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1. INTRODUCTION

This paper deals with C^*-like topologies on the test function algebra $\mathcal{S}_\otimes$, the tensor algebra over the Schwartz space $\mathcal{S}$. These topologies are connected with continuous representations of this algebra.

If R is a $*$-algebra and $\mathcal{D}$ a pre-Hilbert space, then a representation $a \mapsto A(a)$ of R in $\mathcal{D}$ is a homomorphism of R into End $\mathcal{D}$, such that $\langle\phi,A(a)\psi\rangle = \langle A(a^*)\phi,\psi\rangle$ for all $\phi,\psi \in \mathcal{D}$.

For a representation of a Banach $*$-algebra R all operators A(a) are automatically bounded and the representation is uniformly continuous. In fact, for $\phi \in \mathcal{D}$, $f(a) = \langle\phi,A(a)\phi\rangle$ is a positive functional on R and therefore [1] :

$$||A(a)\phi||^2 = \langle\phi,A(a^*a)\phi\rangle = f(a^*a) \leqslant f(1)||a^*a||$$
$$\leqslant ||\phi||^2 \quad ||a||^2 ,$$

which implies $\|A(a)\| \leq \|a\|$. In comparison with this for general topological algebras, one has to face some new problems mainly connected with topologies in algebras of unbounded operators. With this question we deal first in the next section.

2. TOPOLOGIES ON Op^*-ALGEBRAS

For a pre-Hilbert space $\mathcal{D}$ by $\mathcal{L}^+(\mathcal{D})$ we denote the set of all operators $A \in \mathrm{End}\,\mathcal{D}$ for which there exist an operator $A^+ \in \mathrm{End}\,\mathcal{D}$ satisfying $\langle\phi, A\psi\rangle = \langle A^+\phi, \psi\rangle$ for all $\phi, \psi \in \mathcal{D}$. $\mathcal{L}^+(\mathcal{D})$ becomes a $*$-algebra of operators with the involution $A \to A^+$. A $*$-subalgebra $\mathcal{A}$ of $\mathcal{L}^+(\mathcal{D})$ containing the identity I will be called an Op^*-*algebra*. $\mathcal{L}^+(\mathcal{D})$ is the maximal Op^*-algebra over $\mathcal{D}$.

A subset $\mathcal{M} \subset \mathcal{D}$ we call $\mathcal{A}$-*bounded* if $\sup_{\phi \in \mathcal{M}} \|A\phi\| < \infty$ for all operators $A \in \mathcal{A}$. If the Op^*-algebra contains bounded operators only, then the $\mathcal{A}$-bounded sets are precisely the bounded sets in $\mathcal{D}$.

Analogously to the bounded case we can also on Op^*-algebra of unbounded operators define different topologies related to the underlying pre-Hilbert space. We regard four topologies, defined by the following systems of seminorms.

DEFINITION 2.1. -

$\sigma_\mathcal{J}$, weak topology : $\|A\|_{\phi,\psi} = |\langle\phi, A\psi\rangle|$ *for all* $\phi, \psi \in \mathcal{D}$;

$\sigma^{\mathcal{D}}$, strong topology : $||A||^{\phi} = ||A\phi||$ *for all* $\phi \in \mathcal{D}$;

$\tau_{\mathcal{D}}$, uniform topology : $||A||_{\mathcal{M}} = \sup_{\phi,\psi \in \mathcal{M}} |<\phi, A\psi>|$ *for all* $\mathcal{A}$*-bounded sets* $\mathcal{M}$;

$\tau^{\mathcal{D}}$, quasi-uniform topology : $||A||^{\mathcal{M}} = \sup_{\phi \in \mathcal{M}} ||A\phi||$ *for all* $\mathcal{A}$*-bounded sets* $\mathcal{M}$.

