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SPECIAL PERTURBATIONS OF GIBBS STATES

by Gisela A. LASSNER

I. Introduction.

A characteristic of equilibrium states is their stability with respect to exterior perturbations. A perturbation of the system is an exterior action on the system under consideration. If the equilibrium state is the Gibbs state

$$\rho = 1/Z\ e^{-\beta H_o}\ , \qquad (1.1)$$

where H_0 is the Hamiltonian, $\beta=1/kT$ the normed inverse temperature, then an exterior perturbation means an alteration of the Hamiltonian H_o to the perturbed one $H = H_o+V$. Then the perturbed equilibrium state ρ^V is the Gibbs state to the Hamiltonian H. One expects that the perturbed equilibrium state returns to the unperturbed state (1.1), if the perturbation converges to zero in an appropriate sense.

The problem is non trivial for several reasons. On the one hand one get the real equilibrium state ρ only by going over to the thermodynamical limit in (1.1). On the other hand one has to consider very special convergences of the perturbations V. From physical point of view it is evident that not every family of "small perturbations" leads already to small alterations of the equilibrium states. It belongs to the most important results of statistical physics that the equilibrium states for C*-and W*-dynamical systems can be characterized mathematically rigorous by their properties of stability (/1/,5.4). The admissible perturbations are local perturbations /2/, i.e. they converge to zero, and they are localized silmutaneously in a bounded region (of R^3 or of the lattice).

For such perturbations also that the dynamic τ_t^V of the perturbed system as group of automorphism on the algebra of observables converges to the dynamic τ_t of the unperturbed system, if the perturbation V converges in a certain sense to 0. But even for C*-dynamical systems it is not sufficient that V converges in the norm to 0 by fixed locality. The result is only that the perturbed equilibrium state $\rho^{\lambda V}$ converges to ρ , if λ tends to 0 (/1/,5.4.2).

Now it is known that even for simple models like the soluble BCS-Bogoljubov-model of the superconductor the classical treatment in the frame of C*-dynamical systems is not possible /7,8/. At this one has already to turn to more general topological algebras and quasi-algebras /4,5/ for a rigorous mathematical treatment. This makes clear that there will be also similar effects for the choice of topologies on the set of admissible "small" perturbations. These investigations are yet at the very beginning and we shall investigate in this paper only one special class of perturbations.

We shall restrict us to Bose systems with a finite number of freedom and demonstrate at this model a number of characteristics of the continuity behavior of the equilibrium states. Because the algebra of observables will generated by unbounded operators, we shall be confronted with the topological questions connected with these algebras. But since we restrict us to finite number of freedom the Gibbs state is given by the density matrix (1.1). Therefore the entropy $S(\rho) = -\operatorname{tr}\rho\log\rho$ is well defined. As appropriate perturbations we regard only such that the entropy changes continuously in dependence on the perturbations.

II. A class of states.

In this section we introduce a class of states with an appropriate topology, such that the entropy becomes a continuous functional on it.

The following results are not only restricted to Bose systems but they are valid for all Hamiltonians whose eigenvalues satisfy the following increasing condition.

Let be $E_i > 0$ the eigenvalues of H_o and ϕ_i the corresponding normalized eigenvectors. The vectors ϕ_i form an orthonormal basis of the Hilbert space,

$$H_o \ \phi_i = E_i \ \phi_i \ , \quad i=1,2... \quad . \tag{2.1}$$

For our aim the eigenvalues E_i must converge to infinity at least such fast that

$$\sum_{i=1}^{\infty} e^{-E_i} \ i^k < \infty \qquad \text{for all} \quad k > 0. \tag{2.2}$$

These assumptions are satisfied for example if for $i \to \infty$ the eigenvalues F_i grow faster as i^{α} , $\alpha > 0$.
These assumptions are satisfied for the Hamiltonian

$$H_o = H^o - \mu N = \sum_{k=1}^{f} \varepsilon_k a_k \ a_k - \mu N \tag{2.3}$$

of the free Bose system. $\mu < 0$ is the chemical potential, N the number operator and $\varepsilon_k > 0$. Let $|n> \ = \ |n_1,n_2,\ldots,n_f>$ be the occupation number vectors of the Fock space, then

$$E_{|n>} = \sum_{k=1}^{f} n_k \ (\varepsilon_k - \mu) \tag{2.4}$$

are the eigenvalues of H_o and we get

$$\sum_{|n>\neq|o>} E^{-(f+1)} < \infty \ . \tag{2.5}$$

If we number the vectors $|n>$ in any order, $\phi_i = |n>_i$, then the eigenvalues $E_i = E_{|n>_i}$ satisfy (2.2).

