A=E'\times_{(f_*,\gamma)}A=\left\{(v,w)\in E'\times A;\ f_*(v)=\gamma(w)\right\}\subset E'\oplus f^{*}A,$$

$$[\![(X,\sum_j f^j\cdot\xi_j\circ f),(Y,\sum_j g^k\cdot\eta_k\circ f)]\!]=([X,Y],f^j\cdot g^k\cdot[\![\xi_j,\eta_k]\!]\circ f+X(g^k)\cdot\eta_k\circ f-Y(f^j)\cdot\xi_j\circ f)$$

for $f^j, g^k\in\Omega^0(M')$, $\xi_j,\eta_k\in Sec\,A$. The projection onto the first component

$$pr_1:f^{\wedge}A=E'\times_{(f_*,\gamma)}A\longrightarrow E'$$

serves as the anchor.

A nonstrong homomorphism $H:A'\longrightarrow A$ of regular Lie algebroids (over $f:(M',E')\longrightarrow(M,E)$) [17] can be defined smartly as a superposition $A'\xrightarrow{\bar H}f^{\wedge}A\xrightarrow{\kappa}A$ of some strong homomorphism $\bar H$ and the canonical one $\kappa=pr_2$.

Here we write the basic (easy to prove) properties of the operation of the inverse-image:

a) $(g\circ f)^{\wedge}A\cong f^{\wedge}(g^{\wedge}A)$,

b) if $i_x:\{x\}\hookrightarrow M$ is the inclusion, then $i_x{}^{\wedge}A\cong\mathbf{g}_{|x}$ ($\mathbf{g}:=Ker\,\gamma$ and $\mathbf{g}$ is a bundle of Lie algebras; Lie algebras $\mathbf{g}_{|x}$ and $\mathbf{g}_{|y}$ are isomorphic provided that x and y lie on the same leaf of the foliation E).

Definition 1.1. By a *trivial regular Lie algebroid over* (M,E) we shall mean each algebroid isomorphic to $f^{\wedge}A$ for any completely intransitive Lie algebroid A.

Example 1.2. Transitive trivial Lie algebroid. Let a trivial Lie algebroid $f^{\wedge}A$ (where A is a completely intransitive Lie algebroid A on a manifold N) be transitive (this means that it is over (M,TM)) . Then f is a constant mapping, say, $f(x)\equiv y$. Put $\bar y:M\longrightarrow\{y\}$, $x\longmapsto y$, and let $i_y:\{y\}\hookrightarrow N$ be an inclusion. Then

$$f^{\wedge}A\cong\bar y^{\wedge}(i_y{}^{\wedge}A)\cong\bar y^{\wedge}(\mathfrak{g})=TM\times\mathfrak{g}\qquad(\mathfrak{g}=\mathbf{g}_{|y}).$$

Clearly, $\bar y^{\wedge}(\mathfrak{g})$ is a usual trivial transitive Lie algebroid [26], [17].

Example 1.3. Consider two manifolds M and N, the projection $pr_2:M\times N\longrightarrow N$ and a vector bundle of Lie algebras $\mathfrak{f}$ on N considered as a completely intransitive Lie algebroid. Of course, $pr_2:(M\times N,TM\times 0)\longrightarrow(N,0)$ is a homomorphism of foliated manifolds. We see that the inverse-image $pr_2{}^{\wedge}(\mathfrak{f})$ is equal to $(TM\times 0)\oplus pr_2{}^{*}(\mathfrak{f})$. Each cross-section of $pr_2^{\wedge}(\mathfrak{f})$ is a sum of cross-sections of the form $(X, f\cdot\sigma\circ pr_2)$ for $X\in Sec(TM\times 0)$, $f\in\Omega^0(M\times N)$, $\sigma\in Sec\,\mathfrak{f}$. Therefore the structure of a Lie algebra in $Sec\, pr_2^{\wedge}(\mathfrak{f})$ is determined uniquely by demanding that

$$[\![(X, f\cdot\xi\circ pr_2), (Y, g\cdot\eta\circ pr_2)]\!] = ([X,Y], f\cdot g\cdot [\![\xi,\eta]\!]\circ pr_2 + X(g)\cdot\eta\circ pr_2 - Y(f)\cdot\xi\circ pr_2).$$

Example 1.4. Each C^∞ constant dimensional and completely integrable distribution E on a manifold M is a regular Lie algebroid being, of course, trivial.

The fundamental role in the proof of some structural theorems on a local shape of regular Lie algebroids and their properties is played by some theorem concerning global solutions of some system of differential equations, see below.

2. GLOBAL SMOOTH SOLUTION OF SOME SYSTEM OF DIFFERENTIAL EQUATIONS WITH PARAMETERS

Denote the canonical coordinates on $\mathbb{R}^m\times\mathbb{R}^n$ by $(x^1,\dots,x^m,y^1,\dots,y^n)$.

Theorem 2.1. *Let C^∞ functions b^k_i, $a^k_{ri}:\mathbb{R}^m\times\mathbb{R}^n\longrightarrow\mathbb{R}$, $r,k\leqslant q$, $i\leqslant m$, be given. Consider a system of partial differential equations*

$$\frac{\partial z^k}{\partial x^i}(x,y) = -b^k_i(x,y) + \sum_{r=1}^{q} a^k_{ri}(x,y)\cdot z^r, \qquad k\leqslant q,\quad i\leqslant m, \tag{1}$$

satisfying the conditions of local integrability:

$$\begin{aligned}
&\frac{\partial b^k_i}{\partial x^s} - \frac{\partial b^k_s}{\partial x^i} = -\sum_{u=1}^{q} a^k_{ui}\cdot b^u_s + \sum_{u=1}^{q} a^k_{us}\cdot b^u_i \\
&\frac{\partial a^k_{ri}}{\partial x^s} - \frac{\partial a^k_{rs}}{\partial x^i} = \sum_{u=1}^{q} a^k_{us}\cdot a^u_{ri} - \sum_{u=1}^{q} a^k_{ui}\cdot a^u_{rs}
\end{aligned}\qquad i,s\leqslant m,\ r\leqslant q.$$

Then, for an arbitrarily taken C^∞ mapping $g:\mathbb{R}^n\longrightarrow\mathbb{R}^q$, there exists exactly one globally defined C^∞ solution $z:\mathbb{R}^m\times\mathbb{R}^n\longrightarrow\mathbb{R}^q$ such that $z(0,y)=g(y)$, $y\in\mathbb{R}^n$.

Remark. The simple classical theorem asserts the existence and the uniqueness of some C^∞ solution determined in some neighbourhood of an arbitrarily taken point of the form $(0,y)$.

Proof of Th. 2.1. In this proof we use some elementary facts concerning the theory of foliations and the global existence of a solution of some system of ordinary differential equations without parameters.

Put $M=\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^q$ with coordinates (x^i,y^j,z^k) and define 1-forms ω^k on M by

$$\omega^k=dz^k+\sum_{i=1}^{m}\Big(b^k_i(x,y)-\sum_{r=1}^{q}a^k_{ri}(x,y)\cdot z^r\Big)dx^i.$$

Consider the following system of linearly independent C^∞ 1-forms on M:

$$\big(\omega^1,\dots,\omega^q,dy^1,\dots,dy^n\big).$$

1) *The distribution E generated by this system of 1-forms is integrable.*

This results from the following (easy to obtain) equations:

$$d(dy^j)=0,$$
$$d\omega^k=\sum_u \alpha^k_u\wedge\omega^u+\sum_j\beta^k_j\wedge dy^j\quad,$$

in which $\alpha^k_u=\sum_{i=1}^{m}a^k_{ui}\cdot dx^i$, $\beta^k_j=\sum_{i=1}^{m}\Big(-\dfrac{\partial b^k_i}{\partial y^j}+\sum_{r=1}^{q}\dfrac{\partial a^k_{ri}}{\partial y^j}\cdot z^r\Big)\cdot dx^i$. E has the dimension equal to m.

2) *A C^∞ mapping $z:\mathbb{R}^m\times\mathbb{R}^n\longrightarrow\mathbb{R}^q$ is a solution of (1) if and only if, for each point $y_o\in\mathbb{R}^n$, the manifold $L_{y_o}(z):=\{(x,y_o,z(x,y_o));\ x\in\mathbb{R}^m\}$ is an integral of E.*

Indeed, $L_{y_o}(z)$ is an m-dimensional C^∞ manifold with the global trivialization $\bar z_{y_o}:\mathbb{R}^m\longrightarrow\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^q$, $x\longmapsto(x,y_o,z(x,y_o))$. Therefore the tangent space to $L_{y_o}(z)$ at $\bar x:=\bar z_{y_o}(x)$ is spanned by the vectors $v^i:=d(\bar z_{y_o})\big(\frac{\partial}{\partial x^i}\big|_x\big)=\frac{\partial}{\partial x^i}\big|_{\bar x}+\sum_r\frac{\partial z^r}{\partial x^i}\big|_{\bar x}\cdot\frac{\partial}{\partial z^r}\big|_{\bar x}$. The equalities

$dy^j(v^i)=0$ and

$$\omega^k(v^i)=\frac{\partial z^k}{\partial x^i}(x,y_o)+b^k_i(x,y_o)-\sum_{r=1}^{q}a^k_{ri}(x,y_o)\cdot z^r(x,y_o)=0$$

demonstrate our assertion.

3) The space $E_{(x,y,z)}$ lies on the plane OXZ. Besides, for $v=\sum_i a^i\cdot\frac{\partial}{\partial x^i}+\sum_r c^r\cdot\frac{\partial}{\partial z^r}$, we have: $\omega^k(v)=c^k+\sum_i\big(b^k_i-\sum_{r=1}^{q}a^k_{ri}\cdot z^r\big)\cdot a^i$, which implies that

$$\omega^k(v)=0 \;\Leftrightarrow\; c^k=-\sum_i\Big(b_i^k-\sum_{r=1}^{q} a_{ri}^k\cdot z^r\Big)\cdot a^i. \tag{2}$$

As a simple corollary we obtain: if $a^i=0$ for all j, then $v=0$.

4) Let L be the leaf of the distribution E, passing through a point (x_o,y_o,z_o) and take the projection $pr_1:L\longrightarrow\mathbb{R}^m$. Since $pr_{1*}(\sum a^i\cdot\frac{\partial}{\partial x^i}+\sum c^r\cdot\frac{\partial}{\partial z^r})=\sum a^i\cdot\frac{\partial}{\partial x^i}$, 3) above gives that pr_{1*} is an isomorphism, therefore pr_1 is a local diffeomorphism. According to the simple connectedness of $\mathbb{R}^m$, to see that pr_1 is a diffeomorphism, we only need notice the surjectivity of this projection. For the purpose, take arbitrarily a point $x_1\in\mathbb{R}^m$ and choose $\lambda:=x_1-x_o$. Define the embedding

$$\varphi:\mathbb{R}\times\mathbb{R}^q\longrightarrow\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^q,\quad (t,z)\longmapsto(x_o+t\cdot\lambda,y_o,z)$$

and calculate: $\varphi^*(dx^i)=\lambda^i dt$, $\varphi^*\omega^k=dz^k+\sum_i\big(\tilde b_i^k-\sum_{r=1}^{q}\tilde a_{ri}^k\cdot z^r\big)\cdot\lambda^i dt$, $k\leq q$, where $\tilde b_i^k,\tilde a_{ri}^k:\mathbb{R}\longrightarrow\mathbb{R}$ are defined by the formulae $\tilde b_i^k(t)=b_i^k(x_o+t\cdot\lambda,y_o)$, $\tilde a_{ri}^k(t)=a_{ri}^k(x_o+t\cdot\lambda,y_o)$. The 1-forms $\varphi^*\omega^k$ correspond to the system of ordinary differential equations of the first order, being linear nonhomogeneous

$$\frac{\partial z^k}{\partial t}=-\sum_i\tilde b_i^k(t)\cdot\lambda^i+\sum_{r=1}^{q}\Big(\sum_i\tilde a_{ri}^k(t)\cdot\lambda^i\Big)\cdot z^r.$$

Consider the initial condition $z^k(0)=z_o^k$. The well-known classical theorem [27] states that there exists exactly one globally determined (on the whole space $\mathbb{R}$) solution $z=(z^1,\dots,z^q)$ of this system, satisfying the initial condition. As previously, $\tilde L=\{(t,z(t));\ t\in\mathbb{R}\}$ is a maximal integral of the one dimensional distribution determined by the system of 1-forms $(\varphi^*\omega^1,\dots,\varphi^*\omega^q)$. $\kappa:\mathbb{R}\longrightarrow\tilde L$, $t\longmapsto(t,z(t))$, is a global trivialization of $\tilde L$. Now, we prove that $\varphi[\tilde L]$ is an integral manifold of the distribution E. To this end, we notice that the tangent space to the manifold $\tilde L$ at a point $(t,z(t))$ is spanned by the vector $\frac{d}{dt}+\sum_k\tilde c^k\cdot\frac{\partial}{\partial z^k}$ where $\tilde c^k=-\sum_i\tilde b_i^k(t)\cdot\lambda^i+$ $+\sum_{r=1}^{q}\big(\sum_i\tilde a_{ri}^k(t)\cdot\lambda^i\big)\cdot z^r(t)$. It is easy to obtain that $\varphi_*(\frac{d}{dt}+\sum_k\tilde c^k\cdot\frac{\partial}{\partial z^k})=\sum_i\lambda^i\cdot\frac{\partial}{\partial x^i}+$ $+\sum_k\tilde c^k\cdot\frac{\partial}{\partial z^k}$ and $\tilde c^k=-\sum_i\big(b_i^k(x_o+t\cdot\lambda,y_o)-\sum_r a_{ri}^k(x_o+t\cdot\lambda,y_o)\cdot z^r(t)\big)\cdot\lambda^i$. Therefore, according to step 3) above, the vector $\sum_i\lambda^i\cdot\frac{\partial}{\partial x^i}+\sum_k\tilde c^k\cdot\frac{\partial}{\partial z^k}$ lies in the space $E_{|(x_o+t\lambda,y_o,z(t))}$, which is the reason why $\varphi[\tilde L]$ is contained in the maximal integral of E passing through (x_o,y_o,z_o). Then $x_1=pr_1(x_1,y_o,z(1))=pr_1(\varphi(1,z(1)))=$ $=pr_1\circ\varphi\circ\kappa(1)\in pr_1[\tilde L]$.

5) Take into consideration a function $g:\mathbb{R}^n \longrightarrow \mathbb{R}^q$ and a submanifold $N=\{(0,y,g(y));\ y\in\mathbb{R}^n\}$ of M. It is a transverse manifold of the foliation E. Indeed, let $v\in T_{(0,y,g(y))}N\cap E_{|(0,y,g(y))}$. Then $v=\sum_j b^j\cdot\frac{\partial}{\partial y^j}+\sum_k c^k\cdot\frac{\partial}{\partial z^k}=\sum_i a^i\cdot\frac{\partial}{\partial x^i}+\sum_k \bar{c}^k\cdot\frac{\partial}{\partial z^k}$ for some reals a^i, b^j, c^k, $\bar{c}^k$, therefore $a^i=b^j=0$ and, by step 3) above, $v=0$. Denote by L_y the leaf of the foliation E, passing through $(0,y,g(y))$, and define

$$\bar{L}=\bigcup_{y\in\mathbb{R}^n} L_y.$$

$\bar{L}$ is, of course, an embedding submanifold of M. We prove that

$$pr:=pr_{1,2}|\bar{L}:\bar{L}\longrightarrow\mathbb{R}^m\times\mathbb{R}^n$$

(being clearly a smooth bijection, see the previous step of the proof) is a diffeomorphism. Take a point $(x_o,y_o,z_o)\in\bar{L}$ and a vector $v\in T_{(x_o,y_o,z_o)}\bar{L}$ such that $pr_*(v)=0$. The equality $v=0$ is what we need to assert. v is of the form $v=\sum_k c^k\cdot\frac{\partial}{\partial z^k}$. Consider two complete transversals T_0 and T_{x_o} of E determined by the equations $x=0$ and $x=x_o$, respectively, and a diffeomorphism $\varphi:T_{x_o}\longrightarrow T_0$ such that the points (x_o,y,z) and $\varphi(x_o,y,z)$ lie on one of the leaves of E. φ is, clearly, uniquely determined. The vector v is tangent to T_{x_o}. Since φ is of the form $\varphi(x_o,y,z)=(0,y,\tilde{\varphi}(y,z))$ for some function $\tilde{\varphi}$, therefore $w:=\varphi_*(v)$ is of the form $w=\sum_k\tilde{c}^k\frac{\partial}{\partial z^k}$, i.e. its coordinates with respect to the vectors $\frac{\partial}{\partial y^j}$ are zero. On the other hand, $v\in T_{(x_o,y_o,z_o)}\bar{L}\cap T_{(x_o,y_o,z_o)}T_{x_o}=T_{(x_o,y_o,z_o)}(\bar{L}\cap T_{x_o})$ ($\bar{L}\cap T_{x_o}$ is equal to $\varphi^{-1}[N]$ and is a submanifold) and $\varphi(x_o,y_o,z_o)=(0,y_o,g(y_o))$; then $w\in T_{(0,y_o,g(y_o))}N$. However, N is the image of the mapping $\psi:\mathbb{R}^n\longrightarrow\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^q$, $y\longmapsto(0,y,g(y))$, so, $w=\psi_{*y_o}(\bar{w})$ for some $\bar{w}\in T_{y_o}\mathbb{R}^n$. Therefore $0=pr_*(w)=pr_*(\psi_{*y_o}(\bar{w}))=\bar{w}$, which implies $w=0$ and, next, $v=0$.

6) Let $pr_3:\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^q\longrightarrow\mathbb{R}^q$ denote the projection onto the last factor. The mapping $z:\mathbb{R}^m\times\mathbb{R}^n\longrightarrow\mathbb{R}^q$ equal to $z:=pr_3\circ(pr)^{-1}$ is, according to step 2) of our proof, the sought-for solution of system of differential equations (1). ■

3. A REGULAR LIE ALGEBROID OVER $(\mathbb{R}^p\times\mathbb{R}^q, T\mathbb{R}^p\times 0)$ POSSESSES A FLAT CONNECTION

The following theorem generalizes the result of K.Mackenzie [20] concerning transitive Lie algebroids (see also [3]).

Theorem 3.1. *Every regular Lie algebroid over* $(\mathbb{R}^p\times\mathbb{R}^q, T\mathbb{R}^p\times 0)$ *possesses a flat connection.*

We recall [17] that by a connection in a regular Lie algebroid $A=(A,[\![\cdot,\cdot]\!],\gamma)$ over a foliated manifold (M,E) we mean a splitting of the following Atiyah sequence of A: $0\longrightarrow \mathbf{g}\hookrightarrow A\xrightarrow{\gamma} E\longrightarrow 0$, i.e. a homomorphism of vector bundles $\lambda: E\longrightarrow A$ such that $\gamma\circ\lambda = id_E$. A connection λ is flat if $Sec\,\lambda: Sec\,E\longrightarrow Sec\,A$, $X\longmapsto \lambda\circ X$, is a homomorphism of Lie algebras.

Proof of Theorem 3.1. Consider any regular Lie algebroid B over $(\mathbb{R}^p\times\mathbb{R}^q, T\mathbb{R}^p\times 0)$ and its Atiyah sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{g} \arrow[r, hook] & B \arrow[r, "\gamma"] \arrow[d] & T\mathbb{R}^p\times 0 \arrow[r] & 0 \, . \\ & & \mathbb{R}^p\times\mathbb{R}^q & & \end{tikzcd}$$

Assume that on $\mathbb{R}^p\times\mathbb{R}^q$ we have the canonical coordinates $(y^1,\dots,y^p,y^{p+1},\dots,y^{p+q})$. We prove, by induction with respect to $n=1,2,\dots,p$, that

(*) *there exist linearly independent cross-sections* $Y_1,\dots,Y_n$ *of* B *such that*

$$\begin{aligned} &(a)\ \gamma\circ Y_i = \frac{\partial}{\partial y^i},\quad i\leq n,\\ &(b)\ [\![Y_i,Y_j]\!] = 0,\quad i,j\leq n. \end{aligned}\tag{3}$$

Of course, the cross-sections $Y_1,\dots,Y_n$ fulfilling (a) and (b) for $n=p$ give rise to the connection $\lambda: T\mathbb{R}^p\times 0\longrightarrow B$ defined uniquely by demanding that $\lambda\circ\frac{\partial}{\partial y^i}=Y_i$, $i\leq p$. Clearly, λ is flat.

(*) is evidently valid for $n=1$. Let assertion (*) be valid for some number $m\in\{1,\dots,p-1\}$. We prove that it is true for $m+1$. For the purpose, take linearly independent cross-sections $X_1,\dots,X_{q'},Y_1,\dots,Y_m$ of B such that $X_1,\dots,X_{q'}$ form a basis of $\mathbf{g}$ and $Y_1,\dots,Y_m$ fulfil (a) and (b) from (*) for $n=m$. Let $\bar{Y}$ be an arbitrary cross-section of B for which $\gamma\circ\bar{Y}=\frac{\partial}{\partial y^{m+1}}$. We shall find C^∞ functions $z^1,\dots,z^{q'}\in\Omega^0(\mathbb{R}^p\times\mathbb{R}^q)$ such that $[\![Y_i,Y_{m+1}]\!]=0$, $i\leq m$, where $Y_{m+1}:=\sum_{l=1}^{q'}z^l\cdot X_l+\bar{Y}$. To this end, put $[\![Y_i,\bar{Y}]\!]=\sum_{k=1}^{q'}b^k_i\cdot X_k$, $i\leq m$, and $[X_i,X_j]=\sum_{k=1}^{q'}a^k_{ij}\cdot X_k$, $i,j\leq q'$. Then the equations $[\![Y_i,Y_{m+1}]\!]=0$, $i\leq m$, are all equivalent to the following system of differential equations with parameters $y^{m+1},\dots,y^{p+q}$:

$$\frac{\partial z^k}{\partial y^i}(..y^i..,y^{m+1},\dots,y^{p+q})=-b^k_i(..y^i..,y^{m+1},\dots,y^{p+q})+\sum_{r=1}^{q'}a^k_{ri}(..y^i..,y^{m+1},\dots,y^{p+q})\cdot z^r$$

$$k\leq q',\quad i\leq m.$$

The system like this is always uniquely integrable and is locally integrable if and only if the following conditions of local integrability are satisfied:

$$\begin{aligned}\frac{\partial b^k_i}{\partial y^s}-\frac{\partial b^k_s}{\partial y^i}&=-\sum_{u=1}^{q'}a^k_{ui}\cdot b^u_s+\sum_{u=1}^{q'}a^k_{us}\cdot b^u_i\\ \frac{\partial a^k_{ri}}{\partial y^s}-\frac{\partial a^k_{rs}}{\partial y^i}&=\sum_{u=1}^{q'}a^k_{us}\cdot a^u_{ri}-\sum_{u=1}^{q'}a^k_{ui}\cdot a^u_{rs}\,.\end{aligned}\qquad i,s\leq m,\ r\leq q'.$$

However, these conditions hold by the Jacobi identities $[\![[\![Y_s,\bar{Y}]\!],Y_i]\!]+cycl=0$ and $[\![[\![X_r,Y_s]\!],Y_i]\!]+cycl=0$. According to Theorem 2.1, the system has a global solution $(z^1,\dots z^{q'})\in\Omega^0(\mathbb{R}^p\times\mathbb{R}^q,\mathbb{R}^{q'})$ fulfilling an arbitrarily taken initial condition. To prove our theorem, take the system $(Y_1,\dots,Y_m,Y_{m+1})$ of vector fields where $Y_{m+1}=\sum_{l=1}^{q'}z^l\cdot X_l+\bar{Y}$. ■

4. A REGULAR LIE ALGEBROID OVER $(\mathbb{R}^p\times\mathbb{R}^q, T\mathbb{R}^p\times 0)$ IS TRIVIAL

Theorem 4.1. *Every regular Lie algebroid B over $(\mathbb{R}^p\times\mathbb{R}^q,T\mathbb{R}^p\times 0)$ is trivial; more precisely, it is of the form $pr_2^{\wedge}(f)$ for the projection $pr_2:\mathbb{R}^p\times\mathbb{R}^q\longrightarrow\mathbb{R}^q$ and some* *vector bundle* $\mathfrak{f}$ *over* $\mathbb{R}^q$ *of Lie algebras.*

We begin with two lemmas.

Lemma 4.2. (*cf.* [3]) *Let* $Y_1,\dots,Y_p$ *be cross-sections of* B *satisfying conditions* (3) (*for* $n=p$). *Then there exists a basis* $(X_1,\dots,X_{q'})$ *of* $\mathbf{g}$ *such that*

$$[\![Y_i,X_j]\!]=0,\quad i\leq p,\ j\leq q'. \tag{4}$$

Proof. $\mathbf{g}$ being over $\mathbb{R}^p\times\mathbb{R}^q$ is trivial, therefore it possesses a global basis $(\tilde{X}_1,\dots,\tilde{X}_{q'})$ of cross-sections. We find C^∞ functions f^r_j, $j,r\leq q'$, such that

(1°) $det[f^r_j(x)]\neq 0$ for all $x\in\mathbb{R}^p\times\mathbb{R}^q$,
(2°) the cross-sections $X_j=\sum f^r_j\cdot\tilde{X}_r$ satisfy (4) above.

(2°) is equivalent to the following condition:

$$0=[\![Y_i,X_j]\!]=[\![Y_i,\sum_r f^r_j\cdot\tilde{X}_r]\!]=\sum_r f^r_j\cdot[\![Y_i,\tilde{X}_r]\!]+\frac{\partial}{\partial y^i}(f^r_j)\cdot\tilde{X}_r.$$

Since $\gamma\circ[\![Y_i,\tilde{X}_r]\!]=0$, therefore $[\![Y_i,\tilde{X}_r]\!]\in Sec\,\mathbf{g}$, thus $[\![Y_i,\tilde{X}_r]\!]=\sum_{k=1}^{q'}a^k_{ri}\cdot\tilde{X}_k$ for some function $a^k_{ri}\in C^\infty(\mathbb{R}^p\times\mathbb{R}^q)$. Therefore (2°) is further equivalent to

$$0=\sum_k\Big(\sum_r f^r_j\cdot a^k_{ri}+\frac{\partial}{\partial y^i}(f^k_j)\Big)\cdot X_k\ ,$$

i.e. to the conditions $\dfrac{\partial}{\partial y^i}(f^k_j)+\sum\limits_r^{q'} f^r_j\cdot a^k_{ri}=0,\qquad i\leq p,\quad k,j\leq q'.$ Consider the following system of partial differential equations (with parameters $(y^{p+1},\dots,y^{p+q})$)

$$\frac{\partial z^k}{\partial y^i}(..y^i..,y^{p+1},\dots,y^{p+q})=-\sum_{r=1}^{q'}a^k_{ri}(..y^i..,y^{p+1},\dots,y^{p+q})\cdot z^r,\qquad k\leq q',\ i\leq p. \tag{5}$$

The following equations

$$\frac{\partial a^k_{ri}}{\partial y^s}-\frac{\partial a^k_{rs}}{\partial y^i}=\sum_{u=1}^{q'}a^k_{ui}\cdot a^u_{rs}-\sum_{u=1}^{q'}a^k_{us}\cdot a^u_{ri}\ ,\qquad i,s\leq p,\ r\leq q'$$

are conditions of its local integrability. They are equivalent to the true equality: $[\![\tilde{X}_r,[\![Y_i,Y_s]\!]]\!]=0$. Take into consideration q' initial conditions of the form:

$$z^k(0,y)=\delta^k_j\ ,\quad k=1,\dots,q',\ y\in\mathbb{R}^q, \tag{$*j$}$$

indexed by $j=1,\dots,q'$. Let $f^1_j,\dots,f^{q'}_j$ be the solution of (5) defined on $\mathbb{R}^p\times\mathbb{R}^q$ and satisfying condition $(*j)$ (the existence is obtained by Th.2.1). It remains to show condition (1°) above. Assume to the contrary that, at some point $(x_o,y_o)\in\mathbb{R}^p\times\mathbb{R}^q$, $\det[f^r_j(x_o,y_o)]=0$. This means that the vectors $[f^1_j(x_o,y_o),\dots,f^{q'}_j(x_o,y_o)]$, $j\leqslant q'$, are linearly dependent. Changing, if necessary, the numeration of the initial conditions, we may assume that $[f^1_1(x_o,y_o),\dots,f^{q'}_1(x_o,y_o)]=\sum_{j=2}^{q'}C^j\cdot[f^1_j(x_o,y_o),\dots,f^{q'}_j(x_o,y_o)]$. Fix in equations (5) the parameters $(y^{p+1},\dots,y^{p+q})$ by putting y_o instead of them. In the equations just obtained (without parameters) consider the initial condition:

$$z^k(x_o)=\sum_{j=2}^{q'}C^j\cdot f^k_j(x_o,y_o),\quad k\leqslant q'.$$

It is clearly fulfilled by the solution $(f^1_1(\cdot,y_o),\dots,f^{q'}_1(\cdot,y_o))$ and, simultaneously, by the family $g^k=\sum_{j=2}^{q'}C^j\cdot f^k_j(\cdot,y_o)$, $k\leqslant q'$, which is also a solution of the system of differential equations obtained . By the uniqueness of solutions of this system, $f^k_1(\cdot,y_o)=g^k$ for $k\leqslant q'$. In particular, we have $f^k_1(0,y_o)=g^k(0)$, which means that the vector $[f^1_1(0,y_o),\dots,f^{q'}_1(0,y_o)]$ is a linear combination of $[f^1_j(0,y_o),\dots,f^{q'}_j(0,y_o)]$, $2\leqslant j\leqslant q'$, which is not possible. □

Lemma 4.3. *Let cross-sections* $(X_1,\dots,X_{q'},Y_1,\dots,Y_p)$ *of* B *satisfy conditions* (3) *and* (4) *above. Then the structure functions* c^k_{ij} *such that* $[\![X_i,X_j]\!]=\sum c^k_{ij}\cdot X_k$ *are constant on plaques of the foliation* $T\mathbb{R}^p\times 0$, *i.e. on the submanifolds* $\mathbb{R}^p\times\{*\}$.

Proof. Since $[\![Y_s,[\![X_i,X_j]\!]]\!]+cycl=0$, we have

$$0=[\![Y_s,[\![X_i,X_j]\!]]\!]+[\![X_i,[\![X_j,Y_s]\!]]\!]+[\![X_j,[\![Y_s,X_i]\!]]\!]=[\![Y_s,[\![X_i,X_j]\!]]\!]=[\![Y_s,\sum_k c^k_{ij}\cdot X_k]\!]$$
$$=\sum_k c^k_{ij}\cdot[\![Y_s,X_k]\!]+\frac{\partial}{\partial y^s}(c^k_{ij})\cdot X_k=\frac{\partial}{\partial y^s}(c^k_{ij})\cdot X_k,$$

which asserts our lemma. □

Proof of Theorem 4.1. Assume that the cross-sections $(X_1,\dots,X_{q'},Y_1,\dots,Y_p)$ satisfy conditions (3) and (4). The mapping $\lambda:T\mathbb{R}^p\times 0\longrightarrow B$ given by $\lambda\circ\frac{\partial}{\partial y^i}=Y_i$ is a flat connection. Take the embedding $i:\mathbb{R}^q\longrightarrow\mathbb{R}^p\times\mathbb{R}^q$, $y\longmapsto(0,y)$, and put

$$\mathfrak{f}=i^*\mathbf{g}.$$

The system $(\bar X_1,\dots,\bar X_{q'})$ of cross-sections given by $\bar X_i(y)=X_i(0,y)$ serves as a basis of $\mathfrak{f}$. Consider the projection $pr_2:\mathbb{R}^p\times\mathbb{R}^q\longrightarrow\mathbb{R}^q$ and an isomorphism of vector bundles

$$\varphi:pr_2^*(\mathfrak{f})\longrightarrow\mathbf{g},$$

such that $\varphi_{|(x,y)}(\sum a^i\cdot\bar{X}_i(0,y))=\sum a^i\cdot X_i(x,y)$. Next, we shall treat $\mathfrak{f}$ as a completely nontransitive Lie algebroid over $(\mathbb{R}^q,0)$. Our aim is to prove that the mapping

$$F\colon pr_2{}^{\wedge}(\mathfrak{f})=(T\mathbb{R}^p\times 0)\oplus pr_2{}^{*}(\mathfrak{f})\longrightarrow B,\quad (v,w)\longmapsto\lambda(v)+\varphi(w),$$

is an isomorphism of regular Lie algebroids. Of course, F is an isomorphism of vector bundles. It is sufficient to check that

$$\varphi\circ[\![\xi,\nu]\!]=[\![\varphi\circ\xi,\varphi\circ\nu]\!] \tag{5}$$

for $\xi,\nu\in Sec\,pr_2{}^{\wedge}(\mathfrak{f})$ of the form $\xi=(X,f\cdot\sigma\circ pr_2)$ and $\nu=(Y,g\cdot\eta\circ pr_2)$ for $X,Y\in Sec(T\mathbb{R}^p\times 0)$, $\sigma,\eta\in Sec\,\mathfrak{f}$ and $f,g\in C^\infty(\mathbb{R}^p\times\mathbb{R}^q)$. From the definition (see also Ex.1.3) we have

$$[\![\xi,\nu]\!]=[\![(X,f\cdot\sigma\circ pr_2),(Y,g\cdot\eta\circ pr_2)]\!]=([X,Y],f\cdot g\cdot[\![\sigma,\eta]\!]\circ pr_2+X(g)\cdot\eta\circ pr_2-Y(f)\cdot\sigma\circ pr_2).$$

Therefore

$$\begin{aligned}F\circ[\![\xi,\nu]\!]&=\lambda\circ[X,Y]+\varphi\circ(f\cdot g\cdot[\![\sigma,\eta]\!]\circ pr_2+X(g)\cdot\eta\circ pr_2-Y(f)\cdot\sigma\circ pr_2)\\&=[\![\lambda\circ X,\lambda\circ Y]\!]+f\cdot g\cdot\varphi\circ([\![\sigma,\eta]\!]\circ pr_2)+X(g)\cdot\varphi\circ(\eta\circ pr_2)-Y(f)\cdot\varphi\circ(\sigma\circ pr_2).\end{aligned}$$

On the other hand,

$$\begin{aligned}[\![F\circ\xi,F\circ\nu]\!]&=[\![F\circ(X,f\cdot\sigma\circ pr_2),F\circ(Y,g\cdot\eta\circ pr_2)]\!]\\&=[\![\lambda\circ X+f\cdot\varphi\circ(\sigma\circ pr_2),\lambda\circ Y+g\cdot\varphi\circ(\eta\circ pr_2)]\!]\\&=[\![\lambda\circ X,\lambda\circ Y]\!]+[\![f\cdot\varphi\circ(\sigma\circ pr_2),\lambda\circ Y]\!]+[\![\lambda\circ X,g\cdot\varphi\circ(\eta\circ pr_2)]\!]+[\![f\cdot\varphi\circ(\sigma\circ pr_2),g\cdot\varphi\circ(\eta\circ pr_2)]\!]\\&=[\![\lambda\circ X,\lambda\circ Y]\!]+f\cdot g\cdot[\![\varphi\circ(\sigma\circ pr_2),\varphi\circ(\eta\circ pr_2)]\!]+[\![\lambda\circ X,g\cdot\varphi\circ(\eta\circ pr_2)]\!]-[\![\lambda\circ Y,f\cdot\varphi\circ(\sigma\circ pr_2)]\!].\end{aligned}$$

In order to get (5), it will be necessary to observe that

(a) $\varphi\circ([\![\sigma,\eta]\!]\circ pr_2)=[\![\varphi\circ(\sigma\circ pr_2),\varphi\circ(\eta\circ pr_2)]\!]$,

(b) $X(g)\cdot\varphi\circ(\eta\circ pr_2)=[\![\lambda\circ X,g\cdot\varphi\circ(\eta\circ pr_2)]\!]$.

To see (a), write σ and η in the form $\sigma=\sum\sigma^i\cdot\bar{X}_i$, $\eta=\sum\eta^j\cdot\bar{X}_j$, σ^i, $\eta^j\in\Omega^0(\mathbb{R}^q)$, and calculate

$$\begin{aligned}\varphi\circ([\![\sigma,\eta]\!]\circ pr_2)&=\varphi\circ([\sum\sigma^i\cdot\bar{X}_i,\sum\eta^j\cdot\bar{X}_j]\circ pr_2)=\varphi\circ(\sum_{i,j}\sigma^i\circ pr_2\cdot\eta^j\circ pr_2\cdot[\bar{X}_i,\bar{X}_j]\circ pr_2)\\&=\varphi\circ(\sum\sigma^i\circ pr_2\cdot\eta^j\circ pr_2\cdot(\tilde{a}^k_{ij}\cdot\bar{X}_k)\circ pr_2)=\sum_{i,j}\sigma^i\circ pr_2\cdot\eta^j\circ pr_2\cdot a^k_{ij}\cdot X_k\\&=\sum_{i,j}\sigma^i\circ pr_2\cdot\eta^j\circ pr_2\cdot[X_i,X_j]=[\![\sum\sigma^i\circ pr_2\cdot X_i,\sum\eta^j\circ pr_2\cdot X_j]\!]\\&=[\![\varphi\circ(\sigma\circ pr_2),\varphi\circ(\eta\circ pr_2)]\!].\end{aligned}$$

To see (b), write additionally $X=\sum_{k=1}^{p}a^k\cdot\frac{\partial}{\partial y^k}$, $a^k\in\Omega^0(\mathbb{R}^{p+q})$. Then

$$[\![\lambda\circ X,g\cdot\varphi\circ(\eta\circ pr_2)]\!]=[\![\lambda\circ(\sum a^k\cdot\frac{\partial}{\partial y^k}),g\cdot\varphi\circ((\sum\eta^j\cdot\bar{X}_j)\circ pr_2)]\!]$$

$$= [\![\sum a^k\cdot Y_k, g\cdot\sum\eta^j\circ pr_2\cdot X_j]\!] = \sum_{k,j} a^k\cdot\frac{\partial}{\partial y^k}(g\cdot\eta^j\circ pr_2)\cdot X_j$$

$$= \sum_{k,j} a^k\cdot\frac{\partial}{\partial y^k}(g)\cdot\eta^j\circ pr_2\cdot X_j = X(g)\cdot\eta^j\circ pr_2\cdot X_j$$

$$= X(g)\cdot\varphi\circ((\sum\eta^j\cdot\bar{X}_j)\circ pr_2) = X(g)\cdot\varphi\circ(\eta\circ pr_2). \quad\blacksquare$$

Corollary 4.4. *Any transitive Lie algebroid over* $\mathbb{R}^n$ *is isomorphic to the trivial algebroid* $T\mathbb{R}^n\times\mathfrak{g}$ *for some Lie algebra* $\mathfrak{g}$. ■

Because of the trivial fact that each point of a given n-dimensional manifold M possesses a neighbourhood U diffeomorphic to $\mathbb{R}^n$ $\left(\mathbb{R}^n\xrightarrow[\cong]{\varphi}U\overset{i}{\hookrightarrow}M\right)$, any transitive Lie algebroid A over M is locally isomorphic to the trivial algebroid $T\mathbb{R}^n\times\mathfrak{g}$ $\left(\cong\varphi^{\wedge}(i^{\wedge}(A))\right)$ for some Lie algebra $\mathfrak{g}$.

5. REPRESENTATIONS OF THE TRIVIAL LIE ALGEBROID $TR^N\times\mathfrak{G}$ ON A VECTOR BUNDLE

With a real vector bundle $\mathfrak{f}$ over M there is associated a transitive Lie algebroid $A(\mathfrak{f})$ (over M) [17; Sec.1.2] whose fibre over $x\in M$ consists of all $\mathfrak{f}$-*vectors at* x, i.e. linear homomorphisms $l:Sec\,\mathfrak{f}\longrightarrow\mathfrak{f}_{|x}$ for which there exists a vector $u\in T_xM$ such that $l(f\cdot\nu)=f(x)\cdot l(\nu)+u(f)\cdot\nu(x)$, $f\in\Omega^0(M)$ and $\nu\in Sec\,\mathfrak{f}$. The vector u is determined by l uniquely and serves as its anchor. A local trivialization of $A(\mathfrak{f})$ gives the mapping $\bar{\psi}:TU\times End(V)\longrightarrow A(\mathfrak{f})_{|U}$ (V is the typical fibre of $\mathfrak{f}$) defined for a given local trivialization $\psi:U\times V\longrightarrow\mathfrak{f}_{|U}$ of $\mathfrak{f}$ by the formula:

$$\bar{\psi}(v,a)(\nu)=\psi_{|x}(v(\nu_\psi)+a(\nu_\psi(x)))$$

where, for $\nu\in Sec\,\mathfrak{f}$, $\nu_\psi:U\longrightarrow\mathbb{R}$ is a function $x\longmapsto\psi_{|x}^{-1}(\nu_x)$ [17; Lemma 5.4.4].

A cross-section $\xi\in Sec\,A(\mathfrak{f})$ determines a covariant differential operator $\mathcal{L}_\xi:Sec\,(\mathfrak{f})\longrightarrow Sec\,(\mathfrak{f})$ by the formula $\mathcal{L}_\xi(\nu)(x)=\xi_x(\nu)$. The correspondence $\xi\longmapsto\mathcal{L}_\xi$ is 1-1. The bracket $[\![\cdot,\cdot]\!]$ is defined classically (from the point of view of differential operators). The Lie algebra bundle adjoint of $A(\mathfrak{f})$ can be identified with the vector bundle $End(\mathfrak{f})$. Lem.5.4.4 from [17] mentioned above asserts also that $\bar{\psi}$ is an isomorphism of Lie algebroids. In particular, taking $\psi=id_{\mathbb{R}^n\times V}$, we assert that the Lie algebroid $A(\mathbb{R}^n\times V)$ of the trivial vector bundle $\mathfrak{f}=\mathbb{R}^n\times V$ is isomorphic to the trivial algebroid $T\mathbb{R}^n \times End(V)$ via the canonical isomorphism

$$\mathcal{L}: T\mathbb{R}^n \times End(V) \longrightarrow A(\mathbb{R}^n \times V)$$

defined by the formula: $\mathcal{L}_{|x}(v,a)(\nu) = v(\nu) + a(\nu_x)$. Denote by $\mathcal{L}_{(X,\sigma)}$ the differential operator determined by the cross-section $\mathcal{L}\circ(X,\sigma)$ of $A(\mathbb{R}^n \times V)$, where $X \in \mathfrak{X}(\mathbb{R}^n)$ and $\sigma \in \Omega^0(\mathbb{R}^n, End(V))$. Clearly,

$$\mathcal{L}_{(X,\sigma)}: \Omega^0(\mathbb{R}^n, V) \longrightarrow \Omega^0(\mathbb{R}^n, V), \quad \nu \longmapsto X(\nu) + \sigma(\nu). \tag{6}$$

By a *representation* of a regular Lie algebroid A on $\mathfrak{f}$ (both over M) we mean a strong homomorphism $T: A \longrightarrow A(\mathfrak{f})$ of Lie algebroids. T induces a linear homomorphism $T^+: \mathbf{g} \longrightarrow End(\mathfrak{f})$ of vector bundles of Lie algebras [17].

Let $T: A \longrightarrow A(\mathfrak{f})$ be any representation of a regular Lie algebroid A on $\mathfrak{f}$. A cross-section $\nu \in Sec(\mathfrak{f})$ is called *T-invariant* [17] if $T(v)(\nu) = 0$ for all $v \in A$. The space of all T-invariant cross-sections is denoted by $(Sec(\mathfrak{f}))_{I^\circ(T)}$ (or, briefly, by $(Sec(\mathfrak{f}))_{I^\circ}$).

Theorem 5.1. (*cf.* [20]) *Let* $\bar{T}: T\mathbb{R}^n \times \mathfrak{g} \longrightarrow A(\mathfrak{f})$ *be any representation of the trivial Lie algebroid* $T\mathbb{R}^n \times \mathfrak{g}$ *on* $\mathfrak{f}$. *Then, for each* $\bar{T}^+_{|x}$*-invariant vector* $v \in \mathfrak{f}_{|x}$, *there exists exactly one* $\bar{T}$*-invariant cross-section* $\nu \in Sec(\mathfrak{f})$ (*determined globally !*) *such that* $\nu_x = v$.

Proof. A vector bundle $\mathfrak{f}$ over $\mathbb{R}^n$ is trivial, therefore we may assume that $\mathfrak{f} = \mathbb{R}^n \times V$. $\bar{T}$ determines a homomorphism

$$T: T\mathbb{R}^n \times \mathfrak{g} \longrightarrow T\mathbb{R}^n \times End(V)$$

such that $\mathcal{L}\circ T = \bar{T}$. A mapping $\nu: \mathbb{R}^n \longrightarrow V$ (understood as a cross-section of $\mathbb{R}^n \times V$) is $\bar{T}$-invariant if and only if

$$\mathcal{L}_{T\circ(X,\sigma)}(\nu) = 0 \quad \text{for } X \in \mathfrak{X}(\mathbb{R}^n), \ \nu \in \Omega^0(\mathbb{R}^n; \mathfrak{g}).$$

T can be written in the form

$$T\circ(X,\sigma) = T\circ(X,0) + T(0,\sigma) = (X, \omega(X)) + (0, T^+\circ\sigma) = (X, \omega(X) + T^+\circ\sigma)$$

for a 1-form $\omega \in \Omega^1(\mathbb{R}^n; End(V))$. ω and T^+ satisfy the following (easy to verify) identities (cf. [20; p.102]):

$$-d\omega(X,Y) = [\omega(X),\omega(Y)] \tag{7}$$

$$X(T^{+}\circ\sigma) - T^{+}\circ(X(\sigma)) + [\omega(X),T^{+}\circ\sigma] = 0. \tag{8}$$

ν is $\bar{T}$-invariant if and only if

(a) $\mathcal{L}_{T\circ(X,0)}(\nu) = 0$,

(b) $\mathcal{L}_{T\circ(0,\sigma)}(\nu) = 0$.

(a) is equivalent to the condition of the invariance of ν with respect to the "reduced representation"

$$T\mathbb{R}^{n} \hookrightarrow T\mathbb{R}^{n}\times End(V) \xrightarrow{\bar{T}} A(\mathbb{R}^{n}\times V),$$

whereas (b) says that, for each $x\in\mathbb{R}^{n}$, the vector ν_x is $\bar{T}^{+}_{|x}$-invariant. Condition (a) yields that

$$0 = \mathcal{L}_{T\circ(X,0)}(\nu) = \mathcal{L}_{(X,\omega(X))}(\nu) = X(\nu) + \mathcal{L}_{\omega(X)}(\nu),$$

i.e. that the following differential equation

$$X(\nu) = -\mathcal{L}_{\omega(X)}(\nu), \tag{9}$$

called the *differential equation of an invariant cross-section,* is satisfied.

(7) is the condition of the local integrability of this equation. Indeed, taking a basis $w_1,\dots,w_q$ of V and writing $\nu=\sum z^{s}\cdot w_{s}$, we can equivalently exchange equation (9) for the following system of partial differential equations of the first order:

$$\frac{\partial z^{k}}{\partial x^{i}} = -\sum_{r=1}^{q} a_{i}^{r,k}\cdot z^{r}, \quad i\leqslant n, \quad k\leqslant q, \tag{10}$$

where $a_{i}^{r,k}$ are functions such that $\omega(\frac{\partial}{\partial x^{i}}) = \sum_{r=1}^{q} a_{i}^{r,k}\cdot u_{r,k}$, $u_{r,k}$ being the following basis of $End(V)$ $\left(\cong V^{*}\otimes V\right)$: $u_{r,k} = w_{r}^{*}\otimes w_{k}$. Here are the conditions of the local integrability of (10):

$$-\frac{\partial a_{i}^{r,k}}{\partial x^{s}} + \frac{\partial a_{s}^{r,k}}{\partial x^{i}} = \sum_{u=1}^{q} a_{s}^{u,k}\cdot a_{i}^{r,u} - \sum_{u=1}^{q} a_{i}^{u,k}\cdot a_{s}^{r,u}, \quad i,s\leqslant n, \ r,k\leqslant q.$$

They are equivalent to the equalities:

$$0 = T\circ[\![(\frac{\partial}{\partial x^{i}},0),(\frac{\partial}{\partial x^{s}},0)]\!] \ \left(= [\![T\circ(\frac{\partial}{\partial x^{i}},0),T\circ(\frac{\partial}{\partial x^{s}},0)]\!]\right)$$

which say the same as (7) above.

