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This is a preliminary announcement of some results obtained together with Moshe Roitman (University of Haifa, Israel). They will be included with full proofs in a joint paper in preparation.

Troughout, A will be an integral domain and S a multiplicative subset of A.

We denote by $T(A)$ the group of all t-invertible t-ideals of A and by $C(A)$ the Class Group of A, that is $C(A) := T(A)/P(A)$, where $P(A)$ is the group of all principal ideals of A (for the definitions see for example [Bv1] and [BZ]). We have a natural homomorphism of groups $\varphi_S : T(A) \longrightarrow T(A_S)$, defined by $I \longmapsto IA_S$ which induces a homomorphism $\overline{\varphi}_S : C(A) \longrightarrow C(A_S)$.

If A is a Krull domain, then $C(A)$ coincides with the Divisor Class Group of A and we have the following Theorem (cf. [F, Corollaries 7.2 and 7.3]):

Nagata's Theorem.
If A is a Krull domain, then
(i) The natural homomorphism $\overline{\varphi}_S : C(A) \longrightarrow C(A_S)$ is surjective.
(ii) If S is generated by prime elements, then $\overline{\varphi}_S$ is an isomorphism.

It seems natural to ask whether it is possible to generalize these statements by relaxing the Krull assumption on A (see [R]).

Without any assumption on A, if S is generated by primes, then $\overline{\varphi}_S$ is injective [AA, Theorem 2.3]. Hence, in order to obtain our generalization, we have to deal with the surjectivity of the map $\overline{\varphi}_S$, or equivalently with the surjectivity of the map φ_S.

In [R] it is proved that the map φ_S is always surjective when A is a Pseudo v-Multiplication Domain (PVMD).

Our main result is about Archimedean domains. We recall that a domain A is called Archimedean if $\cap_n Aa^n = 0$ for every non-unit $a \in A$.

The following picture shows that Archimedean domains include many domains of current interest, as for example those domains which are completely integrally closed (CIC) or with the ascending chain conditions on principal ideals (ACCP).

$$\begin{tikzcd}
\text{1-dimensional} \arrow[r, Rightarrow] & \text{the Principal Ideal Theorem holds} \arrow[drr, Rightarrow] & & \\
& \text{Krull} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, dotted] & \text{CIC} \arrow[r, Rightarrow] & \text{Archimedean} \\
\begin{array}{l}\text{Noetherian +}\\ \text{integrally closed}\end{array} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \\
& \text{Noetherian} \arrow[r, Rightarrow] & \text{Mori} \arrow[r, Rightarrow] & \text{ACCP} \arrow[uu, Rightarrow]
\end{tikzcd}$$

Theorem. Let S be generated by prime elements. Assume <u>one</u> of the following conditions:

(i) A is 1-dimensional;

(ii) A has the ascending chain condition on the principal ideals;

(iii) A is Archimedean and S is generated by finitely many elements.

Then the map φ_S is surjective.

Even though there are more cases in which the map φ_S is surjective, the following Examples show that the hypotheses of the Theorem are not very far from being the best ones.

Example 1. (cf. [AA and R]) <u>A quasilocal Mori domain</u> A <u>with an element</u> f <u>such that the map</u> $\varphi_f : T(A) \longrightarrow T(A_f)$ <u>is not surjective</u>.

Let (B,M) be the localization of the Krull domain $K[X,Y,U]/(XY-U^2)$, K a field, at the maximal ideal generated by the classes $\bar{X}$, $\bar{Y}$, $\bar{U}$. We have $C(B) = \mathbf{Z}/2\mathbf{Z}$ [F, Example 16.5]. Let $A := k + M$, where k is a proper subfield of K. A is a Mori domain and $C(A) = 0$. It is possible to find an element f (for example $f = \bar{X}+\bar{Y}$) which is prime in B, but not in A, such that $A_f = B_f$. Thus, by Nagata's Theorem, we have $C(A_f) = C(B_f) = C(B) = \mathbb{Z}/2\mathbb{Z}$ and the map $\varphi_f : T(A) \longrightarrow T(A_f)$ is not surjective.

Starting from the same Krull domain (B,M) we can construct more examples:

Example 2. <u>A quasilocal domain</u> A <u>with a prime element</u> p, <u>such that the map</u> $\varphi_p : T(A) \longrightarrow T(A_p)$ <u>is not surjective</u>.

