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STATIONARY STRATEGIES IN TOPOLOGICAL GAMES

by Fred GALVIN

If X is topological space, the Banach-Mazur game BM(X) is played as follows : first Black chooses a nonempty open set $B_o \subseteq X$, then White chooses a nonempty open set $W_o \subseteq B_o$, then Black chooses a nonempty open set $B_1 \subseteq W_o$, and so on . Thus, a play of the game BM(X) is an infinite decreasing sequence $X \supseteq B_o \supseteq W_o \supseteq B_1 \supseteq W_1 \supseteq \ldots \supseteq B_n \supseteq W_n \supseteq \ldots$ of nonempty open sets ; Black wins if the intersection is empty, White wins if it is nonempty. This generalization of the classical Banach-Mazur game [5] is due to J.C. Oxtoby [6] who showed that Black has a winning strategy in BM(X) if and only if X is not a Baire space. (X is a Baire space if every intersection of countably many dense open subsets of X is dense in X).

A space X is said to be "weakly α-favorable" [8] if White has a winning strategy in BM(X). It follows from Oxtoby's result that every weakly α-favorable space is a Baire space. The converse if false : if X is a subset of the real line which meets every nonempty perfect set and whose complement also meets every nonempty perfect set (i.e. , X is a "Bernstein set"), then neither player has a winning strategy in BM(X) ; i.e. , X is a Baire space but is not weakly α-favorable. However,the spaces satisfying the usual hypotheses of the Baire category theorem (e.g., complete metric spaces or locally compact Hausdorff spaces) are not only Baire but in fact weakly α-favorable.

In general a strategy for White in BM(X) may depend on all the previous moves :

$$W_n = \tau(B_o,W_o,B_1,W_1,\ldots,B_n).$$

It is easy to see that such a strategy can be reduced to an equivalent strategy which depends only on the opponent's previous moves :

$$W_n = \tau'(B_o,B_1,\ldots,B_n).$$

A "stationary strategy" (also known as a "positional strategy" or a "tactic"[1]) is a strategy which depends only on the opponent's last move :

$$W_n = \sigma(B_n).$$

A space X is said to be 'α-favorable" [1] if White has a stationary winning strategy in BM(X). It is easy to see that complete metric spaces and locally compact Hausdorff spaces are α-favorable, and for a long time it was an open problem whether all weakly α-favorable spaces are α-favorable. However, G. Debs has recently found a simple example [2] of a (non-regular) Hausdorff space which is weakly α-favorable but not α-favorable, as well as a more complicated example [3] of a Tychonoff space with the same properties.

Now , a "Markov strategy" for White in BM(X) is one which depends only on the opponent's last move and the number of moves that have been made :

$$W_n = \tau(n,B_n).$$

If X is "pseudocomplete" in the sense of J.C. Oxtoby [7] , then it is clear that White has a Markov winning strategy in BM(X). It is not so clear that he has a stationary winning strategy, but this is a consequence of a theorem of F. Galvin and R. Telgårsky [4] : if White has a Markov winning strategy in BM(X), then he also has a stationary winning strategy.

G. Debs [3] and F. Galvin and R. Telgårsky [4] independently observed that, for every weakly σ-favorable space X , White has a winning strategy in BM(X) of the form $W_n = \sigma(W_{n-1},B_n)$; however, we do not know if there is always a winning strategy of the form $W_n = \sigma(B_{n-1},B_n)$.

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