On infinite-dimensional Op*-algebras of *bounded* operators the weak topology is properly weaker than the strong topology and this is properly weaker than the uniform topology which in this case coincides with the quasi-uniform topology and is equal to the usual norm-topology, i.e.

$$\sigma_{\mathcal{D}} \precnsim \sigma^{\mathcal{D}} \precnsim \tau_{\mathcal{D}} = \tau^{\mathcal{D}} = \text{norm-topology} \qquad (2.1).$$

For Op*-algebras of unbounded operators we have in general only the relation

$$\sigma_{\mathcal{D}} \leq \left\{ \begin{matrix} \tau_{\mathcal{D}} \\ \sigma^{\mathcal{D}} \end{matrix} \right\} \leq \tau^{\mathcal{D}} \qquad (2.2)$$

and in consistency with the foregoing one many relations between these four topologies are possible [2] . For example the strong topology may be stronger than the uniform one, more precisely, the relation

$$\sigma_{\mathcal{D}} \precnsim \tau_{\mathcal{D}} \precnsim \sigma^{\mathcal{D}} = \tau^{\mathcal{D}} \qquad (2.3)$$

is possible, as we shall see in section 5.

For a general theory of topological algebras of unbounded operators the uniform topology $\tau_{\mathcal{D}}$ plays an important role as first was outlined in [3].

DEFINITION 2.2. -

An Op^**-algebra* $\mathcal{A}[\tau_{\mathfrak{D}}]$ *equipped with the uniform topology* $\tau_{\mathcal{J}}$ *we call* $\tilde{O}^*$*-algebra and if it is complete* O^**-algebra. A topological* $*$*-algebra* R, *which is algebraically and topologically isomorphic to an* O^**-algebra (resp.* $\tilde{O}^*$*-algebra) we call* AO^**-algebra (resp.* $A\tilde{O}^*$*-algebra).*

The O^*-algebras are generalizations of the C^*-algebras and the AO^*-algebras are generalizations of the B^*-algebras. It is an interesting problem to give an abstract characterization of an $A\tilde{O}^*$-algebra, like the property $||a^*a|| = ||a||^2$ of a B^*-algebra. For barrelled $*$-algebras this problem could be completely solved.

THEOREM 2.3 (Schmüdgen [4]). -

i) In an $A\tilde{O}^*$*-algebra* R *the cone* $K = \text{conv}\,\{a^*a\ ;\ a\in R\}$ *is a normal cone.*

ii) If in a barrelled $*$*-algebra* R *with a unity* e *the cone* K *of positive elements is a normal cone, then* R *is an* $A\tilde{O}^*$*-algebra.*

3. CONTINUOUS REPRESENTATIONS

A representation of a $*$-algebra R is a $*$-homomorphism $a \mapsto A(a)$ of R onto an Op^*-algebra $\mathcal{A} = A(R)$.

DEFINITION 3.1. - *A representation of a topological* $*$*-algebra* R *is said to be* **weakly** *(resp.* strongly, uniformly, quasi-uniformly) continuous, *if the mapping* $a \mapsto A(a)$ *of* R *onto* $\mathcal{A}$ *is continuous with respect to the weak (resp. strong, uniform, quasi-uniform) topology on* $\mathcal{A}$.

As we already remarked in the introduction, any representation of a Banach $*$-algebra is uniformly continuous. In general one has only results of the following type.

LEMME 3.2. - *If* R *is a barrelled* $*$*-algebra and* $a \mapsto A(a)$ *a weakly continuous representation, then this representation is quasi-uniformly continuous and in consequence of (2.2) uniformly continuous also.*