Let A be the localization of the domain $B[Z, m/Z^n]_{m\in M,\ n\geq 0}$ at the maximal ideal generated by $\{Z,\ m/Z^n\}_{m\in M,\ n\geq 0}$. Z is prime in A and we have $A_Z = B[Z]_Z$. The ideal $P := (\bar{X},\bar{U})$ is a t-invertible ideal of B (because it is a height 1 prime) and it is not principal. Thus its extension $PB[Z]_Z = PA_Z$ is a non-principal t-invertible prime ideal of A_Z. Now suppose that $I \in T(A)$ is such that $IA_Z = PA_Z$, then I is not principal and $(A:I)$ is v-finite. Thus we can write $I = \cap_{i=1}{}^{n} Au_i$, u_i in the quotient field of A. We may assume that there are no inclusions among the fractional ideals Au_i. Hence, if $v \in I$, we have $v = a_i u_i$ $(n\geq 2,\ 1\leq i\leq n)$ with a_i not invertible in A for all i. Thus $a_i/Z \in A$ and $v/Z = (a_i/Z)u_i \in Au_i$ for all i. We conclude that $v/Z \in I$ and so $1/Z \in (I:I)$. This is impossible because, if I is t-invertible, then $(I:I) = A$.

Example 3. <u>A completely integrally closed quasilocal domain</u> A <u>with a multiplicative set</u> S, <u>generated by (infinitely many) primes</u>, <u>such that</u> φ_S <u>is not surjective</u>.

Let $D := B[Z_n]_{n\geq 1}$. Consider D as a domain graded over B and denote by d its degree. In the quotient field of D, consider the set
$V := \{\, f/(\Pi_{\text{finite}}\ Z_i)^{d(f)} :\ \ f \in D$ a nonzero homogeneous element $\}$.

Let C be the domain generated over D by V and $A := C_{\underline{m}}$, where $\underline{m}$ is the maximal ideal generated by the non-invertible elements of V. With some work, we can prove that:

a) A is completely integrally closed ;
b) Z_i is prime in C, and hence in A, for all i ;
c) If $P := (\bar{X},\bar{U}) \subset B$, then $PA_S \in T(A_S)$;
d) There exists no t-invertible t-ideal I of A such that $IA_S = PA_S$.

A useful method to study the map φ_S is to investigate how the maximal t-ideals of A behave with respect to S.

We start by observing that, if $H \in T(A_S)$, then $H = (JA_S)_v = (J_vA_S)_v$ for some $J \subset A$ finitely generated. Thus, if $J_v \in T(A)$, then $H = (JA_S)_v = J_vA_S = \varphi_S(J_v)$. It follows that, if φ_S is <u>not</u> surjective, then there exists at least an ideal $J \subset A$ such that $(JA_S)_v \in T(A_S)$, but $J_v \notin T(A)$.

Denote by $t_m(A)$ the set of maximal t-ideals of A. If $J_v \notin T(A)$, then there exists $Q \in t_m(A)$ such that $J(A:J) \subset Q$, that is $(A:J) = (Q:J)$.

We say for brevity that Q is <u>S-bad</u> if $Q \in t_m(A)$ and $(A:J) = (Q:J)$ for some ideal J finitely generated such that $J_vA_S \in T(A_S)$.

We see that:

<u>if there are no S-bad ideals, then the map φ_S is surjective</u>.

For example, if A is a PVMD, then J_v is t-invertible for every finitely generated ideal J of A. Thus, in this case, the map φ_S is always surjective [R, Proposition 2. 14].

We recall that, if $Q \in t_m(A)$, then we have one and only one of the following possibilities (see [G]) :

- Q is t-invertible
- Q is strongly divisorial (that is Q is divisorial and $(A:Q) = (Q:Q)$)
- Q is unitary (that is $(A:Q) = A$)

The key result used to prove the Theorem and to exhibit some more cases in which the map φ_S is surjective is the following:

Lemma. Suppose that Q is S-bad. Then:
(i) If Q is t-invertible, then ht $Q > 1$.

(ii) If $Q \cap S = \emptyset$, then Q is unitary and QA_Q is not a t-ideal.

The Theorem is proved by showing that, in the given hypotheses, every S-bad ideal Q must intersect S . This leads to a contradiction by the Lemma: indeed in this case Q should be a principal prime ideal and so a t-invertible height 1 S-bad ideal.

In a similar way, we can also prove:

Proposition 1. Let S be generated by primes. Assume that A is CIC and ht $Q = 1$ for every $Q \in t_m(A)$ unitary. Then the map φ_S is surjective.

Proposition 2. Let A be a domain in which every maximal t-ideal is divisorial. If $Q \cap S = \emptyset$ for every $Q \in t_m(A)$ such that A_Q is not CIC , then the map φ_S is surjective.

This last Proposition for Mori domains becomes:

Corollary. Let A be a Mori domain. If $Q \cap S = \emptyset$ for any strongly divisorial maximal t-ideal Q , then the map φ_S is surjective.

We note that, when A is CIC and not Krull, then A has at least a unitary maximal t-ideal [G, Corollary 2.8]. Similarly, when A is Mori and not Krull, then A has at least a strongly divisorial maximal t-ideal.

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