The subset of all density matrices with finite entropy is a set of category I in the space of all density matrices /9/. Now we are defining a class $\mathfrak{z}$ of matrices leading to finite entropy.

Let $\mathfrak{Z}$ be the class of all matrices $\rho = (\rho_{ij})$ satisfying the following conditions

$$\mathfrak{Z} \; : \; p(\rho) = \Big(\sum_{i,j} |\rho_{ij}|^2 \, i^4 j^4 \Big)^{1/2} < \infty \, . \tag{2.6}$$

$\mathfrak{Z}$ is even a Banach algebra with respect to the norm $p(\rho)$.

The such introduced norm $p(\rho)$ is equivalent to the Hilbert-Schmidt norm $\| T^2 \rho T^2 \|_{H.S.}$, where $T = (t_{ij})$ is a diagonal operator with elements $t_{ij} = \delta_{ij} i$.

By $\mathfrak{Z}_+$ we denote the set of positive operators in $\mathfrak{Z}$. The density matrices in $\mathfrak{Z}$ are the elements $\rho \in \mathfrak{Z}_+$ with $\mathrm{tr}\rho = 1$. One can prove the following theorem /3/.

<u>THEOREM 2.1.</u>

> *For every density matrix* $\rho \in \mathfrak{Z}_+$ *the entropy* $S(\rho) = -\mathrm{tr}\,\rho \log \rho$ *is finite and if* $\rho \to \rho'$ *with respect to the norm* $P(.)$ *than* $S(\rho)$ *converges to* $S(\rho')$.

Let H_o be the Hamiltonian (2.3) of the free Bose system and $\rho_o = 1/Z\, e^{-\beta H_o}$ the corresponding Gibbs state, then ρ_o belongs to $\mathfrak{Z}$, i.e. $\rho_o \in \mathfrak{Z}$. Indeed one obtains $P(\rho_o)^2 = 1/Z \;\Sigma\, i^8 \, e^{-\beta E_i} < \infty$ for all $\beta > 0$.
$Z_o = \Sigma\, e^{-\beta E_i}$ is the state sum.

III. Perturbations of the Gibbs state.

Our laim is to perturb the Hamiltonian H_o such that the resulting Gibbs state $\rho_v = 1/Z_v \; e^{-\beta(H_o+V)}$ belongs to $\mathfrak{Z}$ and depends continuously on V.

The arising difficulties come from the non-commutativity of the free Hamiltonian H_o with V.

Our operators V are infinite dimensional matrices $V=(v_{ij})$. The diagonal elements should not grow faster than the eigenvalues of the operator H_o and the matrix elements along every row and column must decrease sufficiently fast. Precisely, we introduce the following class of matrices.

DEFINITION 3.4.

Let $\mathfrak{w}$ be the space of all linear operators $V = (v_{nm})$ *satisfying the following condiitons*

$$\| V\|_\alpha = \sup E_i^{-\alpha}|v_{ii}| + \sum_{i\neq j} |v_{ij}| \, i^3 j^3 \tag{3.1}$$

with $0 < \alpha < 1$.

For every $V \in \mathfrak{w}$ we denote by V_D the diagonal part of V and by V_N the rest of V such that

$$V = V_D + V_N . \tag{3.2}$$

Let $\mathfrak{w}_N$ be the class of all operators V with

$$\| V\|_N = \sum_{i,j} |v_{ij}| \, i^3 j^3 < \infty \tag{3.3}$$

and correspondingly let $\mathfrak{w}_D$ be the class of all diagonal operators V with

$$\| V\|_D = \sup E_i^{-\alpha}|v_{ii}| < \infty. \tag{3.4}$$

In general $V \in \mathfrak{w}$ is a unbounded operator. The advantage of the decomposition (3.2) of $V \in \mathfrak{w}$ consists in the fact that we now have as perturbation a sum of an unbounded but diagonal operator V_D and a noncommuting with H_o but bounded operator V_N.