Proof. - Let $\|\cdot\|^{\mathcal{M}}$ be a seminorm of $\tau^{\mathfrak{D}}$ and $U = \{a \in R \ ; \ \|A(a)\|^{\mathcal{M}} \leq 1\}$. U is an absorbing set and futher

$$\begin{aligned} U &= \bigcap_{\phi \in \mathcal{M}} \{ a \ ; \ \|A(a)\phi\| \leq 1 \} \\ &= \bigcap_{\phi \in \mathcal{M}} \ \bigcap_{\psi \in S} \{a \ ; \ |\langle\psi , A(a)\phi\rangle| \leq 1 \} \ , \end{aligned} \tag{3.1}$$

where S is the unit sphere in $\mathfrak{D}$. In consequence of the weak continuity of the representation the sets on the right-hand side on the last line of (3.1) are closed and absolutly convex. Therefore U is closed, absolutly convex and absorbing, i.e. a barrel. Hence U is a neighbourhood, Q.E.D.

LEMMA 3.3. - *Let* R *be a barrelled* $*$*-algebra and* $a \mapsto A(a)$ *a weakly continuous representation. Then the bilinear mapping* $a, b \mapsto A(ab)$ *from* R *onto* A(R) $[\tau_{\mathfrak{D}}]$ *is jointly continuous.*

Proof. - Let $\mathcal{M}$ be an arbitrary $\mathcal{A}$-bounded set, then

$$||A(ab)||_{\mathcal{M}} = \sup_{\phi,\psi\in\mathcal{M}} |\langle\phi, A(a)A(b)\psi\rangle|$$

$$\leq \sup_{\phi\in\mathcal{M}} ||A(a^*)\phi|| \sup_{\psi\in\mathcal{M}} ||A(b)\psi|| = ||A(a^*)||^{\mathcal{M}} ||A(b)||^{\mathcal{M}}.$$

In consequence of the foregoing lemma there is a seminorm p(.) of the topology of R such that $||A(a^*)||^{\mathcal{M}} \leq p(a)$ and $||A(b)||^{\mathcal{M}} \leq p(b)$. Hence $||\nabla A(ab)||_{\mathcal{M}} \leq p(a)\,p(b)$. Q.E.D.

As a corollary of the foregoing Lemma we obtain immediately following theorem.

THEOREM 3.4. - *In a barrelled* $A\tilde{O}^*$*-algebra the multiplication* $a,b \mapsto ab$ *is jointly continuous.*

4. TOPOLOGIES ON THE TEST FUNCTION ALGEBRA

The test function algebra $\mathcal{S}_{\otimes}$ is the algebraic direct sum $\mathcal{S}_{\otimes} = \dot{\sum}_{n=o}^{\infty} \mathcal{S}_n$ where $\mathcal{S}_o = C$ and $\mathcal{S}_n = \mathcal{S}(\mathbb{R}^{d\,n})$ is the Schwartz space of C^{∞}-functions of rapid decrease. The elements of $\mathcal{S}_{\otimes}$ are thus sequences $f = (f_o, f_1, \ldots, f\ , 0, \ldots)$

where all but a finite number of $f_\nu \in \mathcal{S}_\nu$ are equal to zero. We denote the **direct** sum topology on $\mathcal{S}_\otimes$ by τ. $\mathcal{S}_0[\tau]$ is the completion of the tensor **algebra** over $\mathcal{S}_1$ (cf. e.g. [5]). The multiplication is defined by

$$(f\, g)_n(x_1,\ldots,x_n) = \sum_{\mu+\nu=n} f_\mu(x_1,\ldots,x_\mu)\, g_\nu\,(x_{\nu+1},\ldots,x_n) \qquad (4.1)$$

and the involution by

$$(f^*)_n(x_1,\ldots,x_n) = \overline{f_n(x_n,\ldots,x_1)} \qquad (4.2)$$

Let N be an unbounded operator in $L_2(\mathbb{R}^d)$ defining the topology of $\mathcal{S}_1$, **i.e.** the system $||f_1||_k = ||\, N^k\, f_1||_{L_2}$ defines the usual topology of the **Schwartz** space $\mathcal{S}_1$. Then we define in $\mathcal{S}_n$ the seminorms $||f_n||_k = ||N^k_{x_1}\ldots N^k_{x_n} f_n||_{L_2}$ **where** N_{x_ν} is the operator N acting on the variable x_ν.