According to Th.2.1, the initial conditions

$$z^1(0)=z^1_o,\dots,z^q(0)=z^q_o,$$

(uniquely) determine a solution $(z^1,\dots,z^q)$ of (10) defined on whole $\mathbb{R}^n$. It remains to solve the following problem: if the vector $\nu(0)=\sum z^i_o\cdot w_i$ is $(T^+_{|0}:\mathfrak{g}\longrightarrow End(V))$-invariant, then, for each $x\in\mathbb{R}^n$, the vector $\nu(x)=\sum z^i(x)\cdot w_i$ is $T^+_{|x}$-invariant. The invariance of $\nu(x)$ means that $\mathcal{L}_{T^+_{|x}(h)}(\nu_x)=0$ for all $h\in\mathfrak{g}$. Therefore it is sufficient to show that the function $\mathcal{L}_{T^+\circ\bar{h}}(\nu)$ is identically zero for all $h\in\mathfrak{g}$, where $\bar{h}$ denotes the constant function $\mathbb{R}^n\longrightarrow\mathfrak{g}$ always equalling h. Put $\beta=\mathcal{L}_{T^+\circ\bar{h}}(\nu)$ and assume that $\beta(0)=0$. All we need to prove is $X(\beta)=0$ for $X\in\mathfrak{X}(\mathbb{R}^n)$. Using (8) and (9), we have

$$\begin{aligned}
X(\beta)&=X(\mathcal{L}_{T^+\circ\bar{h}}(\nu))=\mathcal{L}_{X(T^+\circ\bar{h})}(\nu)+\mathcal{L}_{T^+\circ\bar{h}}(X(\nu))\\
&=\mathcal{L}_{T^+\circ X(\bar{h})-[\omega X,T^+\circ\bar{h}]}(\nu)+\mathcal{L}_{T^+\circ\bar{h}}(X(\nu))\\
&=\mathcal{L}_{-\omega(X)\circ(T^+\circ\bar{h})+(T^+\circ\bar{h})\circ\omega(X)}(\nu)+\mathcal{L}_{T^+\circ\bar{h}}(X(\nu))\\
&=\mathcal{L}_{-\omega(X)\circ(T^+\circ\bar{h})}(\nu)+\mathcal{L}_{T^+\circ\bar{h}}(\mathcal{L}_{\omega(X)}(\nu)+X(\nu))\\
&=\mathcal{L}_{-\omega(X)}(\beta).
\end{aligned}$$

The linear first order differential equation just obtained $X(\beta)=\mathcal{L}_{-\omega(X)}(\beta)$ is, clearly, fulfilled by the function identically equal to zero. On account of the uniqueness of solutions, we have the conclusion: $\beta\equiv 0$, which ends the proof. ■

As a corollary we obtain

5.2. *For an arbitrary representation $T:A\longrightarrow A(\mathfrak{f})$ of a transitive Lie algebroid A on $\mathfrak{f}$, each invariant cross-section of $\mathfrak{f}$ (defined locally on a connected subset) is uniquely determined by the value at one point. Particularly, if such a cross-section is zero at one point, then it is zero globally.* ■

Remark 5.3. The above theorem can also be checked in a different way, somewhat exceeding the clean theory of Lie algebroids, by proving firstly the following auxiliary theorem 5.3.1 and, secondly, by using Propositions 5.5.2-3 from [20]. These propositions assert that, in the case when a homomorphism T of Lie algebroids is the differential of a μ-homomorphism $F:P\longrightarrow L(\mathfrak{f})$ of principal bundles (i.e. $\mu:G\longrightarrow GL(V)$ is a homomorphism of Lie groups, $L(\mathfrak{f})$ is the $GL(V)$-principal bundle of repers $V\longrightarrow\mathfrak{f}_{|\pi z}$ and $F(z\cdot a)=F(z)\cdot\mu(a)$), P is assumed to be connected, a cross-section ν of f is T-invariant if and only if there exists a μ-invariant vector $w \in V$ such that $\nu_{\pi z} = F(z)(w)$ for all $z \in P$. (Since $F(z)$ is an isomorphism, we have $\nu = 0$ provided ν is zero at at least one point).

Theorem 5.3.1. *A homomorphism $T: T\mathbb{R}^n \times \mathfrak{g} \longrightarrow T\mathbb{R}^n \times End(V)$ of Lie algebroids is the differential of some homomorphism $F: \mathbb{R}^n \times G \longrightarrow \mathbb{R}^n \times GL(V)$ of principal bundles, where G is the connected and simply connected Lie group having $\mathfrak{g}$ as its Lie algebra.*

To prove this, we can give some proposition (auxiliary in this place, but essential in itself).

Proposition 5.3.2. *If $A' \subset A(P)$ is a transitive Lie subalgebroid of the Lie algebroid $A(P)$ of a principal bundle P $(=(P,\pi,M,G,\cdot))$, then there exists a reduction P' of P having A' as its Lie algebroid.*

Proof. Via the canonical projection $\pi^A: TP \longrightarrow A(P)$ [15], we pullback A' to some C^∞ right-invariant involutive distribution Δ on TP $[\Delta_z := (\pi^A_{|z})^{-1}[A'_{|\pi z}],\ z \in P]$. Let P' be a connected maximal integral manifold of Δ. Analogously to part (a) of the proof of Th.1.1 in [12], we assert that $\pi|P': P' \longrightarrow M$ is a coregular surjection. Take the subgroup $G' = \{a \in G;\ R_a[P'] \subset P'\}$, R_a being the right translation by a. By the equalities

$$G' = \left\{a \in G;\ R_a[P'_{|x_o}] \subset P'_{|x_o}\right\} = \left\{a \in G;\ z_o \cdot a \in P'_{|x_o}\right\} = A_{z_o}^{-1}[P'_{|x_o}],$$

where $A_{z_o}: G \longrightarrow P$, $a \longmapsto z_o \cdot a$, $(x_o \in M$ and $z_o \in P'_{|x_o}$ are arbitrarily taken elements), we assert that G' is an immersed submanifold of G. According to the fact that P' is a weak submanifold of P, we easily notice that G' is an immersed Lie subgroup of G with a countable base, and that the induced action $P' \times G' \longrightarrow P'$ is C^∞. Consequently, $(P', \pi|P', M, G', \cdot)$ is a reduction of P to G' whose Lie algebroid equals A'. □

Proof of Th.5.3.1. Let $A = (T\mathbb{R}^n \times \mathfrak{g}) \times (T\mathbb{R}^n \times End(V))$ be the Whitney product of the Lie algebroids $T\mathbb{R}^n \times \mathfrak{g}$ and $T\mathbb{R}^n \times End(V)$ [11] (see also [20; p.108]). A is the Lie algebroid of the Whitney product $(\mathbb{R}^n \times G) \oplus (\mathbb{R}^n \times GL(V))$ $(= \mathbb{R}^n \times (G \times GL(V)))$ of principal bundles $(\mathbb{R}^n \times G)$ and $(\mathbb{R}^n \times GL(V))$. The subbundle $\mathfrak{c} = \{(v, T(v)) \in A;\ v \in T\mathbb{R}^n \times \mathfrak{g}\}$ forms, of course, a Lie subalgebroid and, by Prop.5.5.2, determines a reduction $Q \subset \mathbb{R}^n \times (G \times GL(V))$. There is no problem in seeing that the superposition

$$\kappa : Q \hookrightarrow \mathbb{R}^n \times (G \times GL(V)) \xrightarrow{pr_1} \mathbb{R}^n \times G$$

is a local diffeomorphism onto a simply connected manifold $\mathbb{R}^n \times G$; in consequence, κ is a diffeomorphism. The mapping

$$F : \mathbb{R}^n \times G \xrightarrow{\kappa^{-1}} Q \hookrightarrow \mathbb{R}^n \times (G \times GL(V)) \xrightarrow{pr} \mathbb{R}^n \times GL(V)$$

is the required homomorphism of principal bundles. ∎

6. INVARIANT CROSS-SECTIONS OVER R×M

Using the previous theorems, we prove that the space of global cross-sections of a vector bundle $\mathfrak{f}$ over $\mathbb{R}\times M$, invariant with respect to the representation of a regular Lie algebroid B over $(\mathbb{R}\times M, T\mathbb{R}\times E)$, is canonically isomorphic to the space of cross-sections of the vector bundle $\mathfrak{f}_{|\{t_o\}\times M}$, invariant with respect to the suitable "restricted" representation.

First, we recall the expression: "restricting" - and more precisely, the "inverse-image" - of a representation [17]. Let A be any regular Lie algebroid over (M,E) and $\mathfrak{f}$ any vector bundle over M, whereas $f:(M',E') \longrightarrow (M,E)$ - any morphism of foliated manifolds. By the *inverse-image of a representation* $T: A \longrightarrow A(\mathfrak{f})$ *over* f we mean the representation $f^{*}T : f^{\wedge}(A) \longrightarrow A(f^{*}\mathfrak{f})$ defined as the superposition

$$f^{*}T : f^{\wedge}(A) \xrightarrow{f^{\wedge}T} f^{\wedge}(A(\mathfrak{f})) \xrightarrow{c_{\mathfrak{f}}} A(f^{*}\mathfrak{f})$$

where (*a*) $f^{\wedge}T$ is a homomorphism of Lie algebroids defined by: $f^{\wedge}T(u,v) = (u, T(v))$, $u \in E'$, $v \in A$ $\left(f_{*}(u) = \gamma(v)\right)$,

(*b*) $c_{\mathfrak{f}}$ is the canonical strong isomorphism of Lie algebroids such that, for $(u,l) \in f^{\wedge}(A(\mathfrak{f}))_{|x}$, $w := c_{\mathfrak{f}}(u,l)$ has u as its anchor and satisfies the relation: $w(\nu\circ f) = l(\nu)$ for $\nu \in Sec(\mathfrak{f})$. Obviously, $c_{\mathfrak{f}}^{+}$ appears as the canonical isomorphism of vector bundles $f^{*}(End(\mathfrak{f})) \cong End(f^{*}\mathfrak{f})$, and, furthermore, we can write $(f^{*}T)^{+}_{|x} = T^{+}_{|f(x)}$ for $x \in M'$.

Identifying $g^{\wedge}(f^{\wedge}A)$ with $(f\circ g)^{\wedge}A$ and $g^{*}(f^{*}\mathfrak{f})$ with $(f\circ g)^{*}\mathfrak{f}$ we can write $g^{*}(f^{*}T) = (f\circ g)^{*}T$.

In [17; 2.4.4] the following property of the inverse-image of a representation is given:

— *the linear mapping* $f^*:Sec(\mathfrak{f})\longrightarrow Sec(f^*\mathfrak{f})$, $\nu\longmapsto\nu\circ f$, *can be restricted to the space of cross-sections invariant under* T *and* f^*T, *respectively:*

$$f^*_{I^\circ}:(Sec(\mathfrak{f}))_{I^\circ(T)}\longrightarrow(Sec(f^*\mathfrak{f}))_{I^\circ(f^*T)}.$$

We shall use this notion to the representation T of a regular Lie algebroid B over $(\mathbb{R}\times M, T\mathbb{R}\times E)$ and the mapping $f_{t_o}:M\longrightarrow\mathbb{R}\times M$, $x\longmapsto(t_o,x)$. In this situation, the mapping $(f_{t_o})^*_{I^\circ}$ turns out to be an isomorphism, however, its monomorphy has a more general nature:

Lemma 6.1. *If the saturating of* $f[M']$ *equals* M, *then* $f^*_{I^\circ}$ *is a monomorphism.* [*The saturating is taken with respect to the foliation of* M *determined by* E].

Proof. Assume that $f^*_{I^\circ}(\nu)=0$ for an invariant cross-section ν. This means that, for an arbitrary point $x\in M'$, we have $\nu(f(x))=0$. Let L be the leaf of the foliation of M passing through $f(x)$, and let $i:L\hookrightarrow M$ be the inclusion. According to Th.2.4.4 from [17] mentioned above, we have that $\nu|L\,(=i^*\nu)$ is invariant with respect to the "restricted" representation $i^*T:i^{\wedge}A\longrightarrow A(i^*\mathfrak{f})$. Since $i^{\wedge}A$ is transitive, $\nu|L\equiv0$ on account of 5.2 above. Our assumption concerning the saturation of $f[M']$ implies now the equality $\nu\equiv0$. ■

Here is the aim of this section:

Theorem 6.2. *Let* B *be any regular Lie algebroid over* $(\mathbb{R}\times M, T\mathbb{R}\times E)$ *and* $T:B\longrightarrow A(\mathfrak{f})$ *any representation of* B *on a vector bundle* $\mathfrak{f}$ (*over* $\mathbb{R}\times M$). *Take an arbitrary point* $t_o\in\mathbb{R}$ *and the mapping* $f_{t_o}:M\longrightarrow\mathbb{R}\times M$, $x\longmapsto(t_o,x)$. *Then* $(f_{t_o})^*_{I^\circ}$ *is an isomorphism of vector spaces.*

Proof. On account of Lemma 6.1, it is sufficient to show the surjectivity of $(f_{t_o})^*_{I^\circ}$. Let $\sigma\in Sec(f^*_{t_o}\mathfrak{f})$ be an invariant cross-section. Then, for each $x\in M$, the vector $\sigma(x)\in\mathfrak{f}_{|(t_o,x)}$ is invariant with respect to the representation $T^+_{|(t_o,x)}:\mathbf{g}_{|(t_o,x)}\longrightarrow End(\mathfrak{f}_{|(t_o,x)})$. Consider the embedding $f_x:\mathbb{R}\longrightarrow\mathbb{R}\times M$, $t\longmapsto(t,x)$. Since $Im(f_x)=\mathbb{R}\times\{x\}$ is contained in some leaf of $T\mathbb{R}\times E$, therefore $f_x^{\wedge}(B)$ is a transitive and, by Cor.4.4, trivial Lie algebroid. Th.5.1 yields that the vector $\sigma(x)$ can be uniquely extended to some C^∞ cross-section σ_x of the vector bundle

$f_x^*\mathfrak{f}$, invariant with respect to the representation $f_x^*T: f_x^{\wedge}(B) \longrightarrow A(f_x^*\mathfrak{f})$ $\big(\sigma(x)$ is invariant under $(f_x^*T)^+_{|x}$ because $(f_x^*T)^+_{|x} = T^+_{|f(x)}\big)$. The family $\{\sigma_x;\ x \in M\}$ determines a global cross-section $\sigma^1: \mathbb{R} \times M \longrightarrow \mathfrak{f}$ by the formula: $\sigma^1(t,x) = \sigma_x(t)$. It is evident that $f^*_{t_o}(\sigma^1) = \sigma$. To end the proof, all we need is to show

(*a*) *the smoothness of* σ^1,

(*b*) *the* T*-invariance of* σ^1.

First, we check (*a*). For the purpose, take arbitrarily a point $x_o \in M$ and a simple distinguished open neighbourhood $U \subset M$ of x_o [the domain of some distinguished chart of the foliation $\mathcal{F}$ having E as its tangent bundle]. The foliation $\mathcal{F}_U$ has a global connected transversal manifold, say N, and its leaves are diffeomorphic to a Euclidean space. Then $N' := \{t_o\} \times N$ is a transversal manifold of the distribution $T\mathbb{R} \times E$, see Figure 1 below.

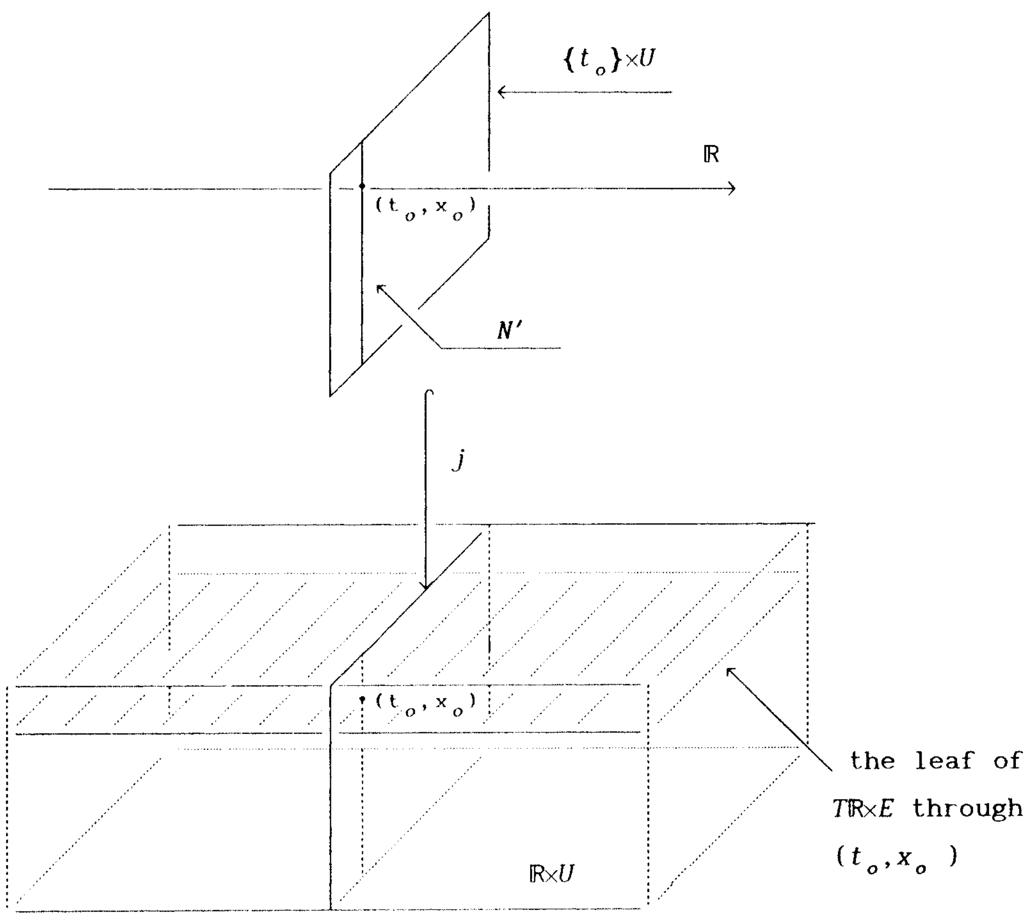


Figure 1.

The cross-section

$$\sigma_o : N' \longrightarrow \mathfrak{f}_{|N'},\quad (t_o, x) \longmapsto \sigma(x),$$

is C^∞ and invariant with respect to the representation $j^* T$, $j: N' \hookrightarrow M' := \mathbb{R}\times U \subset \mathbb{R}\times M$ being the inclusion, moreover, $\sigma' := \sigma^1 | M'$ is some extension of σ_o.

Let $B' := B|M'$. B' is a regular Lie algebroid over $(\mathbb{R}\times U, T\mathbb{R}\times (E_{|U}))$. Leaves of the foliation having $T\mathbb{R}\times (E_{|U})$ as its tangent bundle are of the form $\mathbb{R}\times L$ where L is a leaf of $\mathcal{F}_U$. They are diffeomorphic to a Euclidean space and proper; N' is a global transversal manifold of $T\mathbb{R}\times (E_{|U})$ which cuts each leaf at exactly one point.

It is obvious that, without loss of generality, we may assume that

(1°) $M' = \mathbb{R}^p \times \mathbb{R}^q$,

(2°) B' is over $(\mathbb{R}^p\times\mathbb{R}^q, T\mathbb{R}^p\times 0)$.

(In the context considered above, p is equal to the dimension of the foliation $T\mathbb{R}\times E$).

By the proof of Th.3.1, we assert the existence of global cross-sections $Y_1,\dots,Y_p \in Sec(B')$ such that $\gamma\circ Y_i = \frac{\partial}{\partial y^i}$ and $[Y_i, Y_j] = 0$, $i, j \leqslant p$. Moreover, in the sequel of the proof of our theorem, we can assume that $N' = 0\times\mathbb{R}^q$, $\mathfrak{f} = (\mathbb{R}^p{\times}\mathbb{R}^q)\times V$ and $A(\mathfrak{f}) = T(\mathbb{R}^p{\times}\mathbb{R}^q)\times End(V)$. In our context, a C^∞ cross-section $\sigma_o : 0\times\mathbb{R}^q \longrightarrow \mathfrak{f}_{|0\times\mathbb{R}^q}$, such that $\sigma_o(0,y)$ is invariant with respect to the representation $T^+_{|(0,y)} : \mathbf{g}_{|(0,y)} \longrightarrow End(V)$, is given, and we know that there exists a cross-section $\sigma' : \mathbb{R}^p{\times}\mathbb{R}^q \longrightarrow \mathfrak{f}$ (whose smoothness we are proving) extending σ_o and such that $\sigma' | \mathbb{R}^p{\times}\{y^o\}$ is, for each $y^o \in \mathbb{R}^q$, of the class C^∞ and invariant with respect to the representation $T_{|\mathbb{R}^p\times\{y^o\}}$ of the transitive Lie algebroid $B'_{|\mathbb{R}^p\times\{y^o\}}$ on $\mathfrak{f}_{|\mathbb{R}^p\times\{y^o\}}$ (the cutting $\dots|\mathbb{R}^p{\times}\{y^o\}$ is understood as the inverse-image by the suitable inclusion).

Let $T\circ Y_i = (\frac{\partial}{\partial y^i}, c^i)$ for some $c^i : \mathbb{R}^p{\times}\mathbb{R}^q \longrightarrow End(V)$, $i \leqslant p$. The fact that T is a representation means, particularly, that

$$0 = T[\![Y_i, Y_j]\!] = [\![TY_i, TY_j]\!] = [\![(\frac{\partial}{\partial y^i}, c^i), (\frac{\partial}{\partial y^j}, c^j)]\!]$$
$$= (0, \frac{\partial c_j}{\partial y^i} - \frac{\partial c_i}{\partial y^j} + [c_i, c_j]),$$

i.e. that

$$[c_i, c_j] = \frac{\partial c_i}{\partial y^j} - \frac{\partial c_j}{\partial y^i},\quad i, j \leqslant p. \tag{11}$$

Let $w_1,\dots,w_n$ be a basis of V; write $c_i(x)(w_s) = \sum_{k=1}^{n} c^k_{si}(x)\cdot w_k$, $x\in\mathbb{R}^p{\times}\mathbb{R}^q$. It follows immediately that (11) is equivalent to the following conditions:

$$\frac{\partial c^{k}_{ri}}{\partial y^{s}} - \frac{\partial c^{k}_{rs}}{\partial y^{i}} = \sum_{u=1}^{n} c^{u}_{ri} \cdot c^{k}_{us} - \sum_{u=1}^{n} c^{u}_{rs} \cdot c^{k}_{ui}, \qquad i, s \leqslant p, \quad k, r \leqslant n. \tag{12}$$

The invariance of a cross-section $\tau \in Sec(\mathfrak{f})$ $\left(= \Omega^{0}(\mathbb{R}^{p}\times\mathbb{R}^{q}, V)\right)$ with respect to the representation $T: B' \longrightarrow T(\mathbb{R}^{p}\times\mathbb{R}^{q}) \times End(V)$ means that $\mathcal{L}_{T\circ X}(\tau) = 0$ for all $X \in Sec(B')$, in particular, that $\mathcal{L}_{T\circ Y_{i}}(\tau) = 0$, $i \leqslant p$. According to (6) above, the last condition says that

$$\frac{\partial \tau}{\partial y^{i}} + c_{i}(\tau) = 0, \qquad i \leqslant p,$$

or, equivalently,

$$\frac{\partial \tau^{k}}{\partial y^{i}} = - \sum_{r=1}^{n} c^{k}_{ri} \cdot \tau^{r}, \qquad i \leqslant p. \tag{13}$$

System (13) of differential equations is of the first order with the parameters $(y^{p+1}, \ldots, y^{p+q})$. It is easy to notice that (12) forms conditions of the local integrability of (13). From Th.2.1 it follows that there exists exactly one (globally defined) C^{∞} cross-section $\tilde{\sigma}: \mathbb{R}^{p}\times\mathbb{R}^{q} \longrightarrow \mathfrak{f}$ being a solution of (13) and satisfying the given initial condition $\tilde{\sigma}(0,y) = \sigma_{o}(0,y)$, $y \in \mathbb{R}^{q}$. Of course, $\tilde{\sigma} = \sigma'$, which confirms the smoothness of σ'.

(*b*) follows now trivially. ∎

PART II

THE CHARACTERISTIC CLASSES OF FLAT REGULAR LIE ALGEBROIDS

1. COHOMOLOGY WITH COEFFICIENTS

Let A and $\mathfrak{f}$ be a Lie algebroid and a vector bundle, both over the same manifold, say M. Each element of

$$\Omega_A(M;\mathfrak{f}) = \overset{q\geq 0}{\oplus}\Omega_A^q(M;\mathfrak{f}),$$

where $\Omega_A^q(M;\mathfrak{f}) = Sec(\Lambda^q A^* \otimes \mathfrak{f})$, will be called a ($C^\infty$) *form on the Lie algebroid A, with values in* $\mathfrak{f}$; while, for the trivial vector bundle $\mathfrak{f} = M\times\mathbb{R}$, briefly: a ($C^\infty$) *form on the Lie algebroid A.* A 0-form on A is simply a cross-section of $\mathfrak{f}$. In the case $A=TM$, the space of forms with values in $\mathfrak{f}$ (analogously, of the space of real forms) will be denoted traditionally by $\Omega(M;\mathfrak{f})$ ($\Omega(M)$, respectively). For an involutive C^∞ constant dimensional distribution E on M, $\Omega_E(M;\mathfrak{f})$ consists of the so-called *tangential differential forms on* (M,E) [17], [25].

$\Omega_A(M;\mathfrak{f})$ is a graded module over $\Omega^0(M)$ and a module over the algebra $\Omega_A(M)$ $\big(:=\Omega_A(M;M\times\mathbb{R})\big)$ of forms on A. The structure of the $\Omega_A(M)$-module in $\Omega_A(M;\mathfrak{f})$ is conventionally given under the skew-product $\psi\wedge\Psi$ of forms, $\psi\in\Omega_A(M)$, $\Psi\in\Omega_A(M;\mathfrak{f})$, defined (for the degrees p and q, respectively) by

$$\psi\wedge\Psi(\xi_1,\dots,\xi_{p+q})=\sum_{\substack{\sigma(1)<\dots<\sigma(p)\\ \sigma(p+1)<\dots<\sigma(p+q)}} sgn\,\sigma\cdot\psi(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)})\Psi(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)}), \tag{1}$$

$\xi_i\in Sec\,A$.

Let $f^1,\dots,f^k,f$ be vector bundles over M. An arbitrary k-linear homomorphism of vector bundles $\varphi: f^1\times\dots\times f^k\longrightarrow f$ determines the mapping

$$\varphi_*:\Omega_A(M;f^1)\times\dots\times\Omega_A(M;f^k)\longrightarrow\Omega_A(M;f)$$

defined by the standard formula

$$\varphi_*(\Psi_1,\dots,\Psi_k)(\xi_1,\dots,\xi_m)=\frac{1}{q_1!\cdot\dots\cdot q_k!}\cdot\sum_\sigma sgn\,\sigma\cdot\varphi(\Psi_1(\xi_{\sigma(1)},\dots),\dots,\Psi_k(\dots,\xi_{\sigma(m)})) \tag{2}$$

in which $m=\sum q_i$, $q_i=$ the degree of $\Psi_i\in\Omega_A(M;f^i)$.

Definition 1.1. For a given representation $T:A\longrightarrow A(f)$ of a Lie algebroid A on a vector bundle f, we define three operators

$$\iota_\xi,\ \theta^T_\xi,\ d^T:\Omega_A(M;f)\longrightarrow\Omega_A(M;f),\quad \xi\in Sec\,A,$$

called the *substitution operator*, the *Lie derivative (with respect to ξ)*, and the *exterior derivative*, respectively, by the formulae

(1°) $(\iota_\xi\Psi)(\xi_1,\dots,\xi_{q-1})=\Psi(\xi,\xi_1,\dots,\xi_{q-1})$,

(2°) $(\theta^T_\xi\Psi)(\xi_1,\dots,\xi_q)=\mathcal{L}_{T\circ\xi}(\Psi(\xi_1,\dots,\xi_q))-\sum_{j=1}^{q}\Psi(\xi_1,\dots,[\![\xi,\xi_j]\!],\dots,\xi_q)$,

(3°) $(d^T\Psi)(\xi_0,\dots,\xi_q)=\sum_{j=0}^{q}(-1)^j\mathcal{L}_{T\circ\xi_j}(\Psi(\xi_0,\dots,\hat{j},\dots,\xi_q))+$

$$+\sum_{i<j}(-1)^{i+j}\Psi([\![\xi_i,\xi_j]\!],\xi_0,\dots,\hat{i},\dots,\hat{j},\dots,\xi_q)$$

where $\Psi\in\Omega^q_A(M;f)$ and $\xi_i\in Sec\,A$.

If $T(:=\tilde{\gamma}):A\longrightarrow A(M\times\mathbb{R})$ is the trivial representation, i.e. the one for which $\mathcal{L}_{T\circ\xi}(f)=(\gamma\circ\xi)(f)$ for $f\in\Omega^0(M)$ and $\xi\in Sec\,A$ (under the canonical identification $A(M\times\mathbb{R})\cong TM\times End(\mathbb{R})$ this means that $T(v)=(\gamma(v),0)$), then the operators of the Lie derivative and the exterior derivative, denoted by θ^A_ξ and d^A, are given by

(4°) $(\theta^A_\xi\psi)(\xi_1,\dots,\xi_q)=(\gamma\circ\xi)(\psi(\xi_1,\dots,\xi_q))-\sum_{j=1}^{q}\psi((\xi_1,\dots,[\![\xi,\xi_j]\!],\dots,\xi_q)$,

$$(5^\circ)\quad (d^A\psi)(\xi_o,\dots,\xi_q)=\sum_{j=0}^{q}(-1)^j(\gamma\circ\xi_j)(\psi(\xi_o,\dots,\hat{j},\dots,\xi_q))+$$

$$+\sum_{i<j}(-1)^{i+j}\psi([\![\xi_i,\xi_j]\!],\xi_o,\dots,\hat{i},\dots,\hat{j},\dots,\xi_q).$$

Remark 1.2. Definitions (1°), (4°) and (5°) were first given by L.Maxim-Raileanu in 1976 [22]. Formulae $(1^\circ)\div(3^\circ)$ were obtained by the author [13] in some natural manner for Lie algebroids of Pradines-type groupoids. Independently, they were given (as axioms) by K.Mackenzie [20].

The fundamental properties of the operators ι_ξ, θ^T_ξ, and d^T are given underneath. We first recall that a single representation T determines a number of new ones, for example, $Hom^k(T)$ of A on the space of k-linear homomorphisms $Hom^k(\mathfrak{f};\mathbb{R})$ [17; 2.2.2]. This representation can be generalized as follows:

Let $T^1,\dots,T^k,T$ denote fixed representations of A on vector bundles $\mathfrak{f}^1,\dots,\mathfrak{f}^k,\mathfrak{f}$, respectively. They define a representation $Hom^k(T^1,\dots,T^k;T)$ (briefly, Hom) of A on $Hom^k(\mathfrak{f}^1\times\dots\times\mathfrak{f}^k;\mathfrak{f})$ as the one for which

$$\mathcal{L}_{Hom\circ\xi}(\varphi)(\nu^1,\dots,\nu^k)=\mathcal{L}_{T\circ\xi}(\varphi(\nu^1,\dots,\nu^k))-\sum_i\varphi(\nu^1,\dots,\mathcal{L}_{T^i\circ\xi}\nu^i,\dots,\nu^k),$$

for any k-linear homomorphism $\varphi:\mathfrak{f}^1\times\dots\times\mathfrak{f}^k\longrightarrow\mathfrak{f}$ and for $\nu^i\in Sec(\mathfrak{f}^i)$, $\xi\in Sec\,A$.

Theorem 1.3. (*cf.* [20]) (i) $\iota_{[\![\xi,\eta]\!]}=\theta^T_\xi\circ\iota_\eta-\iota_\eta\circ\theta^T_\xi$,

(ii) $\theta^T_{[\![\xi,\eta]\!]}=\theta^T_\xi\circ\theta^T_\eta-\theta^T_\eta\circ\theta^T_\xi$,

(iii) $\theta^T_\xi=\iota_\xi\circ d^T+d^T\circ\iota_\xi$,

(iv) $d^T\circ d^T=0$,

(v) $d^T\circ\theta^T_\xi=\theta^T_\xi\circ d^T$.

For arbitrarily taken vector bundles $\mathfrak{f}^1,\dots,\mathfrak{f}^k,\mathfrak{f}$ over M and a k-linear homomorphism $\varphi:\mathfrak{f}^1\times\dots\times\mathfrak{f}^k\longrightarrow\mathfrak{f}$ and forms $\Psi_j\in\Omega^{q_j}_A(M;\mathfrak{f}^j)$, we have

$$\text{(vi)}\qquad \iota_\xi(\varphi_*(\Psi_1,\dots,\Psi_k))=\sum_{j=1}^{k}(-1)^{q_1+\dots q_{j-1}}\varphi_*(\Psi_1,\dots,\iota_\xi\Psi_j,\dots,\Psi_k).$$

Let now $T^1,\dots,T^k,T$ denote fixed representations of A on $\mathfrak{f}^1,\dots,\mathfrak{f}^k,\mathfrak{f}$, respectively, and assume that φ is Hom-invariant. Then

$$\text{(vii)}\quad \theta^T_\xi(\varphi_*(\Psi_1,\dots,\Psi_k))=\sum_{j=1}^{k}\varphi_*(\Psi_1,\dots,\theta^{T^j}_\xi\Psi_j,\dots,\Psi_k),$$

(viii) $d^T(\varphi_*(\Psi_1,\ldots,\Psi_k)) = \sum_{j=1}^{k} (-1)^{q_1+\ldots+q_{j-1}}\varphi_*(\Psi_1,\ldots,d^{T^j},\ldots,\Psi_k)$.

In particular, (*taking* $\varphi = \cdot : (M\times\mathbb{R})\times \mathfrak{f} \longrightarrow \mathfrak{f}$, *the multiplication of vectors by reals, being* $Hom(\tilde{\gamma};T)$*-invariant*), *we have, for* $\psi\in\Omega^q_A(M)$ *and* $\Psi\in\Omega_A(M;\mathfrak{f})$, *the following formulae*:

(vi′) $\iota_\xi(\psi\wedge\Psi) = \iota_\xi\psi\wedge\Psi + (-1)^q\psi\wedge\iota_\xi\Psi$,

(vii′) $\theta^T_\xi(\psi\wedge\Psi) = \theta^A_\xi\psi\wedge\Psi + \psi\wedge\theta^T_\xi\Psi$,

(viii′) $d^T(\psi\wedge\Psi) = d^A\psi\wedge\Psi + (-1)^q\psi\wedge d^T\Psi$.

Remark 1.4. These properties were proved by the author [13] for Lie algebroids of Pradines-type groupoids (not all Lie algebroids being taken into account, of course). In all generality, properties (ii)÷(iv) can be found in K.Mackenzie [20, p.200] with the proof "Standard". Now, we give a full proof of this important theorem.

1.5. Proof of Theorem 1.3. For arbitrary $\mathbb{R}$-vector spaces $\mathfrak{A}$ and $\mathfrak{F}$, by $\Omega^q(\mathfrak{A};\mathfrak{F})$ we denote here the $\mathbb{R}$-vector space of q-linear (over $\mathbb{R}$) skew-symmetric mappings $\underbrace{\mathfrak{A}\times\ldots\times\mathfrak{A}}_{q\ \text{times}} \longrightarrow \mathfrak{F}$.

Take a sequence $\mathfrak{A}, \mathfrak{F}_1,\ldots,\mathfrak{F}_k, \mathfrak{F}$ of vector spaces and a k-linear mapping $\cdot : \mathfrak{F}_1\times\ldots\times\mathfrak{F}_k \longrightarrow \mathfrak{F}$. By the formula analogous to (2), we define the skew-symmetric product $\Psi_1\wedge\ldots\wedge\Psi_k\in\Omega(\mathfrak{A};\mathfrak{F})$ of mappings $\Psi_i\in\Omega^{q_i}(\mathfrak{A};\mathfrak{F}_i)$

$$\Psi_1\wedge\ldots\wedge\Psi_k(\xi_1,\ldots,\xi_m) = \frac{1}{q_1!\cdot\ldots\cdot q_k!}\sum_\sigma sgn\,\sigma\cdot\Psi_1(\xi_{\sigma(1)},\ldots)\cdot\ldots\cdot\Psi_k(\ldots,\xi_{\sigma(m)})).$$

Let $\mathfrak{S}:\underbrace{\mathfrak{A}\times\ldots\times\mathfrak{A}}_{m\ \text{times}} \longrightarrow \mathfrak{A}$ be a fixed m-linear mapping, $m\geqslant 0$. For an arbitrary vector space $\mathfrak{F}$ and $\Psi\in\Omega^q(\mathfrak{A};\mathfrak{F})$, $q\geqslant 1$, we define the $q+m-1$-linear mapping $\mathfrak{S}\cdot\Psi:\mathfrak{A}\times\ldots\times\mathfrak{A}\longrightarrow\mathfrak{F}$ by the formula

$$\mathfrak{S}\cdot\Psi(\xi_1,\ldots,\xi_{q+m-1}) = \Psi(\mathfrak{S}(\xi_1,\ldots,\xi_m),\xi_{m+1},\ldots,\xi_{q+m-1}),$$

and next, $\mathfrak{S}*\Psi\in\Omega^{q+m-1}(\mathfrak{A};\mathfrak{F})$ as its "skewing"

$$\mathfrak{S}*\Psi(\xi_1,\ldots,\xi_{q+m-1}) = \sum_{\substack{\sigma(1)<\ldots<\sigma(m)\\ \sigma(m+1)<\ldots<\sigma(q+m-1)}} sgn\,\sigma\cdot(\mathfrak{S}\cdot\Psi)(\xi_{\sigma(1)},\ldots,\xi_{\sigma(q+m-1)}).$$

Of course, for $m=0$ and $\mathfrak{S}=\xi\in\mathfrak{A}$, $\xi*(\cdot)$ is the substitution operator ι_ξ. By arduous, but classical, combinatorial calculations we prove the following lemma (cf. R.Sikorski [31]).

Lemma 1.5.1. (1) *If* $m > 1$, *then* $\mathfrak{S} * (\mathfrak{S} * \Psi) = (\mathfrak{S} * \mathfrak{S}) * \Psi$.

(2) *If* $m > 0$, *then, for* $\Psi_j \in \Omega^{q_j}(\mathfrak{A}; \mathfrak{F}_j)$,

$$\mathfrak{S} * (\Psi_1 \wedge \ldots \wedge \Psi_k) = \sum_{j=1}^{k} (-1)^{q_1 + \cdots q_{j-1}} \Psi_1 \wedge \ldots \wedge (\mathfrak{S} * \Psi_j) \wedge \ldots \wedge \Psi_k.$$

(3) *If* $m = 2$, *then, for* $\xi \in \mathfrak{A}$,

$$(\mathfrak{S} * (\xi * \Psi) + \xi * (\mathfrak{S} * \Psi))(\xi_1, \ldots, \xi_q) = \sum_{j=1}^{q} \Psi(\xi_1, \ldots, \mathfrak{S}(\xi, \xi_j), \ldots, \xi_q). \quad \square$$

Fix now $m=2$ and assume that

(A1) $\mathfrak{S}: \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}$ is skew-symmetric.

Take two vector spaces $\mathfrak{F}$, $\mathfrak{F}'$, a 2-linear mapping $\cdot : \mathfrak{F}' \times \mathfrak{F} \longrightarrow \mathfrak{F}$ and $\mathfrak{J} \in \Omega^1(\mathfrak{A}; \mathfrak{F}')$. For $\xi \in \mathfrak{A}$, we define three operators

$$\iota_\xi,\ \theta_\xi,\ d: \Omega(\mathfrak{A}; \mathfrak{F}) \longrightarrow \Omega(\mathfrak{A}; \mathfrak{F})$$

by the formulae

$(\cdot)$ $\iota_\xi \Psi = \xi * \Psi$,

$(\cdot\cdot)$ $\theta_\xi \Psi = \mathfrak{J}(\xi) \wedge \Psi - (\mathfrak{S} * (\xi * \Psi) + \xi * (\mathfrak{S} * \Psi))$,

$(\cdot\cdot\cdot)$ $d\Psi = \mathfrak{J} \wedge \Psi - \mathfrak{S} * \Psi$.

Lemma 1.5.2. (1) $\theta_\xi \circ \iota_\eta - \iota_\eta \circ \theta_\xi = \iota_{\mathfrak{S}(\xi, \eta)}$,

(2) $\theta_\xi = d \circ \iota_\xi + \iota_\xi \circ d$.

Proof. (1) follows immediately from the definitions.

(2): By 1.5.1(2) [for $k=2$, $\mathfrak{S} = \xi$, $\Psi_1 = \mathfrak{J}$ and $\Psi_2 = \Psi$], we have

$$\begin{aligned}(d \circ \iota_\xi + \iota_\xi \circ d)(\Psi) &= d(\iota_\xi \Psi) + \iota_\xi (d\Psi) \\ &= \mathfrak{J} \wedge (\xi * \Psi) - \mathfrak{S} * (\xi * \Psi) + \xi * (\mathfrak{J} \wedge \Psi) - \xi * (\mathfrak{S} * \Psi) \\ &= - \mathfrak{S} * (\xi * \Psi) + (\xi * \mathfrak{J}) \wedge \Psi) - \xi * (\mathfrak{S} * \Psi) = \theta_\xi \Psi. \quad \square\end{aligned}$$

Assume additionally that a 2-linear mapping $\cdot : \mathfrak{F}' \times \mathfrak{F}' \longrightarrow \mathfrak{F}'$ such that

(A2) $(v' \cdot w') \cdot u = v' \cdot (w' \cdot u)$, $\quad v', w' \in \mathfrak{F}'$, $u \in \mathfrak{F}$,

is given. Then it is easy to see that

$$\mathfrak{J} \wedge (\mathfrak{J} \wedge \Psi) = (\mathfrak{J} \wedge \mathfrak{J}) \wedge \Psi. \tag{3}$$

Lemma 1.5.3. *If*

(A3) $\mathfrak{G} * \mathfrak{G} = 0$,

(A4) $\mathfrak{J} \wedge \mathfrak{J} = \mathfrak{G} * \mathfrak{J}$,

then

(1) $d^2 = 0$, (2) $\theta_\xi \circ d = d \circ \theta_\xi$, (3) $\theta_\xi \circ \theta_\eta - \theta_\eta \circ \theta_\xi = \theta_{\mathfrak{G}(\xi,\eta)}$.

Proof. (1): By 1.5.1(2), (2) and 1.5.1(1), we have
$d^2\Psi = d(\mathfrak{J} \wedge \Psi - \mathfrak{G} * \Psi) = \mathfrak{J} \wedge (\mathfrak{J} \wedge \Psi - \mathfrak{G} * \Psi) - \mathfrak{G} * (\mathfrak{J} \wedge \Psi - \mathfrak{G} * \Psi)$

$= (\mathfrak{J} \wedge \mathfrak{J}) \wedge \Psi - (\mathfrak{G} * \mathfrak{J}) \wedge \Psi + (\mathfrak{G} * \mathfrak{G}) * \Psi = 0.$

(2): Simple by (1) above and 1.5.2(2).

(3): Trivial calculations after using 1.5.1(3), (3) and the fact that the condition $\mathfrak{G} * \mathfrak{G} = 0$ is equivalent to the fulfilling of the Jacobi identity for $\mathfrak{G}$. □

In the end, we assume additionally that k systems of vector spaces and mappings

$$\mathfrak{F}_i, \ \mathfrak{F}'_i, \ \mathfrak{J}_i \in \Omega^1(\mathfrak{A};\mathfrak{F}'_i), \quad \cdot : \mathfrak{F}'_i \times \mathfrak{F}_i \longrightarrow \mathfrak{F}_i, \quad \cdot : \mathfrak{F}'_i \times \mathfrak{F}'_i \longrightarrow \mathfrak{F}'_i, \quad i \leq k,$$

are given, and that, for each $i \leq k$, the mapping $\mathfrak{F}'_i \times \mathfrak{F}_i \longrightarrow \mathfrak{F}_i$ fulfils (A2), and, for a given mapping $\cdot : \mathfrak{F}_1 \times \ldots \times \mathfrak{F}_k \longrightarrow \mathfrak{F}$, $(v_1,\ldots,v_k) \longmapsto v_1 \cdot \ldots \cdot v_k$, the following axiom is satisfied:

(A5) $\mathfrak{J}(\xi) \cdot (v_1 \cdot \ldots \cdot v_k) = \sum_{j=1}^{k} v_1 \cdot \ldots \cdot (\mathfrak{J}_j(\xi) \cdot v_j) \cdot \ldots \cdot v_k$, $\quad \xi \in \mathfrak{A}$, $v_j \in \mathfrak{F}_j$.

Then we have

$$\left.\begin{aligned} \mathfrak{J}(\xi) \wedge (\Psi_1 \wedge \ldots \wedge \Psi_k) &= \sum_{j=1}^{k} \Psi_1 \wedge \ldots \wedge (\mathfrak{J}_j(\xi) \wedge \Psi_j) \wedge \ldots \wedge \Psi_k, \\ \mathfrak{J} \wedge (\Psi_1 \wedge \ldots \wedge \Psi_k) &= \sum_{j=1}^{k} (-1)^{q_1+\cdots+q_{j-1}} \Psi_1 \wedge \ldots \wedge (\mathfrak{J}_j \wedge \Psi_j) \wedge \ldots \wedge \Psi_k. \end{aligned}\right\} \tag{4}$$

Denote by θ^j_ξ, d^j the operators in $\Omega(\mathfrak{A};\mathfrak{F}_j)$ built via $\mathfrak{J}_j$. Then, thanks to **(4)** and 1.5.1(2), we notice the following

Lemma 1.5.4. (1) $\theta_\xi(\Psi_1 \wedge \ldots \wedge \Psi_k) = \sum_{j=1}^{k} \Psi_1 \wedge \ldots \wedge \theta^j_\xi \Psi_j \wedge \ldots \wedge \Psi_k$,

(2) $d(\Psi_1 \wedge \ldots \wedge \Psi_k) = \sum_{j=1}^{k} (-1)^{q_1+\cdots+q_{j-1}} \Psi_1 \wedge \ldots \wedge d^j \Psi_j \wedge \ldots \wedge \Psi_k$. □

To prove Theorem 1.3, we must put $\mathfrak{A}=Sec\,A$, $\mathfrak{S}(\xi,\eta)=[\![\xi,\eta]\!]$ for $\xi,\eta\in Sec\,A$, $\mathfrak{F}=Sec(\mathsf{f})$, $\mathfrak{F}'=Hom_{\mathbb{R}}(\mathfrak{F};\mathfrak{F})$, $\mathfrak{F}'\times\mathfrak{F}\longrightarrow\mathfrak{F}$, $(u,\sigma)\longmapsto u(\sigma)$ - the natural substitute operator, $\mathfrak{J}(\xi)(\sigma)=\mathcal{L}_{T\circ\xi}(\sigma)$ for $\sigma\in\mathfrak{F}$. Then definitions $(\cdot)\div(\cdots)$ agree with $(1^{\circ})\div(3^{\circ})$ from 1.1. Next, put $\mathfrak{F}_j=Sec(\mathsf{f}_j)$, $j\leqslant k$, $\mathfrak{F}'_j=Hom_{\mathbb{R}}(\mathfrak{F}_j;\mathfrak{F}_j)$, $\mathfrak{F}'_j\times\mathfrak{F}_j\longrightarrow\mathfrak{F}_j$ as above, and take the mappings $\cdot:\mathfrak{F}_1\times\ldots\times\mathfrak{F}_k\longrightarrow\mathfrak{F}$ and $\mathfrak{J}_j$ defined by $\sigma_1\cdot\ldots\cdot\sigma_k:=\varphi\circ(\sigma_1\times\ldots\times\sigma_k)$ and $\mathfrak{J}_j(\xi)(\sigma)=\mathcal{L}_{T^j\circ\xi}(\sigma)$, $\sigma\in\mathfrak{F}_j$. Then all assumptions A1÷A5 are satisfied. Assertions (i)÷(viii) of our theorem follow successively from 1.5.2(1), 1.5.3(3), 1.5.2(2), 1.5.3(1), 1.5.3(2), 1.5.1(2) for $m=0$, 1.5.4(1) and 1.5.4(2). ∎

1.6. According to 1.3(iv), $(\Omega_A(M;\mathsf{f}),d^T)$ is a complex; its cohomology spaces will be denoted by $H^n_A(M,T,\mathsf{f})$, $n\geqslant 0$. They generalize the Chevalley-Eilenberg cohomology spaces of a finite-dimensional real Lie algebra $\mathfrak{g}$ (for $A=\mathfrak{g}$) [also those with coefficients, see, for example, [9]] and the de Rham cohomology spaces of a manifold (for $A=TM$).

1.7. $H^0_A(M,T,\mathsf{f})=Ker\,d^{T,0}=\{\sigma\in Sec(\mathsf{f});\ d^T\sigma=0\}$

$$=\left\{\sigma\in Sec(\mathsf{f});\ \underset{\xi\in Sec A}{\forall}\left(\mathcal{L}_{T\circ\xi}(\sigma)=0\right)\right\}=(Sec\,\mathsf{f})_{I^{\circ}}$$

and, by I.5.2, this space, in the case of a transitive Lie algebroid, is finite-dimensional [see also Mackenzie [20, pp.195 and 210]].

For the trivial vector bundle $\mathsf{f}=M\times\mathbb{R}$, the cohomology spaces of the complex $(\Omega_A(M),d^A)$ will be briefly denoted by $H^n_A(M)$, $n=0,1,\ldots$ It is a standard calculation to obtain that $H_A(M,T,\mathsf{f})$ is a module over the algebra $H_A(M)$ under the multiplication $[\psi]\wedge[\Psi]=[\psi\wedge\Psi]$.

Definition 1.8. A form $\Psi\in\Omega_A(M;\mathsf{f})$ will be called a *horizontal form* if $\iota_\nu\Psi=0$ for all $\nu\in Sec\,\mathbf{g}$. The space of horizontal forms will be denoted by $\Omega_{A,i}(M;\mathsf{f})$. Each C^∞ cross-section $\sigma\in Sec(\mathsf{f})$ is a horizontal 0-form on A. By $\Omega_{A,i}(M)$ we denote the space of horizontal forms on the Lie algebroid A, with real values. According to Th.1.3(vi'), $\Omega_{A,i}(M;\mathsf{f})$ is a module over the algebra $\Omega_{A,i}(M)$.

Lemma 1.9. (1) *$\theta^A_\nu\Psi=0$ for $\Psi\in\Omega_{A,i}(M)$, $\nu\in Sec\,\mathbf{g}$.*

(2) *$\Omega_{A,i}(M)$ is stable under d^A.* ∎

1.10. For an arbitrary vector bundle $\mathfrak{f}$, we set

$$\Omega_{A,i,\theta}(M;\mathfrak{f})=\{\Psi\in\Omega_{A,i}(M;\mathfrak{f});\ \theta_\nu^T\Psi=0 \text{ for } \nu\in Sec\,\mathbf{g}\}.$$

By 1.7, we see that $\Omega_{A,i,\theta}(M)=\Omega_{A,i}(M)$.