Now we state the main theorem of the paper.

THEOREM 3.2.

The Gibbs state $\rho_V = 1/Z_v \, e^{-\beta(H_o+V)}$ *varies continuously in* $\mathfrak{z}$, *if* $V = V^*$ *varies continuously in* $\mathfrak{w}$ *with respect to the norm* $\| V\|_\alpha$. *Especially, the entropy* $S_v = -\mathrm{tr}\, \rho_v \log \rho_v$ *depends continuously on* V.

For the proof of Theorem 3.2 we separate first the diagonal part V_D of the perturbation.

LEMMA 3.3.

If V_D *is a diagonal operator of* $\mathfrak{W}$, *then* $e^{-(H_o+V_D)} \in \mathfrak{Z}$ *and for* $\| V_D - V_D' \|_D \to 0$ *follows* $p(e^{-(H_o+V_D)} - e^{-(H_o+V_D')}) \to 0$ *where* V_D' *is a fixed operator.*

The proof is straightforward considering that H_o and V_D are commuting operators /3/ and the diagonal elements v_{ii} increase at most like E_i^{α} .

In the following we denote by $H=H_i+V_D$ the diagonal part of H_o+V , $V \in \mathfrak{W}$. The theorem 3.2 is now reduced to the following theorem.

THEOREM 3.4.

The operator $e^{-(H+V)} \in \mathfrak{Z}$ *depends continuously on* $V = V^* \in \mathfrak{W}_N$.

IV. Proof of theorem 3.4.

To prove theorem 3.4 we shall apply the Trotter product formula /6/

$$\operatorname*{s-lim}_{n \to \infty} (e^{-\beta H/n} e^{-\beta V/n})^n = e^{-\beta(H+V)} \quad . \tag{4.1}$$

First we prove the following fundamental lemma.

LEMMA 4.1.

Let $s_n = (e^{-H/n} e^{-V/n})^n$ *be the operator on the left hand side of (4.1) then* $\| T^2 \; s_n \; T^3 \| \leqslant c$ *for a fixed* c *and all integers* n.

PROOF. We have the following estimations.

$$\| T^2 \; s_n \; T^3 \| = \| T^2 (e^{-H/n} e^{-V/n})^n T^3 \|$$

$$= \| T^2 e^{-H/n} T^{-2+\frac{5}{n}} T^{2-\frac{5}{n}} e^{-V/n} T^{-2+\frac{5}{n}} T^{2-\frac{5}{n}} e^{-H/n} T^{-2+\frac{10}{n}} \; .$$

$$. \; T^{2-\frac{10}{n}} e^{-V/n} T^{-2+\frac{10}{n}} \ldots T^{2-\frac{5n}{n}} e^{-V/n} T^{-2+\frac{5n}{n}} \| \tag{4.2}$$

$$\leq \| T^2 e^{-H/n} T^{-2+\frac{5}{n}} \| \quad \| T^{2-\frac{5}{n}} e^{-V/n} T^{-2+\frac{5}{n}} \| \dots$$

$$\| T^{2-5\frac{n-1}{n}} e^{-H/n} T^{-2+\frac{5n}{n}} \| \quad \| T^{2-\frac{5n}{n}} e^{V/n} T^{-2+\frac{5n}{n}} \| \ .$$

We have divided the norm (4.2) in a product of 2n operator norms.