Now let be (γ_n) a sequence of positive numbers and (k_n) a sequence of **integers**. In $\mathcal{S}_\otimes$ we define the seminorm

$$||f||_{(\gamma_n),(k_n)} = \sum_n \gamma_n ||f_n||_{k_n} \qquad (4.3).$$

The direct sum topology τ of $\mathcal{S}_\otimes$ is defined by the system of all possible **seminorms** (norms) (4.3). Another important topology in $\mathcal{S}_\otimes$ is the topology τ_∞ ([6],[7]) given by the following system of seminorms:

$$||f||_{(\gamma_n),k} = \sum_n \gamma_n ||f_n||_k \qquad (4.4)$$

where (γ_n) runs again over all sequences of positive numbers and k over all integers. The topology τ_∞ is properly weaker than τ but yet a complete topology.

A third important topology in $\mathcal{S}_\otimes$ is the topology $\mathcal{N}$ (cf. [8]) , the strongest locally convex topology on $\mathcal{S}_\otimes$ such that the multiplication on $\mathcal{S}_\otimes$ is a jointly continuous bilinear mapping.

$$m : \mathcal{S}_\otimes[\tau] \times \mathcal{S}_\otimes[\tau] \to \mathcal{S}_\otimes[\mathcal{N}] \qquad (4.5) .$$

Since m is surjective ($\mathcal{S}_0$ has a unit element) this topology exists.

LEMMA 4.1. - *The multiplication* a, b → ab *is*

(i) not jointly continuous in $\mathcal{S}_\otimes[\tau]$,

(ii) jointly continuous in $\mathcal{S}_\otimes[\tau_\infty]$ *and*

(iii) not jointly continuous in $\mathcal{S}_\otimes[\mathcal{N}]$.

i) was proved in [6], [7], [8] and ii) can be shown by a simple estimation [7] . (iii) will be proved in the next section, corollary 5.6.

As an immediate consequence of Lemma 4.1 and the definition of $\mathcal{N}$ we obtain yet the following relation.

$$\tau_\infty \precneqq \mathcal{N} \precneqq \tau \qquad (4.6).$$

From the more or less trivial property i) of the foregoing lemma one gets the following interesting theorem.

THEOREM 4.2. - $\mathscr{S}_{\otimes}[\tau]$ *is not an* AO^**-algebra.*

Proof. - $\mathscr{S}_{\otimes}[\tau]$ is the locally convex direct sum of F-spaces and thus a barrelled space. Therefore the theorem follows from theorem 3.4.

The last theorem is connected with the fact that the direct sum topology τ is bad adapted to the order structure of the $*$-algebra $\mathscr{S}_{\otimes}$, namely the cone $K = \mathrm{conv}\,\{f^*f\ ;\ f \in \mathscr{S}_{\otimes}\}$ of positive elements is not normal with respect to the topology τ. This was directly proved in [7] and gives, in connection with theorem 2.3, (i), another proof for theorem 4.2. (About the normality of cones in semiorderd spaces cr. e.g. [9]).

In contrast to τ the topologies τ_{∞} and $\mathscr{N}$ are "bad" from the point of view of the theory of locally convex spaces, but they are better adapted to the order structure of $\mathscr{S}_{\otimes}$ than τ. Namely it holds the following lemma.

LEMMA 4.3 - *i) Both topologies* τ_{∞} *and* $\mathscr{N}$ *are complete, but neither bornological nor barrelled.*

ii) The cone K *is normal with respect to* τ_{∞} *and* $\mathscr{N}$.