2. HOMOMORPHISMS $\omega^\wedge$, $\Omega^\vee$, AND $(d\omega)^\vee$.

Let $A=(A,[\![\cdot,\cdot]\!],\gamma)$ be an arbitrary regular Lie algebroid over a foliated manifold (M,E), and $\lambda:E\longrightarrow A$ any connection in A, i.e. any splitting of its Atiyah sequence [17; 3.1.1]:

$$0\longrightarrow\mathbf{g}\hookrightarrow A\underset{\overset{\longleftarrow}{\lambda}}{\xrightarrow{\ \gamma\ }}E\longrightarrow 0.$$

Since $\gamma|\mathbf{g}=0$, the linear homomorphism of graded vector spaces

$$\gamma_*:\Omega_E(M;\mathfrak{f})\longrightarrow\Omega_A(M;\mathfrak{f})$$

defined by the formula $\gamma_*(\theta)(x;\ldots v_i\ldots)=\theta(x;\ldots\gamma v_i\ldots)$, $v_i\in A_{|x}$, maps isomorphically $\Omega_E(M;\mathfrak{f})$ onto the space of horizontal forms $\Omega_{A,i}(M;\mathfrak{f})$. The inverse mapping is

$$\lambda_*:\Omega_{A,i}(M;\mathfrak{f})\longrightarrow\Omega_E(M;\mathfrak{f})$$

defined by $\lambda_*(\Psi)(x;\ldots,w_i,\ldots)=\Psi(x;\ldots,\lambda w_i,\ldots)$, $w_i\in E_{|x}$.

For the trivial vector bundle $\mathfrak{f}=M\times\mathbb{R}$, one can easily obtain the equality $d^E=\lambda_*\circ d^A\circ\gamma_*$ which is equivalent to the commutativity of the diagram:

$$\begin{tikzcd}
\Omega_E(M) \arrow[r, "d^E"] \arrow[d, "\cong"', "\gamma^*"] & \Omega_E(M) \arrow[d, "\cong"', "\gamma^*"] \\
\Omega_{A,i}(M) \arrow[r, "d^A"] & \Omega_{A,i}(M)
\end{tikzcd} \qquad (5)$$

Let $\omega:A\longrightarrow\mathbf{g}$ be the connection form of λ (i.e. $\omega|\mathbf{g}=id$ and $\omega|Im\lambda=0$). ω is also treated as a 1-form on the Lie algebroid A, with values at $\mathbf{g}$, $\omega\in\Omega^1_A(M;\mathbf{g})$. The mapping $H=id_A-\omega:A\longrightarrow A$ is to be the *horizontal projection* of vectors from A. It determines the *horizontal projection* of forms

$$H_*:\Omega_A(M;\mathfrak{f})\longrightarrow\Omega_A(M;\mathfrak{f})$$

by $H_*(\Psi)(x;\dots,v_i,\dots)=\Psi(x;\dots,Hv_i,\dots)$, $v_i\in A_{|x}$.

In [17; 3.1.1] there is defined the curvature tensor $\Omega_b\in\Omega^2_E(M;\mathbf{g})$ of λ by $\Omega_b(X_1,X_2)=-\omega([\![\lambda\circ X_1,\lambda\circ X_2]\!])$, $X_i\in Sec\,E$. Now, we define - needed in the sequel - the so-called *curvature form* of λ as a horizontal 2-form on the Lie algebroid A, with values in $\mathbf{g}$, $\Omega\in\Omega^2_{A,i}(M;\mathbf{g})$, by the formula

$$\Omega(\xi_1,\xi_2)=-\omega([\![H\circ\xi_1,H\circ\xi_2]\!]),\quad \xi_i\in Sec\,A.$$

Below, the exterior derivative of forms on the Lie algebroid A, with values in $\mathbf{g}$, [also in the associated vector bundles $\Lambda^k\mathbf{g}^*$, . . .] with respect to the adjoint representation $ad_A:A\longrightarrow A(\mathbf{g})$, $\left(\mathcal{L}_{ad_A\circ\xi}(\nu)=[\![\xi,\nu]\!]\right)$ [17; 2.1.2] [or induced ones] will be shortly denoted by $d^{\mathbf{g}}$.

Proposition 2.1 *(The Maurer-Cartan equation).*

$$\Omega=d^{\mathbf{g}}\omega-\tfrac{1}{2}[\omega,\omega].$$

(The form $[\omega,\omega]$ *is defined via formula (2) for the 2-linear homomorphism* $[\cdot,\cdot]:\mathbf{g}\times\mathbf{g}\longrightarrow\mathbf{g}$).

(**Remark:** The difference here, in comparison with the classical formula for principal bundles- see, for example, [8] - [the sign " - " before the second component], has its roots in the fact that the Lie algebra of the structure Lie group in the principal bundle considered there is taken left, not right).

Proof. Without difficulties we can easily prove (in analogy to [8]) that two forms $\Psi_i\in\Omega^2_A(M;\mathfrak{f})$, $i=1,2$, are equal to each other if and only if (a) $i_\nu\Psi_1=i_\nu\Psi_2$ $\nu\in Sec\,\mathbf{g}$, (b) $H_*(\Psi_1)=H_*(\Psi_2)$.

(a) $i_\nu\Omega=0$ for $\nu\in Sec\,\mathbf{g}$ by the horizontality of Ω; on the other hand, for $\nu\in Sec\,\mathbf{g}$ and $\eta\in Sec\,A$,

$$i_\nu(d^{\mathbf{g}}\omega-\tfrac{1}{2}[\omega,\omega])(\eta)=d^{\mathbf{g}}\omega(\nu,\eta)-\tfrac{1}{2}[\omega,\omega](\nu,\eta)$$
$$=[\![\nu,\omega(\eta)]\!]-[\![\eta,\omega(\nu)]\!]-\omega([\![\nu,\eta]\!])-[\![\omega(\nu),\omega(\eta)]\!]=0.$$

(b) For $\xi_i\in Sec\,A$,

$$H_*(d^{\mathbf{g}}\omega - \tfrac{1}{2}[\omega,\omega])(\xi_1,\xi_2) = (d^{\mathbf{g}}\omega)(H\circ\xi_1, H\circ\xi_2) - \tfrac{1}{2}[\omega,\omega](H\circ\xi_1, H\circ\xi_2)$$

$$= [\![H\circ\xi_1, \omega(H\circ\xi_2)]\!] - [\![H\circ\xi_2, \omega(H\circ\xi_1)]\!] - \omega([\![H\circ\xi_1, H\circ\xi_2]\!]) - [\![\omega(H\circ\xi_1), \omega(H\circ\xi_2)]\!]$$

$$= -\omega([\![H\circ\xi_1, H\circ\xi_2]\!]) = \Omega(H\circ\xi_1, H\circ\xi_2) = (H_*\Omega)(\xi_1,\xi_2). \quad \blacksquare$$

A) Homomorphism $\omega^\wedge$

For each point $x\in M$, the mapping

$$\rho : \mathbf{g}^*_{|x} \longrightarrow A^*_{|x} = \wedge^1 A^*_{|x} \subset \wedge A^*_{|x}$$
$$w^* \longmapsto w^* \circ \omega_{|x}$$

is linear and keeps the property

$$\rho(w^*) \wedge \rho(w^*) = 0 \quad \text{for } w^* \in \mathbf{g}^*_{|x}.$$

$\wedge A^*_{|x}$ is an associative algebra with unit element, therefore, by the universal property of the exterior algebra $\wedge A^*_{|x}$, see [7; p.103], we obtain the existence and the uniqueness of a homomorphism of algebras of degree 0

$$\omega^\wedge_x : \wedge A^*_{|x} \longrightarrow \wedge A^*_{|x}$$

extending ρ and such that $\omega^\wedge_x(1)=1$. Using the canonical duality between the exterior algebra over a vector space and over its dual 1[7; p.104] we have that

$$\langle \omega^\wedge_x(\psi), w_1 \wedge \ldots \wedge w_k \rangle = \langle \psi, \omega(x;w_1) \wedge \ldots \wedge \omega(x;w_k) \rangle$$

for $\psi \in \wedge^k \mathbf{g}^*_{|x}$ and $w_i \in A_{|x}$. We notice that if $\Psi \in Sec\wedge^k \mathbf{g}^*$, then

$$\omega^\wedge(\Psi) : M \longrightarrow \wedge^k A^*, \quad x \longmapsto \omega^\wedge_x(\Psi(x)),$$

is a C^∞ cross-section of $\wedge^k A^*$, i.e. $\omega^\wedge(\Psi) \in \Omega^k_A(M)$.

Of course,

$$\omega^\wedge : \overset{k\geq 0}{\oplus} (Sec\wedge^k \mathbf{g}^*) \longrightarrow \Omega_A(M), \quad \Psi \longmapsto \omega^\wedge(\Psi),$$

is a homomorphism of algebras where the space $\overset{k\geq 0}{\oplus}(Sec\Lambda^k\mathbf{g}^*)$ is equipped with the structure $(\Psi_1,\Psi_2)\longmapsto\Psi_1\wedge\Psi_2$ for which $\Psi_1\wedge\Psi_2$ is defined point by point.

Define a C^∞ 2-linear homomorphism of vector bundles

$$\langle\cdot,\cdot\rangle:\Lambda^k\mathbf{g}^*\times\Lambda^k\mathbf{g}\longrightarrow\mathbb{R}$$

(being, in fact, a duality) via the family of the canonical dualities $\langle\cdot,\cdot\rangle:\Lambda^k\mathbf{g}^*_{|x}\times\Lambda^k\mathbf{g}_{|x}\longrightarrow\mathbb{R}$.

Looking at definition (2) above and treating Ψ as a 0-form on A, with values in $\Lambda^k\mathbf{g}^*$, we can easily assert:

2.2. $\omega^\wedge(\Psi)=\frac{1}{k!}\cdot\langle\Psi,\underbrace{\omega\wedge\ldots\wedge\omega}_{k\ \text{times}}\rangle$ if $\Psi\in Sec\Lambda^k\mathbf{g}^*$ where $\omega\wedge\ldots\wedge\omega$ is defined by formula (2) for the k-linear homomorphism $\wedge:\mathbf{g}\times\ldots\times\mathbf{g}\longrightarrow\Lambda^k\mathbf{g}$, whereas $\langle\Psi,\omega\wedge\ldots\wedge\omega\rangle$ - for the duality $\langle\cdot,\cdot\rangle$. ∎

Lemma 2.3. $\iota_\nu(\omega^\wedge\Psi)=\omega^\wedge(\iota_\nu\Psi)$ *if* $\nu\in Sec\,\mathbf{g}$.

Proof. In view of the obvious equality $\iota_\nu\omega=\nu$, of Th.1.3(vi) and of 2.2 above, we have, for $\Psi\in Sec\Lambda^k\mathbf{g}^*$,

$$\iota_\nu(\omega^\wedge\Psi)=\iota_\nu\Big(\frac{1}{k!}\cdot\langle\Psi,\omega\wedge\ldots\wedge\omega\rangle\Big)=\frac{1}{k!}\cdot\langle\Psi,\iota_\nu(\omega\wedge\ldots\wedge\omega)\rangle$$

$$=\frac{1}{k!}\cdot\langle\Psi,\sum_{j=1}^{k}(-1)^{j-1}\omega\wedge\ldots\wedge\iota_\nu\omega\wedge\ldots\wedge\omega\rangle=\frac{1}{k!}\cdot\langle\Psi,k\cdot\iota_\nu\omega\wedge\omega\wedge\ldots\wedge\omega\rangle$$

$$=\frac{1}{(k-1)!}\cdot\langle\iota_\nu\Psi,\underbrace{\omega\wedge\ldots\wedge\omega}_{k-1\ \text{times}}\rangle=\omega^\wedge(\iota_\nu\Psi).\qquad\blacksquare$$

B) Homomorphism $\Omega^\vee$

Let $\Omega\in\Omega^2_A(M;\mathbf{g})$ be the curvature form of the connection λ under consideration. For each point $x\in M$, the mapping

$$\begin{aligned}\mu:\mathbf{g}^*_{|x}&\longrightarrow\Lambda^2A^*_{|x}\subset\Lambda^{ev}A^*_{|x}\\ w^*&\longmapsto w^*\circ\Omega_{|x}\end{aligned}$$

is linear and keeps the property

$$\mu(u^*)\wedge\mu(w^*)=\mu(w^*)\wedge\mu(u^*) \quad \text{for } u^*,\ w^*\in\mathbf{g}^*_{|x}.$$

$\Lambda^{ev}A^*_{|x}$ is an associative algebra with unit element, therefore, by the universal symmetric algebra property of $\vee\mathbf{g}^*_{|x}$ [7; p.192], there exists a unique homomorphism of algebras of degree 0

$$\Omega^{\vee}_x:\vee\mathbf{g}^*_{|x}\longrightarrow\Lambda^{ev}A^*_{|x}$$

extending μ and such that $\Omega^{\vee}_x(1)=1$.

Lemma 2.4. *Via the canonical dualities* [7; pp. 104, 193], *the homomorphism* $\Omega^{\vee}_x$ *is defined by the formula*

$$\langle\Omega^{\vee}_x(\Gamma),w_1\wedge\ldots\wedge w_{2k}\rangle=\frac{1}{k!}\cdot\frac{1}{2^k}\cdot\sum_\sigma sgn\sigma\cdot\langle\Gamma,\Omega(x;w_{\sigma(1)}\wedge w_{\sigma(2)})\vee\ldots\vee\Omega(x;w_{\sigma(2k-1)}\wedge w_{\sigma(2k)})\rangle$$

for $\Gamma\in\vee^k\mathbf{g}^*_{|x}$ *and* $w_i\in A_{|x}$.

Proof. In view of the linearity with respect to Γ of both sides of the above equality, it is sufficient to check it on the simple tensors $\Gamma=w^*_1\vee\ldots\vee w^*_k$, $w^*_i\in\mathbf{g}^*_{|x}$.

$$\begin{aligned}
&\langle\Omega^{\vee}_x(w^*_1\vee\ldots\vee w^*_k),w_1\wedge\ldots\wedge w_{2k}\rangle\\
&=\langle\Omega^{\vee}_x(w^*_1)\wedge\ldots\wedge\Omega^{\vee}_x(w^*_k),w_1\wedge\ldots\wedge w_{2k}\rangle\\
&=\frac{1}{2^k}\cdot\sum_\sigma sgn\,\sigma\cdot w^*_1(\Omega(x;w_{\sigma(1)}\wedge w_{\sigma(2)}))\cdot\ldots\cdot w^*_k(\Omega(x;w_{\sigma(2k-1)}\wedge w_{\sigma(2k)}))\\
&=\frac{1}{k!}\cdot\frac{1}{2^k}\cdot\sum_\sigma sgn\,\sigma\cdot\sum_\tau w^*_1(\Omega(x;w_{\sigma(2\cdot\tau(1)-1)}\wedge w_{\sigma(2\cdot\tau(1))}))\cdot\ldots\cdot w^*_k(\Omega(x;w_{\sigma(2\cdot\tau(k)-1)}\wedge w_{\sigma(2\cdot\tau(k))}))\\
&=\frac{1}{k!}\cdot\frac{1}{2^k}\cdot\sum_\sigma sgn\,\sigma\cdot perm[\langle w^*_i,\Omega(x;w_{\sigma(2j-1)}\wedge w_{\sigma(2j)})\rangle;\ i,\ j\leqslant k]\\
&=\frac{1}{k!}\cdot\frac{1}{2^k}\cdot\sum_\sigma sgn\,\sigma\cdot\langle w^*_1\vee\ldots\vee w^*_k,\Omega(x;w_{\sigma(1)}\wedge w_{\sigma(2)})\vee\ldots\vee\Omega(x;w_{\sigma(2k-1)}\wedge w_{\sigma(2k)})\rangle. \qquad\blacksquare
\end{aligned}$$

Applying the above lemma, we see that, for $\Gamma\in Sec\vee^k\mathbf{g}^*$, the cross-section

$$\Omega^{\vee}(\Gamma):M\longrightarrow\Lambda^{2k}A^*,\quad x\longmapsto\Omega^{\vee}_x(\Gamma(x)),$$

is C^∞, i.e. $\Omega^{\vee}(\Gamma)\in\Omega^{2k}_A(M)$.

The space $\overset{k\geqslant 0}{\oplus}(Sec\vee^k\mathbf{g}^*)$ forms an algebra in a standard way, and the mapping

$$\Omega^{\vee}:\overset{k\geq 0}{\oplus}(Sec\vee^{k}\mathbf{g}^{*})\longrightarrow\Omega^{ev}_{A}(M)$$
$$\Gamma\longmapsto\Omega^{\vee}(\Gamma)$$

is a homomorphism of algebras.

By standard calculations, we obtain

2.5. $\Omega^{\vee}(\Gamma)=\frac{1}{k!}\cdot\langle\Gamma,\Omega\vee\ldots\vee\Omega\rangle$ for $\Gamma\in Sec\vee^{k}\mathbf{g}^{*}$ (the forms $\Omega\vee\ldots\vee\Omega$ and $\langle\Gamma,\Omega\vee\ldots\vee\Omega\rangle$ are defined by (2) for suitable multilinear homomorphisms). ■

2.6. It is well known that, in the vector space $\Lambda\mathbf{g}^{*}_{|x}$, the classical Chevalley-Eilenberg differential works, see, for example, [9; p.107]. For our purpose, we must slightly modify it by multiplying it by -1 (cf. Remark next to Prop.2.1), i.e. we adopt the following differential:

$$\delta_{x}:\Lambda\mathbf{g}^{*}_{|x}\longrightarrow\Lambda\mathbf{g}^{*}_{|x}$$
$$\langle\delta_{x}(\psi),w_{0}\wedge\ldots\wedge w_{k}\rangle=-\sum_{i<j}(-1)^{i+j}\langle\psi,[w_{i},w_{j}]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge w_{k}\rangle$$

for $\psi\in\Lambda^{k}\mathbf{g}^{*}_{|x}$ $(k\geq 1)$, $w_{i}\in\mathbf{g}_{|x}$, and $\delta_{x}\psi=0$ for $\psi\in\Lambda^{0}\mathbf{g}^{*}_{|x}$. δ_{x} is an antiderivation of degree +1 and, for an arbitrary $k\geq 0$, the induced homomorphism of vector bundles

$$\delta^{k}:\Lambda^{k}\mathbf{g}^{*}\longrightarrow\Lambda^{k}\mathbf{g}^{*}$$

is, obviously, of the C^{∞} class.

Proposition 2.7. $\Omega^{\vee}(w^{*})=\langle w^{*},d^{\mathbf{g}}\omega\rangle-\omega^{\wedge}(\delta(w^{*}))$ *for* $w^{*}\in Sec\,\mathbf{g}^{*}$.

Proof. Applying the Maurer-Cartan equation, we get

$$\Omega^{\vee}(w^{*})=\langle w^{*},\Omega\rangle=\langle w^{*},d^{\mathbf{g}}\omega\rangle-\tfrac{1}{2}\cdot\langle w^{*},[\omega,\omega]\rangle.$$

On the other hand, for $\xi_{1},\xi_{2}\in Sec\,A$,

$$\langle\omega^{\wedge}(\delta w^{*}),\xi_{1}\wedge\xi_{2}\rangle=\langle\delta w^{*},\omega\xi_{1}\wedge\omega\xi_{2}\rangle$$
$$=\langle w^{*},[\![\omega\xi_{1},\omega\xi_{2}]\!]\rangle=\tfrac{1}{2}\cdot\langle\langle w^{*},[\omega,\omega]\rangle,\xi_{1}\wedge\xi_{2}\rangle.\quad ■$$

Definition 2.8. Define the mapping

$$K:\overset{k\geq 0}{\oplus}(Sec\Lambda^{k}\mathbf{g}^{*})\longrightarrow\Omega_{A}(M)$$

by the formula

$$K(\Psi)=\frac{1}{k!}\cdot\langle\Psi, d^{\mathbf{g}}(\underbrace{\omega\wedge\ldots\wedge\omega}_{k\ \text{times}})\rangle-\omega^{\wedge}(\delta\Psi)$$

for $\Psi\in Sec\bigwedge^{k}\mathbf{g}^{*}$.

Of course, by Prop.2.7,

$$K(w^{*})=\Omega^{\vee}(w^{*}) \tag{6}$$

if $w^{*}\in Sec\,\mathbf{g}^{*}$.

Proposition 2.9. *The fundamental formulae for K:*

(1) $$K(w_{1}^{*}\wedge\ldots\wedge w_{k}^{*})=\sum_{s=1}^{k}(-1)^{s-1}K(w_{s}^{*})\wedge\omega^{\wedge}(w_{1}^{*}\wedge\ldots\hat{s}\ldots\wedge w_{k}^{*})$$

for $w_{s}^{*}\in Sec\,\mathbf{g}^{*}$,

(2) $$K(\Psi)=d^{A}(\omega^{\wedge}(\Psi))-\omega^{\wedge}(\delta\Psi)-\frac{1}{k!}\cdot\langle d^{\mathbf{g}}\Psi,\underbrace{\omega\wedge\ldots\wedge\omega}_{k\ \text{times}}\rangle$$

for $\Psi\in Sec\bigwedge^{k}\mathbf{g}^{*}$.

Proof. (1): Applying Th.1.3(viii), we get

$$K(w_{1}^{*}\wedge\ldots\wedge w_{k}^{*})=\frac{1}{k!}\cdot\langle w_{1}^{*}\wedge\ldots\wedge w_{k}^{*},d^{\mathbf{g}}(\omega\wedge\ldots\wedge\omega)\rangle-\omega^{\wedge}(\delta(w_{1}^{*}\wedge\ldots\wedge w_{k}^{*}))$$

$$=\frac{1}{k!}\cdot\langle w_{1}^{*}\wedge\ldots\wedge w_{k}^{*},\sum_{s=1}^{k}(-1)^{s-1}\underbrace{\omega\wedge\ldots\wedge\omega}_{s-1\ \text{times}}\wedge d^{\mathbf{g}}\omega\wedge\ldots\wedge\omega\rangle-\omega^{\wedge}\Big(\sum_{s=1}^{k}(-1)^{s-1}w_{1}^{*}\wedge\ldots\wedge\delta w_{s}^{*}\wedge\ldots\wedge w_{k}^{*}\Big)$$

$$=\frac{1}{k!}\cdot\langle w_{1}^{*}\wedge\ldots\wedge w_{k}^{*},k\cdot(d^{\mathbf{g}}\omega)\wedge\omega\wedge\ldots\wedge\omega\rangle-\sum_{s=1}^{k}(-1)^{s-1}\omega^{\wedge}(\delta w_{s}^{*}\wedge w_{1}^{*}\wedge\ldots\hat{s}\ldots\wedge w_{k}^{*})$$

$$=\frac{1}{(k-1)!}\cdot\langle w_{1}^{*}\wedge\ldots\wedge w_{k}^{*},(d^{\mathbf{g}}\omega)\wedge\omega\wedge\ldots\wedge\omega\rangle-\sum_{s=1}^{k}(-1)^{s-1}\omega^{\wedge}(\delta w_{s}^{*})\wedge\omega^{\wedge}(w_{1}^{*}\wedge\ldots\wedge\hat{s}\ldots\wedge w_{k}^{*}).$$

On the other hand,

$$\sum_{s=1}^{k}(-1)^{s-1}K(w_{s}^{*})\wedge\omega^{\wedge}(w_{1}^{*}\wedge\ldots\hat{s}\ldots\wedge w_{k}^{*})=\sum_{s=1}^{k}(-1)^{s-1}\big(\langle w_{s}^{*},d^{\mathbf{g}}\omega\rangle-\omega^{\wedge}(\delta w_{s}^{*})\big)\wedge\omega^{\wedge}(w_{1}^{*}\wedge\ldots\hat{s}\ldots\wedge w_{k}^{*}).$$

Therefore, it is sufficient to prove the equality

$$\frac{1}{(k-1)!}\cdot\langle w_{1}^{*}\wedge\ldots\wedge w_{k}^{*},(d^{\mathbf{g}}\omega)\wedge\omega\wedge\ldots\wedge\omega\rangle=\sum_{s=1}^{k}(-1)^{s-1}\langle w_{s}^{*},d^{\mathbf{g}}\omega\rangle\wedge\omega^{\wedge}(w_{1}^{*}\wedge\ldots\hat{s}\ldots\wedge w_{k}^{*}).$$

For this purpose, take $x\in M$ and $w_{i}\in A_{|x}$, $i\leqslant k+1$. We have

$$\frac{1}{(k-1)!}\cdot\langle w_1^*\wedge\ldots\wedge w_k^*, (d^{\mathbf{g}}\omega)\wedge\omega\wedge\ldots\wedge\omega\rangle(x;w_1\wedge\ldots\wedge w_{k+1})$$

$$=\frac{1}{(k-1)!}\cdot<(w_1^*\wedge\ldots\wedge w_k^*)(x),((d^{\mathbf{g}}\omega)\wedge\omega\wedge\ldots\wedge\omega)(x;w_1\wedge\ldots\wedge w_{k+1})>$$

$$=<(w_1^*\wedge\ldots\wedge w_k^*)(x),\sum_{i<j}(-1)^{i+j-1}(d^{\mathbf{g}}\omega)(x;w_i\wedge w_j)\wedge\omega(x;w_1)\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge\omega(x;w_{k+1})>$$

$$=\sum_{i<j}(-1)^{i+j-1}\cdot\begin{vmatrix} w_1^*(x;(d^{\mathbf{g}}\omega)(x;w_i\wedge w_j)) & w_1^*(x;\omega(x;w_1)) & \ldots & \hat{i} & \ldots & \hat{j} & \ldots \\ \vdots & \vdots & & \vdots & & \vdots & \\ w_k^*(x;(d^{\mathbf{g}}\omega)(x;w_i\wedge w_j)) & w_1^*(x;\omega(x;w_1)) & \ldots & \hat{i} & \ldots & \hat{j} & \ldots \end{vmatrix}$$

$$=\sum_{i<j}(-1)^{i+j-1}\cdot\sum_{s=1}^{k}(-1)^{s+1}\cdot w_s^*(x;(d^{\mathbf{g}}\omega)(x;w_i\wedge w_j))\cdot$$

$$\cdot<(w_1^*\wedge\ldots\hat{s}\ldots\wedge w_k^*)(x),\omega(x;w_1)\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge\omega(x;w_{k+1})>$$

$$=\sum_{s=1}^{k}(-1)^{s+1}\cdot\langle w_s^*,d^{\mathbf{g}}\omega\rangle\wedge\omega^{\wedge}(w_1^*\wedge\ldots\hat{s}\ldots\wedge w_k^*)(x;w_1\wedge\ldots\wedge w_{k+1}).$$

(2): By Th.1.3(viii) (treating Ψ as a 0-form on the Lie algebroid A, with values in $\wedge^k\mathbf{g}^*$), we have

$$d^A\langle\Psi,\omega\wedge\ldots\wedge\omega\rangle=\langle d^{\mathbf{g}}\Psi,\omega\wedge\ldots\wedge\omega\rangle+\langle\Psi,d^{\mathbf{g}}(\omega\wedge\ldots\wedge\omega)\rangle.$$

Therefore, by 2.2,

$$K(\Psi)=\frac{1}{k!}\cdot\langle\Psi,d^{\mathbf{g}}(\omega\wedge\ldots\wedge\omega)\rangle-\omega^{\wedge}(\delta\Psi)$$

$$=\frac{1}{k!}\cdot[k!\,d^A(\omega^{\wedge}\delta\Psi)-\langle d^{\mathbf{g}}\Psi,\omega\wedge\ldots\wedge\omega\rangle]-\omega^{\wedge}(\delta\Psi)$$

$$=d^A(\omega^{\wedge}\delta\Psi)-\omega^{\wedge}(\delta\Psi)-\frac{1}{k!}\cdot\langle d^{\mathbf{g}}\Psi,\omega\wedge\ldots\wedge\omega\rangle.\ \blacksquare$$

Because of the fact that each cross-section $\Psi\in Sec\wedge^k\mathbf{g}^*$ is locally a sum of cross-sections of the form $w_1^*\wedge\ldots\wedge w_k^*$ for $w_s^*\in Sec\,\mathbf{g}^*$, we get

Corollary 2.10. *If the connection λ considered is flat (i.e. $\Omega=0$), then, according to (6) and Prop.2.9(1), we see that $K\equiv 0$, which means, by definition 2.8 and Prop.2.9(2), that*

$$\omega^{\wedge}(\delta\Psi) = \frac{1}{k!}\cdot\langle\Psi, d^{\mathbf{g}}(\omega\wedge\ldots\wedge\omega)\rangle = d^{A}(\omega^{\wedge}\Psi) - \frac{1}{k!}\cdot\langle d^{\mathbf{g}}\Psi, \omega\wedge\ldots\wedge\omega\rangle. \qquad\blacksquare$$

Remarks 2.11. Keep the assumption $\Omega=0$.

(1). If $\Psi\in Sec\bigwedge^{k}\mathbf{g}^{*}$ is invariant with respect to the representation $ad_{\mathbf{g}}$ of A on $\mathbf{g}$ [i.e. if $\Psi\in(Sec\bigwedge^{k}\mathbf{g}^{*})_{I^{\circ}}$, or, equivalently, if $d^{\mathbf{g}}\Psi=0$, see 1.7], then

$$d^{A}(\omega^{\wedge}\Psi)=0.$$

Indeed, by Cor.2.10, we have $d^{A}(\omega^{\wedge}\Psi)=\omega^{\wedge}(\delta\Psi)$; but, for each point $x\in M$, the tensor $\Psi(x)\in\bigwedge^{k}\mathbf{g}^{*}_{|x}$ is invariant under the canonical representation of the Lie algebra $\mathbf{g}_{|x}$ on $\bigwedge^{k}\mathbf{g}^{*}_{|x}$ (induced by the adjoint one) and such a tensor is a cycle [9; p.186], so $(\delta\Psi)(x)=\delta_{x}(\Psi_{x})=0$. Therefore, there exists a homomorphism of algebras

$$\omega^{\wedge}_{o}:\bigoplus^{k\geq 0}(Sec\bigwedge^{k}\mathbf{g}^{*})_{I^{\circ}}\longrightarrow Z_{A}(M)\subset\Omega_{A}(M),\ \ \Psi\longmapsto\omega^{\wedge}(\Psi),$$

and, next,

$$\begin{array}{l} \omega^{\#}:\bigoplus^{k\geq 0}(Sec\bigwedge^{k}\mathbf{g}^{*})_{I^{\circ}}\longrightarrow Z_{A}(M)\longrightarrow H_{A}(M) \\ \qquad\qquad \Psi \longmapsto [\omega^{\wedge}_{o}(\Psi)] \end{array}.$$

(2). If A is a transitive Lie algebroid, then, in view of Th.I.5.2, each invariant cross-section $\Psi\in(Sec\bigwedge^{k}\mathbf{g}^{*})_{I^{\circ}}$ is determined by the value at an arbitrarily taken point $x_{o}\in M$. Thus, the domain of $\omega^{\#}$ is isomorphic to some subalgebra $B\subset\bigwedge\mathbf{g}^{*}_{|x_{o}}$. If, additionally, $A=A(P)$ for some connected principal bundle $(P,\pi,M,G,\cdot)$, then, according to 5.5.2 from [17], B is isomorphic to the vector space $(\bigwedge\mathfrak{g}^{*})_{I}$ of invariant [with respect to the adjoint representation] vectors. Let $\omega_{P}\in\Omega^{1}(P;\mathfrak{g})$ be the form of the connection on P corresponding to λ. Then, the real-valued form on $A(P)$ $\Theta:=\omega^{\wedge}(\sigma_{V})$ for $v\in(\bigwedge\mathfrak{g}^{*})_{I}$ (for σ_{V}, see [17; 5.5.2]) is precisely the one for which the corresponding right-invariant form $\tilde{\Theta}$ on P is equal to $\langle v,\omega_{P}\wedge\ldots\wedge\omega_{P}\rangle$. Recall that ? $\tilde{\Theta}(z;v_{1}\wedge\ldots\wedge v_{k})=\Theta(\pi z;\pi^{A}_{|z}(v_{1})\wedge\ldots\wedge\pi^{A}_{|z}(v_{k}))$, $z\in P$, $v_{i}\in T_{z}P$ $(\pi^{A}:TP\longrightarrow A(P)$ is the classical projection [15]).

(2′). In particular, for an arbitrary Lie algebra $\mathfrak{g}$ (treated as a trivial Lie algebroid over a point) and for the only connection $\lambda=0$,

$$0\longrightarrow\mathfrak{g}\ \underset{\omega=id}{\overset{=}{\longleftarrow}}\ \mathfrak{g}\ \underset{\lambda=0}{\overset{\gamma}{\rightleftarrows}}\ 0\longrightarrow 0$$

we have $\omega^{\wedge}_{o}:(\bigwedge\mathfrak{g}^{*})_{I}\hookrightarrow\bigwedge\mathfrak{g}^{*}$ is an inclusion and

$$\omega^{\#}:(\Lambda\mathfrak{g}^{*})_{I}\longrightarrow H(\mathfrak{g}).$$
$$\Psi\longmapsto[\Psi]$$

We realize that, for a reductive Lie algebra $\mathfrak{g}$, $\omega^{\#}$ is an isomorphism [9; p.189].

(3). Consider the case of the foliation $(G,\{aH;\,a\in H\})$ of left cosets of a connected Lie group G by a connected nonclosed Lie subgroup $H\subset G$ and let A be the Lie algebroid $A(G;H)$ of this foliation, see [17], [16]. The homomorphism $\omega^{\#}$ has the following form: there exist isomorphisms of algebras α and β such that

$$\begin{array}{ccc}
\overset{k>0}{\oplus}(Sec\Lambda^{k}\mathbf{g}^{*})_{I^{\circ}} & \longrightarrow & H_{A}(G/\overline{H})\\
\alpha\Big\downarrow\cong & & \beta\Big\downarrow\cong\\
\Lambda(\overline{\mathfrak{h}}/\mathfrak{h})^{*} & \longrightarrow & H_{dR}(G/\overline{H})\otimes\Lambda(\overline{\mathfrak{h}}/\mathfrak{h})^{*}\\
\Psi & \longmapsto & 1\otimes\Psi
\end{array}$$

($\overline{\mathfrak{h}}$ is the Lie algebra of the closure $\overline{H}$ of H).

The isomorphism α is built in the following way: via the global trivialization $\varphi:G/\overline{H}\times\overline{\mathfrak{h}}/\mathfrak{h}\xrightarrow{\cong}\mathbf{g}$, see [17; 8.2.4] and [16; 3.2], any cross-section Ψ of $\Lambda^{k}\mathbf{g}^{*}$ determines some $\overline{\mathfrak{h}}/\mathfrak{h}$-valued function $\tilde{\Psi}:G/\overline{H}\longrightarrow\Lambda^{k}(\overline{\mathfrak{h}}/\mathfrak{h})^{*}$. Analogously as in the proof of Prop.8.4.1 from [17], we assert that Ψ is invariant if and only if $\tilde{\Psi}$ is constant. The isomorphism α is defined as follows: $\Psi\longmapsto\tilde{\Psi}(x)$, x being an arbitrary point of $G/\overline{H}$.

The isomorphism β looks as follows: according to [16; Th.3.3], the Lie algebroid A is trivial and an isomorphism of Lie algebroids $\rho:T(G/\overline{H})\times\overline{\mathfrak{h}}/\mathfrak{h}\longrightarrow A$ is given by the formula $\rho(v,[w])=\lambda(v)+\varphi(\pi v,[w])$. Therefore, the superposition

$$\beta:H_{A}(G/\overline{H})\xrightarrow{\cong}H\big(T(G/\overline{H})\times\mathfrak{h}/\overline{\mathfrak{h}}\big)\xrightarrow{\cong}H_{dR}(G/\overline{H})\times\Lambda(\overline{\mathfrak{h}}/\mathfrak{h})^{*}$$

is an isomorphism of algebras. The commutativity of our diagram follows now in a simple way.

C) Homomorphism $(d\omega)^{\vee}$

This section will not be needed till Part III.

$d^{\mathbf{g}}\omega$ at a point $x\in M$ is a 2-linear skew-symmetric tensor $(d^{\mathbf{g}}\omega)_{|x}:A_{|x}\times A_{|x}\longrightarrow\mathbf{g}_{|x}$ understood sometimes equivalently as an element of $\Lambda^{2}A^{*}_{|x}\otimes\mathbf{g}_{|x}$. $(d^{\mathbf{g}}\omega)_{|x}$ defines a linear mapping

$$(d\omega)_x : \mathbf{g}_{|x} \longrightarrow \Lambda^2 A^*_{|x} \subset \Lambda A^*_{|x}$$
$$w^* \longmapsto w^* \circ (d^{\mathbf{g}}\omega)_{|x}$$

having the property

$$(d\omega)_x(w^*_1) \wedge (d\omega)_x(w^*_2) = (d\omega)_x(w^*_2) \wedge (d\omega)_x(w^*_1), \quad w^*_i \in \mathbf{g}^*_{|x}.$$

Therefore, by the universal property of the symmetric algebra $\vee \mathbf{g}^*_{|x}$, see [7; p.], we obtain the existence and the uniqueness of a homomorphism of algebras

$$(d\omega)^{\vee}_x : \vee \mathbf{g}^*_{|x} \longrightarrow \Lambda A^*_{|x}$$

extending $(d\omega)_x$ and such that $(d\omega)^{\vee}_x(1) = 1$.

Lemma 2.12. *Let $\Gamma \in \vee^l \mathbf{g}^*_{|x}$, then for $w_1, \dots, w_{2l} \in A_{|x}$*

$$<(d\omega)^{\vee}_x(\Gamma), w_1 \wedge \dots \wedge w_{2l}> = \frac{1}{l! \cdot 2^l} \cdot \sum_\sigma sgn\,\sigma \cdot <\Gamma, (d^{\mathbf{g}}\omega)_{|x}(w_{\sigma(1)} \wedge w_{\sigma(2)}) \vee \dots$$
$$\dots \vee (d^{\mathbf{g}}\omega)_{|x}(w_{\sigma(2l-1)} \wedge w_{\sigma(2l)})>.$$

Proof. It is sufficient to prove this for a simple tensor $\Gamma = w^*_1 \wedge \dots \wedge w^*_l$:

$$<(d\omega)^{\vee}_x(w^*_1 \wedge \dots \wedge w^*_l), w_1 \wedge \dots \wedge w_{2l}>$$
$$= <(d\omega)_x(w^*_1) \wedge \dots \wedge (d\omega)_x(w^*_l), w_1 \wedge \dots \wedge w_{2l}>$$
$$= \frac{1}{2^l} \cdot \sum_\sigma sgn\,\sigma \cdot (d\omega)_x(w^*_1)(w_{\sigma(1)} \wedge w_{\sigma(2)}) \cdot \dots \cdot (d\omega)_x(w^*_l)(w_{\sigma(2l-1)} \wedge w_{\sigma(2l)})$$
$$= \frac{1}{2^l} \cdot \sum_\sigma sgn\,\sigma \cdot <w^*_1, (d^{\mathbf{g}}\omega)_{|x}(w_{\sigma(1)} \wedge w_{\sigma(2)})> \cdot \dots \cdot <w^*_l, (d^{\mathbf{g}}\omega)_{|x}(w_{\sigma(2l-1)} \wedge w_{\sigma(2l)})>$$
$$= \sum_{\sigma \in I} sgn\,\sigma \cdot \sum_\tau <w^*_1, (d^{\mathbf{g}}\omega)_{|x}(w_{\sigma(2\cdot\tau(1)-1)} \wedge w_{\sigma(2\cdot\tau(1))})> \cdot \dots$$
$$\dots \cdot <w^*_l, (d^{\mathbf{g}}\omega)_{|x}(w_{\sigma(2\cdot\tau(l)-1)} \wedge w_{\sigma(2\cdot\tau(l))})>$$

(where I is the set of all permutations of the sequence $(1,2,\dots,2l)$, such that $\sigma(1)<\sigma(2), \dots, \sigma(2l-1)<\sigma(2l)$, $\sigma(1)<\sigma(3)<\dots<\sigma(2l-1)$)

$$= \frac{1}{l! \cdot 2^l} \cdot \sum_\sigma sgn\,\sigma \cdot <w^*_1 \vee \dots \vee w^*_l, (d^{\mathbf{g}}\omega)_{|x}(w_{\sigma(1)} \wedge w_{\sigma(2)}) \vee \dots \vee (d^{\mathbf{g}}\omega)_{|x}(w_{\sigma(2l-1)} \wedge w_{\sigma(2l)})>. \blacksquare$$

According to this lemma and the fact that the canonical duality $\vee^l \mathbf{g}^* \times \vee^l \mathbf{g}^* \longrightarrow \mathbb{R}$ [defined point by point by: $\big((w^*_1 \vee \dots \vee w^*_l), (w_1 \vee \dots \vee w_l)\big) \longmapsto perm[<w^*_i, w_j>]$] is a C^∞ 2-linear homomorphism of vector bundles, we assert the following

2.13. *For* $\Gamma\in Sec\bigvee^{l}\mathbf{g}^{*}$, *the cross-section*

$$(d\omega)^{\vee}(\Gamma):M\longrightarrow\Lambda^{2l}A^{*},\quad x\longmapsto(d\omega)_{x}^{\vee}(\Gamma_{x}),$$

is a C^{∞} *real* $2l$*-form on* A, *i.e.* $(d\omega)^{\vee}(\Gamma)\in\Omega_{A}^{2l}(M)$, *as well as it is defined by*

$$(d\omega)^{\vee}(\Gamma)=\frac{1}{l!}\cdot\langle\Gamma,d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle. \qquad \blacksquare$$

3. A CONSTRUCTION OF THE CHARACTERISTIC CLASSES OF FLAT REGULAR LIE ALGEBROIDS

Here we construct characteristic classes having the following property:

— *the existence of nontrivial classes among them is a measure of the incompatibility of the flat structure of a given regular Lie algebroid* A *(over* (M,E)*) with a given subalgebroid* B *of* A *(also over* (M,E)*).*

In the case of an integrable transitive Lie algebroid $A=A(P)$, P being any principal bundle, these classes agree with the so-called characteristic classes of the flat principal bundle P [10].

Consider in a given regular Lie algebroid $(A,[\![\cdot,\cdot]\!],\gamma_{A})$ over (M,E) two geometric structures:

(1) a flat connection $\lambda:E\longrightarrow A$,

(2) a subalgebroid $B\subset A$ over (M,E), see the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{g} \arrow[r, hook, shift left] & A \arrow[l, shift left, "\omega"] \arrow[r, shift left, "\gamma_A"] & E \arrow[l, shift left, "\lambda"] \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{h} \arrow[r, hook] \arrow[u, hook] & B \arrow[r, "\gamma_B"] \arrow[u, hook, "j"] & E \arrow[r] \arrow[u, "id"'] & 0
\end{tikzcd} \tag{7}$$

Notice that $\mathbf{h}=\mathbf{g}\cap B$ $(\mathbf{h}:=Ker\,\gamma_{B})$.

The system (A,λ,B) will then be called an *FS-regular Lie algebroid* *(over* (M,E)*)*.

Examples 3.1. (1) A triad (P,P',ω) consisting of a principal bundle P, of an

H-reduction P' and a flat connection in P with a connection form ω determines an FS-transitive Lie algebroid $(A(P),\lambda,A(P'))$ (λ corresponds to ω). For the theory of flat principal bundles with given reductions, see [10].

(2) We recall that both a transitive Lie algebroid $A=(A,[\![\cdot,\cdot]\!],\gamma)$ on M and an involutive distribution $F\subset TM$ give rise to the regular Lie algebroid over (M,F) of the form $A^{F}=\gamma^{-1}[F]\subset A$, see [17; s.1.1.3]. Consider now a triple (A,B,λ) consisting of a transitive Lie algebroid A on M, a transitive Lie subalgebroid B of A and a partially flat connection λ in A, namely, flat over a given involutive distribution $F\subset TM$. The triple

$$(A^{F},B^{F},\lambda|F)$$

is an FS-regular Lie algebroid.

(3) Let now the system (P,P',ω) be given as in Ex.(1) with the difference that ω is assumed to be partially flat, say, over an involutive distribution $F\subset TM$. Such a system (named a *foliated bundle*) is investigated, for example, in [10]. It determines the (nontransitive) FS-regular Lie algebroid $(A(P)^{F},A(P')^{F},\lambda|F)$ written above.

Examples on the ground of the theory of nonclosed Lie subgroups will be given in Ch.7 below.

We construct some characteristic classes of an FS-regular Lie algebroid (A,λ,B), measuring the independence of λ and B, i.e. how far $Im\lambda$ is not contained in B. The construction has a number of steps.

3.2. Let $\mathbf{s}:\mathbf{g}\longrightarrow\mathbf{g}/\mathbf{h}$ be the canonical projection. The form $\varphi(\Psi):=\omega^{\wedge}(\wedge^{k}\mathbf{s}^{*}\circ\Psi)$, where $\Psi\in Sec\wedge^{k}(\mathbf{g}/\mathbf{h})^{*}$, is $\mathbf{h}$-*horizontal*, i.e., equivalently, its restriction to the subalgebroid B - $j^{*}(\omega^{\wedge}(\wedge^{k}\mathbf{s}^{*}\circ\Psi))$ - is horizontal. Indeed, for $\nu\in Sec\,\mathbf{h}$, applying Lemma 2.3, we get

$$\iota_{\nu}(\omega^{\wedge}(\wedge^{k}\mathbf{s}^{*}\circ\Psi))=\omega^{\wedge}(\iota_{\nu}(\wedge^{k}\mathbf{s}^{*}\circ\Psi))=0$$

because the fact that $[\nu]:=\mathbf{s}\circ\nu=0$ yields

$$\langle\iota_{\nu}(\wedge^{k}\mathbf{s}^{*}\circ\Psi),\nu_{1}\wedge\ldots\wedge\nu_{k-1}\rangle=\langle\wedge^{k}\mathbf{s}^{*}\circ\Psi,\nu\wedge\nu_{1}\wedge\ldots\wedge\nu_{k-1}\rangle$$
$$=\langle\Psi,[\nu]\wedge[\nu_{1}]\wedge\ldots\wedge[\nu_{k-1}]\rangle=0.$$

for $\nu_{j}\in Sec\,\mathbf{g}$.

Therefore (see the previous section) there exists a form $\Delta\Psi\in\Omega^{k}_{E}(M)$ such that

$$j^{*}(\omega^{\wedge}(\Lambda^{k}\mathbf{s}^{*}\circ\Psi)) = (\gamma_{B})_{*}(\Delta\Psi).$$

Notice that if λ is a connection in B $\big($i.e. $Im\lambda\subset B\big)$, then $\Delta\Psi=0$. In fact, for $X_{i}\in Sec\,E$,

$$<\Delta\Psi,X_{1}\wedge\ldots\wedge X_{k}> = <(\gamma_{B})_{*}(\Delta\Psi),\lambda\circ X_{1}\wedge\ldots\wedge\lambda\circ X_{k}> = <j^{*}(\omega^{\wedge}(\Lambda^{k}\mathbf{s}^{*}\circ\Psi)),\lambda\circ X_{1}\wedge\ldots\wedge\lambda\circ X_{k}>$$
$$= <\Lambda^{k}\mathbf{s}^{*}\circ\Psi,\omega\circ\lambda\circ X_{1}\wedge\ldots\wedge\omega\circ\lambda\circ X_{k}> = 0.$$

3.3. Put

$$\begin{aligned}\Delta\colon \overset{k\geq 0}{\oplus}(Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*}) &\longrightarrow \Omega_{E}(M).\\ \Psi &\longmapsto \Delta\Psi\end{aligned}$$

Δ being a superposition of homomorphisms of algebras, see the following diagram

$$\begin{tikzcd}
Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*} \arrow[r, "\Delta"] \arrow[dd] \arrow[ddr, "\varphi"] & \Omega_{E}(M) \arrow[d, "(\gamma_{B})_{*}", "\cong"'] \\
 & \Omega_{B,i}(M) \\
Sec\Lambda^{k}\mathbf{g}^{*} \arrow[r, "\omega^{\wedge}"] & \Omega_{A,\mathbf{h}}(M) \arrow[u, "j^{*}"']
\end{tikzcd}$$

$\big($where $\Omega_{A,\mathbf{h}}(M)$ denotes the space of $\mathbf{h}$-horizontal forms on $A\big)$, is itself such a homomorphism.

Directly, Δ is defined by the formula

$$(\Delta\Psi)(x;w_{1}\wedge\ldots\wedge w_{k}) = <\Psi_{x};[\omega(x;\tilde{w}_{1})]\wedge\ldots\wedge[\omega(x;\tilde{w}_{k})]> \tag{8}$$

for $\tilde{w}_{i}\in B_{|x}$ such that $\gamma_{B}(\tilde{w}_{i})=w_{i}$, $w_{i}\in E_{|x}$, $x\in M$.

3.4. Define a representation

$$ad^{\wedge}_{B,\mathbf{g}}\colon B\longrightarrow A(\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*})$$

by the formula

$$<\mathfrak{L}_{ad^{\wedge}_{B,\mathbf{g}}\circ\xi}(\Psi),[\nu_{1}]\wedge\ldots\wedge[\nu_{k}]>$$
$$= (\gamma_{B}\circ\xi)<\Psi,[\nu_{1}]\wedge\ldots\wedge[\nu_{k}]> - \sum_{j=1}^{k}<\Psi,[\nu_{1}]\wedge\ldots\wedge[[\![\xi,\nu_{j}]\!]]\wedge\ldots\wedge[\nu_{k}]>$$

for $\Psi\in Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*}$, $\xi\in Sec\,B$, and $\nu_{j}\in Sec\,\mathbf{g}$.

The correctness of this definition follows from the fact that if one of ν_j's lies in $\mathbf{h}$, then $[\![\xi,\nu_j]\!]$ lies in $\mathbf{h}$, too.

Notice that

$$ad^{\wedge}_{B,\mathbf{g}} = \Lambda^k (ad_{B,\mathbf{g}})^{\natural},$$

where (a)

$$ad_{B,\mathbf{g}} : B \longrightarrow A(\mathbf{g}/\mathbf{h})$$

is a representation given by

$$\mathcal{L}_{ad_{B,\mathbf{g}}\circ\xi}([\nu]) = [\mathcal{L}_{ad_A\circ\xi}(\nu)] \;\big(= [[\![\xi,\nu]\!]]\big)$$

for $\xi\in Sec\,B$ and $\nu\in Sec\,\mathbf{g}$ $\big(ad_A$ is the adjoint representation of A, see $[17; 2.1.2]\big)$,

(b) $(\cdot)^{\natural}$ is the contragredient representation $[17; 2.1.3]$,

(c) $\Lambda^k T$, for an arbitrary representation $T: A \longrightarrow A(\mathbf{f})$, denotes the *skew-symmetric product* of T defined analogously to the symmetric product $[17: 2.2.1]$.