Because of the commutativity of the operators H and T we get in (4.2) a product of n times the norm $\| T^{5/n} e^{-H/n} \|$. It follows from (2.2) that $T^5 e^{-H}$ is a Hilbert-Schmidt operator. Let γ be a constant with

$$\| T^5 e^{-H} \| \leq \gamma \ . \tag{4.3}$$

Since $(T^5 e^{-H})$ is self-adjoint and bounded one has

$$\| T^{5/n} e^{-H/n} \| \leq \gamma^{1/n} \ . \tag{4.4}$$

Further we have in (4.2) a product of n norms of the following kind

$$\| T^{\ell} V T^{-\ell} \| \ , \text{ for } -3 \leq \ell \leq 3. \tag{4.5}$$

For every 1, $-3 < 1 < 3$, we have the following estimation

$$\| T^{\ell} V T^{-\ell} \|_{H.S.} = (\sum_{i,j} |v_{ij}|^2 i^{2\ell} j^{-2\ell})^{1/2} \leq \sum_{i,j} |v_{ij}| i^{\ell} j^{-\ell}$$

$$\leq \sum_{i,j} | v_{ij} | i^3 j^3 = \| V \|_N = \kappa \ . \tag{4.6}$$

On the other hand we get by applying the inequality of the norms $\| V \| \leq \| V \|_{H.S}$ the estimation

$$\| T^{\ell} e^{-V} T^{-\ell} \| \leq \sum_k 1/k! \ \| T^{\ell} V T^{-\ell} \|^k \ , \tag{4.7}$$

and consequently it follows from (4.6) and (4.7)

$$\| T^{\ell} e^{-V/n} T^{-\ell} \| = e^{\kappa/n} \qquad \text{for} \quad -3 \leq \ell \leq 3 \ , \tag{4.8}$$

where κ is a constant. All estimations (4.2), (4.4) and (4.8) together yield

$$\| T^2 s_n T^3 \| \leq \gamma e^{\kappa} = c. \tag{4.9}$$

Further, the following to lemmas will be necessary.

LEMMA 4.2.

Let A_n *a sequence of bounded operators* $\|A_n\| < c = \text{const.}$, *converging strongly to* A *on* $\mathscr{H}$, *and* G *a Hilbert-Schmidt operator. Then* $\|A_nG - AG\|_{H.S.} \to 0$.

This lemma can be shown by standard estimations with Hilbert-Schmidt operators.

LEMMA 4.3.

If $s_n = (e^{-H/n}e^{-V/n})^n$ *converge strongly to* $S = e^{-(H+V)}$, *then* $T^2s_nT^3$ *converges strongly to* T^2ST^3.

PROOF. T and H are commuting and therefore

$$T^2(e^{-H/n}e^{-V/N})^n\, T^3\, \phi = (e^{-H/n}e^{-(T^2V/nT^{-2})})^n\, T^5\, \phi \;.$$

Now T^2VT^{-2} is bounded, then

$$\|T^2VT^{-2}\| \leqslant \|T^2VT^{-2}\|_{H.S.} = (\sum_{i,j} |v_{ij}|^2\, i^4j^{-4})^{1/2}$$

$$\leqslant \sum_{i,j} |v_{ij}|i^2j^{-2} \leqslant \Sigma\, |v_{ij}|\, i^3j^3 = \|V\|_N \;.$$

By applying the Trotter formula (4.1) and the fact that T^2VT^{-2} is bounded the $T^2s_nT^3\,\phi$ converges to $e^{-(H+T^2VT^{-2})}T^5\,\phi$ for $\phi \in \mathscr{D}$. Because $\mathscr{D}$ is dense in Hilbert-space $\mathscr{H}$ and $\|T^2s_nT^3\| \leqslant c$, lemme 4.1, $T^2s_nT^3\,\phi$ converges to $T^2ST^3\,\phi$ for all $\phi \in \mathscr{H}$.

Now we need yet some other facts. First we remark that $\mathfrak{w}_N$ is not only a Banach space but also a Banach algebra.

LEMMA 4.4.

Let $V,W \in \mathfrak{w}_N$, *then the norm* $\|VW\|_N$ *is multipliable, i.e.* $\|VW\|_N \leqslant \|V\|_N\, \|W\|_N$.

It is easy to see that

$$\| VW \|_N = \sum_{i,j} \left| \sum_k v_{ik} w_{kj} \right| i^3 j^3$$

$$\leqslant \sum_{i,k} | v_{ik} | i^3 k^3 \sum_{k,j} | w_{k,j} | j^3 k'^3 \leqslant \| V \|_N \| W \|_N .$$

A consequence of Lemma 4.4 is the following fact.