5. O*- TOPOLOGIES ON $\mathscr{S}_{\otimes}$.

DEFINITION 5.2. - *A topology* ξ *on a* $*$*-algebra* R *is called* O^*-topology (resp. $\tilde{O}^*$-topology), *if* $R[\xi]$ *is an* AO^**-algebra (resp.* $A\hat{O}^*$*-algebra).*

If R carries a normed O^*-topology ξ, then $R[\xi]$ is a B^*-algebra and ξ is the only O^*-topology on R. In general on a $*$-algebra different O^*-topologies can exist, as we see from the following theorem and (4.6).

THEOREM 5.2. - τ_∞ *and* $\mathcal{N}$ *are* O^*-*topologies on* $\mathcal{S}_\otimes$.

For τ_∞ the statement was proved in [6] and the proof for $\mathcal{N}$ will be given in what follows t(theorem 5.5).

We start first with some general considerations. Let R $[\xi]$ be a topological $*$-algebra and ω a positive continuous functional on $R[\xi]$. By $a \mapsto A_\omega(a)$ we denote the GNS-representation associated with ω with the domain $\mathfrak{D}_\omega$ and the cyclic vector Ω_ω ([1],[5]) , uniquely determined up to unitary equivalence by

$$\omega(a) = \langle\Omega_\omega, A_\omega(a)\Omega_\omega\rangle \qquad (5.1).$$

The *universal representation* $a \mapsto A_u(a)$ is the direct sum of all GNS-repres tations,

$$A_u(a) = \sum_\omega \oplus A_\omega(a) \qquad (5.2)$$

defined on the algebraic direct sum

$$\mathfrak{D}_u = \sum_\omega \oplus \mathfrak{D}_\omega , \qquad (5.3)$$

with the natural scalar product. Given any vector $\phi = \sum_\omega \oplus \phi_\omega$ of $\mathfrak{D}_u$ (only finite many components ϕ_ω are different from zero) then

$$A_u(a)\,\phi = \sum_\omega \oplus A_\omega(a)\,\phi_\omega \qquad (5.4).$$

We call a topological $*$-algebra $R[\xi]$ *semi-simple* , if the universal representation is faithfull. $A_u(R) = \mathcal{A}_R$ is then an Op^*-algebra isomorphic to R, which is called the *universal realization* of R.

DEFINITION 5.3. - *Given any semi-simple* $R[\xi]$, *then the uniform topology* $\tau_{\mathfrak{D}_u}$ *on* $\mathcal{A}_R$ *defines by the isomorphism a topology on* R, *which we denote by* ξ_u *and call the* $\tilde{O}^*$-topology generated by ξ.

By simple considerations one can prove the following lemma [2] .

LEMMA 5.4. - *If* ξ *is a barrelled topology on* R, *then* ξ_u *is the strongest* $\tilde{O}^*$*-topology on* R *weaker than* ξ.

E.g. if $C^1[0,1]$ is the topological $*$-algebra of one-times differentiable functions with the natural topology given by the norm $||f||_C=\sup_x|f(x)|$, the strongest C -norm on $C^1[0,1]$.

Now we can state and prove the main theorem of this section.

THEOREM 5.5. - $\mathcal{N}$ *is the strongest* $\tilde{O}^*$*-topology on* $\mathcal{S}_{\oplus}$ *which is weaker than the direct-sum topology* τ , *i.e.* $\mathcal{N} = \tau_u$.

Proof. - i) Let ξ be any $\tilde{O}^*$-topology on $\mathcal{S}_{\oplus}$ weaker than τ , then $\mathcal{S}_{\oplus}[\xi]$ is algebraically and topologically isomorphic to an $\tilde{O}^*$-algebra $\mathcal{A}[\tau_{\mathfrak{D}}]$ by $\mapsto$ A(f) which is therefore a weakly conginuous representation of $\mathcal{S}_{\otimes}[\tau]$. Hence, $(a,b) \mapsto A(ab)$ is by lemma 3.3 a jointly continuous mapping.