3.5. In the space $\overset{k>0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}$ of cross-sections invariant with respect to $ad^{\wedge}_{B,\mathbf{g}}$, we introduce a differential $\bar\delta$ of degree $+1$ defined as follows: for $\Psi\in(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}$ and $\nu_j\in Sec\,\mathbf{g}$, we put

$$\langle\bar\delta\Psi, [\nu_0]\wedge\ldots\wedge[\nu_k]\rangle = -\sum_{i<j}(-1)^{i+j}\langle\Psi, [[\![\nu_i,\nu_j]\!]]\wedge[\nu_0]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge[\nu_k]\rangle.$$

(a). *The correctness of this definition.* If $\nu_{j_o}\in Sec\,\mathbf{h}$ for some index j_o, then

$$\begin{aligned}&\sum_{i<j}(-1)^{i+j}\langle\Psi, [[\![\nu_i,\nu_j]\!]]\wedge[\nu_0]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge[\nu_k]\rangle\\ &= (-1)^{j_o-1}\cdot\sum_{j\neq j_o}\langle\Psi, [\nu_0]\wedge\ldots\hat{j_o}\ldots\wedge[[\![\nu_{j_o},\nu_j]\!]]\wedge\ldots\wedge[\nu_k]\rangle\\ &= 0\end{aligned}$$

by the invariance of Ψ and the equality $\gamma_B\circ\nu_{j_o}=0$.

(b). $\bar\delta\Psi$ *is invariant.* Indeed, for $\xi\in Sec\,B$ and $\nu_j\in Sec\,\mathbf{g}$, we have, by the invariance of Ψ,

$$\begin{aligned}&(\gamma_B\circ\xi)\langle\bar\delta\Psi, [\nu_0]\wedge\ldots\wedge[\nu_k]\rangle\\ &= (\gamma_B\circ\xi)\Big(-\sum_{i<j}(-1)^{i+j}\langle\Psi, [[\![\nu_i,\nu_j]\!]]\wedge[\nu_0]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge[\nu_k]\rangle\Big)\\ &= \sum_{i<j}(-1)^{i+j}\Big(\langle\Psi, [[\![\xi,[\![\nu_i,\nu_j]\!]]\!]]\wedge[\nu_0]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge[\nu_k]\rangle +\end{aligned}$$

$$+\sum_{l\neq i,j}<\Psi,[[\![\nu_i,\nu_j]\!]]\wedge[\nu_0]\wedge\ldots\wedge[[\![\xi,\nu_l]\!]]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge[\nu_k]>\Big)$$

$$=-\sum_l\Big(\sum_{l\neq i<j\neq l}(-1)^{i+j}<\Psi,[[\![\nu_i,\nu_j]\!]]\wedge[\nu_0]\wedge\ldots\wedge[[\![\xi,\nu_l]\!]]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge[\nu_k]>+$$

$$+\sum_{i<l}(-1)^{i+l}\Big(<\Psi,[[\![\nu_i,[\![\xi,\nu_l]\!]]\!]]\wedge[\nu_0]\wedge\ldots\hat{i}\ldots\hat{l}\ldots\wedge[\nu_k]>+$$

$$+\sum_{l<j}(-1)^{l+j}\Big(<\Psi,[[\![\xi,[\![\nu_l,\nu_j]\!]]\!]]\wedge[\nu_0]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\wedge[\nu_k]>\Big)$$

$$=\sum_l<\bar{\delta}\Psi,[\nu_0]\wedge\ldots\wedge[[\![\xi,\nu_l]\!]]\wedge\ldots\wedge[\nu_k]>.$$

(c). It remains to notice that

(i) $\bar{\delta}^2=0$,

(ii) $\bar{\delta}$ is an antiderivation of degree +1.

For this purpose, firstly, for an arbitrary point $x\in M$, we can define a space of tensors $(\Lambda^k(\mathbf{g}_{|x}/\mathbf{h}_{|x})^*)_{I^\circ}$ invariant with respect to the representation of the Lie algebra $\mathbf{h}_{|x}$, induced on $\Lambda^k(\mathbf{g}_{|x}/\mathbf{h}_{|x})^*$ by the representation ad_x^* of $\mathbf{h}_{|x}$ on $(\mathbf{g}_{|x}/\mathbf{h}_{|x})^*$ defined as follows:

$$<ad_x^*(\nu)(\psi),[\mu]>=-<\varphi,[[\nu,\mu]]>$$

for $\nu\in\mathbf{h}_{|x}$, $\varphi\in(\mathbf{g}_{|x}/\mathbf{h}_{|x})^*$ and $\mu\in\mathbf{g}_{|x}$.

Secondly, we define an antiderivation

$$\bar{\delta}_x:(\Lambda(\mathbf{g}_{|x}/\mathbf{h}_{|x})^*)_{I^\circ}\longrightarrow(\Lambda(\mathbf{g}_{|x}/\mathbf{h}_{|x})^*)_{I^\circ}$$

of degree +1 as the one which on elements ψ of degree +1 equals $<\bar{\delta}_x(\psi),[\nu]\wedge[\mu]>=<\psi,[[\nu,\mu]]>$, $\nu,\mu\in\mathbf{g}_{|x}$. It can easily be seen that if $\Psi\in(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}$, then

(1) $\Psi_x\in(\Lambda(\mathbf{g}_{|x}/\mathbf{h}_{|x})^*)_{I^\circ}$,

(2) $(\bar{\delta}\Psi)_x=\bar{\delta}_x(\Psi_x)$.

In consequence, $\bar{\delta}$ fulfils (i) and (ii) in an evident manner. Of course, these properties of $\bar{\delta}$ can also be checked directly.

Definition 3.6. The *relative cohomology algebra of* $\mathbf{g}$ *with respect to* B is defined as the cohomology algebra of the complex $(\overset{k\geqslant 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ},\bar{\delta})$:

$$H(\mathbf{g};B):=H^*\left(\overset{k\geq 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ},\bar{\delta}\right).$$

Proposition 3.7. *The mapping Δ restricted to the invariant cross-sections*

$$\Delta_*\ \left(=\ \Delta_{(A,\lambda,B)}\right):\overset{k\geq 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}\longrightarrow\Omega_E(M),\quad \Psi\longmapsto\Delta\Psi,$$

commutes with the differentials $\bar{\delta}$ and d^E.

Proof. We need to prove the equality

$$\Delta(\bar{\delta}\Psi)=d^E(\Delta\Psi) \tag{9}$$

for invariant cross-sections Ψ.

The fact that $(\gamma_B)_*$ is a monomorphism implies that this equality is equivalent to $(\gamma_B)_*(\Delta(\bar{\delta}\Psi))=(\gamma_B)_*(d^E(\Delta\Psi))$. But, by definition, see 3.2, $(\gamma_B)_*(\Delta(\bar{\delta}\Psi))$ $=j^*(\omega^\wedge(\Lambda^{k+1}\mathbf{s}^*\circ(\bar{\delta}\Psi)))$. On the other hand, applying (5) and the obvious fact $d^B(j^*\Psi)=j^*(d^A\Psi)$, we get

$$(\gamma_B)_*(d^E(\Delta\Psi))=d^B((\gamma_B)_*(\Delta\Psi))=d^B(j^*(\omega^\wedge(\Lambda^k\mathbf{s}^*\circ\Psi)))=j^*(d^A(\omega^\wedge(\Lambda^k\mathbf{s}^*\circ\Psi))).$$

Therefore, to prove (9), it remains to check that the forms $\omega^\wedge(\Lambda^{k+1}\mathbf{s}^*\circ(\bar{\delta}\Psi))$ and $d^A(\omega^\wedge(\Lambda^k\mathbf{s}^*\circ\Psi))$ agree on the cross-sections of B.

Let $\xi_0,\dots,\xi_k\in Sec\,B$; then (see 2.6 and 3.5)

$$\begin{aligned}
&\langle\omega^\wedge(\Lambda^{k+1}\mathbf{s}^*\circ(\bar{\delta}\Psi)),\xi_0\wedge\dots\wedge\xi_k\rangle=\langle\Lambda^{k+1}\mathbf{s}^*\circ(\bar{\delta}\Psi),\omega(\xi_0)\wedge\dots\wedge\omega(\xi_k)\rangle\\
&=\langle\bar{\delta}\Psi,[\omega(\xi_0)]\wedge\dots\wedge[\omega(\xi_k)]\rangle=-\sum_{i<j}(-1)^{i+j}\langle\Psi,[[\![\omega(\xi_i),\omega(\xi_j)]\!]]\wedge\dots\hat{i}\dots\hat{j}\dots\rangle\\
&=-\sum_{i<j}(-1)^{i+j}\langle\Lambda^k\mathbf{s}^*\circ\Psi,[\![\omega(\xi_i),\omega(\xi_j)]\!]\wedge\dots\hat{i}\dots\hat{j}\dots\rangle=\langle\delta\circ\Lambda^k\mathbf{s}^*\circ\Psi,\omega(\xi_0)\wedge\dots\wedge\omega(\xi_k)\rangle\\
&=\langle\omega^\wedge(\delta\circ\Lambda^k\mathbf{s}^*\circ\Psi),\xi_0\wedge\dots\wedge\xi_k\rangle.
\end{aligned}$$

On the other hand, by Prop.2.9(2) and the flatness of λ, we have

$$d^A(\omega^\wedge(\Lambda^k\mathbf{s}^*\circ\Psi))=\omega^\wedge(\delta\circ\Lambda^k\mathbf{s}^*\circ\Psi)+\frac{1}{k!}\cdot\langle d^{\mathbf{g}}(\Lambda^k\mathbf{s}^*\circ\Psi),\underbrace{\omega\wedge\dots\wedge\omega}_{k\ \text{times}}\rangle.$$

So, it remains to show that

$$j^*\langle d^{\mathbf{g}}(\Lambda^k\mathbf{s}^*\circ\Psi),\omega\wedge\dots\wedge\omega\rangle=0.$$

For ξ_j as above, by the invariance of Ψ, we get

$$
\begin{aligned}
&<\langle d^{\mathbf{g}}(\Lambda^{k}\mathbf{s}^{*}\circ\Psi),\omega\wedge\ldots\wedge\omega\rangle,\xi_0\wedge\ldots\wedge\xi_k>\\
&=\sum_{j=0}^{k}(-1)^{j}<d^{\mathbf{g}}(\Lambda^{k}\mathbf{s}^{*}\circ\Psi)(\xi_j),(\omega\wedge\ldots\wedge\omega)(\xi_0\wedge\ldots\hat{j}\ldots\wedge\xi_k)>\\
&=k!\cdot\sum_{j=0}^{k}(-1)^{j}<\mathfrak{L}_{(\Lambda^{k}\mathrm{ad}_{A}^{\natural})\circ\xi_j}(\Lambda^{k}\mathbf{s}^{*}\circ\Psi),\omega(\xi_0)\wedge\ldots\hat{j}\ldots\wedge\omega(\xi_k)>\\
&=k!\cdot\sum_{j=0}^{k}(-1)^{j}\Big((\gamma\circ\xi_j)<\Lambda^{k}\mathbf{s}^{*}\circ\Psi,\omega(\xi_0)\wedge\ldots\hat{j}\ldots\wedge\omega(\xi_k)>-\\
&\qquad\qquad-\sum_{l\neq j}<\Lambda^{k}\mathbf{s}^{*}\circ\Psi,\omega(\xi_0)\wedge\ldots\wedge[\![\xi_j,\omega(\xi_l)]\!]\wedge\ldots\hat{j}\ldots>\Big)\\
&=k!\cdot\sum_{j=0}^{k}(-1)^{j}\Big((\gamma\circ\xi_j)<\Psi,[\omega(\xi_0)]\wedge\ldots\hat{j}\ldots\wedge[\omega(\xi_k)]>-\\
&\qquad\qquad-\sum_{l\neq j}<\Psi,[\omega(\xi_0)]\wedge\ldots\wedge[[\![\xi_j,\omega(\xi_l)]\!]]\wedge\ldots\hat{j}\ldots>\Big)\\
&=0.\ \blacksquare
\end{aligned}
$$

The above Proposition yields as a corollary

Theorem 3.8. *The mapping*

$$
\begin{aligned}
\Delta_{\#}:H(\mathbf{g},B)&\longrightarrow H_E(M)\\
[\Psi]&\longmapsto[\Delta_{*}\Psi]
\end{aligned}
$$

is a correctly defined homomorphism of algebras. ■

$\Delta_{\#}$ is called the *characteristic homomorphism of the FS-regular Lie algebroid* (A,λ,B). Its image $\operatorname{Im}\Delta_{\#}\subset H_E(M)$ is a subalgebra of $H_E(M)$, called the *characteristic algebra of the FS-regular Lie algebroid* (A,λ,B), whereas its elements - the *characteristic classes* of that algebroid.

According to 3.2, the compatibility of λ with B implies the vanishing of $\Delta_{\#}$ [of course, already on the level of forms]. $\Delta_{\#}$ is then a measure of the incompatibility of λ with B.

4. FUNCTORIALITY

Definition 4.1. Let (A',λ',B') and (A,λ,B) be two FS-regular Lie algebroids over (M',E') and (M,E), respectively. By a *homomorphism*

$$H\colon (A',\lambda',B') \longrightarrow (A,\lambda,B)$$

between them we mean a homomorphism $H\colon A' \longrightarrow A$ of regular Lie algebroids, say, over $f\colon (M',E') \longrightarrow (M,E)$, such that

(1) $H\circ\lambda' = \lambda\circ f_*$,

(2) $H[B'] \subset B$.

Notice that $H' = H|B'\colon B' \longrightarrow B$ is then a homomorphism of regular Lie algebroids, too, see the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] \\
 & \mathbf{h}' \arrow[rr, "H'^{+}"] \arrow[ddl] \arrow[ddd, hook] & & \mathbf{h} \arrow[ddl] \arrow[ddd, hook] \\
0 \arrow[d] & & 0 \arrow[d] & \\
\mathbf{g}' \arrow[rr, "H^{+}"] \arrow[dd, hook] & & \mathbf{g} \arrow[dd, hook] & \\
 & B' \arrow[rr, "H'"] \arrow[dl] \arrow[dd, "\gamma_{B'}"] & & B \arrow[dl] \arrow[dd, "\gamma_B"] \\
A' \arrow[rr, "H"] \arrow[ddd, "\gamma_{A'}"] \arrow[uu, bend left, "\omega'"] & & A \arrow[ddd, "\gamma_A"] \arrow[uu, bend left, "\omega"'] & \\
 & E' \arrow[rr, "f_*"] \arrow[ddl, "="] \arrow[d] & & E \arrow[ddl, "="] \arrow[d] \\
 & 0 & & 0 \\
E' \arrow[rr, "f_*"] \arrow[d] \arrow[uuu, bend left, "\lambda'"] & & E \arrow[d] \arrow[uuu, bend left, "\lambda"'] & \\
0 & & 0 &
\end{tikzcd} \tag{10}$$

By the *pullback* of an FS-regular Lie algebroid (A,λ,B) over (M,E) via a mapping $f\colon (M',E') \longrightarrow (M,E)$ we mean the FS-regular Lie algebroid $(f^{\wedge}A,\bar{\lambda},f^{\wedge}B)$ where $\bar{\lambda}$ is the pullback of the connection λ, see definition 3.2.1 from [17].

Notice that $pr_2\colon f^{\wedge}A = E' \times_{(f_*,\gamma)} A \longrightarrow A$ is a homomorphism of FS-regular Lie algebroids, called *canonical*. In view of the equality $\bar{H}\circ\lambda' = \bar{\lambda}$, any homomorphism $H\colon (A',\lambda',B') \longrightarrow (A,\lambda,B)$ of FS-regular Lie algebroids can be represented in the form of a superposition of a strong homomorphism with the canonical one:

$$(A',\lambda',B') \xrightarrow{\bar{H}} (f^{\wedge}A,\bar{\lambda},f^{\wedge}B) \xrightarrow{pr_2} (A,\lambda,B).$$

4.2 Let $H:(A',\lambda',B')\longrightarrow(A,\lambda,B)$ be a homomorphism of FS-regular Lie algebroids, see diagram (10). We define the pullback

$$H^{+*}:Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*}\longrightarrow Sec\Lambda^{k}(\mathbf{g}'/\mathbf{h}')^{*}$$

by the formula

$$\langle H^{+*}(\Psi)_{x},[w'_{1}]\wedge\ldots\wedge[w'_{k}]\rangle=\langle\Psi_{f(x)},[H^{+}(w'_{1})]\wedge\ldots\wedge[H^{+}(w'_{k})]\rangle$$

where $\Psi\in Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*}$, $x\in M$, $w'_{i}\in\mathbf{g}'_{|x}$.

Proposition 4.2.1. (1). *H^{+*} maps the invariant cross-sections into the invariant ones.*

(2). *H^{+*} restricted to the invariant cross-sections commutes with the differentials $\bar{\delta}'$ and $\bar{\delta}$.*

Proof. It is enough to prove the proposition in two cases of H: of a strong homomorphism and of the canonical one.

(a). Assume that H is a strong homomorphism of FS-regular Lie algebroids over (M,E).

(1). Let $\xi'\in Sec\,B'$ and $\nu'_{j}\in Sec\,\mathbf{g}'$. Seeing diagram (10), we have

$$\begin{aligned}
&(\gamma_{B'}\circ\xi')\langle H^{+*}\Psi,[\nu'_{1}]\wedge\ldots\wedge[\nu'_{k}]\rangle\\
&=(\gamma_{B}\circ H'\circ\xi')\langle\Psi,[H^{+}\circ\nu'_{1}]\wedge\ldots\wedge[H^{+}\circ\nu'_{k}]\rangle\\
&=\sum_{j=1}^{k}\langle\Psi,[H^{+}\circ\nu'_{1}]\wedge\ldots\wedge[[\![H'\circ\xi',H^{+}\circ\nu'_{j}]\!]]\wedge\ldots\wedge[H^{+}\circ\nu'_{k}]\rangle\\
&=\sum_{j=1}^{k}\langle H^{+*}\Psi,[\nu'_{1}]\wedge\ldots\wedge[[\![\xi',\nu'_{j}]\!]]\wedge\ldots\wedge[\nu'_{k}]\rangle.
\end{aligned}$$

(2) A very easy proof of the equality $\bar{\delta}'\circ H^{+*}(\Psi)=H^{+*}\circ\bar{\delta}(\Psi)$ for an invariant Ψ will be omitted.

(b). Consider now the canonical homomorphism $pr_{2}:(f^{\wedge}A,\bar{\lambda},f^{\wedge}B)\longrightarrow(A,\lambda,B)$ of FS-regular Lie algebroids over $f:(M',E')\longrightarrow(M,E)$. Identify the vector bundles $f^{*}(\mathbf{g}/\mathbf{h})\cong f^{*}\mathbf{g}/_{f^{*}\mathbf{h}}$. Then, of course, $H^{+*}\Psi=f^{*}\Psi$, and, by the standard calculations, we assert the following equality (cf. [17; 2.3.2]):

$$f^{*}(ad_{B,\mathbf{g}})=ad_{f^{\wedge}B,f^{*}\mathbf{g}}\,. \tag{11}$$

(11) and the fact that $f^*(\wedge^k T)=\wedge^k(f^* T)$ for any representation T (cf. [17; 2.3.3]) yield

$$f^*(ad^{\wedge}_{B,\mathbf{g}})=f^*(\wedge^k(ad_{B,\mathbf{g}})^{\natural})=\wedge^k(ad_{f^\wedge B,f^*\mathbf{g}})^{\natural}=ad^{\wedge}_{f^\wedge B,f^*\mathbf{g}}.$$

Proposition **(1)** follows now from [17; 2.4.4].

To prove proposition **(2)**, it is sufficient to show that

$$<\bar{\delta}(f^*\Psi),[\nu_0\circ f]\wedge\ldots\wedge[\nu_k\circ f]>=<f^*(\bar{\delta}\Psi),[\nu_0\circ f]\wedge\ldots\wedge[\nu_k\circ f]>$$

for an invariant cross-section Ψ and $\nu_j\in Sec\mathbf{g}$.

$$\begin{aligned}&<\bar{\delta}(f^*\Psi),[\nu_0\circ f]\wedge\ldots\wedge[\nu_k\circ f]>\\&=-\sum_{i<j}(-1)^{i+j}<f^*\Psi,[[\![\nu_i\circ f,\nu_j\circ f]\!]]\wedge[\nu_0\circ f]\wedge\ldots\hat{i}\ldots\hat{j}\ldots>\\&=-\sum_{i<j}(-1)^{i+j}<\Psi,[[\![\nu_i,\nu_j]\!]]\wedge[\nu_0]\wedge\ldots\hat{i}\ldots\hat{j}\ldots>\circ f\\&=<\bar{\delta}\Psi,[\nu_0]\wedge\ldots\wedge[\nu_k]>\circ f=<f^*(\bar{\delta}\Psi),[\nu_0\circ f]\wedge\ldots\wedge[\nu_k\circ f]>.\quad\blacksquare\end{aligned}$$

4.2.2. As a corollary we obtain that H^{+*} determines a homomorphism of algebras

$$H^{+*}:H(\mathbf{g},B)\longrightarrow H(\mathbf{g}',B').$$

Proposition 4.3 (*The functoriality of Δ_**).

Let (A',λ',B') and (A,λ,B) be two FS-regular Lie algebroids over (M',E') and (M,E), respectively, and let $H:(A',\lambda',B')\longrightarrow(A,\lambda,B)$ be a homomorphism between them over $f:(M',E')\longrightarrow(M,E)$. Then the following diagram

$$\begin{array}{ccc}H(\mathbf{g},B)&\xrightarrow{\Delta_\#}&H_E(M)\\ \Big\downarrow{\scriptstyle H^{+\#}}&&\Big\downarrow{\scriptstyle f^*}\\ H(\mathbf{g}',B')&\xrightarrow{\Delta'_\#}&H_{E'}(M')\end{array}$$

commutes.

Proof. It is sufficient to show the commutativity of the diagram on the level of forms; this means - the equality:

$$(\gamma_{B'})_*(f^*(\Delta_*\Psi))=j'^*(\omega'^{\wedge}(\wedge^k\mathbf{s}'^*\circ H^{+*}\Psi))$$

for an invariant Ψ.

Let $x\in M'$ and $w_j\in B'_{|x}$. By (7) and seeing diagram (10), we have:

$$(\gamma_{B'})_*(f^*(\Delta_*\Psi))(x,w_1\wedge\ldots\wedge w_k)=(f^*(\Delta_*\Psi))(x,\gamma_{B'}(w_1)\wedge\ldots\wedge\gamma_{B'}(w_k))$$
$$=(\Delta_*\Psi)(f(x);f_*(\gamma_{B'}(w_1))\wedge\ldots\wedge f_*(\gamma_{B'}(w_k)))=(\Delta_*\Psi)(f(x);(\gamma_B(H'(w_1))\wedge\ldots\wedge(\gamma_B(H'(w_k)))$$
$$=\langle\Psi_{f(x)};[\omega(f(x);H'(w_1))]\wedge\ldots\wedge[\omega(f(x);H'(w_k))]\rangle$$
$$=\langle\Psi_{f(x)};[H^+(\omega'(x;w_1))]\wedge\ldots\wedge[H^+(\omega'(x;w_k))]\rangle$$
$$=\langle(H^{+*}\Psi)(x);[\omega'(x;w_1)]\wedge\ldots\wedge[\omega'(x;w_k)]\rangle=\langle\wedge^k\mathbf{s}'^*\circ H^{+*}\Psi(x);\omega'(x;w_1)\wedge\ldots\wedge\omega'(x;w_k)\rangle$$
$$=\ \omega'^\wedge(\wedge^k\mathbf{s}'^*\circ H^{+*}\Psi)(x;w_1\wedge\ldots\wedge w_k)=j'^*(\omega'^\wedge(\wedge^k\mathbf{s}'^*\circ H^{+*}\Psi))(x;w_1\wedge\ldots\wedge w_k).\qquad\blacksquare$$

5. THE DEPENDENCE OF $\Delta_\#$ ON A SUBALGEBROID

Let $(A,[\![\cdot,\cdot]\!],\gamma)$ be a given regular Lie algebroid with the Atiyah sequence $0\longrightarrow\mathbf{g}\hookrightarrow A\xrightarrow{\gamma}E\longrightarrow 0$ and consider the algebroid$(T\mathbb{R}\times A,[\![\cdot,\cdot]\!]',id\times\gamma)$ – the product of the trivial Lie algebroid $T\mathbb{R}$ with A [21]. Its Atiyah sequence is

$$0\longrightarrow 0\times\mathbf{g}\hookrightarrow T\mathbb{R}\times A\xrightarrow{id\times\gamma}T\mathbb{R}\times E\longrightarrow 0.$$

For the mapping $f_t:M\longrightarrow\mathbb{R}\times M,\quad x\longmapsto(t,x),$ take the pullback $f_t^\wedge(T\mathbb{R}\times A)$. Notice that $f_t^\wedge(T\mathbb{R}\times A)=\{(\gamma(w),0,w)\in E\times(T\mathbb{R}\times A);\,w\in A\}$, and that the homomorphism

$$F_t:A\longrightarrow T\mathbb{R}\times A,\quad w\longmapsto(\theta_t,w),$$

(θ_t being the null tangent vector at $t\in\mathbb{R}$) of regular Lie algebroids (see the proof of Th.4.3.1 in [17]) is represented in the form of the canonical superposition

$$F_t:A\xrightarrow{\bar{F}_t}f_t^\wedge(T\mathbb{R}\times A)\xrightarrow{pr_2}T\mathbb{R}\times A\tag{12}$$

(see [17; s.1.1]). It is not difficult to see that

5.1. $\bar{F}_t:A\longrightarrow f_t^\wedge(T\mathbb{R}\times A),\quad w\longmapsto(\gamma(w),0,w),$ *is an isomorphism of regular Lie algebroids.* ■

Definition 5.2. Two Lie algebroids $B_0,B_1\subset A$ (both over (M,E)) are said to be *homotopic* if there exists a Lie subalgebroid $B\subset T\mathbb{R}\times A$ over $(\mathbb{R}\times M,T\mathbb{R}\times E)$ such that the isomorphism $\bar{F}_t$ maps B_t onto $f_t^{\wedge}(B)$ for $t=0,1$ (equivalently, if, for $v\in A$, we have: $v\in B_t \Leftrightarrow (\theta_t, v)\in B$).

B is called *joining* B_0 to B_1.

Remarks 5.3. (1). The Lie algebra bundles adjoint of homotopic Lie subalgebroids need not be identical, see the example below.

(2). Let a Lie subalgebroid $B\subset T\mathbb{R}\times A$ join B_0 to B_1 and let $B_t := \bar{F}_t^{-1}[f_t^{\wedge}(B)]\subset A$ for $t\in\mathbb{R}$. It turns out that B is not uniquely determined by the family $\{B_t; t\in\mathbb{R}\}$, see the following example.

Example 5.4. Consider a trivial principal bundle $P=M\times G$, a C^{∞} curve $a:M\longrightarrow G$ and a closed nontrivial Lie subgroup H of G, $H\neq G$. Let $\mathfrak{h}$ and $\mathfrak{g}$ be the Lie algebras of H and G, respectively. Then, for each $t\in\mathbb{R}$, $P_t := M\times(a_t\cdot H)\subset M\times G$ is an H-reduction of P whose Lie algebroid - which is easy to obtain - equals $B_t = TM\times Ad_{a_t}[\mathfrak{h}]\subset TM\times\mathfrak{g}$. Consequently, A_t [and also its Lie algebra bundle $\mathbf{g}_t = M\times Ad_{a_t}[\mathfrak{h}]$] depends on t in general. Define a vector subbundle $B\subset T\mathbb{R}\times(TM\times\mathfrak{g})$ as follows:

$$B_{|(t,x)} = \{(u, v, R_{a_t^{-1}}(a_{*t}(u)) + Ad_{a_t}(w));\ u\in T_t\mathbb{R},\ v\in T_xM,\ w\in\mathfrak{h}\}.$$

B is a transitive Lie subalgebroid (of the product of Lie algebroids $T\mathbb{R}\times(TM\times\mathfrak{g})$) joining the family $\{B_t; t\in\mathbb{R}\}$.

If, additionally, G is abelian, then $B_t\equiv const$, but B depends on the curve a; therefore B is not uniquely determined by the family $\{B_t; t\in\mathbb{R}\}$.

5.5. We compare the relation of homotopic subbundles of a principal bundle P with the relation of homotopic subalgebroids of $A(P)$.

Let $P=(P,\pi,M,G,\cdot)$ be a G-principal bundle over a manifold M. It determines a new G-principal bundle $\mathbb{R}\times P = (\mathbb{R}\times P, id\times\pi, \mathbb{R}\times M, G, \cdot')$ with the action $(t,z)\cdot' a = (t, z\cdot a)$. For an arbitrary $t\in\mathbb{R}$, the mapping

$$\begin{aligned} F_t : P &\longrightarrow f_t^{*}(\mathbb{R}\times P) \quad \left(= M_{(f_t,\, id\times\pi)}(\mathbb{R}\times P)\right) \\ z &\longmapsto (\pi z, (t,z)) \end{aligned}$$

is an isomorphism of G-principal bundles.

Take a Lie subgroup $H\subset G$ (nonclosed and disconnected in general). Two H-reductions $P_t\subset P$, $t=0,1$, are said to be *homotopic* [10] if there exists an H-reduction $\bar{P}\subset\mathbb{R}\times P$ such that F_t maps P_t onto $f_t^{*}(\bar{P})$ for $t=0,1$. $\bar{P}$ is called *joining* P_0 to P_1.

5.5.1. Notice that $\bar{P}$ is determined uniquely by the family of H-reductions $P_t = F_t^{-1}[f_t^*(\bar{P})]$, $t \in \mathbb{R}$ [which follows from the observation: $z \in P_t \Leftrightarrow (t,z) \in \bar{P}$].

5.5.2. If H is closed and P_t are defined by C^∞ cross-sections $\sigma_t : M \longrightarrow P/H$ for $t = 0, 1$, of the associated bundle $P/H \longrightarrow M$, then, P_0 and P_1 are homotopic if and only if σ_0 and σ_1 are homotopic in the usual sense (via cross-sections, of course).

Proposition 5.5.3. *If $P_t \xhookrightarrow{i_t} P$, $t = 0,1$, are homotopic H-reductions of P, then the Lie subalgebroids $B_0 := di_0[A(P_0)]$ and $B_1 := di_1[A(P_1)]$ of $A(P)$ are homotopic. The converse theorem is not true unless P_t and G are connected.*

Proof. Let $P_0, P_1 \subset P$ be two H-reductions of P. Assume that they are homotopic, and that $\bar{P} \subset \mathbb{R} \times P$ is a joining H-reduction. Then $B := \varphi[A(\bar{P})] \subset T\mathbb{R} \times A(P)$,

$$\varphi : A(\mathbb{R} \times P) = T(\mathbb{R} \times P)/G \ni [(v,w)] \longmapsto (v, [w]) \in T\mathbb{R} \times TP/G = T\mathbb{R} \times A(P)$$

being the canonical isomorphism, is a Lie algebroid joining B_0 to B_1. Indeed, one can easily see that $\bar{F}_t : A(P) \longrightarrow f_t^\wedge(T\mathbb{R} \times A(P))$ equals the superposition

$$A(P) \xrightarrow{dF_t} A(f_t^*(\mathbb{R} \times P)) \cong f_t^\wedge(A(\mathbb{R} \times P)) \cong f_t^\wedge(T\mathbb{R} \times A(P))$$

and then maps B_t onto $f_t^\wedge(B)$ for $t = 0, 1$.

Conversely, assume that the Lie subalgebroids B_0 and B_1 are homotopic, say, via a joining Lie subalgebroid B of $T\mathbb{R} \times A(P)$. This means that $\bar{F}_t$ maps B_t onto $f_t^\wedge(B)$ for $t = 0, 1$. Let $\bar{P} \subset \mathbb{R} \times P$ be the arbitrarily taken connected H-reduction corresponding to the Lie subalgebroid $\varphi^{-1}[B] \subset A(\mathbb{R} \times P)$, see I.5.3.2. Put $\tilde{P}_t := F_t^{-1}[f_t^*(\bar{P})]$, $t \in \mathbb{R}$. By its construction, $\{\tilde{P}_t, t \in \mathbb{R}\}$ is a family of homotopic H-reductions. Of course, P_t and $\tilde{P}_t$ are, for $t = 0, 1$, two H-reductions corresponding to the same Lie subalgebroid B_t.

If P_t is connected, then, according to the fact that P_t and $\tilde{P}_t$ are integral manifolds of the same G-right invariant distribution on P (see I.5.3.2), we notice that $\tilde{P}_t = R_g[P_t]$ for a point $g \in G$. If, additionally, G is connected, g can be joined to the unit $e \in G$, say, by a C^∞ family g_s, $s \in \mathbb{R}$. The family $\tilde{P}_{t,s} := R_{g_s}[P_t]$, $s \in \mathbb{R}$, determines a homotopy between $\tilde{P}_t$ and P_t, $t = 0, 1$. Therefore P_0 and P_1 are homotopic. ■

5.6. For the further investigations, we fix

- a regular Lie algebroid $A = (A, [\![\cdot,\cdot]\!], \gamma)$ over (M,E),
- a flat connection $\lambda : E \longrightarrow A$ in it,

- two Lie subalgebroids B_0, $B_1 \subset A$, both over (M,E), homotopic to each other via a joining Lie algebroid $B \subset T\mathbb{R} \times A$.

λ determines a flat connection in $T\mathbb{R} \times A$ of the form $id \times \lambda : T\mathbb{R} \times E \longrightarrow T\mathbb{R} \times A$. This implies that the triad

$$(T\mathbb{R} \times A, id \times \lambda, B) \tag{13}$$

is an FS-regular Lie algebroid. Besides, we have that

$$F_t : (A, \lambda, B_t) \longrightarrow (T\mathbb{R} \times A, id \times \lambda, B)$$

is a homomorphism of FS-regular Lie algebroids.

Proposition 5.7. *The characteristic homomorphisms* $\Delta_{t\#}$, $t = 0, 1$, *of FS-regular Lie algebroids* (A, λ, B_t) *are related to each other by the commutativity of the following diagram*:

$$\begin{tikzcd}
H(\mathbf{g}, B_0) \arrow[dr, "\Delta_{0\#}"] & \\
H(0\times\mathbf{g}, B) \arrow[u, "F_0^{+\#}"] \arrow[d, "F_1^{+\#}"'] & H_E(M) \\
H(\mathbf{g}, B_1) \arrow[ur, "\Delta_{1\#}"'] &
\end{tikzcd} .$$

Proof. By the functoriality of the characteristic homomorphisms of FS-regular Lie algebroids, we get the commutative diagram

$$\begin{tikzcd}
H(\mathbf{g}, B_0) \arrow[r, "\Delta_{0\#}"] & H_E(M) \\
H(0\times\mathbf{g}, B) \arrow[u, "F_0^{+\#}"] \arrow[d, "F_1^{+\#}"'] \arrow[r, "\Delta_{\#}"] & H_{T\mathbb{R}\times E}(\mathbb{R}\times M) \arrow[u, "f_0^{\#}"'] \arrow[d, "f_1^{\#}"] \\
H(\mathbf{g}, B_1) \arrow[r, "\Delta_{1\#}"] & H_E(M)
\end{tikzcd}$$

where $\Delta_{\#}$ is the characteristic homomorphism of (13).

Since $f_0^{\#} = f_1^{\#}$ (see the proof of Th.4.3.1 from [17]) and $f_0^{\#}$ is an isomorphism (because f_0 and f_1 are homotopic in the category of foliated manifolds and each of them is a homotopic equivalence in this category) therefore so $f_1^{\#}=(f_0^{\#})^{-1}$, which implies our proposition. ■

Notice that if $F_t^{+\#}$, $t=0,1$, are isomorphisms, then $\Delta_{0\#}$ and $\Delta_{1\#}$ can be interpreted as *equivalence homomorphisms* in the sense of the following definition.

Definition 5.8. Let $B_0, B_1 \subset A$ be two Lie subalgebroids of a flat regular Lie algebroid A (all the three over (M,E)). We say that the characteristic homomorphisms $\Delta_{t\#}:H(\mathbf{g},B_t)\longrightarrow H(E)$, $t=0,1$, corresponding to B_0 and B_1, respectively, are *equivalent* if there exists an isomorphism of algebras $\alpha:H(\mathbf{g},B_0)\xrightarrow{\cong}H(\mathbf{g},B_1)$ such that $\Delta_{0\#}=\Delta_{1\#}\circ\alpha$.

Theorem 5.9. *If B_0 and B_1 are homotopic, then $\Delta_{0\#}$ and $\Delta_{1\#}$ are equivalent.*

Proof. Recall that $F_t=pr_2\circ\bar{F}_t$, see (12). $\bar{F}_t$ is an isomorphism of FS-regular Lie algebroids, therefore

$$\bar{F}_t^{+\#}:H(f_t^{*}(0\times\mathbf{g}),f_t^{\wedge}B)\xrightarrow{\cong}H(\mathbf{g},B_t)$$

is an isomorphism of algebras. It remains to consider the homomorphism $pr_2^{+\#}:H(0\times\mathbf{g},B)\longrightarrow H(f_t^{*}(0\times\mathbf{g}),f_t^{\wedge}B)$. Identifying (via the canonical isomorphism) the vector bundles $f_t^{*}(0\times\mathbf{g})/_{f_t^{*}\mathbf{h}}\cong f_t^{*}(0\times\mathbf{g}/\mathbf{h})$, we get (cf. the proof of Prop.5.2.1)

(i) $f_t^{\wedge}(ad^{\wedge}_{B,0\times\mathbf{g}})=ad^{\wedge}_{f_t^{\wedge}B,f_t^{*}(0\times\mathbf{g})}$,

(ii) $pr_2^{+\#}:\overset{k\geqslant 0}{\oplus}(Sec\wedge^{k}(0\times\mathbf{g}/\mathbf{h})^{*})_{I^{\circ}}\longrightarrow\overset{k\geqslant 0}{\oplus}(Sec\wedge^{k}f_t^{*}(0\times\mathbf{g}/\mathbf{h})^{*})_{I^{\circ}}$ is the usual pullback $\Psi\longmapsto f_t^{*}\Psi$.

Theorem 5.9 follows now from Th.I.6.2. ■

6. COMPARISON WITH THE CHARACTERISTIC CLASSES OF A FLAT PRINCIPAL FIBRE BUNDLE

Given:

(a) a G-principal fibre bundle $P=(P,\pi,M,G,\cdot)$,

(b) a flat connection in P with a connection form ω,

(c) a closed Lie subgroup $H\subset G$ and an H-reduction $P'\subset P$,

let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Of course, $i:P'\hookrightarrow P$ is an $(H\hookrightarrow G)$-homomorphism of principal bundles and its differential $di:A(P')\longrightarrow A(P)$, see [14], [15], [20; p.289], is a monomorphism of the corresponding transitive Lie algebroids, see the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{g} & \hookrightarrow & A(P) & \xrightarrow{\gamma} & TM & \longrightarrow & 0 \\ & & \Big\uparrow{\scriptstyle (di)^{+}} & & \Big\uparrow{\scriptstyle di} & & \Big\uparrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathbf{h} & \hookrightarrow & A(P') & \xrightarrow{\gamma'} & TM & \longrightarrow & 0 \end{array} .$$

Identify $A(P')$ with $Im(di)$ and $\mathbf{h}$ with $Im(di)^{+}$. Then, for each $z\in P'_{|x}$, the isomorphism $\hat{z}:\mathfrak{g}\longrightarrow\mathbf{g}_{|x}$, $v\longmapsto[A_{z*v}]$ $\left(A_{z}:G\longrightarrow P,\ a\longmapsto za\right)$, see [17; s.5.1], maps $\mathfrak{h}$ onto $\mathbf{h}_{|x}$ and determines an isomorphism $[\hat{z}]:\mathfrak{g}/\mathfrak{h}\xrightarrow{\cong}(\mathbf{g}/\mathbf{h})_{|x}$. It is worth recalling that

(**) *$\hat{z}$ is an isomorphism of Lie algebras provided that $\mathfrak{g}$ is the right Lie algebra of G,*

see [15], [17].

According to 3.4 above, we have a representation $ad_{A(P'),\mathbf{g}}:A(P')\longrightarrow A(\mathbf{g}/\mathbf{h})$ such that $\mathcal{L}_{ad_{A(P'),\mathbf{g}}\circ\xi}([v])=[[\![\xi,v]\!]]$, $\xi\in SecA(P')$, $v\in Sec\,\mathbf{g}$, and a representation induced by it $ad^{\wedge}_{A(P'),\mathbf{g}}:A(P')\longrightarrow A(\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*})$. Consider auxiliarily the representation $Ad_{P',\mathbf{g}}$ of the principal bundle P' on the $\mathfrak{g}/\mathfrak{h}$-vector bundle $\mathbf{g}/\mathbf{h}$, defined by $Ad_{P',\mathbf{g}}:P'\longrightarrow L(\mathbf{g}/\mathbf{h})$, $z\longmapsto[\hat{z}]$, and the representation $Ad^{\wedge}_{P',\mathbf{g}}:P'\longrightarrow L(\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*})$ induced by it (cf. [17; 5.3.2]). By the same argument as in the proof of Th.5.4.3 in [17], to see that $ad_{A(P'),\mathbf{g}}$ is the differential of $Ad_{P',\mathbf{g}}$, we must only notice an analogous fact concerning the representations of Lie algebras and of Lie groups: $\mathfrak{h}\longrightarrow End(\mathfrak{g}/\mathfrak{h})$, $v\longmapsto[ad_{\mathfrak{g}}(v)]$, and $H\longrightarrow GL(\mathfrak{g}/\mathfrak{h})$, $h\longrightarrow[Ad_{G}(h)]$. By this, $ad^{\wedge}_{A(P'),\mathbf{g}}$ is the differential of $Ad^{\wedge}_{P',\mathbf{g}}$.

Therefore, according to [17; Props 5.5.2-3], we have a monomorphism

$$\kappa:(\Lambda(\mathfrak{g}/\mathfrak{h})^{*})_{I}\xrightarrow{\cong}\overset{k\geqslant 0}{\oplus}(Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*})_{I}\subset\overset{k\geqslant 0}{\oplus}(Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*})_{I^{\circ}}$$

defined by the formula $\kappa(\psi)(x)=Ad^{\wedge}_{P',\mathbf{g}}(z)(\psi)$, $z\in P'_{|x}$, and being an isomorphism when P' is connected.

It is needful to verify that κ commutes with the differentials d^H and $\bar{\delta}$ acting on the spaces $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I$ and $\overset{k\geqslant 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}$, respectively (notice that the spaces of cohomology of these are domains of the characteristic homomorphisms). The differential $\bar{\delta}$ in $\overset{k\geqslant 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}$ is defined in 3.5 above, whereas, the differential d^H in $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I$ must be defined by the formula

$$\langle d^H(\psi),[w_1]\wedge\ldots\wedge[w_k]\rangle=\sum_{i<j}(-1)^{i+j}\langle\psi,[[w_i,w_j]]\wedge[w_1]\wedge\ldots\hat{i}\ldots\hat{j}\ldots\rangle$$

$w_1,\ldots,w_k\in\mathfrak{g}$; here $[w_i,w_j]$ is the bracket in the left Lie algebra of G [we get it following the fact that this differential must be the one for which the canonical isomorphism $G_{\Omega^*(G/H)}\cong((\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I$ (also $(\Lambda\mathfrak{g}^*)^H\cong(\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I$) should be an isomorphism of DG-algebras, see [10]].

Taking account of remark (**) above, the equality $\bar{\delta}\circ\kappa^k=\kappa^{k+1}\circ d^H$ may now be obtained immediately.

Theorem 6.1. *The characteristic homomorphisms* $\Delta_\#:H(\mathfrak{g},H)\longrightarrow H_{dR}(M)$ *of the triad* (P,P',ω) (*see* [10]) *and* $\Delta_\#:H(\mathbf{g},A(P'))\longrightarrow H_{dR}(M)$ *of the FS-transitive Lie algebroid* $(A(P),\lambda,A(P'))$ (λ *corresponds to* ω) *are related by the following commutative diagram:*

$$\begin{tikzcd}
H(\mathfrak{g},H) \arrow[dr, "\Delta_\#"] \arrow[dd, "\kappa_\#"'] & \\
& H_{dR}(M) \\
H(\mathbf{g},A(P')) \arrow[ur, "\Delta_\#"'] &
\end{tikzcd}$$

Proof. We prove the commutativity of this diagram on the level of forms. For the purpose, consider the diagram

$$\begin{tikzcd}
(\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I \arrow[dr, "\Delta"] \arrow[dd, "\kappa"] & & \\
& \Omega(M) \arrow[r, tail, "\pi'^*"] \arrow[d, "\gamma'_*", "\cong"'] & \Omega(P') \\
\overset{k\geqslant 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ} \arrow[ur, "\Delta"] \arrow[dr, "\varphi"'] & \Omega_{A(P'),i}(M) \arrow[r, hook] & \Omega_{A(P')}(M) \arrow[u, "\rho"'] \\
& \Omega_{A(P),\mathbf{h}}(M) \arrow[u, "(di)^*"'] &
\end{tikzcd}$$

in which

(a) $\varphi(\Psi)=(\omega^{A})^{\wedge}(\wedge^{k}\mathbf{s}^{*}\circ\Psi)$ $\left(=\frac{1}{k!}\cdot\langle\wedge^{k}\mathbf{s}^{*}\circ\Psi,\omega^{A}\wedge\ldots\wedge\omega^{A}\rangle\right)$, $\omega^{A}:A(P)\longrightarrow\mathbf{g}$ being the connection form corresponding to λ,

(b) $\Omega_{A(P),\mathbf{h}}(M)$ denotes the space of $\mathbf{h}$-horizontal forms on $A(P)$,

(c) ρ maps real forms on $A(P')$ into right-invariant forms on P', $\Theta\longmapsto\tilde{\Theta}$, see remark 2.11(2) above.

We recall that, for $\psi\in(\wedge^{k}(\mathfrak{g}/\mathfrak{h})^{*})_{I}$, the form $\Delta(\psi)\in\Omega^{k}(M)$ is defined uniquely in such a way that $\pi'^{*}(\Delta\psi)=\frac{1}{k!}\cdot i^{*}\langle\wedge^{k}s^{*}(\psi),\omega\wedge\ldots\wedge\omega\rangle$ where $i:P'\hookrightarrow P$, whereas $s:\mathfrak{g}\longrightarrow\mathfrak{g}/\mathfrak{h}$ and $\pi':P'\longrightarrow M$ are the canonical projections. On the other hand, $\Delta(\Psi)$ for $\Psi\in(Sec\wedge^{k}(\mathbf{g}/\mathbf{h})^{*})_{I^{\circ}}$ is given as one for which $\gamma'_{*}(\Delta(\Psi))=(di)^{*}(\varphi(\Psi))$.

Therefore, to end the proof, we need to assert the equality $i^{*}\langle\wedge^{k}s^{*}(\psi),\omega\wedge\ldots\wedge\omega\rangle=\rho\circ(di)^{*}(\langle\wedge^{k}\mathbf{s}^{*}(\kappa\psi),\omega^{A}\wedge\ldots\wedge\omega^{A}\rangle)$ only. Thanks to the relation $\omega^{A}_{|x}\circ\pi^{A}_{|z}=\hat{z}\circ\omega_{|z}$ [15; Ch.4], [15], we get, for $w_{i}\in T_{z}P'\subset T_{z}P$,

$$\begin{aligned}
&\rho\circ(di)^{*}\left(\langle\wedge^{k}\mathbf{s}^{*}(\kappa\psi),\omega^{A}\wedge\ldots\wedge\omega^{A}\rangle\right)(z;w_{1}\wedge\ldots\wedge w_{k})\\
&=\langle\wedge^{k}\mathbf{s}^{*}(\kappa\psi),\omega^{A}\wedge\ldots\wedge\omega^{A}\rangle(\pi z;\pi^{A}_{|z}(w_{1})\wedge\ldots\wedge\pi^{A}_{|z}(w_{k}))\\
&=k!\cdot(\wedge^{k}\mathbf{s}^{*}(\kappa\psi))(\pi z;\omega^{A}(\pi z;\pi^{A}_{|z}(w_{1}))\wedge\ldots\wedge\omega^{A}(\pi z;\pi^{A}_{|z}(w_{k}))\\
&=k!\cdot(\wedge^{k}\mathbf{s}^{*}(\kappa\psi))(\pi z;\hat{z}(\omega(z;w_{1}))\wedge\ldots\wedge\hat{z}(\omega(z;w_{k})))\\
&=k!\cdot\langle\wedge^{k}[\hat{z}]^{-1*}(\psi),[\hat{z}(\omega(z;w_{1}))]\wedge\ldots\wedge[\hat{z}(\omega(z;w_{k}))]\rangle\\
&=k!\cdot\langle\psi,[\omega(z;w_{1})]\wedge\ldots\wedge[\omega(z;w_{k})]\rangle\\
&=\langle\wedge^{k}s^{*}(\psi),\omega\wedge\ldots\wedge\omega\rangle(z;w_{1}\wedge\ldots\wedge w_{k})\\
&=i^{*}\langle\wedge^{k}s^{*}(\psi),\omega\wedge\ldots\wedge\omega\rangle(z;w_{1}\wedge\ldots\wedge w_{k}).\qquad\blacksquare
\end{aligned}$$

6.2. The tangential characteristic classes of a partially flat principal bundle.

Consider now Ex.3.1(2), i.e. a triple (A,B,λ) consisting of a transitive Lie algebroid A on M, a transitive Lie subalgebroid B of A and a partially flat connection λ in A, namely, flat over a given involutive distribution $F\subset TM$. The characteristic homomorphism $\Delta^{F}_{\#}:H(\mathbf{g},B^{F})\longrightarrow H_{F}(M)$ of the FS-regular Lie algebroid $(A^{F},B^{F},\lambda|F)$ will also be called the *tangential characteristic homomorphism* of the system (A,B,λ) and the cohomology classes from its image - the *tangential characteristic classes* of the system

(A,B,λ).