LEMMA 4.5.

If $V \in \mathfrak{w}_N$, *then* $e^{-V} \in \mathfrak{w}_N$ *and* e^{-V} *depends continuously on* W. *Futher-more,* $\| e^{-V} \|_N \leqslant e^{-\| V \|_N}$.

Finally we need yet the next lemma.

LEMMA 4.6.

Let $W \in \mathfrak{w}_N$ *and* $\rho \in \mathfrak{z}$, *then* $\rho W \in \mathfrak{z}$ *and it is continuous in both factors.*

If $W \in \mathfrak{w}_N$ then we must estimate the norm $p(\rho W)$ (2.6).

$$p(\rho W)^2 = \sum_{i,j} \left| \sum_k \rho_{ik} w_{kj} \right|^2 i^4 j^4 \leqslant$$

$$\leqslant \sum_{i,j} \left(\sum_k |\rho_{ik}|^2 \sum_{k'} |w_{k',j}|^2 \right) i^4 j^4 \leqslant p(\rho)^2 \| W \|_N^2 .$$

Now we can give the proof of Theorem 3.4 :

Let be V, V' $\in \mathfrak{w}_N$ and V' a fixed operator, then we have to show that

$$\| T^2 (e^{-(H+V} e^{-(H+V')}) T^2 \|_{H.S.} \to 0 \quad \text{for} \quad V \to V'. \tag{4.10}$$

We get the following inequality

$$\| T^2(e^{-(H+V)} - e^{-(H+V')})T^2 \|_{H.S} \tag{4.11}$$

$$\| T^2(e^{-(H+V)}T^3T^{-1} - T^2(e^{-H/n}e^{-V/n})^nT^3T^{-1} \|_{H.S.} \quad \text{(I)}$$

$$\| T^2(e^{-(H+V')}T^3T^{-1} - T^2(e^{-H/n}e^{-V'/n})^nT^3T^{-1} \|_{H.S.} \quad \text{(II)}$$

$$\| T^2((e^{-H/n}e^{-V/n})^n - (e^{-H/n}e^{-V'/n})^n)T^2 \|_{H.S.} \quad . \quad \text{(III)}$$

That the first and second therm of (4.11) will be less then $\varepsilon/3$ for sufficiently large n is a consequence of Lemma 4.2 and lemma 4.3 remembering that T^{-1} is a Hilbert-Schmidt operator. It remains to estimate term (III) of (4.11) for this fixed n. For this aim we use the identity

$$(e^{-H/n}e^{-V/n})^n - (e^{-H/n}e^{-V'/n})^n =$$

$$= \sum_{i=1}^{n} \{(e^{-H/n}e^{-V'/n})^{i-1} [e^{-H/n}(e^{-V/n} - e^{-V'/n})]. \tag{4.12}$$

$$. (e^{-H/n}e^{-V/n})^{n-i}\} \quad .$$

Now we obtain for (III)

$$\| T^2((e^{-H/n}e^{-V/n})^n - (e^{-H/n}e^{-V'/n})^n)T^2 \|_{H.S.}$$

$$\leqslant \sum_{i=1}^{n} \| T^2(e^{-H/n}e^{-V/n})^{i-1}T^{-2} \|_{H.S.} \cdot \| T^{-2}(e^{-H/n}e^{-V/n})^{n-i}T^2 \|_{H.S.} \cdot$$

$$. \| T^2 \ e^{-H/n}(e^{-V/n} - e^{-V'/n})T^2 \|_{H.S.} \quad .$$

$$\leqslant \sum_{i=1}^{n} \eta \ \beta \ \|T^2 \ e^{-H/n}(e^{-V/n} - e^{-V'/n})T^2\|_{H.S.} \quad . \tag{4.13}$$

where η, β are constant . That the first two norms in (4.13) are constant is a consequence of Lemma 4.6. The third norm in (4.13) will be less $\varepsilon/3$ if V is sufficiently near to V'. This follows from lemma 4.6 and lemma 4.5.

Now the main theorem 3.4 is completely proved.

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