$$\mathcal{S}_{\otimes}[\tau] \times \mathcal{S}_{\otimes}[\tau] \rightarrow \mathcal{A}[\tau_{\mathcal{D}}] \cong \mathcal{S}_{\otimes}[\xi] \qquad (5.6)\ .$$

From the definition of $\mathcal{N}$ then it follows that $\mathcal{N}$ is stronger than ξ.

ii) Now we show $\mathcal{N}$ to be an O^*-topology. Since $\mathcal{N}$ is a topology for which the cone K is normal (lemma 4.3) it is defined by seminorms

$$p_M(f) = \sup_{\omega \in M} |\omega(f)| \qquad (5.7),$$

where the sets M are weakly bounded sets of $\mathcal{N}$-continuous positive functionals ([9], chap. V). Let $a \mapsto A_u(a)$ be the universal representation of $\mathcal{S}_{\otimes}[\tau]$ onto $\mathcal{A}_u$ and M any set of (5.7). We put $\mathcal{M} = \{\Omega_\omega ; \omega \in M\} \subset \mathcal{D}_u$. $\mathcal{M}$ is $\mathcal{t}_u$-bounded. In fact

$$\begin{aligned} \sup_{\Omega_\omega \in \mathcal{M}} ||A_u(f)\Omega_\omega||^2 &= \sup_{\omega \in M} < \Omega_\omega, A_u(f^* f)\Omega_\omega > \\ &= \sup_{\omega \in M} \omega(f^* f) < \infty \end{aligned} \qquad (5.8)\ .$$

Further, any seminorm (5.7) of $\mathcal{N}$ can be estimated in the following way (see (5.1-4)) :

$$\begin{aligned} p_M(f) &= \sup_{\Omega_\omega \in \mathcal{M}} |<\Omega_\omega , A_u(f)\Omega_\omega>| \\ &\leq \sup_{\phi,\psi \in \mathcal{M}} |<\phi , A_u(f)\psi>| = || A_u(f)||_{\mathcal{M}} \end{aligned} \qquad (5.9).$$

(5.9) means that any seminorm of $\mathcal{N}$ can be estimated by a seminorm of $\tau_u = \tau_{\mathcal{D}_u}$. Thus, τ_u is stronger than $\mathcal{N}$. Together with i) this yields $\mathcal{N} = \tau_u$. Q.E.D..

In [8], theorem 1, it was proved that for any seminorm q(f) of τ there is a positive continuous functional ω such that

$$q(f) \leq \omega(f^* f)^{1/2} = ||A_u(f)\,\Omega_\omega|| \qquad (5.10).$$

A consequence of that is $\tau \leq \sigma^{\mathcal{D}_u}$ and since τ is barreled we have $\tau = \sigma^{\mathcal{D}_u} = \tau^{\mathcal{D}_u}$ (lemma 3.2).

Therefore

$$\sigma_{\mathcal{D}_u} \precneqq \tau_{\mathcal{D}_u} \precneqq \sigma^{\mathcal{D}_u} = \tau^{\mathcal{D}_u} \qquad (5.11),$$

which is one possible relation between the four topologies of definition 2.1, already mentioned in (2.1). $\tau_{\mathcal{D}_u} = \mathcal{N}$ is different from $\tau^{\mathcal{D}_u} = \tau$ since $\mathcal{S}_0[\mathcal{N}]$ is an AO -algebra but not $\mathcal{S}_0[\tau]$.

Since in an $\tilde{O}^*$-algebra $\mathcal{A}[\tau_{\mathcal{D}}]$ the multiplication $(a,b) \mapsto ab$ is jointly continuous if and only if $\tau_{\mathcal{D}} = \tau^{\mathcal{D}}$ ([3], theorem 3.2), we obtain from (5.11) yet the

COROLLARY 5.6. - *The multiplication* $(f,g) \to f \cdot g$ *is not jointly continuous with respect to* $\mathcal{N} = \tau_{\mathcal{D}_u}$.

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