Let now the system (P,P',ω) be given as in Ex.3.1(3). It determines the FS-regular Lie algebroid $(A(P)^F, A(P')^F, \lambda|F)$, and via this a characteristic homomorphism

$$\Delta_{\#}^{F}: H(\mathbf{g}, A(P')^{F}) \longrightarrow H_{F}(M),$$

called the *characteristic homomorphism* of the system (P,P',ω). The cohomology classes from the image of $\Delta_{\#}^{F}$ should be called the *tangential characteristic classes* of the system (P,P',ω). By construction, they measure the independence of ω and P' - exactly the same as the exotic characteristic classes of a partially flat principal bundle [10]. To investigate this more precisely, we shall devote a separate work.

7. THE CASE OF A TC-FOLIATION

This chapter is devoted to giving a class of the FS-regular Lie algebroids coming from TC-foliations (exactly on the ground of the theory of nonclosed Lie subgroups) whose characteristic homomorphisms are not trivial.

Fix an arbitrary TC-foliation $(M,\mathcal{F})$ with the basic fibration $\pi_b: M \longrightarrow W$ and denote by $A(M,\mathcal{F}) = (A(M,\mathcal{F}), [\![\cdot,\cdot]\!], \gamma)$ its Lie algebroid; see [17; Ch.7] for notations and terminology. $A(M,\mathcal{F})$ is a transitive Lie algebroid on the basic manifold W.

A) Interpretations of various objects

In [18] there are given interpretations of a foliation of the basic manifold W and a partial connection in $A(M,\mathcal{F})$. Namely, any distribution F on the basic manifold W determines a subbundle $\bar{F} := \alpha^{-1}[\beta^{-1}[A(M,\mathcal{F})^F]]$ $\left(= \pi_{b*}^{-1}[F]\right)$ of TM where $A(M,\mathcal{F})^F = \gamma^{-1}[F]$ and

7.1 [18; 2.1.1] *The correspondence* $F \longmapsto \bar{F}$ *establishes a bijection between involutive* C^∞ *distribution on* W *and distributions* $\bar{F}$ *on* M *such that* (1) $E_b \subset \bar{F}$, (2) *the space* $Sec(\bar{F}) \cap L(M,\mathcal{F}))$ *generates at each point* $x \in M$ *the entire tangent space* $\bar{F}_{|x}$, (3) $\bar{F}$ *is involutive.* ■

Each distribution $\bar{F}$ on M satisfying conditions (1)÷(3) above is called an *involutive $\mathcal{F}$-distribution.*

By a *partial connection* over F in a transitive Lie algebroid $A=(A,[\![\cdot,\cdot]\!],\gamma)$ over M, F being an involutive distribution on M, we mean [18; 1.2.1] any linear homomorphism $\lambda: F\longrightarrow A$ such that $\gamma\circ\lambda=id_F$, i.e. any connection in the regular Lie algebroid A^F. Assume further that $A=A(M,\mathcal{F})$ as above. Let $F\subset TW$ be any involutive distribution and $\lambda: F\longrightarrow A(M,\mathcal{F})^F$ - any partial connection in $A(M,\mathcal{F})$ over F. Put $\bar{C}^\lambda:=\alpha^{-1}[\beta^{-1}[Im\lambda]]$.

7.2 [18; 2.1.2] *The correspondence $\lambda\longmapsto\bar{C}^\lambda$ establishes a bijection between partial connections in $A(M,\mathcal{F})$ over F and distributions $\bar{C}\subset TM$ such that (1) $E_b\cap\bar{C}=E$, (2) $E_b+\bar{C}=\bar{F}$ $\left(\bar{F}=\pi_{b*}^{-1}[F]\right.$, see $\left.[18; 2.1.1]\right)$, (3) $L(M,\mathcal{F})\cap Sec\bar{C}$ generates at each point $x\in M$ the entire vector space $\bar{C}_{|x}$.*

In particular, such a distribution $\bar{C}$ exists and is C^∞. A partial connection λ is flat if and only if the corresponding distribution $\bar{C}^\lambda$ is involutive. ■

Each distribution $\bar{C}$ on M satisfying (1)÷(3) above is called a *partial $\mathcal{F}$-connection over involutive $\mathcal{F}$-distribution $\bar{F}$.*

Now, we give interpretations of Lie subalgebroids, of the Lie algebroid $T\mathbb{R}\times A(M,\mathcal{F})$ and of the relation of homotopy between Lie subalgebroids.

Consider a transitive Lie subalgebroid $B\subset A(M,\mathcal{F})$. Via the family of canonical isomorphisms $\beta_x: Q_{|x}\longrightarrow A_{|\bar{x}}$, and epimorphisms $\alpha_x: T_xM\longrightarrow Q_{|x}$, $x\in M$, $\bar{x}=\pi_b(x)$, we can define a family of vector subspaces

$$\bar{B}_{|x}:=\alpha_x^{-1}[\beta_x^{-1}[B_{|\bar{x}}]]\subset T_xM,\ x\in M,$$

which constitutes a vector subbundle $\bar{B}$ of TM.

Lemma 7.3 $\left(\textbf{An interpretation of Lie subalgebroids of } A(M,\mathcal{F})\right)$. *The correspondence $B\longmapsto\bar{B}$ establishes a bijection between transitive Lie subalgebroids B of $A(M,\mathcal{F})$ and vector subbundles $\bar{B}$ of TM such that*

(1) $E\subset\bar{B}$,

(2) $E_b+\bar{B}=TM$,

(3) *the Lie algebra $Sec(\bar{B})\cap L(M,\mathcal{F})$ generates, at each point $x\in M$, the entire space $\bar{B}_{|x}$.*

The very easy proof will be omitted. ■

Each vector subbundle $\bar{B}$ of TM satisfying (1)÷(3) above will be called an *$\mathcal{F}$-subalgebroid.*

We now assert that the Lie algebroid $T\mathbb{R}\times A(M,\mathcal{F})$ is isomorphic to the Lie algebroid $A(\mathbb{R}\times M,\tilde{\mathcal{F}})$ of the foliation $(\mathbb{R}\times M,\tilde{\mathcal{F}}):=(\mathbb{R},\mathcal{F}_d)\times(M,\mathcal{F})$ being the product of the discrete foliation $\mathcal{F}_d$ of $\mathbb{R}$ with the given foliation $(M,\mathcal{F})$. First of all, we notice that the tangent bundle $\tilde{E}$ of $\tilde{\mathcal{F}}$ equals $\tilde{E}=0\times E\subset T\mathbb{R}\times TM$ $\left(=T(\mathbb{R}\times M)\right)$ and the basic fibration $\tilde{\pi}_b$ of $\tilde{\mathcal{F}}$ equals $\tilde{\pi}_b=id\times\pi_b:\mathbb{R}\times M\longrightarrow\mathbb{R}\times W$. Of course, the leaves of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_b$ through $(t,x)\in\mathbb{R}\times M$ are equal to $\tilde{L}_{(t,x)}=\{t\}\times L_x$ and $\tilde{L}_{b(t,x)}=\{t\}\times L_{bx}$, respectively. Finally, we see that $\tilde{Q}=T(\mathbb{R}\times M)_{/\tilde{E}}\cong T\mathbb{R}\times Q$.

Theorem 7.4 $\left(\textbf{An interpretation of the Lie algebroid } T\mathbb{R}\times A(M,\mathcal{F})\right)$. *If* $A(\mathbb{R}\times M,\tilde{\mathcal{F}})$ $\left(=\tilde{Q}_{/\approx}\right)$ *is the space of the Lie algebroid of the foliation* $(\mathbb{R}\times M,\tilde{\mathcal{F}})$, *then the mapping*

$$\psi: A(\mathbb{R}\times M,\tilde{\mathcal{F}})\longrightarrow T\mathbb{R}\times A(M,\mathcal{F}),\quad [(v,\overline{w})]\longmapsto(v,[\overline{w}]),$$

$v\in T\mathbb{R}$, $\overline{w}\in Q$, *is an isomorphism of Lie algebroids* $\left(\textit{for a definition of the equivalence relation } \approx, \textit{ see } [17;\ \text{s.}7.2]\right)$.

We start with the following

Lemma 7.5. *The canonical equivalence relation* $\approx$ *in* $\tilde{Q}$ *is given by*

$$(v,\overline{w})\approx(v',\overline{w}')\quad\Leftrightarrow\quad v=v' \text{ and } \overline{w}\approx\overline{w}'$$

for $v,\ v'\in T\mathbb{R}$ *and* $\overline{w},\ \overline{w}'\in Q$.

Proof of the Lemma. A real number $a\in\mathbb{R}$ and a transverse field $\zeta\in l(M,\mathcal{F})$ determine a cross-section of $\tilde{Q}$ of the form

$$\mathbb{R}\times M\ni(t,x)\longmapsto(a\cdot\frac{\partial}{\partial t}_{|t},\zeta(x))\in T\mathbb{R}\times Q\cong\tilde{Q}. \tag{14}$$

Clearly, to prove this lemma, it is sufficient to show that (14) is a transverse field for $\tilde{\mathcal{F}}$. Let $\zeta=\overline{X}$ for a foliate vector field $X\in L(M,\mathcal{F})$. We perceive that the vector field $(a\cdot\frac{\partial}{\partial t},X)$ on $\mathbb{R}\times M$ is an $\tilde{\mathcal{F}}$-foliate vector field. For the purpose, take arbitrarily a field $Y\in\mathfrak{X}(\tilde{\mathcal{F}})$. Obviously, Y is tangent to the submanifold $\{t\}\times M$ for each $t\in\mathbb{R}$, and $Y|\{t\}\times M$ is tangent to the foliation $(\{t\}\times M,\{\{t\}\times L;\ L\in\mathcal{F}\})$. Write

$$[(a\cdot\frac{\partial}{\partial t},X),Y]=[(0,X,),Y]+[(a\cdot\frac{\partial}{\partial t},0),Y].$$

The field $[(0,X,),Y]|\{t\}\times M=[(0,X)|\{t\}\times M,Y|\{t\}\times M]$ is tangent to $\tilde{\mathcal{F}}$ because X is foliate. To investigate the second component, take any simply distinguished open set $U\subset M$ equipped with distinguished local coordinates $(x^1,\dots,x^p,y^1,\dots,y^q)$ for $\mathcal{F}$. It is evident that $(x^1,\dots,x^p,t,y^1,\dots,y^q)$ are distinguished local coordinates for $\tilde{\mathcal{F}}$ on $\mathbb{R}\times U$. Therefore $Y|\mathbb{R}\times U=\sum_i a^i(x,t,y)\cdot\frac{\partial}{\partial x^i}$ and, by this equality, the field

$[(a\cdot\frac{\partial}{\partial t},0),Y]|\mathbb{R}\times U$ equals $[a\cdot\frac{\partial}{\partial t},\sum_i a^i(x,t,y)\cdot\frac{\partial}{\partial x^i}]=a\cdot\frac{\partial a^i}{\partial t}(x,t,y)\cdot\frac{\partial}{\partial x^i}$, thus it is tangent to $\tilde{\mathcal{F}}_{\mathbb{R}\times U}$. This gives that $[(a\cdot\frac{\partial}{\partial t},X),Y]$ is tangent to $\tilde{\mathcal{F}}$, and that $(a\cdot\frac{\partial}{\partial t},X)$ is $\tilde{\mathcal{F}}$-foliate. The fact that (14) is a transverse field is implied now immediately. □

The lemma above sets up that ψ is an isomorphism of vector bundles.

Proof of Th.7.4. Since the anchor $\tilde{\gamma}:A(\mathbb{R}\times M,\tilde{\mathcal{F}})\longrightarrow T\mathbb{R}\times TW$ is defined by $\tilde{\gamma}([(v,\bar{w})])=(v,\gamma([\bar{w}]))$, $v\in T\mathbb{R}$, $\bar{w}\in Q$, we see that the diagram

$$\begin{tikzcd} A(\mathbb{R}\times M,\tilde{\mathcal{F}}) \arrow[rr, "\psi"] \arrow[dr, "\tilde{\gamma}"'] & & T\mathbb{R}\times A \arrow[dl, "id\times\gamma"] \\ & T\mathbb{R}\times W & \end{tikzcd}$$

commutes. To prove that $Sec\,\psi:Sec\,A(\mathbb{R}\times M,\tilde{\mathcal{F}})\longrightarrow Sec(T\mathbb{R}\times A(M,\mathcal{F}))$ is a homomorphism of Lie algebras, it is sufficient to show that the following mapping

$$\kappa:L(\mathbb{R}\times M,\tilde{\mathcal{F}})\longrightarrow Sec(T\mathbb{R}\times A(M,\mathcal{F})),\quad X\longmapsto\psi\circ c(\bar{X}),$$

is such a homomorphism $\big(c:l(\mathbb{R}\times M,\tilde{\mathcal{F}})\longrightarrow Sec\,A(\mathbb{R}\times M,\tilde{\mathcal{F}})$ is an isomorphism described in [17; Prop.7.2.2]$\big)$. First of all, we observe that a vector field $X\in\mathfrak{X}(\mathbb{R}\times M)$ is $\tilde{\mathcal{F}}$-foliate if and only if $X=f\cdot\frac{\partial}{\partial t}+X_o$ for an $\tilde{\mathcal{F}}$-basic function f and $X_o\in\mathfrak{X}(\mathbb{R}\times M)$ such that $X_o(t,\cdot)\in\mathfrak{X}(M)$ is $\mathcal{F}$-foliate. Let $X=f\cdot\frac{\partial}{\partial t}+X_o$ be an $\tilde{\mathcal{F}}$-foliate vector field. Then

$$\psi\circ c(\bar{X})(t,\bar{x})=\bar{f}(t,\bar{x})\cdot\frac{\partial}{\partial t}_{|t}+c(\overline{X_o(t,\cdot)})(\bar{x})$$

$\big($where $\bar{f}\in\Omega^o(\mathbb{R}\times W)$ is a function such that $f=\bar{f}\circ\tilde{\pi}_b\big)$. Since, for $X=f\cdot\frac{\partial}{\partial t}+X_o$ and $Y=g\cdot\frac{\partial}{\partial t}+Y_o$ belonging to $L(\mathbb{R}\times M,\tilde{\mathcal{F}})$, we have,

$$\begin{aligned}[X,Y] &= [f\cdot\frac{\partial}{\partial t}+X_o,g\cdot\frac{\partial}{\partial t}+Y_o]\\ &=\big(f\cdot\frac{\partial g}{\partial t}-g\cdot\frac{\partial f}{\partial t}+X_o(g)-Y_o(f)\big)\cdot\frac{\partial}{\partial t}+g\cdot[X_o,\frac{\partial}{\partial t}]-f\cdot[Y_o,\frac{\partial}{\partial t}]+[X_o,Y_o]\end{aligned}$$

after taking account of the equalities

$$[X_o,\frac{\partial}{\partial t}](t,x)=-\frac{\partial}{\partial t}(X_o(t,x)),$$

$$\overline{X_o(g)}=(\gamma\circ\psi\circ c(\overline{X_o}))(\bar{g}),$$

$$\psi\circ c(\overline{[X_o,Y_o]})(t,\bar{x})=[\![c(\overline{X_o(t,\cdot)}),c(\overline{Y_o(t,\cdot)})]\!](\bar{x}),$$

we get

$$\kappa([X,Y]) = \psi\circ c(\overline{[X,Y]}) = \Big(\bar f\cdot\frac{\partial\bar g}{\partial t} - \bar g\cdot\frac{\partial \bar f}{\partial t} + (\gamma\circ\psi\circ c(\overline{X_o})(\bar g) - (\gamma\circ\psi\circ c(\overline{Y_o})(\bar f)\Big)\cdot\frac{\partial}{\partial t}$$

$$- \bar g\cdot\frac{\partial}{\partial t}(\psi\circ c(\overline{X_o}) + \bar f\cdot\frac{\partial}{\partial t}(\psi\circ c(\overline{Y_o}) + [\![\psi\circ c(\overline{X_o}),\psi\circ c(\overline{Y_o})]\!]$$

$$= [\![\psi\circ c(\bar X),\psi\circ c(\bar Y)]\!] = [\![\kappa(X),\kappa(Y)]\!],$$

according to the definition of the bracket in the Lie algebra $Sec(T\mathbb{R}\times A(M,\mathcal{F}))$ [21]. ■

Let $B_0, B_1 \subset A(M,\mathcal{F})$ be two Lie subalgebroids of $A(M,\mathcal{F})$. Denote by $\bar B_0$, $\bar B_1 \subset TM$ the vector subbundles of TM for $\mathcal{F}$ corresponding to B_0, B_1, respectively (see Lemma 7.3). We recall (Def.5.2) that B_0 is homotopic to B_1 if and only if there exists a transitive Lie subalgebroid $B\subset T\mathbb{R}\times M(M,\mathcal{F})$ such that $v\in B_t \Leftrightarrow (\theta_t, v)\in B$ for $t=0,1$. The following proposition is a simple consequence of the definitions.

Proposition 7.6 (An interpretation of the relation of homotopy between Lie subalgebroids). *B_0 and B_1 are homotopic if and only if there exists an involutive subbundle $\bar B\subset T\mathbb{R}\times TM$ such that*

(1) $0\times E\subset \bar B$,

(2) $\bar B + (0\times E_b) = T\mathbb{R}\times TM$,

(3) *the Lie algebra $Sec\bar B\cap L(\mathbb{R}\times M,\tilde{\mathcal{F}})$ generates at each point $(t,x)\in\mathbb{R}\times M$ the entire space $\bar B_{|(t,x)}$,*

(4) $v\in \bar B_t \Leftrightarrow (\theta_t, v)\in \bar B$ *for* $t=0,1$. ■

B) The characteristic homomorphism of a partially flat Lie algebroid of a TC-foliation

In this section we describe in the language of a TC-foliation $(M,\mathcal{F})$ the characteristic homomorphism of a flat regular [in particular, transitive] Lie algebroid of the form $(A(M,\mathcal{F})^F, B^F, \lambda|F)$, where, $B\subset A(M,\mathcal{F})$ is a transitive Lie subalgebroid of A, F is an involutive distribution on the basic manifold W, and λ is a connection in $A(M,\mathcal{F})$ whose part lying over the distribution F is flat. Denote by γ^F and γ_1^F the anchors in $A(M,\mathcal{F})^F$ and in B^F, respectively, whereas by ω^F the connection form of $\lambda|F: F\longrightarrow A(M,\mathcal{F})^F$, being de facto the restriction of the connection form $\omega: A(M,\mathcal{F})\longrightarrow \mathbf{g}$ of λ.

By $(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I_F^\circ}$ we denote the space of cross-sections of $\Lambda^k(\mathbf{g}/\mathbf{h})^*$ invariant with respect to a suitable representation of the regular Lie algebroid B^F, see Ch.3;

this means that $\Gamma\in(Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*})_{I_F^\circ}$ if and only if $\Gamma\in Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*}$ and

$$(\gamma\circ\xi)\langle\Gamma,[\nu_1]\wedge\ldots\wedge[\nu_k]\rangle=\sum_{j=1}^{k}\langle\Gamma,[\nu_1]\wedge\ldots\wedge[[\![\xi,\nu_j]\!]]\wedge\ldots\wedge[\nu_k]\rangle$$

for $\xi\in Sec(B^F)$, $\nu_j\in Sec\,\mathbf{g}$.

Now, recall that the characteristic homomorphism of the flat regular Lie algebroid $(A(M,\mathcal{F})^F,B^F,\lambda|F)$ is - on the level of forms - given by the formula

$$\Delta_*^F:\overset{k\geqslant 0}{\oplus}(Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*})_{I_F^\circ}\longrightarrow\Omega_F(W)$$

$$\langle(\Delta_*^F\Gamma)(\bar{x};w_1\wedge\ldots\wedge w_k)\rangle=\langle\Gamma(\bar{x}),[\omega^F(\bar{x};\tilde{w}_1)]\wedge\ldots\wedge[\omega^F(\bar{x};\tilde{w}_k)]\rangle$$

where $\bar{x}\in W$ and $w_i\in F_{|\bar{x}}$, while $\tilde{w}_i\in B^F_{|\bar{x}}$ are vectors such that $\gamma_1^F(\tilde{w}_i)=w_i$. Next, we recall that Δ_*^F commutes with differentials and gives rise to a homomorphism of algebras $\Delta_\#^F:H(\mathbf{g},B^F)\longrightarrow H_F(W)$. The homomorphism $\Delta_\#^F$ vanishes if the Lie subalgebroid B^F can be homotopically changed to one which contains $Im(\lambda|F)$.

C) The case of a TC-foliation of left cosets of a Lie group

Here we give a more detailed description of the examined homomorphism $\Delta_\#$ of the Lie algebroid $A(G;H)$ of the TC-foliation $(G,\mathcal{F})$ of a connected Lie group G by left cosets of a connected nonclosed Lie subgroup $H\subset G$. The Lie algebroid $A(G;H)$ was precisely examined in the works by the author [17], [16]. In [18] there are given interpretations of conditions (3) from 7.1.and 7.2 above to that $\bar{F}$ and $\bar{C}$ are $\bar{H}$-right-invariant.

Proposition 7.7 (**An interpretation of transitive Lie subalgebroids of** $A(G;H)$). *A necessary and sufficient condition for an involutive* C^∞ *distribution* $\bar{B}$ *on* G *to be an* $\mathcal{F}$*-subalgebroid is the realization of the conditions*: (1) $E\subset\bar{B}$, (2) $E_b+\bar{B}=TG$, (3) $\bar{B}$ *is* $\bar{H}$*-right-invariant* [*i.e.* $\bar{B}_{|gt}=R_t[\bar{B}_{|g}]$, $g\in G$, $t\in\bar{H}$]. ■

The proof of this Proposition, being analogous to that for Prop.7.3.1 from [17], will be omitted.

Example 7.8. Let $\mathfrak{h}$, $\bar{\mathfrak{h}}$, $\mathfrak{g}$ denote, as usual, the Lie algebras of H, $\bar{H}$ and G, respectively. Let $\mathfrak{b}\subset\mathfrak{g}$ be a Lie subalgebra such that (1) $\mathfrak{h}\subset\mathfrak{b}$, (2) $\bar{\mathfrak{h}}+\mathfrak{b}=\mathfrak{g}$, then, by the same argument as in example 7.4.7 from [17], we assert that the G-left-invariant distribution $\bar{B}_{\mathfrak{b}}\subset TG$ determined by $\mathfrak{b}$ (i.e. the one tangent to the foliation $\{gF;\,g\in G\}$, F being the connected Lie subgroup with the Lie algebra equalling $\mathfrak{b}$) is a transitive $\mathcal{F}$-subalgebroid.

It seems to be interesting that $\mathfrak{b}$ can be interpreted as a "connection", but in another Lie algebroid. Namely, let H_1 be the connected Lie subgroup of G whose Lie algebra equals $\mathfrak{b}\cap\bar{\mathfrak{h}}$. Of course, $\mathfrak{h}\subset\mathfrak{b}\cap\bar{\mathfrak{h}}\subset\bar{\mathfrak{h}}$, therefore $H\subset H_1\subset\bar{H}$, thereby $\bar{H}_1=\bar{H}$. Then, it is clear (see [17; Ex.8.4.7]) that $\bar{B}_{\mathfrak{b}}$ is an $\mathcal{F}_1$-connection where $\mathcal{F}_1$ is the foliation of left cosets of G by H_1.

7.9 (An interpretation of the Lie algebroid $T\mathbb{R}\times A(G;H)$). Seeing that the foliation $(\mathbb{R}\times G,\tilde{\mathcal{F}}):=(\mathbb{R},\mathcal{F}_d)\times(G,\mathcal{F})$ is equal to the foliation of the Lie group $\mathbb{R}\times G$ (being the product of the additive Lie group of reals, with G) by left cosets of a Lie subgroup $\theta\times H$, θ being the null Lie subgroup $\theta=\{0\}$ of $\mathbb{R}$, we assert that the Lie algebroid $T\mathbb{R}\times A(G;H)$ is isomorphic - according to Th.7.4 - to the Lie algebroid $A(\mathbb{R}\times G;\theta\times H)$.

7.10 (An interpretation of the relation of homotopy between transitive Lie subalgebroids of $A(G;H)$). Assume that $\bar{B}_0$, $\bar{B}_1\subset TG$ are two transitive $\mathcal{F}$-subalgebroids and let $\bar{B}\subset T\mathbb{R}\times TG$ $(=T(\mathbb{R}\times G))$ be a transitive $\tilde{\mathcal{F}}$-subalgebroid joining $\bar{B}_0$ to $\bar{B}_1$. Thanks to Prop. 7.7, we may equivalently change condition (3) from 7.6 above - assuming that $\bar{B}$ is a C^∞ subbundle - to

(3′) $\bar{B}$ is $\theta\times\bar{H}$-right-invariant.

Definition 7.11. Two Lie subalgebras $\mathfrak{b}_t\subset\mathfrak{g}$, $t=0,1$, fulfilling

$$\mathfrak{h}\subset\mathfrak{b}_t \quad\text{and}\quad \bar{\mathfrak{h}}+\mathfrak{b}_t=\mathfrak{g} \tag{15}$$

for $t=0,1$ will be called *homotopic* if the corresponding transitive Lie subalgebroids $\bar{B}_{\mathfrak{b}_0}$ and $\bar{B}_{\mathfrak{b}_1}$ are homotopic.

Exercise 7.12. We present some sufficient conditions for two Lie subalgebras to be homotopic. Consider $T\mathbb{R}\times\mathfrak{g}$ as a trivial Lie algebroid on $\mathbb{R}$.

(1) *Assume that two Lie subalgebras $\mathfrak{b}_t\subset\mathfrak{g}$, $t=0,1$, fulfilling (15) for $t=0,1$ are given. If there exists a transitive Lie subalgebroid $\bar{B}_o\subset T\mathbb{R}\times\mathfrak{g}$ such that*

(i) *its isotropy Lie algebras $\mathfrak{b}_t$ fulfil (15) for each $t\in\mathbb{R}$,*

(ii) *$(id\times Ad(h))[\bar{B}_{o|t}]=\bar{B}_{o|t}$ for $t\in\mathbb{R}$ and $h\in H$,*

then $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are homotopic.

(2) *Let two Lie subalgebras $\mathfrak{b}_0$ and $\mathfrak{b}_1$ of $\mathfrak{g}$ fulfilling (15) for $t=0,1$ be given. Then they are homotopic if there exists a C^∞ vector subbundle* **b** *of the trivial vector bundle $\mathbb{R}\times\mathfrak{g}$ over $\mathbb{R}$, such that*

(i) *the fibre* $\mathbf{b}_{|t}$ *is a Lie subalgebra of* $\mathfrak{g}$ *fulfilling* (15) *for each* $t\in\mathbb{R}$,

(ii) $\mathbf{b}_{|t}=\mathfrak{b}_t$ *for* $t=0,1$,

(iii) *there exists a* C^∞ *mapping* $\nu:\mathbb{R}\longrightarrow\mathfrak{g}$ *realizing the conditions*:

(1°) $-\dfrac{\partial\mu}{\partial t}+[\mu,\nu]\in Sec\,\mathbf{b}$ *for each* $\mu\in Sec\,\mathbf{b}$,

(2°) $Ad(h)\circ\nu-\nu\in Sec\,\mathbf{b}$ *for each* $h\in H$.

In spite of these two propositions, the problem of the finding of two different but homotopic Lie subalgebras is open. This is, however, a side problem.

7.13 (**The characteristic homomorphism for a transitive case**). In this section we calculate the characteristic homomorphism of the FS-transitive Lie algebroid $(A(G;H),B,\lambda)$ in which

(i) $B=B_{\mathfrak{b}}$ is the Lie subalgebroid of $A(G;H)$ determined by a Lie subalgebra $\mathfrak{b}\subset\mathfrak{g}$ fulfilling (1) $\mathfrak{h}\subset\mathfrak{b}$, (2) $\bar{\mathfrak{h}}+\mathfrak{b}=\mathfrak{g}$, see Ex.7.8 above,

(ii) λ is the flat connection determined by a Lie subalgebra $\mathfrak{c}\subset\mathfrak{g}$ fulfilling (1) $\mathfrak{c}+\bar{\mathfrak{h}}=\mathfrak{g}$, (2) $\mathfrak{c}\cap\bar{\mathfrak{h}}=\mathfrak{h}$, see Example 7.4.7 from [17].

(According to [16] for such a Lie subalgebra $\mathfrak{c}$ to exist, $\pi_1(G)$ must be infinite).

Denote by γ_A and γ_B the anchors in $A(G;H)$ and in B, respectively.

7.13.A (**The domain of the characteristic homomorphism** $\Delta_\#$). Recall that [17; 8.2.4] $\varphi:G/\bar{H}\times\bar{\mathfrak{h}}/\mathfrak{h}\longrightarrow\mathbf{g}$, $(\bar{g},[w])\longmapsto[\bar{X}_w(g)]$, $g\in\pi_b^{-1}(\bar{g})$, is a global trivialization of the Lie algebra bundle $\mathbf{g}$ of $A(G;H)$, and that the typical fibre $\bar{\mathfrak{h}}/\mathfrak{h}$ of this bundle is an abelian Lie algebra [X_w stands for the left-invariant vector field on G generated by a vector w], [17: 8.1.3]. The equalities $\mathbf{h}=\mathbf{g}\cap B$ and $dim(\bar{\mathfrak{h}}\cap\mathfrak{b}/\mathfrak{h})=rank\,\mathbf{h}$ yield that φ induces a global trivialization $\varphi^1:G/\bar{H}\times\bar{\mathfrak{h}}/\mathfrak{h}\longrightarrow\mathbf{h}$ of the bundle $\mathbf{h}$. Next, φ and φ^1 give a global trivialization $\varphi^2:G/\bar{H}\times\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b})\longrightarrow\mathbf{g}/\mathbf{h}$ of the bundle $\mathbf{g}/\mathbf{h}$. Using φ^2, we can modify

(a) any cross-section $\nu\in Sec\,\mathbf{g}/\mathbf{h}$ to the $\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b})$-valued function

$$\tilde{\nu}:G/\bar{H}\longrightarrow\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b}),$$

(b) analogously, via the canonically induced global isomorphism $\bigwedge^k(\mathbf{g}/\mathbf{h})^*\cong G/\bar{H}\times\bigwedge^k(\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b}))^*$ – any cross-section $\Psi\in Sec\bigwedge^k(\mathbf{g}/\mathbf{h})^*$ to the function $\tilde{\Psi}:G/\bar{H}\longrightarrow\bigwedge^k(\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b}))^*$.

One can easily see that

$$\langle\Psi,[c(\bar{X}_{w_1})]\wedge\ldots\wedge[c(\bar{X}_{w_k})]\rangle(\bar{g})=\langle\tilde{\Psi}(g),[w_1]\wedge\ldots\wedge[w_k]\rangle$$

for $\bar{g}\in G/\bar{H}$ and $w_i\in\bar{\mathfrak{h}}$; here $[c(\bar{X}_{w_i})]$ denotes the cross-section of $\mathbf{g}/\mathbf{h}$ determined by $c(\bar{X}_{w_i})\in Sec\,\mathbf{g}$, i.e. $[c(\bar{X}_{w_i})]=\mathbf{s}\circ c(\bar{X}_{w_i})$, see 3.1 above.

Analogously to the proof of Prop.7.4.1 from [17] we assert that:

— *Let* $\Psi\in Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*$. *Then* Ψ *is invariant* [*see* 3.3-4 *above*] *if and only if* $\tilde{\Psi}$ *is constant.*

As a corollary we obtain that

$$\chi\colon \overset{k\geqslant 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}\xrightarrow{\cong}\Lambda(\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b}))^*,\qquad(16)$$
$$\psi\longmapsto\hat{\Psi}\ \ (=\text{the value of }\tilde{\Psi})$$

is an isomorphism of algebras.

Notice also that

$$<\hat{\Psi},[w_1]\wedge\ldots\wedge[w_k]>=<\Psi(\bar{g}),[\varphi_{\bar{g}}([w_1])]\wedge\ldots\wedge[\varphi_{\bar{g}}([w_k])]\qquad(17)$$

for an arbitrary $\bar{g}\in G/\bar{H}$.

In the space $\overset{k\geqslant 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}$, the differential $\bar{\delta}$ defined in 3.4 above works. Via χ we can carry $\bar{\delta}$ over to the space $\Lambda(\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b}))^*$ and obtain a differential $\bar{\bar{\delta}}$.

We can easily obtain that $\bar{\bar{\delta}}=0$ [*hence* $\bar{\delta}=0$]. For the purpose, take $\hat{\Psi}\in\Lambda^k(\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b}))^*$ and let $\hat{\Psi}=\chi(\Psi)$ for $\Psi\in(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}$. For $w_0,\ldots,w_k\in\bar{\mathfrak{h}}$, we have, by (17),

$$<\bar{\bar{\delta}}\hat{\Psi},[w_0]\wedge\ldots\wedge[w_k]>=<(\bar{\delta}\Psi)^{\wedge},[w_0]\wedge\ldots\wedge[w_k]>=<\bar{\delta}\Psi,[c(\bar{X}_{w_0})]\wedge\ldots\wedge[c(\bar{X}_{w_k})]>(\bar{g})$$
$$=-\sum_{i<j}(-1)^{i+j}<\Psi,[[\![c(\bar{X}_{w_i}),c(\bar{X}_{w_j})]\!]]\wedge[c(\bar{X}_{w_0})]\wedge\ldots\hat{i}\ldots\hat{j}\ldots>(\bar{g})$$
$$=-\sum_{i<j}(-1)^{i+j}<\Psi,[c(\bar{X}_{[w_i,w_j]})]\wedge\ldots\hat{i}\ldots\hat{j}\ldots>(\bar{g})$$
$$=-\sum_{i<j}(-1)^{i+j}<\hat{\Psi},[[w_i,w_j]]\wedge\ldots\hat{i}\ldots\hat{j}\ldots>$$
$$=0$$

because $[w_i,w_j]\in\mathfrak{h}\subset\mathfrak{b}\cap\bar{\mathfrak{h}}$ [$\bar{\mathfrak{h}}/\mathfrak{h}$ is abelian!]. As a corollary we obtain an isomorphism of algebras

$$H(\mathbf{g},B) \cong H(\Lambda(\bar{\mathfrak h}/(\bar{\mathfrak h}\cap\mathfrak b))^*,0)=\Lambda(\bar{\mathfrak h}/(\bar{\mathfrak h}\cap\mathfrak b))^*$$

and the fact that the forms from $Im\Delta_*$ are closed.

7.13.B (**The characteristic homomorphism**). Take into account the connection λ determined by the G-left-invariant distribution $\bar B\subset TG$ generated by $\mathfrak c$, see 7.13. Let ω be its connection form. The conditions $\mathfrak c+\bar{\mathfrak h}=\mathfrak g$ and $\mathfrak c\cap\bar{\mathfrak h}=\mathfrak h$ determine a decomposition $\mathfrak g/\mathfrak h=\bar{\mathfrak h}/\mathfrak h\oplus\mathfrak c/\mathfrak h$. Define $\omega_o:\mathfrak g\longrightarrow\bar{\mathfrak h}/\mathfrak h$ as the linear mapping being the superposition

$$\omega_o:\mathfrak g\longrightarrow\mathfrak g/\mathfrak h=\bar{\mathfrak h}/\mathfrak h\oplus\mathfrak c/\mathfrak h\xrightarrow{pr_1}\bar{\mathfrak h}/\mathfrak h.$$

Take also the canonical linear homomorphism $\rho:\bar{\mathfrak h}/\mathfrak h\longrightarrow\bar{\mathfrak h}/(\bar{\mathfrak h}\cap\mathfrak b)$ and put

$$\omega_1=\rho\circ\omega_o:\mathfrak g\longrightarrow\bar{\mathfrak h}/(\bar{\mathfrak h}\cap\mathfrak b).$$

Let $L_g:TG\longrightarrow TG$ denote [as usual] the differential of the left translation by the element $g\in G$. Since the left translation by g is an automorphism of the foliation $\mathcal F_b=\{a\bar H;\ a\in G\}$, therefore L_b determines an automorphism $\bar L_g$ of the vector bundle $T(G/\bar H)$. Identify canonically $T_{\bar e}(G/\bar H)$ with $\mathfrak g/\bar{\mathfrak h}$. In particular, we have a linear isomorphism $\bar L_g:\mathfrak g/\mathfrak h\longrightarrow T_{\bar g}(G/\bar H)$. Without any substantial difficulties one can obtain the commutativity of the following diagram

$$\begin{tikzcd}
 & \mathbf{g}_{|\bar g} & A(G;H)_{|\bar g} \arrow[l, "\omega_{|\bar g}"'] & B_{|\bar g} \arrow[l, hook'] \arrow[r, "(\gamma_B)_{|\bar g}"] & T_{\bar g}(G/\bar H) \\
 & & T_gG \arrow[u, "(\beta\circ\alpha)_{|g}"] & \bar B_{|g} \arrow[l, hook'] \arrow[u, "(\beta\circ\alpha)_{|g}"'] \arrow[ur, "\pi_{b*g}"'] & \\
\bar{\mathfrak h}/(\bar{\mathfrak h}\cap\mathfrak b) & \bar{\mathfrak h}/\mathfrak h \arrow[l, "\rho"'] \arrow[uu, "\varphi_{|g}"] & \mathfrak g \arrow[l, "\omega_o"'] \arrow[u, "L_g"'] \arrow[ll, bend left, "\omega_1"] & \mathfrak b\subset\mathfrak g \arrow[l, hook'] \arrow[u, "L_g"'] \arrow[r] \arrow[r, bend right, "\text{epimorphism}"'] & \mathfrak g/\bar{\mathfrak h} \arrow[uu, "\bar L_g"']
\end{tikzcd}$$

for an arbitrary element $g\in G$.

Recall that $\Delta_*(\Psi)=\Delta\Psi\in\Omega(G/\bar H)$ is, for $\Psi\in(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ}$, defined by formula (8) (see 3.2 above).

Lemma 7.13.B.1. *$\Delta_*(\Psi)$ is, for Ψ as above, a G-left-invariant form on $G/\bar H$ (i.e. $\Delta_*(\Psi)\in\Omega_I(G/\bar H)$ under the notation of [8]) such that its value $\Delta_*(\Psi)_{\bar e}\in\Lambda^k(\mathfrak g/\bar{\mathfrak h})$ at $\bar e$ is* *equal to*

$$\langle \Delta_*(\Psi)_{\bar{e}}, [w_1]\wedge\ldots\wedge[w_k]\rangle = \langle\hat{\Psi}, \omega_1(\tilde{w}_1)\wedge\ldots\wedge\omega_1(\tilde{w}_k)\rangle \tag{18}$$

where $\tilde{w}_i \in \mathfrak{b}$ *are vectors such that* $[\tilde{w}_i]=[w_i]$ $\left(\in \mathfrak{g}/\bar{\mathfrak{h}}\right)$.

Proof. We prove the equality

$$\langle \Delta_*(\Psi)_{\bar{g}}, \bar{L}_g([w_1])\wedge\ldots\wedge\bar{L}_g([w_k])\rangle = \langle\hat{\Psi}, \omega_1(\tilde{w}_1)\wedge\ldots\wedge\omega_1(\tilde{w}_k)\rangle,$$

$g\in G$, which, in particular, implies (18) as well as the equality $((\bar{L}_g)^*(\Delta_*(\Psi)))_{\bar{e}} = \Delta_*(\Psi)_{\bar{e}}$. This last implies, of course, the G-left-invariance of $\Delta_*(\Psi)\in\Omega(G/\bar{H})$.

At first, we notice (see the diagram above) that if $v=\bar{L}_g([w])$ $\left(\in T_{\bar{g}}(G/\bar{H})\right)$ for $w\in\mathfrak{g}$, then, for $\tilde{v}\in B_{|\bar{g}}$ fulfilling $(\gamma_B)(\tilde{v})=v$, we can put $\tilde{v}:=[L_g(\tilde{w})]$, i.e. $\tilde{v}:=\beta\circ\alpha(L_g(\tilde{w}))$ where $\tilde{w}\in\mathfrak{b}$ is a vector such that $[\tilde{w}]=[w]$ $\left(\in\mathfrak{g}/\bar{\mathfrak{h}}\right)$. Therefore, according to the diagram above and equality (17), we have

$$\begin{aligned}
&\langle \Delta_*(\Psi)_{\bar{g}}, \bar{L}_g([w_1])\wedge\ldots\wedge\bar{L}_g([w_k])\rangle \\
&= \langle \Psi_{\bar{g}}, [\omega(\bar{g};\overline{[L_g(\tilde{w}_1)]})]\wedge\ldots\wedge[\omega(\bar{g};\overline{[L_g(\tilde{w}_k)]})]\rangle \\
&= \langle\hat{\Psi}, \omega_1(\tilde{w}_1)\wedge\ldots\wedge\omega_1(\tilde{w}_k)\rangle. \quad \blacksquare
\end{aligned}$$

Corollary 7.13.B.2. There exists a homomorphism of algebras $\hat{\Delta}_*$ making the following diagram commutative:

$$\begin{tikzcd}
H(\mathbf{g},B) = \overset{k\geq 0}{\oplus}(Sec\Lambda^k(\mathbf{g}/\mathbf{h})^*)_{I^\circ} \arrow[r] \arrow[rr, bend left, "\Delta_*"] & \Omega_I(G/\bar{H}) \subset & \Omega(G/\bar{H}) \\
\Lambda(\bar{\mathfrak{h}}/(\bar{\mathfrak{h}}\cap\mathfrak{b})) \arrow[u, "\cong"'] \arrow[r, "\hat{\Delta}_*"] & (\Lambda(\mathfrak{g}/\bar{\mathfrak{h}}))_I \arrow[u, "\cong"', "L"] &
\end{tikzcd}$$

$(\Lambda(\mathfrak{g}/\bar{\mathfrak{h}})^*)_I$ denotes here the DG-algebra of vectors invariant with respect to the adjoint representation $Ad^{\perp}:\bar{H}\longrightarrow GL(\Lambda(\mathfrak{g}/\bar{\mathfrak{h}})^*)$, see [8; PropXI]. The forms from $Im\hat{\Delta}_*$ are closed and $\hat{\Delta}_*$ is defined by the equality

$$\langle\hat{\Delta}_*(\hat{\Psi}), [w_1]\wedge\ldots\wedge[w_k]\rangle = \langle\hat{\Psi}, \omega_1(\tilde{w}_1)\wedge\ldots\wedge\omega_1(\tilde{w}_k)\rangle \tag{19}$$

for $\hat{\Psi}\in\Lambda^k(\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{b}))^*$ and $w_i\in\mathfrak{g}$, where $\tilde{w}_i\in\mathfrak{b}$ are vectors such that $[\tilde{w}_i]=[w_i]$ $(\in\mathfrak{g}/\mathfrak{h})$. ■

Put $\hat{\Delta}_\#$ as the superposition

$$\hat{\Delta}_\#:\Lambda(\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{b}))^*\longrightarrow Z((\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I)\longrightarrow H((\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I)\ \left(\cong H_I(G/\bar{H})\right).$$

From the above we obtain the fundamental (for the situation considered) diagram

$$\begin{tikzcd}
H(\mathbf{g},B) \arrow[r, "\Delta_\#"] & H_{dR}(G/\bar{H}) & \\
\Lambda(\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{b}))^* \arrow[u, "\cong"] \arrow[r, "\hat{\Delta}_\#"] & H(\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I) \arrow[u] & \cong\ H_I(G/\bar{H})\ .
\end{tikzcd}$$

If G is compact, then the right arrow is an isomorphism [8].

Theorem 7.13.B.3. *$\hat{\Delta}_\#$ is trivial if and only if $\mathfrak{c}\subset\mathfrak{b}$.*

Proof. (a) *If $\mathfrak{c}\subset\mathfrak{b}$, then $\hat{\Delta}_\#$ is trivial.* We prove the triviality of $\hat{\Delta}_*$ provided that $\mathfrak{c}\subset\mathfrak{b}$. The epimorphy of $\mathfrak{b}\hookrightarrow\mathfrak{g}\longrightarrow\mathfrak{g}/\mathfrak{h}$, as well as (19), imply that it is sufficient to show the equality $\omega_1(w)=0$ for $w\in\mathfrak{b}$. For this purpose, take an arbitrary point $w\in\mathfrak{b}$ and write $w=w_1+w_2$ for $w_1\in\mathfrak{h}$ and $w_2\in\mathfrak{c}$ $(\subset\mathfrak{b})$. Then $w_1=w-w_2\in\mathfrak{b}$, so $\omega_1(w)=\rho(w_1)=0$.

(b) *If $\mathfrak{c}\not\subset\mathfrak{b}$, then $\hat{\Delta}_\#$ is not trivial.* Assume $\mathfrak{c}\not\subset\mathfrak{b}$. Take $w\in\mathfrak{c}\backslash\mathfrak{b}$ and let $\tilde{w}\in\mathfrak{b}$ *be a vector such that* $[w]=[\tilde{w}]$ in $\mathfrak{g}/\mathfrak{h}$. Of course, $\tilde{w}-w\in\mathfrak{h}\backslash(\mathfrak{h}\cap\mathfrak{b})$ and $\bar{w}=(\tilde{w}-w)+w\in\mathfrak{h}+\mathfrak{c}$. Take a covector $\hat{\Psi}\in(\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{b}))^*$ such that $\hat{\Psi}([\tilde{w}-w])\neq 0$. Then

$$\hat{\Delta}_*(\hat{\Psi})([w])=\langle\hat{\Psi},\omega_1(\tilde{w})\rangle=\langle\hat{\Psi},[\tilde{w}-w]\rangle\neq 0. \tag{20}$$

Since $Z((\mathfrak{g}/\mathfrak{h})^*)_I\longrightarrow H^1((\Lambda(\mathfrak{g}/\mathfrak{h})^*)_I)$ is a monomorphism, (20) implies that $\hat{\Delta}_\#(\hat{\Psi})\neq 0$. ■

Then, for compact G, each case $\mathfrak{c}\not\subset\mathfrak{b}$ is the source of the nontrivial characteristic homomorphism of a flat transitive Lie algebroid on the ground of TC-foliations.

Problem 7.13.B.4. The nontriviality of $\Delta_\#$ means the impossibility of the homotopic changing of a Lie subalgebroid to contain the connection. Does the homomorphism $\hat{\Delta}_\#$

possess this property ?

7.14. The characteristic homomorphism for some nontransitive case

If $\mathfrak{f}\subset\mathfrak{g}$ is a Lie subalgebra such that $\bar{\mathfrak{h}}\subset\mathfrak{f}$, then, [18; 2.3.1] the G-left-invariant distribution $\bar{F}(\mathfrak{f})\subset TG$ determined by $\mathfrak{f}$ is an involutive $\mathcal{F}$-distribution, i.e. corresponds to some involutive distribution $F(\mathfrak{f})$ on the homogeneous space $G/\bar{H}$ and then to some foliation $\bar{\mathcal{F}}$ of $G/\bar{H}$. The leaves of $\bar{\mathcal{F}}$ are of the form $\{a\cdot\bar{H};\ a\in L\}$ where L is a leaf of $\bar{\mathcal{F}}$. $F(\mathfrak{f})$ is G-left-invariant and generated by $\mathfrak{f}/\bar{\mathfrak{h}}$, besides, $\operatorname{codim} F(\mathfrak{f})=\operatorname{codim}\mathfrak{f}$.

If $\mathfrak{c}\subset\mathfrak{g}$ is a Lie subalgebra such that $\bar{\mathfrak{h}}\cap\mathfrak{c}=\mathfrak{h}$ and $\bar{\mathfrak{h}}+\mathfrak{c}=\mathfrak{f}$ ($\mathfrak{f}$ as above), then [18; 2.3.2] the G-left-invariant distribution $\bar{C}(\mathfrak{c})$ on G is a partial $\mathcal{F}$-connection over $\bar{F}(\mathfrak{f})$ and, then, determines some partial flat connection $\lambda_{\mathfrak{c}}$ in $A(G;H)$ over $F(\mathfrak{f})$.

In this section we calculate the characteristic homomorphism of the FS-regular Lie algebroid $(A(G;H)^{F(\mathfrak{f})}, B_{\mathfrak{b}}^{F(\mathfrak{f})}, \lambda_{\mathfrak{c}})$ in which

(1) $F(\mathfrak{f})$ is the involutive distribution on $G/\bar{H}$ determined by a Lie subalgebra $\mathfrak{f}\subset\mathfrak{g}$ such that $\bar{\mathfrak{h}}\subset\mathfrak{f}$, described in [18; 2.3.1],

(2) $B_{\mathfrak{b}}$ is the Lie subalgebroid of $A(G;H)$ determined by a Lie subalgebra $\mathfrak{b}\subset\mathfrak{g}$ such that $\mathfrak{h}\subset\mathfrak{b}$, $\bar{\mathfrak{h}}+\mathfrak{b}=\mathfrak{g}$, see Example 7.8,

(3) $\lambda_{\mathfrak{c}}$ is a flat partial connection in $A(G;H)$ over $F(\mathfrak{f})$, determined by a Lie subalgebra $\mathfrak{c}\subset\mathfrak{g}$ such that $\bar{\mathfrak{h}}\cap\mathfrak{c}=\mathfrak{h}$ and $\bar{\mathfrak{h}}+\mathfrak{c}=\mathfrak{f}$, see [18; 2.3.2].

It is simpler to give such examples in comparison with the transitive case. For example, $\mathfrak{f}$ may be the Lie algebra of a maximal torus in G.

Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{g} \arrow[r, hook] & A(G;H)^{F(\mathfrak{f})} \arrow[r, "\gamma"] & F(\mathfrak{f}) \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{h} \arrow[r, hook] \arrow[u, hook] & B_{\mathfrak{b}}^{F(\mathfrak{f})} \arrow[r, "\gamma_1"] \arrow[u, hook] & F(\mathfrak{f}) \arrow[r] \arrow[u, equal] & 0\ .
\end{tikzcd}$$

The FS-regular Lie algebroid considered has the characteristic homomorphism $\Delta_{\#}^{F(\mathfrak{f})}: H(\mathbf{g};B_{\mathfrak{b}}^{F(\mathfrak{f})})\longrightarrow H_{F(\mathfrak{f})}(G/\bar{H})$.

7.14.A). The domain of $\Delta_{\#}$. We are interested in the representation $ad^{\wedge}_{B_{\mathfrak{b}}^{F(\mathfrak{f})},\mathbf{g}}: B_{\mathfrak{b}}^{F(\mathfrak{f})}\longrightarrow A(\Lambda(\mathbf{g}/\mathbf{h})^{*})$ defined in 2.3. Of course, this representation is the restriction of $ad^{\wedge}_{B_{\mathfrak{b}},\mathbf{g}}$ to the Lie subalgebroid $B^{F(f)}_{\mathfrak{b}}$ and will be denoted here (for brevity) by $ad^{\wedge}_{F}$. Let $\left(Sec\wedge(\mathbf{g}/\mathbf{h})^{*}\right)_{I^{\circ}(F)}$ be the space of $ad^{\wedge}_{F}$-invariant cross-sections. By the definition, we have:

$\Psi\in\left(Sec\wedge(\mathbf{g}/\mathbf{h})^{*}\right)_{I^{\circ}(F)}$ if and only if

$$(\gamma_1\circ\xi)\langle\Psi,[\nu_1]\wedge\ldots\wedge[\nu_k]\rangle=\sum_{j=1}^{k}\langle\Psi,[\nu_1]\wedge\ldots\wedge[[\![\xi,\nu_j]\!]]\wedge\ldots\wedge[\nu_k]\rangle$$

for any $\xi\in SecB^{F(f)}_{\mathfrak{b}}$ and $\nu_1,\ldots,\nu_k\in Sec\,\mathbf{g}$.

Consider analogously to 7.13A) the canonical isomorphism of vector bundles $\wedge^{k}(\mathbf{g}/\mathbf{h})^{*}\longrightarrow G/\bar{H}\times\wedge^{k}(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}$ and denote by $\tilde{\Psi}$ the function on $G/\bar{H}$ with values in $\wedge^{k}(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}$ corresponding to a cross-section $\Psi\in Sec\wedge^{k}(\mathbf{g}/\mathbf{h})^{*}$.

Proposition 7.14.A. *$\Psi\in\left(Sec\wedge^{k}(\mathbf{g}/\mathbf{h})^{*}\right)_{I^{\circ}(F)}$ if and only if $\Psi=\sum f^{i}\Psi_{i}$ for $f^{i}\in\Omega^{o}_{b}(M,\mathcal{F})$, and $\Psi_{i}\in Sec\wedge^{k}(\mathbf{g}/\mathbf{h})^{*}$ such that $\tilde{\Psi}_{i}$ are constant.*

Proof. A cross-section $\Psi\in Sec\wedge^{k}(\mathbf{g}/\mathbf{h})^{*}$ for which $\tilde{\Psi}$ is constant is invariant with respect to $ad^{\wedge}_{B_{\mathfrak{b}},\mathbf{g}}$ (by the same argument as in [17; 7.4.1]) so, thereby, with respect to $ad^{\wedge}_{F}$. A cross-section $\Psi=\sum f^{i}\Psi_{i}$, where f^{i} and Ψ_{i} are as in the text of our proposition, is $ad^{\wedge}_{F}$-invariant because, for $v\in B^{F(f)}_{\mathfrak{b}|x}$, we have

$$ad^{\wedge}_{F}(v)(\sum f^{i}\Psi_{i})=\sum f^{i}(x)\cdot ad^{\wedge}_{F}(v)(\Psi_{i})+(\gamma_1(v))(f^{i})\cdot\Psi_{i}(x)=0.$$

Let $\Psi_1,\ldots,\Psi_k\in Sec\wedge(\mathbf{g}/\mathbf{h})^{*}$ be cross-sections such that $\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k$ are constant and their values $\hat{\psi}_1,\ldots,\hat{\psi}_k$ form a basis of $\wedge(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}$. It is evident that each cross-section $\Psi\in Sec\wedge(\mathbf{g}/\mathbf{h})^{*}$ is of the form $\Psi=\sum f^{i}\Psi_{i}$ for some $f^{i}\in\Omega^{o}(G/\bar{H})$ and Ψ_{i} as above. For $\xi\in SecB^{F_f}_{\mathfrak{b}}$,

$$\mathcal{L}_{ad^{\wedge}_{F}\circ\xi}(\sum f^{i}\Psi_{i})=\sum\left(f^{i}\mathcal{L}_{ad^{\wedge}_{F}\circ\xi}(\Psi_{i})+(\gamma_1\circ\xi)(f^{i})\Psi_{i}\right)=\sum(\gamma_1\circ\xi)(f^{i})\Psi_{i}$$

Therefore, when Ψ is invariant, we have $(\gamma_1\circ\xi)(f^{i})\Psi_{i}=0$ for each i; equivalently, $X(f^{i})=0$ for each $X\in\mathfrak{X}(\mathcal{F})$, this means that $f^{i}\in\Omega^{o}_{b}(M,\mathcal{F})$. ∎

The mapping

$$\begin{aligned}\kappa:\left(Sec\wedge(\mathbf{g}/\mathbf{h})^{*}\right)_{I^{\circ}(F)}&\longrightarrow\Omega^{o}_{b}(M,\mathcal{F})\cdot\wedge(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}\\ f^{i}\Psi_{i}&\longmapsto f^{i}\cdot\hat{\psi}_{i}\end{aligned}$$

is a correctly defined isomorphism of vector spaces (identifying these spaces).

In the space $\left(Sec\Lambda^{k}(\mathbf{g}/\mathbf{h})^{*}\right)_{I^{\circ}(F)}$ the differential $\bar{\delta}$ defined in 3.4 works. Via κ we can carry out $\bar{\delta}$ to the space $\Omega^{\circ}_{b}(M,\mathcal{F})\cdot\Lambda(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}$ obtaining a differential $\bar{\bar{\delta}}$. An analogous reasoning as in 7.13A) yields that $\bar{\bar{\delta}}=0$ [hence $\bar{\delta}=0$]. As a corollary we obtain the equality

$$H(\mathbf{g},B^{F_{\mathfrak{f}}}_{\mathfrak{b}})=\Omega^{\circ}_{b}(M,\mathcal{F})\cdot\Lambda(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}.$$

7.14.B). The characteristic homomorphism. Let ω be the connection form of the connection λ_{c} under consideration. The conditions $\bar{\mathfrak{h}}\cap\mathfrak{b}=\mathfrak{h}$ and $\bar{\mathfrak{h}}+\mathfrak{b}=\mathfrak{f}$ determine a decomposition $\mathfrak{f}/\mathfrak{h}=\bar{\mathfrak{h}}/\mathfrak{h}\oplus\mathfrak{c}/\mathfrak{h}$.

Define $\omega_{o}:\mathfrak{f}\longrightarrow\bar{\mathfrak{h}}/\mathfrak{h}$ as the superposition

$$\mathfrak{f}\longrightarrow\mathfrak{f}/\mathfrak{h}=\bar{\mathfrak{h}}/\mathfrak{h}\oplus\mathfrak{c}/\mathfrak{h}\xrightarrow{pr_{1}}\bar{\mathfrak{h}}/\mathfrak{h}$$

and put $\omega_{1}=\rho\circ\omega_{o}$ where $\rho:\bar{\mathfrak{h}}/\mathfrak{h}\longrightarrow\bar{\mathfrak{h}}/\bar{\mathfrak{h}}\cap\mathfrak{b}$ is the canonical linear homomorphism. Analogously to 7.13B we obtain

Proposition 7.14.B.1. *The homomorphism $\Delta^{F(\mathfrak{f})}_{\#}$, on the level of forms, is defined by the formula*

$$\Delta_{*}\left(\sum f^{i}\hat{\psi}_{i}\right)(\bar{g};\hat{L}_{g}[w_{1}]\wedge\ldots\wedge\hat{L}_{g}[w_{k}])=\sum f^{i}(\bar{g})\langle\hat{\psi}_{i},\omega_{1}(\tilde{w}_{1})\wedge\ldots\wedge\omega_{1}(\tilde{w}_{k})\rangle$$

for $f^{i}\in\Omega^{\circ}_{b}(M,\mathcal{F})$, $\hat{\psi}_{i}\in\Lambda^{k}(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}$, $w_{1},\ldots,w_{k}\in\mathfrak{f}$, where $\tilde{w}_{1},\ldots,\tilde{w}_{k}\in\mathfrak{b}\cap\mathfrak{f}$ are vectors such that $[w_{i}]=[\tilde{w}_{i}]$ $\left(\in\mathfrak{f}/\bar{\mathfrak{h}}\right)$. The form $\Delta_{*}(\hat{\psi})$ for $\hat{\psi}\in\Lambda(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}$ is G-left-invariant. ∎

Define auxiliarily the homomorphism of algebras

$$\Delta^{1}_{\#}:\Lambda(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*}\longrightarrow H_{F(\mathfrak{f}),I}(G/\bar{H}),\quad\hat{\psi}\longmapsto\Delta_{\#}(\hat{\psi}),$$

where $H_{F(\mathfrak{f}),I}(G/\bar{H})$ is the cohomology algebra of the complex $\Omega_{F(\mathfrak{f}),I}(G/\bar{H})$ of the G-left-invariant tangential forms.

Between $\Delta_{\#}$ and $\Delta^{1}_{\#}$ there is a relation shown in the following diagram

$$\begin{tikzcd}
\Omega^{\circ}_{b}(M,\mathcal{F})\cdot\Lambda(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^{*} \arrow[r, "id\times\Delta^{1}_{\#}"] \arrow[rr, bend left, "\Delta_{\#}"] & \Omega^{\circ}_{b}(M,\mathcal{F})\cdot H_{F(\mathfrak{f}),I}(G/\bar{H}) \arrow[r] & H_{F(\mathfrak{f})}(G/\bar{H})
\end{tikzcd}.$$

If G is compact, then the canonical inclusion $\Omega_{F(\mathfrak{f}),I}(G/\bar{H})\hookrightarrow\Omega_{F(\mathfrak{f})}(G/\bar{H})$ induces a monomorphism on cohomologies $H_{F(\mathfrak{f}),I}(G/\bar{H})\rightarrowtail H_{F(\mathfrak{f})}(G/\bar{H})$ [19], therefore the nontriviality of $\Delta^1_{\#}$ implies the same for $\Delta_{\#}$. One can also formulate the problem analogous to 7.13.B.4.

Proposition 7.14.B.2. *$\Delta^1_{\#}$ is trivial if and only if $\mathfrak{c}\subset\mathfrak{b}$.*

Proof. (a) *If $\mathfrak{c}\subset\mathfrak{b}$, then $\Delta^1_{\#}$ is trivial.* The epimorphy of $\mathfrak{b}\cap\mathfrak{f}\longrightarrow\mathfrak{f}/\bar{\mathfrak{h}}$ implies that it is sufficient to show the equality $\omega_1|\mathfrak{b}\cap\mathfrak{f}=0$. We do it in the same way as in 7.13.B.3.

(b) *If $\mathfrak{c}\not\subset\mathfrak{b}$, then $\Delta^1_{\#}$ is not trivial.* Analogously to 7.13.B.3 we prove the existence of $\hat{\psi}\in(\bar{\mathfrak{h}}/_{\bar{\mathfrak{h}}\cap\mathfrak{b}})^*$ such that $\Delta^1_{\#}(\hat{\psi})\neq 0$. Since the action of G on $G/\bar{H}$ is transitive, $\Omega^0_{F(\mathfrak{f}),I}(G/\bar{H})\cong\mathbb{R}$ and $B(\Omega^1_{F(\mathfrak{f}),I}(G/\bar{H})=0$, which yields the monomorphism $Z(\Omega^1_{F(\mathfrak{f}),I}(G/\bar{H}))\rightarrowtail H^1_{F(\mathfrak{f}),I}(G/\bar{H})$. Therefore $\Delta^1_{\#}(\hat{\psi})\neq 0$. ■

PART III

III. THE CHARACTERISTIC CLASSES OF PARTIALLY FLAT REGULAR LIE ALGEBROIDS

1. THE WEIL ALGEBRA OF $\mathbf{g}$

A) Preliminary definitions and properties

We return to the general consideration of a regular Lie algebroid A over M with the Atiyah sequence $0 \longrightarrow \mathbf{g} \hookrightarrow A \xrightarrow{\gamma} E \longrightarrow 0$, equipped with a connection λ having ω as its connection form. We have:

$\mathbf{g}$ is a vector bundle of Lie algebras,

$\Lambda\mathbf{g}^*_{|x}$ is an anticommutative graded algebra; $(\Lambda\mathbf{g}^*_{|x})^k := \Lambda^k\mathbf{g}^*_{|x}$, $x \in M$,

$V\mathbf{g}^*_{|x}$ is an (anti)commutative graded algebra over the graded vector space $\mathbf{g}^*_{|x}$ with elements of degree two only, i.e. $(V\mathbf{g}^*_{|x})^{2l} = V^l\mathbf{g}^*_{|x}$ and $(V\mathbf{g}^*_{|x})^{2l+1} = 0$,

$W\mathbf{g}_{|x} := \Lambda\mathbf{g}^*_{|x} \otimes V\mathbf{g}^*_{|x}$ is the anticommutative (bi)graded tensor product of the anticommutative graded algebras. The bidegree $(W\mathbf{g}_{|x})^{k,2l} = \Lambda^k\mathbf{g}^*_{|x} \otimes V^l\mathbf{g}^*_{|x}$ leads one, as usual, to the total degree $(W\mathbf{g}_{|x})^r = \bigoplus_{k+2l=r}(W\mathbf{g}_{|x})^{k,2l}$. $W\mathbf{g}_{|x}$ as an algebra is generated by 1, $w^* \otimes 1$ and $1 \otimes w^*$, for $w^* \in \mathbf{g}^*_{|x}$.

Put

$$(W\mathbf{g})^{k,2l} := \wedge^k\mathbf{g}^* \otimes \vee^l\mathbf{g}^*,$$

$$(W\mathbf{g})^r := \bigoplus_{k+2l=r}(W\mathbf{g})^{k,2l},$$

$$(\mathcal{W}\mathbf{g})^{k,2l} := Sec(W\mathbf{g})^{k,2l},$$

$$(\mathcal{W}\mathbf{g})^r := Sec(W\mathbf{g})^r \quad \left(= \bigoplus_{k+2l=r}(\mathcal{W}\mathbf{g})^{k,2l}\right),$$

$$\mathcal{W}\mathbf{g} := \bigoplus_r (\mathcal{W}\mathbf{g})^r.$$

$\mathcal{W}\mathbf{g}$ is a bigraded algebra with the multiplication defined point by point. It is called the *Weil algebra of the bundle* $\mathbf{g}$ *of Lie algebras*. Each element of $\mathcal{W}\mathbf{g}$ is locally [even globally, which can be proved by using the paracompactness of M] a sum of cross-sections of the form $\psi_1\wedge\ldots\wedge\psi_k\otimes\Gamma_1\vee\ldots\vee\Gamma_l$, ψ_i, $\Gamma_j\in Sec\,\mathbf{g}^*$, $k,\ l\geqslant 0$.

In the above, k, l, r are nonnegative integers.

Remark 1.1. Under the gradation considered, the homomorphism $(d\omega)^\vee_x : \vee\mathbf{g}^*_{|x} \longrightarrow \wedge A^*_{|x}$ defined in Chapter 2C is of degree 0. Analogously, introducing the "point by point" structure of an algebra in $\overset{l\geqslant 0}{\otimes} Sec\vee^l\mathbf{g}^*$ and the gradation as above, we see that

$$(d\omega)^\vee : \overset{l\geqslant 0}{\otimes} Sec\vee^l\mathbf{g}^* \longrightarrow \Omega_A(M)$$

is a homomorphism of algebras of degree 0.

Three fundamental operators i, d, θ in $\mathcal{W}\mathbf{g}$, as well as the mapping $\mathbf{k}:\mathcal{W}\mathbf{g}\longrightarrow\Omega_A(M)$, will be introduced in two steps passing through some isomorphisms $\psi_x : W\mathbf{g}_{|x} \longrightarrow W\mathbf{g}_{|x}$, $x\in M$ [i.e. some change of variables]. This method, due to G.Andrzejczak [2], enables us to define and prove the property of these objects in the clear and technically lucid manner. The main value is that the differential d is then defined by one simple formula.

We begin with defining some auxiliary objects $\bar{k}$, $\bar{i}$, $\bar{d}$, $\bar{\theta}$.

1.2. Without any difficulties we can show that, for each point $x\in M$, there exists exactly one homomorphism

$$\bar{\mathbf{k}}_x : W\mathbf{g}_{|x} \longrightarrow \wedge A^*_{|x}$$

of algebras of degree 0 such that $\bar{\mathbf{k}}_x(1)=1$, $\bar{\mathbf{k}}_x(w^*\otimes 1)=\omega^\wedge_x(w^*)$ and $\bar{\mathbf{k}}_x(1\otimes w^*)=(d\omega)^\vee_x(w^*)$ when $w^*\in\mathbf{g}^*_{|x}$. $\bar{\mathbf{k}}_x$ is directly defined by the formula

$$\bar{\mathbf{k}}_x(\psi_x\otimes\Gamma_x)=\omega^\wedge_x(\psi_x)\wedge(d\omega)^\vee_x(\Gamma_x)$$

for $\psi_x \in \Lambda \mathbf{g}^*_{l\,x}$ and $\Gamma_x \in \vee \mathbf{g}^*_{l\,x}$. The homomorphisms $\bar{\mathbf{k}}_x$, $x \in M$, thanks to this formula, give rise to the homomorphism

$$\bar{\mathbf{k}} : W\mathbf{g} \longrightarrow \Omega_A(M)$$

of algebras of degree 0 defined point by point: $\bar{\mathbf{k}}(\Psi \otimes \Gamma)_x = \bar{\mathbf{k}}_x(\Psi_x \otimes \Gamma_x)$, $\Psi \in \overset{k \geq 0}{\oplus} Sec \Lambda^k \mathbf{g}^*$, $\Gamma \in \overset{l \geq 0}{\oplus} Sec \vee^l \mathbf{g}^*$. It has the property

$$\bar{\mathbf{k}}(\Psi \otimes \Gamma) = \omega^{\wedge}(\Psi) \wedge (d\omega)^{\vee}(\Gamma) \tag{1}$$

for Ψ and Γ as above.

Lemma 1.3. *For each point $x \in M$ and for $v_x \in \mathbf{g}_{l\,x}$, there exists exactly one antiderivation $\bar{i}_{x,v_x} : W\mathbf{g}_{l\,x} \longrightarrow W\mathbf{g}_{l\,x}$ of degree -1 such that*

(1) $\bar{i}_{x,v_x}(w^* \otimes 1) = \langle w^*, v_x \rangle$,

(2) $\bar{i}_{x,v_x}(1 \otimes w^*) = -(w^* \circ ad_{v_x}) \otimes 1$, $w^* \in \mathbf{g}^*_{l\,x}$.

It has the properties

(i) $\bar{i}_{x,v_x} | (W\mathbf{g}_{l\,x})^{0,0} = 0$,

(ii) $\bar{i}_{x,v_x}(1 \otimes w_1^* \vee \ldots \vee w_l^*) = -\sum_i w_i^* \circ ad_{v_x} \otimes w_1^* \vee \ldots \hat{i} \ldots \vee w_l^*$, $l \geq 1$,

(iii) $\bar{i}_{x,v_x}(\Psi_x \otimes \Gamma_x) = i_{v_x}(\Psi_x) \otimes \Gamma_x + (-1)^k \Psi_x \otimes 1 \cdot \bar{i}_{x,v_x}(1 \otimes \Gamma_x)$ when $\Psi_x \in \Lambda^k \mathbf{g}^*_{l\,x}$ and $\Gamma_x \in \vee \mathbf{g}^*_{l\,x}$,

(iv) $\bar{i}_{x,v_x}[(W\mathbf{g}_{l\,x})^{k,2l}] \subset (W\mathbf{g}_{l\,x})^{k-1,2l} \oplus (W\mathbf{g}_{l\,x})^{k+1,2(l-1)}$.

Proof. *Uniqueness.* The *uniqueness* of $\bar{i}_{x,v_x}$ is evident because every antiderivation is uniquely determined by the values on generators. Properties (i) ÷ (iv) of each antiderivation $\bar{i}_{x,v_x}$ fulfilling (1) and (2) above are evident.

Existence. First step. For $l \geq 1$, there exists exactly one linear mapping

$$\tilde{i}^{\,l}_{x,v_x} : \vee^l \mathbf{g}^*_{l\,x} \longrightarrow \mathbf{g}^*_{l\,x} \otimes \vee^{l-1} \mathbf{g}^*_{l\,x}$$

such that $\tilde{i}^{\,l}_{x,v_x}(w_1^* \vee \ldots \vee w_l^*) = \sum_i w_i^* \circ ad_{v_x} \otimes w_1^* \vee \ldots \hat{i} \ldots \vee w_l^*$. It has the property

$$\tilde{i}^{\,m+n}_{x,v_x}(\Gamma_{1x} \vee \Gamma_{2x}) = \tilde{i}^{\,m}_{x,v_x}(\Gamma_{1x}) \cdot 1 \otimes \Gamma_{2x} + 1 \otimes \Gamma_{1x} \cdot \tilde{i}^{\,n}_{x,v_x}(\Gamma_{2x})$$

for $\Gamma_{1x} \in \vee^m \mathbf{g}^*_{l\,x}$ and $\Gamma_{2x} \in \vee^n \mathbf{g}^*_{l\,x}$.

Second step. For $k \geq 0$ and $l \geq 1$, there exists exactly one linear mapping

$$\bar{i}^{k,2l}_{x,\nu_x}:(W\mathbf{g}_{|x})^{k,2l}\longrightarrow(W\mathbf{g}_{|x})^{k-1,2l}\oplus(W\mathbf{g}_{|x})^{k+1,2(l-1)}$$

such that $\bar{i}^{k,2l}_{x,\nu_x}(\Psi_x\otimes\Gamma_x)=i_{\nu_x}(\Psi_x)\otimes\Gamma_x+(-1)^k\Psi_x\otimes 1\cdot\tilde{i}^{\,l}_{x,\nu_x}(\Gamma_x)$ when $\Psi_x\in\wedge^k\mathbf{g}^*_{|x}$ and $\Gamma_x\in\vee\mathbf{g}^*_{|x}$.

Third step. Accept, additionally,

$$\bar{i}^{0,0}_{x,\nu_x}=0 \quad\text{and}\quad \bar{i}^{k,0}_{x,\nu_x}:(W\mathbf{g}_{|x})^{k,0}\longrightarrow(W\mathbf{g}_{|x})^{k-1,0},\ \Psi_x\otimes 1\longmapsto i_{\nu_x}\Psi_x\otimes 1,\quad \text{for } k\geqslant 1.$$

All the linear mappings $\bar{i}^{k,2l}_{x,\nu_x}$, $k,\ l\geqslant 0$, together define the operator

$$\bar{i}_{x,\nu}=\sum_{k,l\geqslant 0}\bar{i}^{k,2l}_{x,\nu_x}:W\mathbf{g}_{|x}\longrightarrow W\mathbf{g}_{|x}.$$

Of course, $\bar{i}_{x,\nu_x}$ satisfies (1) and (2).

It remains to show that $\bar{i}_{x,\nu_x}$ is an antiderivation of degree -1, i.e. that $\bar{i}_{x,\nu_x}(\Theta_1\cdot\Theta_2)=\bar{i}_{x,\nu_x}(\Theta_1)\cdot\Theta_2+(-1)^r\Theta_1\cdot\bar{i}_{x,\nu_x}(\Theta_2)$ for $\Theta_1\in(W\mathbf{g}_{|x})^r$, $\Theta_2\in W\mathbf{g}_{|x}$, which is easy to obtain by considering elements Θ_i homogeneous with respect to the bigradation only. ■

For a cross-section $\nu\in Sec\,\mathbf{g}$ and for $\Theta\in W\mathbf{g}$, the formula

$$M\ni x\longmapsto\bar{i}_{x,\nu_x}(\Theta_x)$$

defines an element $\bar{i}_\nu(\Theta)$ of $W\mathbf{g}$ and

$$\bar{i}_\nu:W\mathbf{g}\longrightarrow W\mathbf{g},\ \Theta\longmapsto\bar{i}_\nu(\Theta),$$

is an antiderivation of degree -1. The smoothness of $\bar{i}_\nu(\Theta)$, according to properties (i)÷(iii) from Lemma 1.3, follows from the smoothness in the cases $\Theta=\Psi\otimes 1$ where $\Psi\in Sec\wedge^k\mathbf{g}^*$, and $\Theta=1\otimes\Gamma$ where $\Gamma\in Sec\,\mathbf{g}^*$, which is easy to investigate. $\bar{i}_\nu$ has the property

$$\bar{i}_\nu(\Psi\otimes\Gamma)=i_\nu\Psi\otimes\Gamma\ +\ (-1)^k\Psi\otimes 1\cdot\bar{i}_\nu(1\otimes\Gamma)\tag{2}$$

for $\Psi\in Sec\wedge^k\mathbf{g}^*$ and $\Gamma\in Sec\vee^l\mathbf{g}^*$.

Lemma 1.4. *For each point $x\in M$, there exists exactly one antiderivation $\bar{d}_x:W\mathbf{g}_{|x}\longrightarrow W\mathbf{g}_{|x}$ of degree +1 such that*

(1) $\bar{d}_x(w^*\otimes 1)=1\otimes w^*$,

(2) $\bar{d}_x(1\otimes w^*)=0$, $w^*\in\mathbf{g}^*_{l\,x}$.

It has the properties

(i) $\bar{d}_x|(W\mathbf{g}_{l\,x})^{0,0}=0$,

(ii) $\bar{d}_x(w^*_1\wedge\ldots\wedge w^*_k\otimes 1)=\sum_i(-1)^{i+1}w^*_1\wedge\ldots\hat{i}\ldots\wedge w^*_k\otimes w^*_i$, $k\geqslant 1$,

(iii) $\bar{d}_x(\Psi_x\otimes\Gamma_x)=\bar{d}_x(\Psi_x\otimes 1)\cdot 1\otimes\Gamma_x$ *when* $\Psi_x\in\Lambda\mathbf{g}^*_{l\,x}$ *and* $\Gamma_x\in\mathrm{V}\mathbf{g}^*_{l\,x}$; *in particular*, $\bar{d}_x(1\otimes\Gamma_x)=0$,

(iv) $\bar{d}_x[(W\mathbf{g}_{l\,x})^{k,2l}]\subset(W\mathbf{g}_{l\,x})^{k-1,2(l+1)}$,

(v) $\bar{d}_x$ *is a differential, i.e.* $\bar{d}_x\circ\bar{d}_x=0$.

Proof. The uniqueness of $\bar{d}_x$ and properties (i)÷(v) are evident.

Existence. First step. For $k\geqslant 1$, there exists exactly one linear mapping $\tilde{d}^k_x:\Lambda^k\mathbf{g}^*_{l\,x}\longrightarrow(\Lambda^{k-1}\mathbf{g}^*_{l\,x})\otimes\mathbf{g}^*_{l\,x}$ such that

$$\tilde{d}^k_x(w^*_1\wedge\ldots\wedge w^*_k)=\sum_i(-1)^{i+1}w^*_1\wedge\ldots\hat{i}\ldots\wedge w^*_k\otimes w^*_i.$$

It has the property $\tilde{d}^{m+n}_x(\Psi_{1x}\wedge\Psi_{2x})=(\tilde{d}^m_x\Psi_{1x})\cdot\Psi_{2x}\otimes 1+(-1)^m\Psi_{1x}\otimes 1\cdot\tilde{d}^n_x\Psi_{2x}$ when $\Psi_{1x}\in\Lambda^m\mathbf{g}^*_{l\,x}$ and $\Psi_{2x}\in\Lambda^n\mathbf{g}^*_{l\,x}$.

Second step. For $k\geqslant 1$ and $l\geqslant 1$, there exists exactly one linear mapping $\bar{d}^{k,2l}_x:(W\mathbf{g}_{l\,x})^{k,2l}\longrightarrow(W\mathbf{g}_{l\,x})^{k-1,2(l+1)}$ such that $\bar{d}^{k,2l}_x(\Psi_x\otimes\Gamma_x)=(\tilde{d}^k_x\Psi_x)\cdot 1\otimes\Gamma_x$ for $\Psi_x\in\Lambda^k\mathbf{g}^*_{l\,x}$ and $\Gamma_x\in\mathrm{V}^l\mathbf{g}^*_{l\,x}$.

Third step. Add $\bar{d}^{0,0}_x=0$ and put $\bar{d}_x=\sum_{k,l\geqslant 0}\bar{d}^{k,2l}_x:W\mathbf{g}_{l\,x}\longrightarrow W\mathbf{g}_{l\,x}$. Of course, $\bar{d}_x$ satisfies (1) and (2). It remains to show that $\bar{d}_x$ is an antiderivation of degree +1 which is easy to obtain by considering elements homogeneous with respect to the bigradation. ∎

All homomorphisms $\bar{d}_x$, $x\in M$, define point by point a homomorphism

$$\bar{d}:W\mathbf{g}\longrightarrow W\mathbf{g}$$

being an antiderivation of degree +1 and a differential. It has the property

$$\bar{d}(\Psi\otimes\Gamma)=\bar{d}(\Psi\otimes 1)\cdot 1\otimes\Gamma,\quad \Psi\in Sec\Lambda^k\mathbf{g}^*,\quad \Gamma\in Sec\mathrm{V}^l\mathbf{g}^*. \tag{3}$$

Lemma 1.5. *For each point* $x\in M$ *and for* $v_x\in\mathbf{g}_{l\,x}$, *there exists exactly one* *derivation* $\bar{\theta}_{x,\nu_x}:W\mathbf{g}_{|x}\longrightarrow W\mathbf{g}_{|x}$ *of degree* 0 *such that*

(1) $\bar{\theta}_{x,\nu_x}(w^*\otimes 1)=-w^*\circ ad_{\nu_x}\otimes 1$,

(2) $\bar{\theta}_{x,\nu_x}(1\otimes w^*)=1\otimes(-w^*\circ ad_{\nu_x})$.

It has the property

(i) $\bar{\theta}_{x,\nu_x}(\Psi_x\otimes\Gamma_x)=(\theta^{\wedge}_x\Psi_x)\otimes\Gamma_x\ +\ \Psi_x\otimes(\theta^{\vee}_x\Gamma_x)$ *when* $\Psi_x\in\wedge\mathbf{g}^*_{|x}$ and $\Gamma_x\in\vee\mathbf{g}^*_{|x}$, *where* $\theta^{\wedge}_x$ *and* $\theta^{\vee}_x$ *denote the only derivations in the algebras* $\wedge\mathbf{g}^*_{|x}$ and $\Gamma_x\in\vee\mathbf{g}^*_{|x}$, *respectively, induced by* $-ad^*_{\nu_x}:\mathbf{g}^*_{|x}\longrightarrow\mathbf{g}^*_{|x}$.

Proof. The uniqueness and property (i) are evident. Formula (i) gives the sought-for operator. ■

For $\nu\in Sec\,\mathbf{g}$ and $\Theta\in W\mathbf{g}$, the formula $M\ni x\longmapsto\bar{\theta}_{x,\nu_x}(\Theta_x)$ defines an element of $W\mathbf{g}$ and

$$\bar{\theta}_\nu:W\mathbf{g}\longrightarrow W\mathbf{g},\quad \Theta\longmapsto\bar{\theta}_\nu(\Theta),$$

is a derivation of degree 0.

The adjoint representation $ad_A:A\longrightarrow A(\mathbf{g})$, according to 2.1.3 and 2.2.1 from [17], determines a representation of A on each associated vector bundle such as $\wedge^k\mathbf{g}$, $\vee^1\mathbf{g}$, $\wedge^k\mathbf{g}^*\otimes\vee^1\mathbf{g}^*$, etc. It will be denoted - for brevity - by *ad*.

Lemma 1.6. (1). *The linear operator*

$$\mathcal{L}_{ad\circ\xi}:W\mathbf{g}\longrightarrow W\mathbf{g},$$

$\xi\in Sec\,A$, *is a differentiation of the Weil algebra* $W\mathbf{g}$.

(2). $\mathcal{L}_{ad\circ\nu}=\bar{\theta}_\nu$ for $\nu\in Sec\,\mathbf{g}$.

Proof. Trivial calculations on simple tensors. ■

The relationships between the operators $\bar{i}_\nu$, $\bar{d}$, $\bar{\theta}_\nu$, $\mathcal{L}_{ad\circ\xi}$ are the following:

1.7. (1) $\mathcal{L}_{ad\circ\xi}\circ\bar{d}=\bar{d}\circ\mathcal{L}_{ad\circ\xi}$,

(2) $\bar{i}_\nu\circ\bar{d}\ +\ \bar{d}\circ\bar{i}_\nu=\bar{\theta}_\nu$.

Indeed, $\mathcal{E}_1:=\mathcal{L}_{ad\circ\xi}\circ\bar{d}-\bar{d}\circ\mathcal{L}_{ad\circ\xi}$ is an antiderivation, whereas $\mathcal{E}_2:=\bar{i}_\nu\circ\bar{d}+\bar{d}\circ\bar{i}_\nu$ is a derivation, of the Weil algebra $W\mathbf{g}$, therefore to prove (1) and (2) it is sufficient to show that $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = \bar{\theta}_\nu$ on the cross-sections $\Psi \otimes 1$ and $1 \otimes \Psi$, $\Psi \in Sec\,\mathbf{g}^*$, which is trivial.

Proposition 1.8. *For* $\Theta \in Sec\Lambda^k\mathbf{g}^* \otimes V^l\mathbf{g}^*$,

$$d^A\bar{\mathbf{k}}(\Theta) = \bar{\mathbf{k}}\bar{d}(\Theta) + \frac{1}{k!\cdot l!}\cdot\langle d^{\mathbf{g}}\Theta, (\omega\wedge\ldots\wedge\omega)\otimes(d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega)\rangle.$$

Lemma 1.8.1. (1) *For* $\Psi \in Sec\Lambda^k\mathbf{g}^*$,

$$\bar{\mathbf{k}}\bar{d}(\Psi\otimes 1) = \frac{1}{k!}\cdot\langle\Psi, d^{\mathbf{g}}(\omega\wedge\ldots\wedge\omega)\rangle.$$

(2) *For* $\Psi \in Sec\Lambda^k\mathbf{g}^*$ *and* $\Gamma \in SecV^l\mathbf{g}^*$,

$\langle d^{\mathbf{g}}(\Psi\otimes\Gamma), \omega\wedge\ldots\wedge\omega\otimes d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle$

$= \langle d^{\mathbf{g}}\Psi, \omega\wedge\ldots\wedge\omega\rangle\wedge\langle\Gamma, d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle + (-1)^k\langle\Psi,\omega\wedge\ldots\wedge\omega\rangle\wedge\langle d^{\mathbf{g}}\Gamma, d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle.$

Proof. (1): Thanks to the linearity of both sides with respect to Ψ, it is sufficient to show this on the simple tensor Ψ of the form $\Psi = \psi_1\wedge\ldots\wedge\psi_k$ where $\psi_i \in Sec\,\mathbf{g}^*$.

$\bar{\mathbf{k}}\bar{d}(\psi_1\wedge\ldots\wedge\psi_k\otimes 1) = \bar{\mathbf{k}}\big(\sum_i(-1)^{i+1}\psi_1\wedge\ldots\hat{i}\ldots\wedge\psi_k\otimes\psi_i\big)$

$= \sum_i(-1)^{i+1}\omega^\wedge(\psi_1\wedge\ldots\hat{i}\ldots\wedge\psi_k)\wedge(d\omega)^\vee(\psi_i).$

On the other hand, for $x\in M$ and $v_i \in A_{|x}$ (by II.1.3 and II.2.2 above),

$\frac{1}{k!}\cdot\langle\psi_1\wedge\ldots\wedge\psi_k, d^{\mathbf{g}}(\omega\wedge\ldots\wedge\omega)\rangle(x;v_1\wedge\ldots\wedge v_{k+1})$

$= \frac{1}{k!}\cdot\langle\psi_1\wedge\ldots\wedge\psi_k, \sum_i(-1)^{i+1}\underbrace{\omega\wedge\ldots\wedge\omega}_{i-1\text{ times}}\wedge d^{\mathbf{g}}\omega\wedge\ldots\wedge\omega)\rangle(x;v_1\wedge\ldots\wedge v_{k+1})$

$= \frac{1}{(k-1)!}\cdot\langle\psi_1\wedge\ldots\wedge\psi_k, d^{\mathbf{g}}\omega\wedge\underbrace{\omega\wedge\ldots\wedge\omega}_{k-1\text{ times}}\rangle(x;v_1\wedge\ldots\wedge v_{k+1})$

$= \frac{1}{2\cdot(k-1)!}\cdot\sum_\sigma sgn\sigma\,\langle\psi_{1x}\wedge\ldots\wedge\psi_{kx}, d^{\mathbf{g}}\omega(x;v_{\sigma(1)}\wedge v_{\sigma(2)})\wedge\omega(x;v_{\sigma(3)})\wedge\ldots\wedge\omega(x;v_{\sigma(k+1)})\rangle$

$$= \frac{1}{2\cdot(k-1)!}\cdot\sum_\sigma sgn\sigma\cdot\begin{vmatrix} \langle\psi_{1x}, d^{\mathbf{g}}\omega(x;v_{\sigma(1)}\wedge v_{\sigma(2)})\rangle & \ldots & \langle\psi_{kx}, d^{\mathbf{g}}\omega(x;v_{\sigma(1)}\wedge v_{\sigma(2)})\rangle \\ \langle\psi_{1x},\omega(x;v_{\sigma(3)})\rangle & \ldots & \langle\psi_{kx},\omega(x;v_{\sigma(3)})\rangle \\ \vdots & & \vdots \\ \langle\psi_{1x},\omega(x;v_{\sigma(k+1)})\rangle & \ldots & \langle\psi_{kx},\omega(x;v_{\sigma(k+1)})\rangle \end{vmatrix}$$

$$=\frac{1}{2\cdot(k-1)!}\cdot\sum_{\sigma} sgn\,\sigma\cdot\sum_{i}(-1)^{i+1}(d\omega)^{\vee}(\psi_i)(x;v_{\sigma(1)}\wedge v_{\sigma(2)})\cdot$$

$$\cdot\langle\psi_1\wedge\ldots\hat{i}\ldots\wedge\psi_k,\omega\wedge\ldots\wedge\omega\rangle(x;v_{\sigma(3)}\wedge\ldots\wedge v_{\sigma(k+1)})$$

$$=\sum_{i}(-1)^{i+1}(d\omega)^{\vee}(\psi_i)\wedge\omega^{\wedge}(\psi_1\wedge\ldots\hat{i}\ldots\wedge\psi_k)(x;v_1\wedge\ldots\wedge v_{k+1})$$

$$=\Big(\sum_{i}(-1)^{i+1}\omega^{\wedge}(\psi_1\wedge\ldots\hat{i}\ldots\wedge\psi_k)\wedge(d\omega)^{\vee}(\psi_i)\Big)(x;v_1\wedge\ldots\wedge v_{k+1}).$$

(2): For $x\in M$ and $v_i\in A_{|x}$, we have

$$\langle d^{\mathbf{g}}(\Psi\otimes\Gamma),\omega\wedge\ldots\wedge\omega\otimes d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle(x;v_1\wedge\ldots\wedge v_{k+2l+1})$$

$$=\langle d^{\mathbf{g}}\Psi\otimes\Gamma+\Psi\otimes d^{\mathbf{g}}\Gamma,\omega\wedge\ldots\wedge\omega\otimes d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle(x;v_1\wedge\ldots\wedge v_{k+2l+1})$$

$$=\frac{1}{k!\cdot(2l)!}\cdot\sum_{\sigma} sgn\,\sigma\cdot\langle d^{\mathbf{g}}\Psi(x;v_{\sigma(1)})\otimes\Gamma_x+\Psi_x\otimes d^{\mathbf{g}}\Gamma(x;v_{\sigma(1)}),$$

$$(\omega\wedge\ldots\wedge\omega)(x;v_{\sigma(2)}\wedge\ldots)\otimes(d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega)(x;\ldots\wedge v_{\sigma(k+2l+1)})\rangle$$

$$=\frac{1}{k!\cdot(2l)!}\cdot\sum_{\sigma} sgn\sigma\langle d^{\mathbf{g}}\Psi(x;v_{\sigma(1)}),(\omega\wedge\ldots\wedge\omega)(x;v_{\sigma(2)}\wedge\ldots\wedge v_{\sigma(k+1)})\rangle\cdot$$

$$\cdot\langle\Gamma_x,(d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega)(x;v_{\sigma(k+2)}\wedge\ldots\wedge v_{\sigma(k+2l+1)})\rangle$$

$$+\frac{1}{k!\cdot(2l)!}\cdot\sum_{\sigma} sgn\sigma\cdot\langle\Psi_x,(\omega\wedge\ldots\wedge\omega)(x;v_{\sigma(2)}\wedge\ldots\wedge v_{\sigma(k+1)})\rangle\cdot$$

$$\cdot\langle d^{\mathbf{g}}\Gamma(x;v_{\sigma(1)}),\langle\Gamma_x,(d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega)(x;v_{\sigma(k+2)}\wedge\ldots\wedge v_{\sigma(k+2l+1)})\rangle$$

$$=\Big(\langle d^{\mathbf{g}}\Psi,\omega\wedge\ldots\wedge\omega\rangle\wedge\langle\Gamma,d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle+$$

$$+(-1)^{k}\langle\Psi,\omega\wedge\ldots\wedge\omega\rangle\wedge\langle d^{\mathbf{g}}\Gamma,d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle\Big)(x;v_1\wedge\ldots\wedge v_{k+2l+1}).\qquad\square$$

Proof of Prop.1.8. It is sufficient to consider $\Theta=\Psi\otimes\Gamma$ for $\Psi\in Sec\bigwedge^{k}\mathbf{g}^{*}$ and $\Gamma\in Sec\bigvee^{l}\mathbf{g}^{*}$. According to (1), Th.II.1.3, II.2.2, II.2.13, the lemma above and (3),

$$d^{A}\bar{\mathbf{k}}(\Psi\otimes\Psi)=d^{A}(\omega^{\wedge}\Psi\wedge(d\omega)^{\vee}\Gamma)$$

$$=d^{A}(\tfrac{1}{k!}\cdot\langle\Psi,\omega\wedge\ldots\wedge\omega\rangle)\wedge(d\omega)^{\vee}\Gamma)+(-1)^{k}\omega^{\wedge}\Psi\wedge d^{A}(\tfrac{1}{l!}\cdot\langle\Gamma,d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle)$$

$$=\big[\langle d^{\mathbf{g}}\Psi,\omega\wedge\ldots\wedge\omega\rangle+\langle\Psi,d^{\mathbf{g}}(\omega\wedge\ldots\wedge\omega)\rangle\big]\wedge(d\omega)^{\vee}\Gamma+$$

$$+(-1)^{k}\cdot\frac{1}{k!\cdot l!}\cdot\langle\Psi,\omega\wedge\ldots\wedge\omega\rangle\wedge\langle d^{\mathbf{g}}\Gamma,d^{\mathbf{g}}\omega\vee\ldots\vee d^{\mathbf{g}}\omega\rangle$$

$$= \bar{\mathbf{k}}\bar{d}(\Psi\otimes 1)\wedge\bar{\mathbf{k}}(1\otimes\Gamma)+\frac{1}{k!\cdot l!}\cdot\left[\langle d^{\mathfrak{g}}\Psi,\omega\wedge\ldots\wedge\omega\rangle\wedge\langle\Gamma,d^{\mathfrak{g}}\omega\vee\ldots\vee d^{\mathfrak{g}}\omega\rangle+\right.$$

$$\left.+(-1)^{k}\cdot\langle\Psi,\omega\wedge\ldots\wedge\omega\rangle\wedge\langle d^{\mathfrak{g}}\Gamma,d^{\mathfrak{g}}\omega\vee\ldots\vee d^{\mathfrak{g}}\omega\rangle\right]$$

$$=\bar{\mathbf{k}}\bar{d}(\Psi\otimes\Gamma)+\frac{1}{k!\cdot l!}\cdot d^{\mathfrak{g}}(\Psi\otimes\Gamma),\omega\wedge\ldots\wedge\omega\otimes d^{\mathfrak{g}}\omega\vee\ldots\vee d^{\mathfrak{g}}\omega\rangle. \quad\blacksquare$$

Let $(W\mathbf{g})^{k,2l}_{I^\circ}$ denote the space of cross-sections invariant with respect to the "adjoint" representation of A on $(W\mathbf{g})^{k,2l}$ $\left(=\wedge^k\mathbf{g}^*\otimes\vee^l\mathbf{g}^*\right)$. Put

$$(W\mathbf{g})_{I^\circ}=\bigoplus_{k,l\geq 0}(W\mathbf{g})^{k,2l}_{I^\circ}.$$

$(W\mathbf{g})^{0,0}_{I^\circ}$ is equal to $\Omega^0_b(M,\mathcal{F})$, of course.

The following follows easily from 1.6.(1).

1.9. $(W\mathbf{g})_{I^\circ}$ *is a subalgebra of the Weil algebra* $W\mathbf{g}$. ■

1.7.1 implies

1.10. $\bar{d}$ *maps invariant elements of* $W\mathbf{g}$ *into invariant ones, defining an antiderivation of algebras*

$$\bar{d}_{o}:(W\mathbf{g})_{I^\circ}\longrightarrow(W\mathbf{g})_{I^\circ}. \quad\blacksquare$$

Whereas 1.8 and II.1.7 yield

1.11. $\bar{\mathbf{k}}$ *restricted to the invariant cross-sections*

$$\bar{\mathbf{k}}_{o}:(W\mathbf{g})_{I^\circ}\longrightarrow\Omega_A(M)$$

commutes with the differentials $\bar{d}_{o}$ *and* d^A, *giving - on cohomologies - a homomorphism*

$$\bar{\mathbf{k}}_{o\#}:H\big((W\mathbf{g})_{I^\circ},\bar{d}_{o}\big)\longrightarrow H_A(M)$$

of algebras. ■

However, $\bar{\mathbf{k}}_{o\#}$ is unimportant because the space $H\big((W\mathbf{g})_{I^\circ},\bar{d}_{o}\big)$ is trivial:

$$H\big((W\mathbf{g})_{I^\circ},\bar{d}_{o}\big)\cong(W\mathbf{g})^0_{I^\circ}\quad\left(=(W\mathbf{g})^{0,0}_{I^\circ}=\Omega^0_b(M,\mathcal{F})\right).$$

This follows from the fact that some chain homotopy joining id to 0 is defined by the family of invariant linear homomorphisms of vector bundles

$$c^{k,0} = 0 : \Lambda^k \mathbf{g}^* \longrightarrow \Lambda^{k+1}\mathbf{g}^*, \quad k \geqslant 0,$$

and

$$c^{k,l} : \Lambda^k \mathbf{g}^* \otimes \vee^l \mathbf{g}^* \longrightarrow \Lambda^{k+1}\mathbf{g}^* \otimes \vee^{l-1}\mathbf{g}^*, \quad l > 0,$$

$$w_1^* \wedge \ldots \wedge w_k^* \otimes \Gamma_1 \vee \ldots \vee \Gamma_l \longmapsto \frac{(-1)^k}{k+1} \cdot \sum_{s=1}^{l} w_1^* \wedge \ldots \wedge w_k^* \wedge \Gamma_s \otimes \Gamma_1 \vee \ldots \hat{s} \ldots \vee \Gamma_l .$$

B) The change of variables in $W\mathbf{g}_{lx}$

Proposition 1.12. *There exists exactly one isomorphism*

$$\varphi_x : W\mathbf{g}_{lx} \longrightarrow W\mathbf{g}_{lx}$$

of algebras of degree 0 *such that*

(1) $\varphi_x(1) = 1$,

(2) $\varphi_x(w^* \otimes 1) = w^* \otimes 1$,

(3) $\varphi_x(1 \otimes w^*) = 1 \otimes w^* - \delta_x w^* \otimes 1, \quad w^* \in \mathbf{g}^*_{lx}$,

where δ_x *denotes the differential in the algebra* $\Lambda\mathbf{g}^*_{lx}$, *defined in* II.2.6.

Proof. The uniqueness is evident. To prove the existence, take two linear mappings $\tilde{\varphi}_{x+}, \tilde{\varphi}_{x-} : \vee\mathbf{g}^*_{lx} \longrightarrow W\mathbf{g}_{lx}$ satisfying the conditions

(i) $\tilde{\varphi}_{x\pm}(1) = 1$,

(2) $\tilde{\varphi}_{x\pm}(\Gamma_{1x} \vee \ldots \vee \Gamma_{l\,x}) = \prod_{i=1}^{l} \big(1 \otimes \Gamma_{ix} \pm \delta_x(\Gamma_{ix}) \otimes 1\big), \quad \Gamma_{ix} \in \mathbf{g}^*_{lx}, \quad l \geqslant 1.$

Such mappings exist and are exactly the only ones. They are homomorphisms of algebras of degree 0 [the degree $\Gamma_x = 2$ for $\Gamma_x \in \mathbf{g}^*_{lx}$] and fulfil

$$\tilde{\varphi}_{x\pm}[\vee^l \mathbf{g}^*_{lx}] \subset \bigoplus_{m=0}^{l} (\Lambda^{2(l-m)} \mathbf{g}^*_{lx} \otimes \vee^m \mathbf{g}^*_{lx}).$$

Clearly, there exist two linear mappings $\varphi_{x+}, \varphi_{x-} : W\mathbf{g}_{lx} \longrightarrow W\mathbf{g}_{lx}$ such that

$(4_\pm)$ $\varphi_{x\pm}(\Psi_x \otimes \Gamma_x) = \Psi_x \otimes 1 \cdot \tilde{\varphi}_{x\pm}(\Gamma_x), \quad \Psi_x \in \Lambda\mathbf{g}^*_{lx}, \ \Gamma_x \in \vee\mathbf{g}^*_{lx}$.

They are of degree 0, are homomorphisms of algebras (which can be easy to prove by considering tensors bihomogeneous only), and fulfil the property

$(3_\pm)$ $\varphi_{x\pm}(1\otimes w^*) = 1\otimes w^* \pm \delta_x w^*\otimes 1$, $w^*\in \mathbf{g}^*_{|x}$.

To end the proof, put $\varphi_x := \varphi_{x-}$. To see that φ_x is an isomorphism, we check the equalities $\varphi_{x-}\circ\varphi_{x+} = id$, $\varphi_{x+}\circ\varphi_{x-} = id$. Both sides of these are homomorphisms of algebras, therefore it is sufficient to notice them on the generators, which is trivial. ∎

All the isomorphisms φ_x, $x\in M$, establish an isomorphism of algebras

$$\varphi : W\mathbf{g} \longrightarrow W\mathbf{g},$$

$\varphi(\Theta)(x) = \varphi_x(\Theta_x)$, $x\in M$. By the proof above,

$$\varphi^{-1}(\Psi\otimes 1) = \Psi\otimes 1 \quad\text{and}\quad \varphi^{-1}(1\otimes\Psi) = 1\otimes\Psi + \delta\Psi\otimes 1$$

hold for $\Psi\in Sec\,\mathbf{g}^*$.

Besides, φ_x establish linear homomorphisms of vector bundles

$$\varphi^1 : \Lambda\mathbf{g}^*\otimes\vee^1\mathbf{g}^* \xrightarrow{\cong} \Lambda\mathbf{g}^*\otimes\vee^1\mathbf{g}^*,$$

$$\varphi^{k,2l} : \Lambda^k\mathbf{g}^*\otimes\vee^l\mathbf{g}^* \longrightarrow \bigoplus_{m=0}^{l}\left(\Lambda^{k+2(l-m)}\mathbf{g}^*_{|x}\otimes\vee^m\mathbf{g}^*_{|x}\right).$$

The following equality holds

$$\varphi^{k,2l}\circ(\Psi\otimes\Gamma) = \Psi\otimes 1\cdot\left(\varphi^{0,2l}\circ(1\otimes\Gamma)\right), \qquad \Psi\in Sec\Lambda^k\mathbf{g}^*, \quad \Gamma\in Sec\vee^l\mathbf{g}^*. \tag{4}$$

Proposition 1.13. *$\varphi^{k,2l}$ is an invariant homomorphism.*

Proof. It is needed (see II.Ch.1) to prove only that

$$\mathscr{L}_{ad\circ\xi}(\varphi^{k,2l}\circ\Theta) = \varphi^{k,2l}\circ\mathscr{L}_{ad\circ\xi}\Theta \tag{5}$$

for $\xi\in Sec\,A$ and $\Theta\in Sec\Lambda^k\mathbf{g}^*\otimes\vee^l\mathbf{g}^*$. As usual, it is sufficient to consider $\Theta = \psi_1\wedge\ldots\wedge\psi_k\otimes\Gamma_1\vee\ldots\vee\Gamma_l$, ψ_i, $\Gamma_j\in Sec\,\mathbf{g}^*$. By (4) and 1.6(1),

$$\begin{aligned}
&\mathscr{L}_{ad\circ\xi}\left(\varphi^{k,2l}\circ\psi_1\wedge\ldots\wedge\psi_k\otimes\Gamma_1\vee\ldots\vee\Gamma_l\right)\\
&=\mathscr{L}_{ad\circ\xi}\left(\psi_1\wedge\ldots\wedge\psi_k\otimes 1\cdot(\varphi^{0,2l}\circ(1\otimes\Gamma_1\vee\ldots\vee\Gamma_l))\right)\\
&=\mathscr{L}_{ad\circ\xi}(\psi_1\wedge\ldots\wedge\psi_k\otimes 1)\cdot(\varphi^{0,2l}\circ(1\otimes\Gamma_1\vee\ldots\vee\Gamma_l)) +\\
&\qquad\qquad +\psi_1\wedge\ldots\wedge\psi_k\otimes 1\cdot\mathscr{L}_{ad\circ\xi}\left(\varphi^{0,2}\circ(1\otimes\Gamma_1)\cdot\ldots\cdot 1\otimes\Gamma_l)\right)
\end{aligned}$$

$$=\sum_i \psi_1\wedge\ldots\wedge\mathcal{L}_{ad\circ\xi}(\psi_i)\wedge\ldots\wedge\psi_k\otimes 1\cdot\varphi^{0,21}\circ(1\otimes\Gamma_1\vee\ldots\vee\Gamma_l)+$$

$$+\psi_1\wedge\ldots\wedge\psi_k\otimes 1\cdot\sum_i\varphi^{0,2}\circ(1\otimes\Gamma_1)\cdot\ldots\cdot\mathcal{L}_{ad\circ\xi}\big(\varphi^{0,2}\circ(1\otimes\Gamma_i)\big)\cdot\ldots\cdot\varphi^{0,2}\circ(1\otimes\Gamma_l).$$

On the other hand,

$$\varphi^{k,21}\circ\mathcal{L}_{ad\circ\xi}\big(\psi_1\wedge\ldots\wedge\psi_k\otimes\Gamma_1\vee\ldots\vee\Gamma_l\big)$$

$$=\varphi^{k,21}\circ\big(\mathcal{L}_{ad\circ\xi}(\psi_1\wedge\ldots\wedge\psi_k)\otimes\Gamma_1\vee\ldots\vee\Gamma_l+\psi_1\wedge\ldots\wedge\psi_k\otimes\mathcal{L}_{ad\circ\xi}(\Gamma_1\vee\ldots\vee\Gamma_l)\big)$$

$$=\varphi^{k,21}\circ\big(\sum_i\psi_1\wedge\ldots\wedge\mathcal{L}_{ad\circ\xi}(\psi_i)\wedge\ldots\wedge\psi_k\otimes\Gamma_1\vee\ldots\vee\Gamma_l+$$

$$+\psi_1\wedge\ldots\wedge\psi_k\otimes\sum_i\Gamma_1\vee\ldots\vee\mathcal{L}_{ad\circ\xi}(\Gamma_i)\vee\ldots\vee\Gamma_l\big)$$

$$=\sum_i\psi_1\wedge\ldots\wedge\mathcal{L}_{ad\circ\xi}(\psi_i)\wedge\ldots\wedge\psi_k\otimes 1\cdot\varphi^{0,21}\circ(1\otimes\Gamma_1\vee\ldots\vee\Gamma_l)+$$

$$+\psi_1\wedge\ldots\wedge\psi_k\otimes 1\cdot\sum_i\varphi^{0,2}\circ(1\otimes\Gamma_1)\cdot\ldots\cdot\varphi^{0,2}\circ(1\otimes\mathcal{L}_{ad\circ\xi}(\Gamma_i))\cdot\ldots\cdot\varphi^{0,2}\circ(1\otimes\Gamma_l).$$

What lacks here to prove the veracity of (5) is the equality

$$\mathcal{L}_{ad\circ\xi}\big(\varphi^{0,2}\circ(1\otimes\Gamma)\big)=\varphi^{0,2}\circ(1\otimes\mathcal{L}_{ad\circ\xi}\Gamma)\quad\text{for }\Gamma\in Sec\,\mathbf{g}^*.\tag{6}$$

However,

$$\mathcal{L}_{ad\circ\xi}\big(\varphi^{0,2}\circ(1\otimes\Gamma)\big)=1\otimes\mathcal{L}_{ad\circ\xi}\Gamma-\mathcal{L}_{ad\circ\xi}(\delta\Gamma)\otimes 1,$$

whereas

$$\varphi^{0,2}\circ(1\otimes\mathcal{L}_{ad\circ\xi}\Gamma)=1\otimes\mathcal{L}_{ad\circ\xi}\Gamma-\delta(\mathcal{L}_{ad\circ\xi}\Gamma)\otimes 1,$$

therefore (6) follows from the following lemma. ∎

Lemma 1.14. $\mathcal{L}_{ad\circ\xi}(\delta\Gamma)=\delta(\mathcal{L}_{ad\circ\xi}\Gamma)$ *for* $\Gamma\in Sec\,\mathbf{g}^*$.

Proof. For $\nu_1,\nu_2\in Sec\,\mathbf{g}$ and the Jacobi identity,

$$<\mathcal{L}_{ad\circ\xi}(\delta\Gamma),\nu_1\wedge\nu_2>=\ (\gamma\circ\xi)<\delta\Gamma,\nu_1\wedge\nu_2>-<\delta\Gamma,[\![\xi,\nu_1]\!]\wedge\nu_2>-<\delta\Gamma,\nu_1\wedge[\![\xi,\nu_2]\!]>$$

$$=(\gamma\circ\xi)<\Gamma,[\![\nu_1,\nu_2]\!]>-<\Gamma,[\![[\![\xi,\nu_1]\!],\nu_2]\!]>-<\Gamma,[\![\nu_1,[\![\xi,\nu_2]\!]]\!]>$$

$$=(\gamma\circ\xi)<\Gamma,[\![\nu_1,\nu_2]\!]>-<\Gamma,[\![\xi,[\![\nu_1,\nu_2]\!]=<\mathcal{L}_{ad\circ\xi}\Gamma,[\![\nu_1,\nu_2]\!]>$$

$$=<\delta(\mathcal{L}_{ad\circ\xi}\Gamma),\nu_1\wedge\nu_2>.$$ ∎

Corollaries 1.15. **(1).** $\varphi^1:\Lambda\mathbf{g}^*\otimes\vee^1\mathbf{g}^*\longrightarrow\Lambda\mathbf{g}^*\otimes\vee^1\mathbf{g}^*$ is an invariant isomorphism of vector bundles, therefore $(\varphi^1)^{-1}$ is invariant, too.

(2). $\varphi(\Theta)$ is an invariant element of $W\mathbf{g}$ whenever $\Theta\in W\mathbf{g}$ is invariant.

(3). $\varphi_0:(W\mathbf{g})_{I^\circ}\longrightarrow(W\mathbf{g})_{I^\circ}$, the restriction of φ to invariant cross-sections, is an isomorphism of algebras.

C). Operators i_ν, d, θ_ν and their properties

We define the fundamental operators i_ν, d, θ_ν in $W\mathbf{g}$ in such a way that the following three diagrams commute

$$\begin{array}{ccc} W\mathbf{g} & \xrightarrow{\varphi} & W\mathbf{g} \\ {\scriptstyle i_\nu, d, \theta_\nu}\downarrow & & \downarrow{\scriptstyle \bar{i}_\nu, \bar{d}, \bar{\theta}_\nu} \\ W\mathbf{g} & \xrightarrow{\varphi} & W\mathbf{g} \end{array} .$$

Of course, one can execute this procedure on each level of $x\in M$ to obtain the operators i_{x,ν_x}, d_x, θ_{x,ν_x} on $W\mathbf{g}_{|x}$, with the relations $i_{x,\nu_x}(\Theta_x)=i_\nu(\Theta)(x)$, etc.

Proposition 1.16. *The fundamental properties of the operators* i_ν, d, θ_ν *are as follows*:

(1) $\theta_\nu=\bar{\theta}_\nu$,

(2) $\mathcal{L}_{ad\circ\xi}\circ d=d\circ\mathcal{L}_{ad\circ\xi}$,

(3) $i_\nu\circ d+d\circ i_\nu=\theta_\nu$.

Proof. (1): θ_ν and $\bar{\theta}_\nu$ are derivations, therefore it is sufficient to show the equality $\theta_\nu(\Theta)=\bar{\theta}_\nu(\Theta)$ for the cross-sections $\Theta=\Psi\otimes 1$ and $\Theta=1\otimes\Psi$, $\Psi\in Sec\,\mathbf{g}^*$:

$$\theta_\nu(\Psi\otimes 1)=\varphi^{-1}\circ\bar{\theta}_\nu\circ\varphi(\Psi\otimes 1)=\theta_\nu^\wedge(\Psi)\otimes 1=\bar{\theta}_\nu(\Psi\otimes 1),$$

$$\begin{aligned}\theta_\nu(1\otimes\Psi)&=\varphi^{-1}\circ\bar{\theta}_\nu\circ\varphi(1\otimes\Psi)=\varphi^{-1}\circ\bar{\theta}_\nu(1\otimes\Psi-\delta\Psi\otimes 1)\\ &=\varphi^{-1}(1\otimes\theta_\nu\Psi-\theta_\nu^\wedge(\delta\Psi)\otimes 1)=1\otimes\theta_\nu^\vee\Psi+\delta(\theta_\nu\Psi)\otimes 1-\theta_\nu^\wedge(\delta\Psi)\otimes 1)\\ &=1\otimes\theta_\nu^\vee\Psi=\bar{\theta}_\nu(1\otimes\Psi)\end{aligned}$$

by [9; p.175 (5.3)].

(2): Evident because $\mathcal{L}_{ad\circ\xi}$ commutes with φ, φ^{-1} and $\bar{d}$.

(3): $i_\nu \circ d + d \circ i_\nu = \varphi^{-1} \circ \bar{i}_\nu \circ \bar{d} \circ \varphi + \varphi^{-1} \circ \bar{d} \circ \bar{i}_\nu \circ \varphi = \varphi^{-1} \circ \left(\bar{i}_\nu \circ \bar{d} + \bar{d} \circ \bar{i}_\nu \right) \circ \varphi$

$= \varphi^{-1} \circ \bar{\theta}_\nu \circ \varphi = \theta_\nu$. ■

Proposition 1.17. (1). i_ν *is an antiderivation of degree -1 defined uniquely by the conditions*

(1°) $i_\nu(\Psi \otimes 1) = i_\nu \Psi$,

(2°) $i_\nu(1 \otimes \Psi) = 0, \quad \Psi \in Sec\, \mathbf{g}^*$.

It has the property

(i) $i_\nu(\Psi \otimes \Gamma) = i_\nu(\Psi) \otimes \Gamma$ for $\Psi \in Sec \wedge^k \mathbf{g}^*$, $\Gamma \in Sec \vee^l \mathbf{g}^*$.

(2). d *is an antiderivation of degree +1 defined uniquely by the conditions*

(1°) $d(\Psi \otimes 1) = 1 \otimes \Psi + \delta\Psi \otimes 1$,

(2°) $d(1 \otimes \Psi)$ *is an element of* $(W\mathbf{g})^{1,2}$ $(= Sec\, \mathbf{g}^* \otimes \mathbf{g})$ *such that*

$$i_\nu \circ d(1 \otimes \Psi) = \theta_\nu \Psi \quad \text{for } \nu \in Sec\, \mathbf{g}.$$

Proof. (1) and (2) follow from 1.16(3). The rest is trivial. ■

The families of operators i_{x,ν_x}, d_x, θ_{x,ν_x}, indexed by $x \in M$, give rise, for $k, l \geqslant 0$, the linear homomorphisms of vector bundles

$$i_\nu^{k,2l} : \wedge^k \mathbf{g}^* \otimes \vee^l \mathbf{g}^* \longrightarrow \wedge^{k-1} \mathbf{g}^* \otimes \vee^l \mathbf{g}^*,$$

$$\theta_\nu^{k,2l} : \wedge^k \mathbf{g}^* \otimes \vee^l \mathbf{g}^* \longrightarrow \wedge^k \mathbf{g}^* \otimes \vee^l \mathbf{g}^*,$$

$$d_\nu^{k,2l} : \wedge^k \mathbf{g}^* \otimes \vee^l \mathbf{g}^* \longrightarrow \wedge^{k+1} \mathbf{g}^* \otimes \vee^l \mathbf{g}^* \oplus \wedge^{k-1} \mathbf{g}^* \otimes \vee^{l+1} \mathbf{g}^*.$$

1.16(2) implies

1.18. d *maps invariant elements of* $W\mathbf{g}$ *into invariant ones, defining an antiderivation*

$$d_0 : (W\mathbf{g})_{I^\circ} \longrightarrow (W\mathbf{g})_{I^\circ}. \quad ■$$

$\varphi_0 : (W\mathbf{g})_{I^\circ} \longrightarrow (W\mathbf{g})_{I^\circ}$ commuting with $\bar{d}_0$ and d_0 gives an isomorphism

$$\varphi_{0\#} : H\left((W\mathbf{g})_{I^\circ}, d_0\right) \xrightarrow{\cong} H\left((W\mathbf{g})_{I^\circ}, \bar{d}_0\right),$$

therefore $H\big((W\mathbf{g})_{I^\circ}, d_o\big)$ is trivial according to 1.12.

1.19. The cross-sections $\Theta \in W\mathbf{g}$, for which $i_\nu \Theta = 0$ for each $\nu \in Sec\,\mathbf{g}$, are called *horizontal* (or more precisely, $\mathbf{g}$-*horizontal*). Since i_ν is an antiderivation, all horizontal cross-sections form a subalgebra of $W\mathbf{g}$ denoted by $(W\mathbf{g})_i$. This construction can be executed on each level of $x \in M$ to obtain the algebra $(W\mathbf{g}_{|x})_i$. Of course, $\Theta \in (W\mathbf{g})_i \Leftrightarrow \Theta_x \in (W\mathbf{g}_{|x})_i$ for each $x \in M$.

Lemma 1.19.1. $(W\mathbf{g})_i = \overset{l\geq 0}{\oplus}(W\mathbf{g})^{0,2l}$ $(\cong \overset{l\geq 0}{\oplus} Sec\vee^l \mathbf{g}^*)$; *equivalently*, $(W\mathbf{g}_{|x})_i = \mathbb{R} \otimes \vee \mathbf{g}^*_{|x}$ *for each* $x \in M$. *In consequence, each nontrivial homogeneous element of* $(W\mathbf{g})_i$ *has an even degree.*

Proof. Let $\psi_x = \sum_j \psi_x^j \otimes \Gamma_x^j \in (W\mathbf{g}_{|x})_i$, $\psi_x^j \in \wedge \mathbf{g}^*_{|x}$ and $\Gamma_x^j \in \vee \mathbf{g}^*_{|x}$; the linear independence of Γ_x^j can be assumed [7; p.7]. Since $0 = i_{x,\nu_x}(\psi_x) = \sum_j i_{x,\nu_x}(\psi_x^j) \otimes \Gamma_x^j$, therefore [7; p.7], $i_{x,\nu_x}(\psi_x^j) = 0$ for each $\nu_x \in \mathbf{g}_{|x}$. But $\bigcap_{\nu_x \in \mathbf{g}_{|x}} Ker\, i_{x,\nu_x} = \mathbb{R}$ [7; p.117], then we obtain $\psi_x^j = r^j \in \mathbb{R} \subset \wedge \mathbf{g}^*_{|x}$. ■

1.6.2, 1.16(1)-(3) yield

1.20. *d maps invariant and (simultaneously) horizontal elements of* $W\mathbf{g}$ *into such elements, defining the antiderivation*

$$d_{i,o} : (W\mathbf{g}_{|x})_{i,I^\circ} \longrightarrow (W\mathbf{g}_{|x})_{i,I^\circ}.$$

D) The mapping k

Put

$$\mathbf{k} = \bar{\mathbf{k}} \circ \varphi : W\mathbf{g} \longrightarrow \Omega_A(M).$$

It is a homomorphism of algebras.

1.21. $\mathbf{k}_o : (W\mathbf{g})_{I^\circ} \longrightarrow \Omega_A(M)$, *the restriction of* $\mathbf{k}$ *to the invariant cross-sections commutes with the differentials* d_o *and* d^A.

Proof. For $\Theta \in (W\mathbf{g})_{I^\circ}$ we have, by 1.15(2),

$$d^A\circ\mathbf{k}_o(\Theta)=d^A\circ\bar{\mathbf{k}}_o\circ\varphi(\Theta)=\bar{\mathbf{k}}\circ\bar{d}\circ\varphi(\Theta)=\bar{\mathbf{k}}\circ\varphi\circ d(\Theta)=\mathbf{k}_o\circ d_o(\Theta). \qquad \blacksquare$$

Proposition 1.22. $\mathbf{k}(\Psi\otimes\Psi)=\omega^\wedge(\Psi)\wedge\Omega^\vee(\Gamma)$ *for* $\Psi\in\overset{k\geq 0}{\oplus}Sec\bigwedge^k\mathbf{g}^*$, $\Gamma\in\overset{1\geq 0}{\oplus}Sec\bigvee^1\mathbf{g}^*$.

Proof. $\mathbf{k}(\Psi\otimes\Psi)=\mathbf{k}(\Psi\otimes 1\cdot 1\otimes\Gamma)=\mathbf{k}(\Psi\otimes 1)\wedge\mathbf{k}(1\otimes\Gamma)=\omega^\wedge(\Psi)\wedge\mathbf{k}(1\otimes\Gamma)$. It remains to verify that $\mathbf{k}(1\otimes\Gamma)=\Omega^\vee(\Gamma)$. But the mappings $\Gamma\longmapsto\mathbf{k}(1\otimes\Gamma)$ and $\Gamma\longmapsto\Omega^\vee(\Gamma)$ are homomorphisms of algebras such that $1\longmapsto 1$, therefore it is sufficient to check the equality for $\Gamma=\Psi\in Sec\,\mathbf{g}^*$. 1.6 yields

$$\mathbf{k}(1\otimes\Psi)=\bar{\mathbf{k}}\circ\varphi(1\otimes\Psi)=\bar{\mathbf{k}}(1\otimes\Psi-\delta\Psi\otimes 1)$$
$$=(d\omega)^\vee(\Psi)\ -\omega^\wedge(\delta\Psi)=\langle\Psi,d^{\mathbf{g}}\omega\rangle-\omega^\wedge(\delta\Psi)=\Omega^\vee(\Psi). \qquad \blacksquare$$

1.23. $i_\nu\circ\mathbf{k}=\mathbf{k}\circ i_\nu$ *for* $\nu\in Sec\,\mathbf{g}$.

Proof. By the horizontality of the form from $Im\,\Omega^\vee$ [which easily follows from the horizontality of Ω], the property Th.II.1.3(vi′) of the substitution operator $i_\nu:\Omega_A(M)\longrightarrow\Omega_A(M)$, Lemma II.2.3 and 1.17(1)(i) above, we get, for $\Psi\in Sec\bigwedge^k\mathbf{g}^*$ and $\Gamma\in\overset{1\geq 0}{\oplus}Sec\bigvee^1\mathbf{g}^*$,

$$i_\nu\circ\mathbf{k}(\Psi\otimes\Psi)=i_\nu(\omega^\wedge(\Psi)\wedge\Omega^\vee(\Gamma))=i_\nu(\omega^\wedge(\Psi))\wedge\Omega^\vee(\Gamma)+(-1)^k\omega^\wedge(\Psi)\wedge i_\nu(\Omega^\vee(\Gamma))$$
$$=\omega^\wedge(i_\nu(\Psi))\wedge\Omega^\vee(\Gamma)=\mathbf{k}(i_\nu(\Psi)\otimes\Gamma)=\ \mathbf{k}(i_\nu(\Psi\otimes\Gamma)).$$

Our proposition now follows from the linearity of $\mathbf{k}$ and i_ν. $\blacksquare$

Remark 1.24 (*The Chern-Weil homomorphism of A, revisited*). A consequence of 1.22 is that $\mathbf{k}$ maps horizontal elements of $W\mathbf{g}$ into horizontal real forms on A, giving a homomorphism of algebras

$$\mathbf{k}_i:(W\mathbf{g})_i\longrightarrow\Omega_{A,i}(M).$$

This mapping is defined by the formula

$$\mathbf{k}_i(1\otimes\Gamma)=\Omega^\vee(\Gamma),\quad \Gamma\in\overset{1\geq 0}{\oplus}Sec\bigvee^1\mathbf{g}^*.$$

Consider the further restriction of $\mathbf{k}$,

$$\mathbf{k}_{o,i}:(W\mathbf{g})_{I^\circ,i}\longrightarrow\Omega_{A,i}(M)$$

where $(W\mathbf{g})_{I^\circ,i}$ denotes the algebra of elements horizontal and invariant simultaneously. We prove that

$$d|(W\mathbf{g})_{I^\circ,i}=0. \tag{7}$$

Let $0\neq\Theta\in(W\mathbf{g})_{I^\circ,i}$. By 1.20, $d\Theta\in(W\mathbf{g})_{I^\circ,i}$. But Θ has an even degree (see 1.19.1), whereas d is an antiderivation of degree +1, therefore $d\Theta$ has an odd degree. Using 1.19.1 once again, we assert that $d\Theta=0$.

According to 1.21 and (7), the forms from $Im\,\mathbf{k}_{o,i}$ are d^A-closed. Take into account the isomorphism $\lambda_*:\Omega_{A,i}(M)\longrightarrow\Omega_E(M)$ II.s.2. It maps d^A-closed forms into d^E-closed forms, see II.(5). By the above, there exists a homomorphism of algebras

$$\begin{array}{ccc} \overset{l\geq 0}{\oplus}(Sec\vee^l\mathbf{g}^*)_{I^\circ} & \longrightarrow & H_E(M) \\ \Gamma & \longmapsto & [\lambda_*(\Omega^\vee(\Gamma)] \end{array}\quad . \tag{8}$$

However,

$$\lambda_*(\Omega^\vee(\Gamma))=\lambda_*(\tfrac{1}{l!}\cdot\langle\Gamma,\Omega\vee\ldots\vee\Omega\rangle)=\tfrac{1}{l!}\cdot\langle\Gamma,\lambda_*\Omega\vee\ldots\vee\lambda_*\Omega\rangle=\tfrac{1}{l!}\cdot\langle\Gamma,\Omega_b\vee\ldots\vee\Omega_b\rangle,$$

therefore $[\lambda_*(\Omega^\vee(\Gamma))]=[\frac{1}{l!}\cdot\langle\Gamma,\Omega_b\vee\ldots\vee\Omega_b\rangle]=h_A(\Gamma)$ according to [17; Ch.4], which means that (8) is the Chern-Weil homomorphism of the regular Lie algebroid A.

2. REGULAR LIE ALGEBROIDS AND IDEALS

Take two vector bundles F' and F on a (paracompact, for recalling) manifold M, such that $F'\subset F$, and define (see [18; s.1.1]), for $p\geq 1$,

$$I_{\wedge^k F'}:=\bigcup_{x\in M} I_{\wedge^k(F'_{|x})}\subset\wedge F.$$

$I_{\wedge^k F'}$ is a vector subbundle of $\wedge F$ and the space of global cross-sections $Sec(I_{\wedge^p F'})$ sets up an ideal in the algebra $Sec(\wedge F)$; besides, $Sec(I_{\wedge^k F'})=(Sec\,I_{F'})^k$, $k>1$.

Let $E'\subset E\subset TM$ be two C^∞ constant dimensional distributions on M, and suppose E to be integrable. Denote by $E'^\perp$ the vector subbundle of E^* consisting of all covectors vanishing on E'. Using the above (for $F=E^*$, $F'=E'^\perp$, $p=1$), we obtain an ideal I in the algebra $\Omega_E(M)$ $(=Sec\wedge E^*)$ of tangential differential forms, generated by 1-forms vanishing on E'. Standard calculations give the following

2.1 (**The Frobenius Theorem for subdistributions**). *E' is involutive if and only if the ideal I is differential, i.e. $d^E[I]\subset I$.* ■

Consider a regular Lie algebroid $(A,[\![\cdot,\cdot]\!],\gamma)$ over a foliated manifold (M,E) and an involutive subdistribution $E'\subset E$. This produces a new regular Lie algebroid $(A',[\![\cdot,\cdot]\!],\gamma|A')$ in which $A'=\gamma^{-1}[E']$.

In the sequel, the symbols $A'^{\perp}$ and $E'^{\perp}$ are understood with respect to the canonical dualities $A^*\times A\longrightarrow\mathbb{R}$ and $E^*\times E\longrightarrow\mathbb{R}$ (see [18]). Consider the ideal

$$Sec(I_{\wedge^k(A'^{\perp})})\subset\Omega_A(M)\ \left(=Sec\wedge A^*\right)$$

being the k-power of the ideal of real forms on the Lie algebroid A, vanishing on A'. Since $\Psi\in Sec\,A'^{\perp}$ if and only if $\Psi=\gamma^*\theta$ *for some* $\theta\in Sec\,E'^{\perp}$, we obtain that each form $\Psi\in Sec(I_{\wedge^k(A'^{\perp})})$ is globally of the form

$$\Psi=\sum_{i=1}^{l}\gamma^*(\theta_i^1\wedge\ldots\wedge\theta_i^k)\wedge\Psi_i$$

for an integer l, $\theta_i^j\in Sec(E'^{\perp})$ and $\Psi_i\in\Omega_A(M)$. This fact, 2.1, and the equalities $d^A\gamma_*=\gamma_*d^E$ (II.(5)), $i_\xi\gamma_*=\gamma_*i_{\gamma\circ\xi}$, $\theta_\xi^A\gamma_*=\gamma_*\theta_{\gamma\circ\xi}$ for $\xi\in Sec\,A'$, make the following proposition obvious

2.2. *The ideal $Sec(I_{\wedge^k(A'^{\perp})})$ is closed with respect to the operators d^A, i_ξ, θ_ξ for $\xi\in Sec\,A'$.* ■

The monomorphy of $\wedge\gamma^*:\wedge E^*\longrightarrow\wedge A^*$, the equality

$$i_{w_1\wedge\ldots\wedge w_{h-k+1}}(\gamma_*\theta)=\gamma_*i_{\gamma w_1\wedge\ldots\wedge\gamma w_{h-k+1}}\theta$$

for $\theta\in\Omega_E(M)$ and $w_i\in Sec\,A$, and theorem 1.1.1 from [18] (see also [1]) imply

2.3. $\theta\in Sec(I_{\wedge^k(E'^{\perp})})\Leftrightarrow\gamma_*\theta\in Sec(I_{\wedge^k(A'^{\perp})})$. ■

Recall that [18; 1.2.1] by a *partial connection* in A over E' we mean any connection $\lambda':E'\longrightarrow A'$ in the regular Lie algebroid $A'=\gamma^{-1}[E']$.

If λ' is flat, then the pair (A,λ') is called a *partially flat regular Lie algebroid.* Any foliated principal bundle [10, p.20], gives in a natural manner, a partially flat regular Lie algebroid.

A connection $\lambda : E \longrightarrow A$ in A is said to be *adapted* to λ' when $\lambda' = \lambda|E'$ (an adapted connection always exists).

Assume that A is equipped with a connection λ and a partial connection λ' over E'. Let Ω and Ω' (Ω_b and Ω'_b) be the curvature forms (the curvature tensors) of these connections [see II.Sec.2 and [17; 3.1.1]]. According to the equality $\Omega = \gamma_* \Omega_b$, Prop.1.2.3 from [18] and 2.3 above, we assert

2.4. *If* λ *is adapted to* λ', *then*
(a) λ' *is flat if and only if* $\langle v^*, \Omega_{|x}\rangle \in I^{(2)}_{\Lambda^1(A'^{\perp}_{|x})}$ *for any* $x \in M$ *and* $v^* \in \mathbf{g}^*_{|x}$,
(b) λ *is basic if and only if* $\langle v^*, \Omega_{|x}\rangle \in I^{(2)}_{\Lambda^2(A'^{\perp}_{|x})}$ *for any* $x \in M$ *and* $v^* \in \mathbf{g}^*_{|x}$. ∎

Pass to the Weil algebras $W\mathbf{g}_{|x}$ and $W\mathbf{g}$. $W\mathbf{g}_{|x}$ has a standard even decreasing filtration by ideals

$$\mathbf{F}^{2p}(W\mathbf{g}_{|x}) := I_{\mathbb{R}\otimes \vee^p \mathbf{g}^*_{|x}} \quad \left(= \Lambda \mathbf{g}^*_{|x} \otimes \vee^{\geqslant p}\mathbf{g}^*_{|x}\right).$$

These, for all $x \in M$, define an even decreasing filtration by ideals of the Weil algebra $W\mathbf{g}$

$$\mathbf{F}^{2p}(W\mathbf{g}) := \left\{\Theta \in W\mathbf{g};\ \forall x \in M,\ \Theta_x \in \mathbf{F}^{2p}(W\mathbf{g}_{|x})\right\}$$
$$= \bigoplus^{1 \geqslant p} Sec(\Lambda \mathbf{g}^* \otimes \vee^1 \mathbf{g}^*).$$

The algebras $\Lambda A^*_{|x}$ and $\Lambda E^*_{|x}$ possess decreasing filtrations by ideals

$$\mathbf{F}^p(\Lambda A^*_{|x}) = I_{\Lambda^p(A'^{\perp}_{|x})}, \quad \mathbf{F}^p(\Lambda E^*_{|x}) = I_{\Lambda^p(E'_{|x})}$$

which determine decreasing filtrations by ideals of the algebras $\Omega_A(M)$ and $\Omega_E(M)$

$$\mathbf{F}^p(\Omega_A(M)) = \left\{\Psi \in \Omega_A(M);\ \forall x \in M,\ \Psi_x \in \mathbf{F}^p(\Lambda A^*_{|x})\right\}$$
$$= Sec\ I_{\Lambda^p(A'^{\perp})}$$
$$F^p(\Omega_E(M)) = \left\{\theta \in \Omega_E(M);\ \forall x \in M,\ \theta_x \in F^p(\Lambda E^*_{|x})\right\}$$
$$= Sec I_{\Lambda^p(E'^{\perp})}.$$

Proposition 2.5. *Let* (A, λ') *be a partially flat regular Lie algebroid and* λ *an adapted connection. Then the homomorphism* $\mathbf{k} : W\mathbf{g} \longrightarrow \Omega_A(M)$ (*defined for* λ) *is* *filtration-preserving in the sense that*

$$\mathbf{k}[\mathrm{F}^{2p}(W\mathbf{g})] \subset \Omega^{p}_{A}(M), \quad p \geq 0.$$

Moreover, if λ is basic, then

$$\mathbf{k}[\mathrm{F}^{2p}(W\mathbf{g})] \subset \Omega^{2p}_{A}(M), \quad p \geq 0.$$

Proof. Of course, it is sufficient to verify that $\mathbf{k}_x : W\mathbf{g}_{|x} \longrightarrow \Lambda A^*_{|x}$ preserves the filtrations. Since $I_{\mathbb{R}\otimes \vee^p \mathbf{g}^*_{|x}} = (I_{\mathbb{R}\otimes \mathbf{g}^*_{|x}})^p$, therefore $\mathrm{F}^{2p}(W\mathbf{g}_{|x}) = (\mathrm{F}^2(W\mathbf{g}_{|x}))^p$. On the other hand, $\mathbf{k}_x$ is a homomorphism of algebras, thus we need only to check that

(a) $\mathbf{k}_x[\mathrm{F}^2(W\mathbf{g}_{|x})] \subset \mathrm{F}^1(\Lambda A^*_{|x})$,

whereas, in the case of a basic connection, that

(b) $\mathbf{k}_x[\mathrm{F}^2(W\mathbf{g}_{|x})] \subset \mathrm{F}^2(\Lambda A^*_{|x})$.

$\mathrm{F}^2(W\mathbf{g}_{|x})$, $\mathrm{F}^1(\Lambda A^*_{|x})$ and $\mathrm{F}^2(\Lambda A^*_{|x})$ are ideals and $\mathrm{F}^2(W\mathbf{g}_{|x})$ equals $I_{\mathbb{R}\otimes\mathbf{g}^*_{|x}}$, so it suffices to check that

(a′) $\mathbf{k}_x(1\otimes w^*) \in I^{(2)}_{\Lambda^1(A'^{\perp}_{|x})}$, $\quad w^* \in \mathbf{g}^*_{|x}$,

(b′) for a basic connection λ, that $\mathbf{k}_x(1\otimes w^*) \in I^{(2)}_{\Lambda^2(A'^{\perp}_{|x})}$, $\quad w^* \in \mathbf{g}^*_{|x}$.

However, $\mathbf{k}_x(1\otimes w^*) = \langle w^*, \Omega_{|x}\rangle$, thereby (a′) and (b′) follow from 2.4(a)-(b). ∎

Corollary 2.6. *Let the situation be as in the previous proposition. If $q = \operatorname{rank} E/E'$ [i.e. q equals the codimension of $\mathcal{F}'$ with respect to $\mathcal{F}$; $\mathcal{F}'$ and $\mathcal{F}$ being the foliations determined by E and E', respectively], then*

$$\mathbf{k}[\mathrm{F}^{2p}(W\mathbf{g})] = 0 \quad \textit{for} \quad p \geq q+1.$$

If λ is, in addition, basic, then

$$\mathbf{k}[\mathrm{F}^{2p}(W\mathbf{g})] = 0 \quad \textit{for} \quad p \geq [q/2]+1.$$

Proof. Clearly, $q = \operatorname{rank} A/A' = \dim(A'^{\perp}_{|x})$ for each $x \in M$, which gives $\Lambda^p(A'^{\perp}_{|x}) = 0$ for $p \geq q+1$ and, in consequence, $\mathrm{F}^p(\Omega(A)) = 0$ for $p \geq q+1$; then **2.5** implies $\mathbf{k}[\mathrm{F}^{2p}(W\mathbf{g})] = 0$ for such a p. Under the additional assumption concerning λ, $\mathbf{k}[\mathrm{F}^{2p}(W\mathbf{g})] = 0$ for $2p \geq q+1$, i.e. for $p \geq [q/2]+1$. ∎

The filtration of $W\mathbf{g}$ in the intersection with the subalgebra $\overset{1\geq 0}{\oplus}\mathrm{Sec}(\vee^1\mathbf{g}^*)_{I^\circ}$, gives a filtration of this last:

$$F^{2p}({}^{1\geqslant 0}_{\oplus}Sec(\vee^1\mathbf{g}^*)_{I^\circ}) := ({}^{1\geqslant 0}_{\oplus}Sec(\vee^1\mathbf{g}^*)_{I^\circ}) \cap F^{2p}(W\mathbf{g})$$
$$= {}^{1\geqslant p}_{\oplus}Sec(\vee^1\mathbf{g}^*)_{I^\circ} \ .$$

Notice also, see 2.3, that the isomorphism $\gamma_*:\Omega_E(M)\longrightarrow\Omega_{A,i}(M)$ preserves the filtrations. As a corollary we obtain the so-called "*Vanishing Bott's Phenomenon*" [18] because, keeping the assumptions from the previous corollary, we have that the homomorphism of algebras

$${}^{1\geqslant 0}_{\oplus}(Sec\vee^1\mathbf{g}^*)_{I^\circ} \xrightarrow{\mathbf{k}_{i,o}} \Omega_{A,i}(M) \xrightarrow{\lambda_*} \Omega_E(M),$$

and further, passing to the cohomology, the Chern-Weil homomorphism [17]

$$h_A: {}^{1\geqslant 0}_{\oplus}(Sec\vee^1\mathbf{g}^*)_{I^\circ} \longrightarrow H_E(M)$$

of the Lie algebroid A preserve the filtrations and, then, $Pont^p(A)=0$ *for* $p>2\cdot(q+1)$, whereas if λ' extends to a basic connection, then $\mathrm{Pont}^p(A)=0$ for $p>q+1$.

3. THE TRUNCATED WEIL ALGEBRA

Definition 3.1. By the *symmetric truncated algebra* over a vector space $\mathfrak{g}$ we shall mean the space $\vee^{\leqslant 1}\mathfrak{g}^*$ with the canonical even gradation, and with the structure of an (anti)commutative graded algebra such that

$$(u_1^*\vee\ldots\vee u_k^*)\cdot(v_1^*\vee\ldots\vee v_s^*) = \begin{cases} u_1^*\vee\ldots\vee u_k^*\vee v_1^*\vee\ldots\vee v_s^* & \text{when } k+s\leqslant l,\\ 0 & \text{when } k+s>l.\end{cases}$$

This algebra can be constructed isomorphically as a quotient algebra $(\vee\mathfrak{g}^*)/I_{\vee^{>l}\mathfrak{g}^*}$ of the symmetric algebra $\vee\mathfrak{g}^*$ by the ideal generated by $\vee^{l+1}\mathfrak{g}^*$. The mapping

$$\vee^{\leqslant l}\mathfrak{g}^*\longrightarrow(\vee\mathfrak{g}^*)/I_{\vee^{>l}\mathfrak{g}^*},\quad w^*\longmapsto[w^*],$$

establishes the canonical isomorphism of algebras. The canonical projection $\pi_l:\vee\mathfrak{g}^*\longrightarrow\vee^{\leqslant l}\mathfrak{g}^*$ is, of course, a homomorphism of algebras.

Denote by

$$(W\mathfrak{g})_l := \Lambda\mathfrak{g}^* \otimes V^{<l}\mathfrak{g}^*$$

the anticommutative graded tensor product of the anticommutative graded algebras. It is called the *truncated Weil algebra* of the vector space $\mathfrak{g}$.

Return to the consideration of a regular Lie algebroid A over (M,E), with the Atiyah sequence $0 \longrightarrow \mathbf{g} \hookrightarrow A \xrightarrow{\gamma} E \longrightarrow 0$. Notice that, for each $x \in M$,

$$(W\mathbf{g}_{|x})_l \cong (W\mathbf{g}_{|x})/\mathbf{F}^{2(l+1)}(W\mathbf{g}_{|x})$$

$(\theta_x \longmapsto [\theta_x]$ establishes the canonical isomorphism$)$ and, by the relation $d_x[(W\mathbf{g}_{|x})^{k,2s}] \subset (W\mathbf{g}_{|x})^{k+1,2s} \oplus (W\mathbf{g}_{|x})^{k-1,2(s+1)}$, d_x defines a new differential $[d_x]_l : (W\mathbf{g}_{|x})_l \longrightarrow (W\mathbf{g}_{|x})_l$. Writing $d_x = d'_x + d''_x$ where $d'_x[W^{k,2s}] \subset W^{k+1,2s}$ and $d''_x[W^{k,2s}] \subset W^{k-1,2(s+1)}$, we assert that

$$[d_x]_l(\varphi_x \otimes \Gamma_x) = \begin{cases} d_x(\varphi_x \otimes \Gamma_x) \text{ when } \Gamma_x \in V^{<l}\mathbf{g}^*_{|x}\,, \\ d'_x(\varphi_x \otimes \Gamma_x) \text{ whem } \Gamma_x \in V^{l}\mathbf{g}^*_{|x}\,. \end{cases}$$

Put

$$(W\mathbf{g})_l := \Lambda\mathbf{g}^* \otimes V^{<l}\mathbf{g}^* \text{ and } (W\mathbf{g})_l := Sec(W\mathbf{g})_l.$$

Of course,

$$(W\mathbf{g})_l \cong (W\mathbf{g})/\mathbf{F}^{2(l+1)}(W\mathbf{g})$$

$(\Theta \longmapsto [\Theta]$ establishes the canonical isomorphism$)$.

$(W\mathbf{g})_l$ will be called the *truncated Weil algebra* of the vector bundle $\mathbf{g}$.

The family $[d_x]_l$, $x \in M$, determines an endomorphism $[d]_l : (W\mathbf{g})_l \longrightarrow (W\mathbf{g})_l$ and, denoted by the same letter, a differential

$$[d]_l : (W\mathbf{g})_l \longrightarrow (W\mathbf{g})_l.$$

For $s < l$, the projection $(W\mathbf{g})_l \longrightarrow (W\mathbf{g})_s$ is a homomorphism of algebras commuting with the differentials $[d]_l$ and $[d]_s$.

Take the canonical adjoint representation ad of A on $(W\mathbf{g})_l$ and denote by $(W\mathbf{g})_{l,I^\circ}$ the space $($de facto, the subalgebra of $(W\mathbf{g})_l)$ of invariant cross-sections.

$(W\mathbf{g})_{l,I^\circ}$ is stable under the operator $[d]_l$. Indeed, let Θ be a bihomogeneous element of $(W\mathbf{g})_{l,I^\circ}$. Then $d\Theta$ is invariant, in particular, $d'\Theta$ is invariant; $[d]_l\Theta$, being equal to $d\Theta$ or $d'\Theta$, is invariant, too.

Let λ be any connection in A and let $\mathbf{k}\colon W\mathbf{g}\longrightarrow\Omega(A)$ be the homomorphism of algebras determined by λ.

Proposition 3.2. *Assume that* $\mathbf{k}[F^{2(l+1)}(W\mathbf{g})]=0$. *Then*

(1) *there exists a homomorphism of algebras* $[\mathbf{k}]_l\colon(W\mathbf{g})_l\longrightarrow\Omega_A(M)$ *such that the following diagram*

$$\begin{tikzcd} (W\mathbf{g})_l \arrow[r, "{[\mathbf{k}]_l}"] & \Omega_A(M) \\ W\mathbf{g} \arrow[u, "\pi"] \arrow[ur, "\mathbf{k}"'] & \end{tikzcd}$$

commutes $\big(\pi$ *being the canonical projection*$\big)$,

(2) $[\mathbf{k}]_l$ *is equal to the restriction* $\mathbf{k}|(W\mathbf{g})_l$,

(3) $[\mathbf{k}]_l$ *restricted to the invariant cross-sections* $(W\mathbf{g})_{l,I^\circ}$ *commutes with the differentials* $[d]_l$ *and* d^A, *defining a homomorphism of algebras*

$$[\mathbf{k}]_{l\#}\colon H\big((W\mathbf{g})_{l,I^\circ},[d]_l\big)\longrightarrow H_A(M).$$

The class $[\mathbf{k}]_{l\#}[\Theta]$ *for* $\Theta\in(Sec\bigwedge^k\mathbf{g}^*\otimes\bigvee^l\mathbf{g}^*)_{I^\circ}$, $s\le l$, *has the form* $\frac{1}{k!s!}\cdot\langle\Theta,\underbrace{\omega\wedge\ldots\wedge\omega}_{k\text{ times}}\otimes\underbrace{\Omega\vee\ldots\vee\Omega}_{s\text{ times}}\rangle$ *as its representative.*

Proof. (1) and (2) are evident.

(3): Let $\Theta\in(W\mathbf{g})_{l,I^\circ}$ $(\subset(W\mathbf{g})_{I^\circ})$. By 1.21,

$$d^A\circ[\mathbf{k}]_l(\Theta)=d^A\circ\mathbf{k}(\Theta)=\mathbf{k}\circ d(\Theta)=[\mathbf{k}]_l\circ\pi\circ d(\Theta)=[\mathbf{k}]_l\circ[d]_l\circ\pi(\Theta)=[\mathbf{k}]_l\circ[d]_l(\Theta).$$

The last sentence is a consequence of II.2.2, II.2.5 and 1.22. ∎

Example 3.3. Assume that A is equipped with a flat partial connection λ' over $E'\subset E$ $\big($as in 2.5$\big)$ and let $q=rank(E/E')$. According to Corollary 2.6, $\mathbf{k}[F^{2(q+1)}(W\mathbf{g})]=0$ for an adapted connection λ, and $\mathbf{k}[F^{2([q/2]+1)}(W\mathbf{g})]=0$ for a basic connection λ. Prop.3.2 produces in these situations the homomorphisms of algebras $[\mathbf{k}]_{q'}\colon(W\mathbf{g})_{q'}\longrightarrow\Omega_A(M)$ for $q'\geqslant q$ and $q'\geqslant[q/2]$, respectively, and next, the corresponding homomorphisms on cohomologies. The homomorphism $[\mathbf{k}]_{q\#}\colon H\big((W\mathbf{g})_q,[d]_q\big)\longrightarrow H_A(M)$ generalizes the $\omega_\#$ described in II.2.11: in the case when $E'=E$, i.e. when λ' is a flat connection in A, we have $q=0$ and $[\mathbf{k}]_0=\omega_0^\wedge$.

Let $H: A_1 \longrightarrow A_2$ be a homomorphism of regular Lie algebroids over $f:(M,E_1) \longrightarrow (M,E_2)$. Define the pullback

$$H^{+*}: W(\mathbf{g}_2) \longrightarrow W(\mathbf{g}_1)$$

in the standard way: $H^{+*}(\Psi)(x) = H_x^{+*}(\Psi(f(x)))$, $x \in M$, where $H_x^{+*} = \wedge H^{+*}_{|x} \otimes \vee H^{+*}_{|x}$. It is clear that H^{+*} is a homomorphism of algebras of the bidegree (0,0).

Proposition 3.4. *The pullback H^{+*} has the following properties:*

(1) $i_{x,v} \circ H_x^{+*} = H_x^{+*} \circ i_{f(x),H^+(v)}$ *for* $v \in \mathbf{g}_{1|x}$, $x \in M_1$;
in consequence, H^{+*} *maps* $\mathbf{h}_2$*-horizontal elements into* $\mathbf{h}_1$*-horizontal ones.*

(2) $\delta \circ H^{+*}(\Theta) = H^{+*} \circ \delta(\Theta)$ *for* $\Theta \in Sec \wedge \mathbf{g}_2^*$, *where* δ*'s denote the differentials described in* II.2.6.

(3) $\varphi \circ H^{+*} = H^{+*} \circ \varphi$ *where* φ*'s denote the change of variables, see chapter* 1.B.

(4) $d \circ H^{+*} = H^{+*} \circ d$.

(5) $[d]_{l_1} \circ H^{+*} = H^{+*} \circ [d]_{l_2}$ for $l_1 \leq l_2$,

(6) $\mathcal{L}_{ad(v)} \circ H^{+*} = H_x^{+*} \circ \mathcal{L}_{ad(Hv)}$ *for* $v \in A_{1|x}$; *in consequence,* H^{+*} *maps invariant elements into invariant ones.*

Proof. (1): By the homomorphy of H_x^{+*} and the antiderivativity of $i_{x,v}$'s, it is sufficient to check the equality for the elements of $W(\mathbf{g}_{2|f(x)})$ of the forms $\theta \otimes 1$ and $1 \otimes \theta$, where $\theta \in \mathbf{g}^*_{2|f(x)}$.

$$i_{x,v} \circ H_x^{+*}(\theta \otimes 1) = i_{x,v}(H_x^{+*}\theta \otimes 1) = i_{x,v}(H_x^{+*}\theta) = \langle\theta, H^+(v)\rangle = i_{f(x),H^+(v)}(\theta)$$
$$= H_x^{+*} \circ i_{f(x),H^+(v)}(\theta).$$

$$i_{x,v} \circ H_x^{+*}(1 \otimes \theta) = i_{x,v}(1 \otimes H_x^{+*}\theta) = 0 = H_x^{+*} \circ i_{f(x),H^+(v)}(1 \otimes \theta).$$

(2): δ's are antiderivations, therefore it is sufficient to consider $\Theta \in Sec\, \mathbf{g}_2^*$. For $x \in M_1$ and $w_i \in \mathbf{g}_{1|x}$,

$$\langle(\delta \circ H^{+*}\Theta)_x, w_0 \wedge w_1\rangle = \langle(H^{+*})_x, [w_0,w_1]\rangle = \langle\Theta_{f(x)}, H_x^+[w_0,w_1]\rangle = \langle\Theta_{f(x)}, [H_x^+(w_0), H_x^+(w_1)]\rangle$$

$$= \langle(\delta\Theta)_{f(x)}, H_x^+(w_0) \wedge H_x^+(w_1)\rangle = \langle(H^{+*} \circ \delta(\Theta))_x, w_0 \wedge w_1\rangle.$$

(3): By (2) above,

$$\varphi \circ H^{+*}(1) = H^{+*} \circ \varphi(1),$$

$$\varphi\circ H^{+*}(\Theta\otimes 1)=\varphi(H^{+*}\Theta\otimes 1)=H^{+*}\Theta\otimes 1=H^{+*}\circ\varphi(\Theta\otimes 1),$$
$$\varphi\circ H^{+*}(1\otimes\Theta)=\varphi(1\otimes H^{+*}\Theta)=1\otimes H^{+*}\Theta-\delta(H^{+*}\Theta)\otimes 1=1\otimes H^{+*}\Theta-H^{+*}(\delta\Theta)\otimes 1$$
$$=H^{+*}(1\otimes\Theta-\delta\Theta\otimes 1)=H^{+*}\circ\varphi(1\otimes\Theta).$$

The general formula follows from the homomorphy of φ and H^{+*}.

(4): Thanks to the previous property, (4) follows from the equality $\bar{d}\circ H^{+*}=H^{+*}\circ\bar{d}$ which can be checked trivially.

(5): (4) yields $d'\circ H^{+*}=H^{+*}\circ d'$ and the two imply (5) immediately.

(6): First, we show, for $\Theta\in Sec\,\mathbf{g}_2^*$, that

$$<\mathcal{L}_{ad^\natural\circ\xi_1}(H^{+*}\Theta),\nu_1>=<H^{+*}(\mathcal{L}_{ad^\natural\circ\xi_2}\Theta),\nu_1>$$

where $\nu_1\in Sec\,\mathbf{g}_1$ is a cross-section for which there exists $\nu_2\in Sec\,\mathbf{g}_2$ fulfilling $H^+\circ\nu_1=\nu_2\circ f$. For the purpose, we notice [17] that $H^+\circ[\![\xi_1,\nu_1]\!]=[\![\xi_2,\nu_2]\!]\circ f$. Thus

$$<\mathcal{L}_{ad^\natural\circ\xi_1}(H^{+*}\Theta),\nu_1>=(\gamma_1\circ\xi_1)<H^{+*}\Theta,\nu_1>-<H^{+*}\Theta,[\![\xi_1,\nu_1]\!]>$$
$$=(\gamma_2\circ\xi_2)<\Theta,\nu_2>\circ f-<\Theta,[\![\xi_2,\nu_2]\!]>\circ f=<\mathcal{L}_{ad^\natural\circ\xi_2}\Theta,\nu_2>\circ f=<H^{+*}(\mathcal{L}_{ad^\natural\circ\xi_2}\Theta),\nu_1>.$$

Lemma 1.6(1) leads now to the equality

$$<\mathcal{L}_{ad\circ\xi_1}(H^{+*}\Psi),\nu_1>=<H^{+*}(\mathcal{L}_{ad\circ\xi_2}\Psi),\nu_1>,\tag{9}$$

$\Psi\in W\mathbf{g}_2$, where ν_1, ξ_i are as above.

The equality $\mathcal{L}_{ad\circ\xi_1}(H^{+*}\Psi)=H^{+*}(\mathcal{L}_{ad\circ\xi_2}\Psi)$ follows in an evident manner from those written for a strong homomorphism and for the canonical one. In each of these cases, this follows from (9) and the observation which says

— *for arbitrarily taken $x\in M_1$ and $v\in\mathbf{g}_{1|x}$, there exist local cross-sections ν_1 and ν_2 such that $\nu_1(x)=v$ and ν_1 and ν_2 fulfil the required condition $H^+\circ\nu_1=\nu_2\circ f$.*

By the analogous reasoning, we assert equality (6). ■

Corollary 3.5. *H^{+*} maps $\mathbf{h}_2$-horizontal and invariant elements into $\mathbf{h}_1$-horizontal and invariant ones, defining, for $l_2\geq l_1$, a homomorphism of algebras*

$$H^{+*}:(W\mathbf{g}_2)_{l_2,\mathbf{h}_2,I^\circ}\longrightarrow(W\mathbf{g}_1)_{l_1,\mathbf{h}_1,I^\circ}$$

commuting with the differentials (i.e. $[d]_{l_1}\circ H^{+*}=H^{+*}\circ[d]_{l_2}$). ■

Let us assume that in A_i we are given some regular Lie subalgebroid B_i over (M,E_i), $i=1,2$, and that $H[B_1]\subset B_2$. In the standard way, one can define the pullback

$$[H]^{+*}:W(\mathbf{g}_2,\mathbf{h}_2)\longrightarrow W(\mathbf{g}_1,\mathbf{h}_1)$$

$\left([H]^{+*}(\Psi)(x)=\Lambda[H_x^+]^*\otimes\mathrm{V}H_x^+(\Psi(x))\right.$ where $[H_x^+]^*:\mathbf{g}_1/\mathbf{h}_1\longrightarrow\mathbf{g}_2/\mathbf{h}_2$ is the induced linear homomorphism$\left.\right)$. Since the diagram

$$\begin{tikzcd}
W(\mathbf{g}_1,\mathbf{h}_1)_{I_1,I^\circ} \arrow[r, "\cong"] & W(\mathbf{g}_1)_{I_1,\mathbf{h}_1,I^\circ} \\
W(\mathbf{g}_2,\mathbf{h}_2)_{I_2,I^\circ} \arrow[r, "\cong"] \arrow[u, "{[H]^{+*}}"] & W(\mathbf{g}_2)_{I_2,\mathbf{h}_2,I^\circ} \arrow[u, "H^{+*}"']
\end{tikzcd}$$

commutes, we obtain - by the above - that $[H]^{+*}$ commutes with the differentials, i.e.

$$H^{+*}\circ[d]_{I_2,\mathbf{h}_2}=[d]_{I_1,\mathbf{h}_1}\circ[H^{+*}],$$

giving a homomorphism on cohomologies

$$[H]^{+*}:H(W(\mathbf{g}_2,\mathbf{h}_2)_{I_2,I^\circ})\longrightarrow H(W(\mathbf{g}_1,\mathbf{h}_1)_{I_1,I^\circ}).$$

4. CHARACTERISTIC HOMOMORPHISM - ITS CONSTRUCTION

Here we construct some characteristic homomorphism of a partially flat regular Lie algebroid, being a generalization of the one constructed in II for a flat regular Lie algebroid.

Consider, in a given regular Lie algebroid A over (M,E), two geometric structures:

(1) a partial flat connection λ' over an involutive subdistribution $E'\subset E$,

(2) a subalgebroid $B\subset A$ over (M,E), see the following diagram:

$$\begin{tikzcd}
 & \gamma^{-1}[E'] \arrow[d, equal] & & \\
0 \arrow[r] & \mathbf{g} \arrow[r, hook] \arrow[d, equal] & A' \arrow[r, shift left, "\gamma"] \arrow[d, hook] & E' \arrow[l, shift left, "\lambda'"] \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbf{g} \arrow[r, hook] & A \arrow[r, "\gamma_A"] & E \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{g} \arrow[r, hook] \arrow[u, hook] & B \arrow[r, "\gamma_B"] \arrow[u, hook, "j"] & E \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

The system (A,B,λ') will be called a *PFS-regular Lie algebroid* $\big($*over* $(M,E,E')\big)$.

The construction of the characteristic homomorphism of a *PFS*-regular Lie algebroid has, as in the case of an *FS*-regular Lie algebroid, a number of steps.

4.1. Let $\mathbf{s}:\mathbf{g}\longrightarrow\mathbf{g}/\mathbf{h}$ denote, as in II.3.1, the canonical projection. Put, for a positive integer l

$$W(\mathbf{g};\mathbf{h})_l := \Lambda(\mathbf{g}/\mathbf{h})^*\otimes\mathrm{V}^{<l}\mathbf{g}^* \quad\text{and}\quad W(\mathbf{g};\mathbf{h})_l := Sec\, W(\mathbf{g};\mathbf{h})_l .$$

$W(\mathbf{g};\mathbf{h})_l$ with the natural structure of an algebra will be called the *truncated relative Weil algebra*.

The representation $ad^{\wedge}_{B,\mathbf{g}}$ of B on $\Lambda(\mathbf{g}/\mathbf{h})^*$ described in II.3.3, together with the representation $ad|B$ of B on $\mathrm{V}^{<l}\mathbf{g}^*$ $\big($being the restriction to B of the adjoint representation of A on $\mathrm{V}^{<l}\mathbf{g}^*\big)$, yields the representation of B on $W(\mathbf{g};\mathbf{h})_l$ denoted also - for brevity - by ad.

For an arbitrary $\xi\in Sec\,B$, the differential operator $\mathcal{L}_{ad\circ\xi}:W(\mathbf{g};\mathbf{h})_l\longrightarrow W(\mathbf{g};\mathbf{h})_l$ is a differentiation of the truncated relative Weil algebra $W(\mathbf{g};\mathbf{h})_l$, from which we obtain that the space $W(\mathbf{g};\mathbf{h})_{l,I^\circ}$ of invariant cross-sections is a subalgebra of $W(\mathbf{g};\mathbf{h})_l$.

The monomorphisms

$$\Lambda\mathbf{s}^*:\Lambda(\mathbf{g}/\mathbf{h})^*\longrightarrow\Lambda\mathbf{g}^* \text{ and } \Lambda\mathbf{s}^*\otimes id^l:\Lambda(\mathbf{g}/\mathbf{h})^*\otimes\mathrm{V}^{<l}\mathbf{g}^*\longrightarrow\Lambda\mathbf{g}^*\otimes\mathrm{V}^{<l}\mathbf{g}^*$$

of vector bundles are invariant with respect to the representations considered of the Lie algebroid B, which is easy to see by the definitions. As a corollary from the above and the monomorphy of $\Lambda\mathbf{s}^*\otimes id^l$ we obtain that $\Lambda\mathbf{s}^*\otimes id^l\circ\Psi$, $\Psi\in W(\mathbf{g};\mathbf{h})_l$, is an invariant cross-section if and only if Ψ is invariant, and that

$$W(\mathbf{g};\mathbf{h})_{1,I^\circ} \ni \Psi \longmapsto \wedge\mathbf{s}^* \otimes id^1 \circ \Psi \in W(\mathbf{g})_{1,I^\circ}$$

is a homomorphism of algebras. On the other hand, a cross-section Ψ' of $W(\mathbf{g})_1$ is of the image of some cross-section of the bundle $W(\mathbf{g};\mathbf{h})_1$ if and only if Ψ' is $\mathbf{h}$-horizontal (i.e. if and only if $i_\nu\Psi = 0$ for $\nu \in Sec\,\mathbf{h}$, where i_ν is the operator defined in Section 1.C), so

$$W(\mathbf{g};\mathbf{h})_{1,I^\circ} \longrightarrow W(\mathbf{g})_{1,\mathbf{h},I^\circ}, \quad \Psi \longmapsto \wedge\mathbf{s}^* \otimes id^1 \circ \Psi,$$

is an isomorphism of algebras.

4.2. The subspace $W(\mathbf{g})_{1,\mathbf{h},I^\circ}$ is stable under the differential $[d]_1 : W(\mathbf{g})_1 \longrightarrow W(\mathbf{g})_1$. Indeed, for an invariant element Ψ' of $W(\mathbf{g})_1$, we have $i_\nu \circ d(\Psi') = -d \circ i_\nu(\Psi')$ by 1.6(2), 1.16(1), 1.16(3), and, in consequence, $i_\nu \circ d'(\Psi') = -d' \circ i_\nu(\Psi')$. Therefore, for a bihomogeneous element $\Psi' \in W(\mathbf{g})_1$, $i_\nu \circ [d]_1(\Psi') = i_\nu(d\Psi') = -d(i_\nu\Psi') = 0$ or $i_\nu \circ [d]_1(\Psi') = i_\nu(d'\Psi') = -d'(i_\nu\Psi') = 0$, see Section 3. This enables us to define the differential $d_{1,\mathbf{h}} : W(\mathbf{g};\mathbf{h})_{1,I^\circ} \longrightarrow W(\mathbf{g};\mathbf{h})_{1,I^\circ}$ in such a way that the following diagram commutes:

$$\begin{tikzcd}
W(\mathbf{g};\mathbf{h})_{1,I^\circ} \arrow[r, "\cong"] \arrow[d, "d_{1,\mathbf{h}}"] & W(\mathbf{g})_{1,\mathbf{h},I^\circ} \arrow[d, "{[d]_1}"] \\
W(\mathbf{g};\mathbf{h})_{1,I^\circ} \arrow[r, "\cong"] & W(\mathbf{g})_{1,\mathbf{h},I^\circ}
\end{tikzcd} \tag{10}$$

4.3. Consider any connection $\lambda : E \longrightarrow A$ in A and let the homomorphism $\mathbf{k} : W\mathbf{g} \longrightarrow \Omega_A(M)$ be constructed for λ. The form $\varphi(\Psi) := [\mathbf{k}]_1(\wedge\mathbf{s}^* \otimes id^1 \circ \Psi)$, $\Psi \in W(\mathbf{g};\mathbf{h})_1$, is $\mathbf{h}$-horizontal, which follows in an easy way from 1.17 and 1.23. Therefore, the form $j^*(\varphi(\Psi)) \in \Omega_B(M)$ is horizontal. Then (see II.2) there exists a form $\Delta\Psi \in \Omega_E(M)$ such that

$$(\gamma_B)_*(\Delta\Psi) = j^*([\mathbf{k}]_1(\wedge\mathbf{s}^* \otimes id^1 \circ \Psi)).$$

4.4. Remark. One can easily check that if λ is a connection in B, then, for $\Psi \in Sec\wedge(\mathbf{g}/\mathbf{h})^* \otimes \vee^1\mathbf{g}^*$,

$$\Delta\Psi = \begin{cases} 0 & \text{when } k > 0, \\ \lambda_*(\Omega^\vee\Psi) & \text{when } k = 0. \end{cases}$$

4.5. Let $q = rank(E/E')$ and let λ be adapted to λ'. Defined in the above manner, the mapping

$$\Delta_{q'}: W(\mathbf{g};\mathbf{h})_{q'} \longrightarrow \Omega_E(M), \quad \Psi \longmapsto \Delta\Psi,$$

$q' \geq q$ (and $q' \geq [q/2]$ in the case of a basic connection), is a homomorphism of algebras, see Example 3.3, and the diagram

$$\begin{tikzcd}
W(\mathbf{g};\mathbf{h})_{q'} \arrow[rr, "\Delta_{q'}"] \arrow[dd, dash] \arrow[ddrr, "\varphi"] & & \Omega_E(M) \arrow[d, "(\gamma_B)_*", "\cong"'] \\
& & \Omega_{B,i}(M) \\
(W\mathbf{g})_{q'} \arrow[r, "{[\mathbf{k}]_{q'}}"] & \Omega_A(M) \supset & \Omega_{A,\mathbf{h}}(M) \arrow[u, "j^*"', "\cong"]
\end{tikzcd} \tag{11}$$

commutes.

Proposition 4.6. *The mapping $\Delta_{q'}$ restricted to the invariant cross-sections*

$$\Delta_{q'*}: W(\mathbf{g};\mathbf{h})_{q',I^\circ} \longrightarrow \Omega_E(M)$$

commutes with the differentials $d_{q',\mathbf{h}}$ and d^E.

Proof. j and γ_B are homomorphisms of regular Lie algebroids; then, according to II.(5), the commutativity of j^* with the differentials d^A and d^B, and seeing the last diagram and the definition of $d_{1,\mathbf{h}}$, we notice that it is sufficient to have the commuting of $[\mathbf{k}]_{q'}: W(\mathbf{g})_{q',\mathbf{h},I^\circ} \longrightarrow \Omega_A(M)$ with $[d]_{q'}$ and d^A, but this follows from 3.2(3). ∎

Theorem 4.7. *The mapping*

$$\Delta_{q'\#}: H\big(W(\mathbf{g};\mathbf{h})_{q',I^\circ}, d_{q',\mathbf{h}}\big) \longrightarrow H_E(M)$$
$$[\Psi] \longmapsto [\Delta_{q'*}\Psi]$$

is a correctly defined homomorphism of algebras. ∎

4.8. If λ is basic, then the following diagram commutes:

$$\begin{tikzcd}[column sep=small]
H\big(W(\mathbf{g};\mathbf{h})_{q,I^\circ}\big) \arrow[d] \arrow[dr, "\Delta_{q\#}"] & \\
& H_E(M) \\
H\big(W(\mathbf{g};\mathbf{h})_{[q/2],I^\circ}\big) \arrow[ur, "\Delta_{[q/2]\#}"'] &
\end{tikzcd}$$

in which the vertical arrow is a homomorphism of algebras, induced by the projection.

$\Delta_{q\#}$ (also $\Delta_{[q/2]\#}$ for a basic connection) is called the *characteristic homomorphism* of the *PFS*-regular Lie algebroid (A,B,λ'). Its image is a subalgebra of $H_E(M)$ called the *characteristic algebra* of the *PFS*-regular Lie algebroid (A,B,λ'), whereas its elements - the *characteristic classes* of this algebroid.

5. THE FUNCTORIALITY AND OTHER PROPERTIES

Let (A_1,B_1,λ_1') and (A_2,B_2,λ_2') be two *PFS*-regular Lie algebroids over (M_1,E_1,E_1') and (M_2,E_2,E_2'), respectively.

Definition 5.1. By a *homomorphism*

$$H:(A_1,B_1,\lambda_1')\longrightarrow(A_2,B_2,\lambda_2')$$

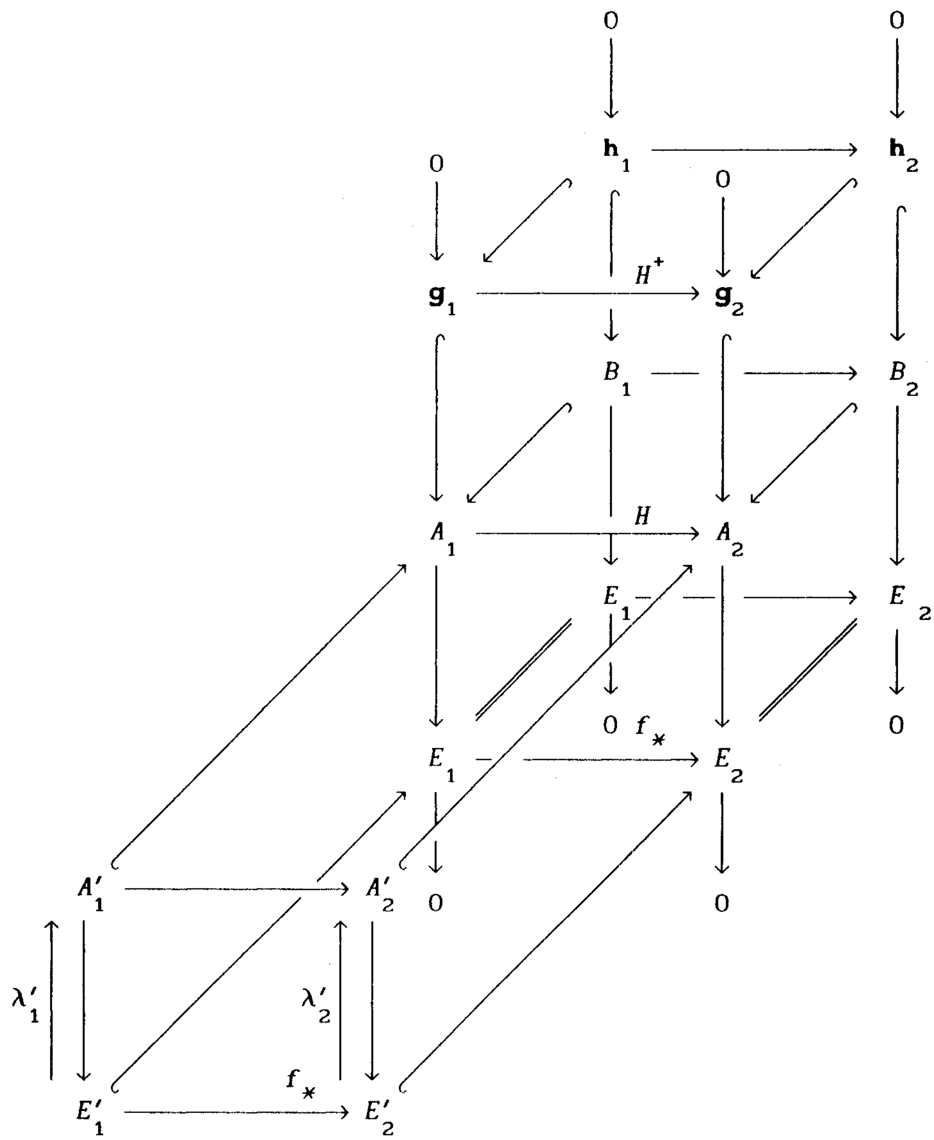
between them we mean a homomorphism $H:A_1\longrightarrow A_2$ of regular Lie algebroids, say over $f:(M_1,E_1)\longrightarrow(M_2,E_2)$, such that

(1) $f_*[E_1']\subset E_2'$,

(2) $H\circ\lambda_1'=\lambda_2'\circ f_*|E_1'$,

(3) $H[B_1]\subset B_2$.

In the sequel, the notations of some objects related to A_1, B_1 and H will be the same as in the following diagram:



(12)

By the *pullback* of a *PFS*-regular Lie algebroid (A,B,λ') over (M,E,E') via a mapping $f:(M_1,E_1,E'_1)\longrightarrow(M,E,E')$, i.e. a mapping $f:M_1\longrightarrow M$ such that $f_*[E_1]\subset E$ and $f_*[E'_1]\subset E'$, we mean the *PFS*-regular Lie algebroid $(f^\wedge A,f^\wedge B,\bar{\lambda}')$ where $\bar{\lambda}':E'_1\longrightarrow f^\wedge A_1$ is the pullback of λ' [17; 3.2.1]: $\bar{\lambda}'(v)=(v,\lambda'(f_*v))$, $v\in E'_1$. Proposition [17; 3.2.2] gives the flatness of $\bar{\lambda}'$. The canonical homomorphism $pr_2:f^\wedge A\longrightarrow A$ is a homomorphism of *PFS*-regular Lie algebroids.

Each homomorphism $H:(A_1,B_1,\lambda'_1)\longrightarrow(A_2,B_2,\lambda'_2)$ of *PFS*-regular Lie algebroids can be represented in the form of a superposition of a strong homomorphism with the canonical one:

$$(A_1,B_1,\lambda'_1)\xrightarrow{\bar{H}}(f^\wedge A_2,f^\wedge B_2,\lambda'_2)\xrightarrow{pr_2}(A_2,B_2,\lambda'_2).$$

Theorem 5.2. (*The functoriality of* $\Delta_{q\#}$).

Let (A_1,B_1,λ'_1) *and* (A_2,B_2,λ'_2) *be two PFS-regular Lie algebroids over* (M_1,E_1,E'_1) and (M_2,E_2,E'_2), *respectively; put* $q_1=rank(E_1/E'_1)$. *Let* $H:(A_1,B_1,\lambda'_1)\longrightarrow(A_2,B_2,\lambda'_2)$ *be a homomorphism between them over* $f:(M_1,E_1,E'_1)\longrightarrow(M_2,E_2,E'_2)$. *Assume that the adapted connections* λ_1 *and* λ_2, *such that* $H\circ\lambda_1=\lambda_2\circ f_*$, *are given. Then the following diagram*

$$\begin{array}{ccc} H(W(\mathbf{g}_2,\mathbf{h}_2)_{\max(q_1,q_2)},I^\circ) & \xrightarrow{\Delta_{\max(q_1,q_2)\#}} & H(E_2) \\ \big\downarrow{\scriptstyle [H]^{+\#}} & & \big\downarrow{\scriptstyle f^{\#}} \\ H(W(\mathbf{g}_1,\mathbf{h}_1)_{q_1},I^\circ) & \xrightarrow{\Delta_{q_1\#}} & H(E_1) \end{array}$$

commutes.

Proof. From the commutativity of diagrams (10) and (11) it follows that it is sufficient to check the same for the diagram

$$\begin{array}{ccc} W(\mathbf{g}_2) & \xrightarrow{\mathbf{k}_2} & \Omega_{A_2}(M) \\ \big\downarrow{\scriptstyle H^{+*}} & & \big\downarrow{\scriptstyle H^{*}} \\ W(\mathbf{g}_1) & \xrightarrow{\mathbf{k}_1} & \Omega_{A_1}(M) \end{array}$$

in which $\mathbf{k}_2$ and $\mathbf{k}_1$ are defined for λ_2 and λ_1, respectively. Using the equalities (a) $H^+_*\omega_1=H^{+*}\omega_2$ and (b) $H^+_*\Omega_1=H^{+*}\Omega_2$ $\big(\omega_i$ and Ω_i being the connection form and the curvature form of λ_i, $i=1,2\big)$, one can prove $\big($without any difficulties$\big)$ the commutativity at each point $x\in M$, considering the generators 1, $1\otimes\theta$, $\theta\otimes 1$, $\theta\in\mathbf{g}^*_{2|f(x)}$, only. Equality (a) is evident, whereas (b) follows from $\big[17;3.2.2\big]$ and the horizontality of the curvature forms. ■

5.3 Theorem $\big($*The independence of* $\Delta_{q'\#}$ *of an adapted connection*$\big)$.

For any PFS-regular Lie algebroid (A,B,λ') *over* (M,E,E'), *the characteristic homomorphism* $\Delta_{q'\#}:H(W(\mathbf{g},\mathbf{h})_{q'},I^\circ)\longrightarrow H_E(M)$, $q'\geq rank(E/E')$, *is independent of the choice of an adapted connection.*

Proof. Let us consider any two connections $\lambda_0,\lambda_1:E\longrightarrow A$ adapted to λ' and the connection $\lambda:T\mathbb{R}\times E\longrightarrow T\mathbb{R}\times A$ in $T\mathbb{R}\times A$ defined by $\lambda_{|(t,x)}(v,w)=(v,\lambda_0(w)\cdot(1-t)+\lambda_1(w)\cdot t)$, $(v,w)\in T_t\mathbb{R}\times E_{|x}$. λ is adapted to the flat partial connection $id\times\lambda':T\mathbb{R}\times E'\longrightarrow T\mathbb{R}\times A'$. Of course, the system $(T\mathbb{R}\times A,T\mathbb{R}\times B,id\times\lambda')$ is a PFS-regular Lie algebroid and λ is an adapted connection. One can prove that the connection form $\omega:T\mathbb{R}\times A\longrightarrow 0\times\mathbf{g}$ of λ equals

$\omega_{(t,x)}(v,w)=(0,\omega_0(w)\cdot(1-t)+\omega_1(w)\cdot t)$, $(v,w)\in T_t\mathbb{R}\times A_{|x}$, where ω_0 and ω_1 are the connection forms of λ_0 and λ_1, respectively. The homomorphisms $F_i:A\longrightarrow T\mathbb{R}\times A$, $i=0,1$, of regular Lie algebroids (over $f_i:M\longrightarrow\mathbb{R}\times M$, $x\longmapsto(i,x)$), defined in II.5, give homomorphisms $F_i:(A,B,\lambda')\longrightarrow(T\mathbb{R}\times A,T\mathbb{R}\times B,id\times\lambda')$ of *PFS*-regular Lie algebroids such that $F_i\circ\lambda_i=\lambda\circ f_{i*}$, $i=0,1$. The principle of functoriality (Theorem 5.2) ensures the commutativity of the diagrams

$$\begin{array}{ccc}
H(W(0\times\mathbf{g},0\times\mathbf{h})_{q'}) & \xrightarrow{\Delta_{q'\#}} & H_{T\mathbb{R}\times E}(\mathbb{R}\times M)\\
\Big\downarrow [F_i]^{+\#} & & \Big\downarrow f_i^{\#}\\
H(W(\mathbf{g},\mathbf{h})_{q'}) & \xrightarrow{\Delta_{iq'\#}} & H_E(M)
\end{array},$$

$i=0,1$. Since $f_0^{\#}=f_1^{\#}$ (see the proof of Th.4.3.1 from [17]) and the superposition $A\xrightarrow{F_1}T\mathbb{R}\times A\xrightarrow{G}A$, $G:=pr_2$, of homomorphisms of regular Lie algebroids being equal to id_A, gives $[F_i]^{+\#}\circ[G]^{+\#}=id$ (G does not determine a *PFS*-homomorphism, but this is no problem), therefore we have

$$\begin{aligned}
\Delta_{0q'\#}&=\Delta_{0q'\#}\circ[F_0]^{+\#}\circ[G]^{+\#}=f_0^{\#}\circ\Delta_{0q'\#}\circ[G]^{+\#}=f_1^{\#}\circ\Delta_{0q'\#}\circ[G]^{+\#}\\
&=\Delta_{1q'\#}\circ[F_1]^{+\#}\circ[G]^{+\#}=\Delta_{1q'\#}. \qquad \blacksquare
\end{aligned}$$

Definition 5.4. Let us consider two *PFS*-regular Lie algebroids (A,B_i,λ'), $i=0,1$ (which differ only in subalgebroids) over (M,E,E'). By analogy with definition II.5.8, we say that the characteristic homomorphisms $\Delta_{iq'\#}:H(W(\mathbf{g},\mathbf{h}_i)_{q'},_{I^\circ})\longrightarrow H_E(M)$, $i=0,1$, $q'\geqslant rank(E/E')$, are *equivalent* if there exists an isomorphism of algebras $\alpha:H(W(\mathbf{g},\mathbf{h}_0)_{q'},_{I^\circ})\longrightarrow H(W(\mathbf{g},\mathbf{h}_1)_{q'},_{I^\circ})$, such that $\Delta_{0q'\#}=\Delta_{1q'\#}\circ\alpha$.

Theorem 5.5. *If B_0 and B_1 are homotopic (for definition, see* II.5.2*), then $\Delta_{0q'\#}$ and $\Delta_{1q'\#}$ are equivalent.*

Proof. By the same argument as in the proof of Prop.II.5.7, we assert that $\Delta_{0q'\#}$ and $\Delta_{1q'\#}$ are related via the commutative diagram:

$$\begin{array}{ccc}
H(W(\mathbf{g},\mathbf{h}_0)_{q'}) & & \\
\uparrow [F_0]^{+\#} & \searrow \Delta_{0q'\#} & \\
H(W(0\times\mathbf{g},\mathbf{h}\ )_{q'}) & & H_E(M) \\
\downarrow [F_1]^{+\#} & \nearrow \Delta_{1q'\#} & \\
H(W(\mathbf{g},\mathbf{h}_1)_{q'}) & & .
\end{array}$$

It remains to show that $[F_i]^{+\#}$ is an isomorphism of algebras, $i=0,1$. We do it as in the proof of Th.II.5.9:

For F_i being equal to the superposition $pr_2\circ\bar{F}_i$ (in which $\bar{F}_i$ is an isomorphism), the problem reduces to the consideration of the canonical projection $pr_2:(f_i^{\wedge}(T\mathbb{R}\times A),f_i^{\wedge}(B),(id\times\lambda')^{-})\longrightarrow(T\mathbb{R}\times A,B,id\times\lambda')$, more exactly, to the investigation of the homomorphism

$$pr_2^{+*}:W(0\times\mathbf{g},\mathbf{h})_{q',I^\circ}\longrightarrow W(f_i^*(0\times\mathbf{g}),f_i^*\mathbf{h})_{q',I^\circ}.$$

After the canonical identification

$$f_i^*\left(\wedge(0\times\mathbf{g}/\mathbf{h})^*\otimes\vee^{<q'}(0\times\mathbf{g})^*\right)\cong\wedge(f_i^*(0\times\mathbf{g})/f_i^*\mathbf{h})^*\otimes\vee^{<q'}f_i^*(0\times\mathbf{g})^*,$$

according to II.(11) (see Chapter II.4) and the fact that $f^*(\otimes^k T)=\otimes^k(f^*T)$ for any representation T (cf. [17; 2.3.3] and the proof of Prop.II.4.2.1), we obtain that $f_i^{\wedge}(ad)=ad$ (the ad's denote the canonical representations induced by the adjoint one), and that $pr_2^{+*}\Psi=f_i^*\Psi$. As in the proof of Th.II.5.9, the rest follows from Prop.I.6.2. ∎

5.6. A comparison with the tangential classes of partially flat principal bundles.

A *PFS*-regular Lie algebroid (A,B,λ') over (M,E,E') determines an *FS*-regular Lie algebroid (A',B',λ') over (M,E') in which $A'=\gamma_A^{-1}[E']$, $B'=\gamma_B^{-1}[E']$. With these objects we have associated some homomorphisms: $\Delta_{q'*}:W(\mathbf{g},\mathbf{h})_{q',I_B^\circ}\longrightarrow\Omega_E(M)$ and $\Delta_*:W(\mathbf{g},\mathbf{h})_{0,I_{B'}^\circ}\longrightarrow\Omega_{E'}(M')$ (see 4.7 and II.3.7). The indices B and B' at the letter I indicate the regular Lie algebroid with respect to which the invariant elements are taken. A simple relation between $\Delta_{q'*}$ and Δ_* is described by the following diagram:

$$
\begin{array}{ccc}
(Sec\Lambda(\mathbf{g}/\mathbf{h})^{*})_{I^{\circ}_{B}} \subset W(\mathbf{g},\mathbf{h})_{q',I^{\circ}_{B}} & \xrightarrow{\Delta_{q'*}} & \Omega_{E}(M) \\
\big\downarrow & & \big\downarrow \\
(Sec\Lambda(\mathbf{g}/\mathbf{h})^{*})_{I^{\circ}_{B'}} & \xrightarrow{\Delta_{*}} & \Omega_{E'}(M') .
\end{array}
$$

The problem what the relation looks like on the level of cohomologies will not be investigated here. We only notice that each element $\Psi \in (Sec\Lambda(\mathbf{g}/\mathbf{h})^{*})_{I^{\circ}_{B}}$ being a cycle in $W(\mathbf{g},\mathbf{h})_{q',I^{\circ}_{B}}$ $\big($i.e with respect to $[d]_{q'}\big)$ is a cycle in $W(\mathbf{g},\mathbf{h})_{o}$ $\big($i.e. with respect to $\bar{\delta}\big)$; the converse fact is not true in general, which may be the source of the characteristic classes $\big($in $H_{E'}(M')\big)$ measuring the concordance of λ' with B, which can not be obtained by $\Delta_{q'}$.

6. A COMPARISON WITH THE CHARACTERISTIC CLASSES OF FOLIATED BUNDLES

Let us be given:

(a) a G-principal fibre bundle $P=(P,\pi,M,G.\cdot)$,

(b) a flat partial connection in P over an involutive distribution $F\subset TM$,

(c) a closed Lie subgroup $H\subset G$ and an H-reduction $P'\subset P$.

In other words, we are given some foliated principal bundle with a reduction, considered, for example, in [10]. As usual, let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. In [10], to such a bundle there corresponds a characteristic homomorphism $\hat{\Delta}_{1q'\#}:H(\mathfrak{g},H)_{q'})\longrightarrow H_{dR}(M)$ $\big($denoted there by $\Delta\big)$ where $q'\geqslant codim\,\mathcal{F}$, $\mathcal{F}$ being the *foliation* determined by F, and

$$H(\mathfrak{g},H)_{q'}=\big(\Lambda\mathfrak{g}^{*}\otimes V^{\leqslant q'}\mathfrak{g}^{*}\big)_{H}\overset{S}{\cong}\big(\Lambda(\mathfrak{g}/\mathfrak{h})^{*}\otimes V^{\leqslant q'}\mathfrak{g}^{*}\big)_{I_{H}}$$

is the truncated relative Weil algebra constructed isomorphically as the subalgebra of the truncated algebra $\Lambda(\mathfrak{g}/\mathfrak{h})^{*}\otimes V^{\leqslant q'}\mathfrak{g}^{*}$, consisting only of those elements which are invariant with respect to the representation $Ad^{q'}$ of H induced by the restriction to H of the adjoint representation $Ad_{G}:G\longrightarrow GL(\mathfrak{g})$. The differential $d_{q'}$ in $W(\mathfrak{g},H)_{q'}$, defined in the standard way, comes from the differential, denoted here by $d^{L}_{\mathfrak{g}}$, in the

Weil algebra $W\mathfrak{g}=\wedge\mathfrak{g}^{*}\otimes\vee\mathfrak{g}^{*}$, defined as follows: we treat $\mathfrak{g}$ as a left Lie algebra of G (with the bracket denoted by $[\cdot,\cdot]^{L}$) and $d^{L}_{\mathfrak{g}}:W\mathfrak{g}\longrightarrow W\mathfrak{g}$ is the antiderivation of total degree +1 such that $d^{L}_{\mathfrak{g}}(w^{*}\otimes 1)=1\otimes w^{*}+d_{\wedge}w^{*}\otimes 1$ and $i_{\nu}d^{L}_{\mathfrak{g}}(1\otimes w^{*})=\theta^{L}_{\mathfrak{g}}w^{*}$ for $\nu\in\mathfrak{g}$, $w^{*}\in\mathfrak{g}^{*}$ ($d_{\wedge}$ is the Chevalley-Eilenberg differential, whereas $\theta^{L}_{\mathfrak{g}}w^{*}=-w^{*}\circ ad^{L}_{\mathfrak{g}}$ where $ad^{L}_{\mathfrak{g}}(\mu)=[\nu,\mu]^{L}$, $\mu\in\mathfrak{g}$). In the sequel, as opposed to the left Lie algebra, the bracket in the right Lie algebra of G will be denoted by $[\cdot,\cdot]^{R}$; there is a relation $[\nu,\mu]^{L}=-[\nu,\mu]^{R}$, and we recall once again that, for $z\in P_{|x}$, $\hat{z}:\mathfrak{g}\longrightarrow\mathbf{g}_{|x}$ is an isomorphism of Lie algebras when $\mathfrak{g}$ is equipped with the right structure.

The partial connection in P determines a partial connection λ' in the transitive Lie algebroid $A(P)$, and the system obtained $(A(P),A(P'),\lambda')$ is a *PFS*-transitive Lie algebroid. In 4..7 the characteristic homomorphism $\Delta_{q'\#}:H(W(\mathbf{g},\mathbf{h})_{q',I^{\circ}})\longrightarrow\Omega_{dR}(M)$ is obtained ($\mathbf{g}$ and $\mathbf{h}$ being the Lie algebra bundles adjoint of $A(P)$ and $A(P')$, respectively). We compare $\hat{\Delta}_{q'\#}$ with $\Delta_{q'\#}$. For the purpose, consider the adjoint representation $Ad_{P}:P\longrightarrow L(\mathbf{g})$ [17; 5.3.2] and the representation $Ad^{q'}_{P',\mathbf{g}}:P'\longrightarrow L(W(\mathbf{g},\mathbf{h})_{q'})$, $Ad^{q'}_{P',\mathbf{g}}=Ad^{\wedge}_{P',\mathbf{g}}\otimes\vee^{<q'}(Ad_{P}|P')^{\natural}$ (for $Ad^{\wedge}_{P',\mathbf{g}}$ see Chapter II.6), induced by it. As in Chapter II.6, we notice that the differential of $Ad^{q'}_{P',\mathbf{g}}$ is equal to the representation $ad^{q'}_{A(P'),\mathbf{g}}:A(P')\longrightarrow A(W(\mathbf{g},\mathbf{h})_{q'})$ defined by $ad^{q'}_{A(P'),\mathbf{g}}=ad^{\wedge}_{A(P'),\mathbf{g}}\otimes\vee^{<q'}(ad_{A(P')}|A(P'))^{\natural}$. Propositions 5.5.2-3 from [17] give a monomorphism

$$\bar{\kappa}:(\wedge(\mathfrak{g}/\mathfrak{h})^{*}\otimes\vee^{<q'}\mathfrak{g}^{*})_{I_{H}}\xrightarrow{\cong}(Sec\wedge(\mathbf{g}/\mathbf{h})^{*}\otimes\vee^{<q'}\mathbf{g}^{*})_{I}\hookrightarrow(Sec\wedge(\mathbf{g}/\mathbf{h})^{*}\otimes\vee^{<q'}\mathbf{g}^{*})_{I^{\circ}}$$

defined by the formula $\bar{\kappa}(\psi)(x)=Ad^{q'}_{P',\mathbf{g}}(z)(x)$, $z\in P'_{|x}$, i.e. $\bar{\kappa}(\psi)(x)=(\wedge[\hat{z}]^{*-1}\otimes\vee^{<q'}\hat{z}^{*-1})(\psi)$, being an isomorphism when P' is connected.

Theorem 6.1. *$\bar{\kappa}\circ s$ commutes with the differentials $d_{q'}$ and $d_{q',\mathbf{h}}$, giving the commutative diagram*

$$\begin{tikzcd}
H(W(\mathfrak{g},H)_{q'}) \arrow[dd, "(\bar{\kappa}\circ s)_{\#}"'] \arrow[dr, "\hat{\Delta}_{q'\#}"] & \\
& H_{dR}(M) \\
H(W(\mathbf{g},\mathbf{h})_{q',I^{\circ}}) \arrow[ur, "\Delta_{q'\#}"'] &
\end{tikzcd}$$

Proof. The evident commuting of the diagram

$$\begin{array}{ccc}
(\wedge\mathfrak{g}^*\otimes V^{<q'}\mathfrak{g}^*)_{\mathbf{h},I_H} & \xrightarrow{s} & (\wedge(\mathfrak{g}/\mathfrak{h})^*\otimes V^{<q'}\mathfrak{g}^*)_{I_H} \\
\Big\downarrow{\scriptstyle\kappa} & & \Big\downarrow{\scriptstyle\bar\kappa} \\
(Sec\wedge\mathbf{g}^*\otimes V^{<q'}\mathbf{g}^*)_{\mathbf{h},I^\circ} & \xrightarrow{\bar s} & (Sec\wedge(\mathbf{g}/\mathbf{h})^*\otimes V^{<q'}\mathbf{g}^*)_{I^\circ}
\end{array}$$

in which $\kappa(\psi)(x)=(\wedge\hat{z}^{*-1}\otimes V^{<q'}\hat{z}^{*-1})(\psi)$, $z\in P'_{|x}$, and of diagram (10), implies that the commutativity of $\bar\kappa\circ s$ with the differentials follows from that for κ. On the other hand, this fact concerning κ can be reduced, in an easy way, to the commutativity of $\kappa_z:W\mathfrak{g}\longrightarrow W\mathbf{g}_{|x}$ $\left(=\wedge\hat{z}^{*-1}\otimes V\hat{z}^{*-1}\right)$ with $d^L_{\mathfrak{g}}$ and d_x. There are two ways to establish this.

The first way. $d^L_{\mathfrak{g}}$ is the differential for which the following diagram

$$\begin{array}{ccc}
W\mathfrak{g} & \xrightarrow{\varphi} & W\mathfrak{g} \\
\Big\downarrow{\scriptstyle d^L_{\mathfrak{g}}} & & \Big\downarrow{\scriptstyle\bar d} \\
W\mathfrak{g} & \xrightarrow{\varphi} & W\mathfrak{g}
\end{array}$$

commutes where φ is the isomorphism of algebras, defined uniquely on the generators by: $\varphi(w^*\otimes 1)=w^*\otimes 1$ and $\varphi(1\otimes w^*)=1\otimes w^*-d_\wedge w^*\otimes 1$, whereas $\bar d$ is an antiderivation defined by $\bar d(w^*\otimes 1)=1\otimes w^*$, $\bar d(1\otimes w^*)=0$. To see this, we calculate $\varphi^{-1}(w^*\otimes 1)=w^*\otimes 1$ and $\varphi^{-1}(1\otimes w^*)=1\otimes w^*+d_\wedge w^*\otimes 1$ $\left(=d^L_{\mathfrak{g}}(w^*\otimes 1)\right)$; next,

$$\varphi\circ d^L_{\mathfrak{g}}\circ\varphi^{-1}(w^*\otimes 1)=\varphi\circ d^L_{\mathfrak{g}}(w^*\otimes 1)=1\otimes w^*,$$
$$\varphi\circ d^L_{\mathfrak{g}}\circ\varphi^{-1}(1\otimes w^*)=\varphi\circ d^L_{\mathfrak{g}}\circ d^L_{\mathfrak{g}}(w^*\otimes 1)=0.$$

Therefore it remains to show that (1) $\kappa_z\circ\varphi=\varphi_x\circ\kappa_z$ and (2) $\kappa_z\circ\bar d=\bar d\circ\kappa_z$. (2) is trivial, whereas (1) needs the equality

$$\kappa_z(d_\wedge w^*)=\delta(\kappa_z w^*),\quad w^*\in\mathfrak{g}^*. \tag{13}$$

To prove this, take $v,w\in\mathbf{g}_{|x}$.

$$\kappa_z(d_\wedge w^*)(v,w)=d_\wedge w^*(\hat{z}^{-1}(v),\hat{z}^{-1}(w))=-w^*([\hat{z}^{-1}(v),\hat{z}^{-1}(w)]^L)$$
$$=w^*([\hat{z}^{-1}(v),\hat{z}^{-1}(w)]^R)=w^*(\hat{z}^{-1}([v,w]))=(\kappa_z w^*)([v,w])=\delta(\kappa_z w^*)(v,w).$$

The second way (direct).

(a) $\kappa_z\circ d^L_{\mathfrak{g}}(w^*\otimes 1)=\kappa_z(1\otimes w^*+d_\wedge w^*\otimes 1)$

$=1\otimes\kappa_z w^*+\kappa_z(d_\wedge w^*)\otimes 1$ $\quad$ (by (37))

$$= 1 \otimes \kappa_z w^* + \delta(\kappa_z w^*) \otimes 1 = d_x \circ \kappa_z (w^* \otimes 1).$$

(b) To prove that $\kappa_z \circ d^L_{\mathfrak{g}}(1 \otimes w^*) = d_x \circ \kappa_z(1 \otimes w^*)$, take $\nu \in \mathbf{g}_{|x}$. Since $i_\nu \circ d_x = -\theta_\nu$ (Prop. 1.17), it is sufficient to verify the equality $i_\nu \circ \kappa_z \circ d^L_{\mathfrak{g}}(1 \otimes w^*) = -\theta_\nu \circ \kappa_z(1 \otimes w^*)$. To this end, we immediately notice that $i_\nu \circ \kappa_z = \kappa_z \circ i_{\hat{z}^{-1}(\nu)}$ and $\kappa_z \circ \theta^L_{\hat{z}^{-1}(\nu)} = -\theta_\nu \circ \kappa_z$. Now, we can calculate

$$i_\nu \circ \kappa_z \circ d^L_{\mathfrak{g}}(1 \otimes w^*) = \kappa_z \circ i_{\hat{z}^{-1}(\nu)} \circ d^L_{\mathfrak{g}}(1 \otimes w^*) = \kappa_z(1 \otimes \theta^L_{\hat{z}^{-1}(\nu)} w^*)$$
$$= -1 \otimes \theta_\nu \circ \kappa_z(w^*) = -\theta_\nu \circ \kappa_z(1 \otimes w^*).$$

At present, we pass to the second part of our theorem. We can write a diagram analogous to the one in the proof of Th.II.6.1. Analogously, we assert that we need the equality

$$j^*(\mathbf{k}_{q'}(\omega_P)(\theta)) = \rho \circ (di)^* [\mathbf{k}]_{q'}(\tilde{\kappa}(\theta))$$

where $\tilde{\kappa}$ is the superposition

$$W(\mathfrak{g}, H)_{q'} \longrightarrow W(\mathbf{g}; \mathbf{h})_{q', I^\circ} \longrightarrow W(\mathbf{g})_{q', \mathbf{h}, I^\circ},$$

while ω_P is the connection form of an adapted connection. We see that the last equality is as good as the commuting of the following simple diagram:

$$\begin{array}{ccc} W\mathfrak{g} & \xrightarrow{k_z} & \Lambda T^*_z P \\ \big\downarrow \kappa_z & & \big\uparrow \Lambda(\pi^{A\,*}_{|z}) \\ W\mathbf{g}_{|x} & \xrightarrow{k_x} & \Lambda A_{|x} \end{array} \tag{14}$$

for any $x \in M$ and $z \in P_{|x}$. In this diagram, k_z is a homomorphism of algebras fulfilling $(k(\omega_P)(\Theta))_z = k_z(\theta)$; in other words, $k_z(\psi \otimes \Gamma) = \omega^\wedge_{|z}(\psi) \wedge \Omega^\vee_{|z}(\Gamma)$, $\psi \in \Lambda\mathfrak{g}^*$, $\Gamma \in \vee\mathfrak{g}^*$, where $\omega^\wedge_{|z} : \Lambda\mathfrak{g}^* \longrightarrow \Lambda T^*_z P$ and $\Omega^\vee_{|z} : \vee\mathfrak{g}^* \longrightarrow \Lambda T^*_z P$ are homomorphisms of algebras constructed in the classical manner, Ω being the connection form of ω_P. The commutativity of (14) can be checked trivially on the generators when one only knows the relations $\omega_{|x} \circ \pi^A_{|z} = \hat{z} \circ \omega_{P|z}$, $\Omega_{|z} \circ \Lambda^2 \pi^A_{|z} = \hat{z} \circ \Omega_{|z}$ (ω is the connection form in $A(P)$ corresponding to ω_P). ∎

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