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# APPLICATIONS OF FACTOR CATEGORIES TO COMPLETELY INDECOMPOSABLE MODULES

by Manabu HARADA

In this note we assume the reader is familiar to elementary properties of rings and modules. In some sense we can understand that the theory of categories is a generalization of the theory of rings. Especially, additive categories $\underline{A}$ have very similar properties to rings from their definitions.

From this point of view, we shall define an ideal $\underline{C}$ in $\underline{A}$ and a factor category $\underline{A}/\underline{C}$ of $\underline{A}$ with respect to $\underline{C}$ (see Chapter 1), which is analogous to factor modules or rings. The purpose of this lecture is to apply those factor categories to completely indecomposable modules.

First, we take an artinian ring R. The radical J(R) of R is a very important tool to study structures of R. Since R/J(R) is a semi-simple and artinian ring, we know useful properties of R/J(R). In order to study structures of R, we contrive to lift those properties to R. The idea in this note is closely related to the above situation.

Let R be a ring with identity and $\{M_\alpha\}_I$ a set of completely indecomposable right R-modules. In Chapter 1 we define the induced category $\underline{A}$ from $\{M_\alpha\}$ , which is a full sub-additive category in the category $\underline{M}_R$ of all right R-modules and define a special ideal $\underline{J}'$ of $\underline{A}$. Then $\underline{A}/\underline{J}'$ is *a abelian Grothendieck and completely reducible category* (Theorem 1.4.8), which is nearly equal to $\underline{M}_S$, where S is a semi-simple artinian ring. In this note we frequently make use of this theorem. Especially, in Chapter 2 we shall prove the Krull-Remak-Schmidt-Azumaya' theorem by virtue of this theorem, (see below).

Let $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ be any sets of completely indecomposable modules such that $M = \sum_I \oplus M_\alpha = \sum_J \oplus N_\beta$ . Then we consider the following properties :

I) *There exists a one-to-one mapping* $\phi$ *of* I *to* J *such that* $M_\alpha \approx N_{\phi(\alpha)}$ *and hence,* $|I| = |J|$ , *where* $|I|$ *means the cardinal of* I.

II) *(Take out (some components)) For any subset* I' *of* I, *there exists a one-to-one mapping* $\psi$ *of* I' *into* J *such that* $M_{\alpha'} \approx N_{\psi(\alpha')}$ *for* $\alpha' \in I'$ *and* $M = \sum_{\alpha' \in I'} \oplus N_{\psi(\alpha')} \oplus \sum_{\alpha'' \in I-I'} \oplus M_{\alpha''}$ .

II') *(Put into) For any subset* I' *of* I, *there exists a one-to-one mapping* $\psi$ *of* I' *into* J *such that* $M_{\alpha'} \approx N_{\psi(\alpha')}$ *for* $\alpha' \in I'$ *and*
$M = \sum_{\alpha' \in I'} \oplus M_{\alpha'} \oplus \sum_{\beta' \in J-\psi(I')} \oplus N_{\beta'}$ .

III) *Every direct summand of* M *is also a direct sum of completely indecomposable modules.*

M has always the properties I), II) and II') if I' in II) and II') are finite, which we call the Krull-Remak-Schmidt-Azumaya' theorem. If it is allowed to take any subset I', in II) or II'), then it is clear that II) and II') are equal to each other.

G. Azumaya [1] proved the avove II) and II') step by step and proved I) with II) and II'), provided I' is finite. We shall prove them independently and its proof suggests us how we can drop the assumption of finiteness on I' in the Azumaya' theorem. This argument is very much owing to the factor category $\underline{A}/\underline{J}'$ . The idea of dropping the assumption of finiteness gives us a definition of locally semi-T-nilpotency of the set of $\{M_\alpha\}_I$ (see Chapter 2), which is a generalization of T-nilpotency defined by H. Bass [2] .

On the other hand, the exchange property is very impcrtant to study decompositions of modules (cf. [4] ). In this note we shall slightly change its definition as follows : Let M be an R-module and N a direct summand of M. We suppose that for any decomposition $M = \sum_I \oplus K_\beta$ with $|I| \leqslant a$, we have a new decomposition ; $M = N \oplus \sum_I \oplus K_\beta'$ , where $K_\beta' \subseteq K_\beta$ for all $\beta \in I$. In this case, we say N has the a-*exchange property in* M. If N has the a-exchange property in M for any cardinal a, we say N has *the exchange property in* M. Furthermore, we define a new concept in

Chapter 3. Let K be a submodule of M and $K = \sum_{J'} \oplus K_{\gamma'}$ . If for any finite subset J' of J $\sum_{J'} \oplus K_{\gamma'}$ is a direct summand of M, we call K a *locally direct summand* of M (with respect to the decomposition $K = \sum_{J} \oplus K_{\gamma}$). It is clear that if all $K_\gamma$ are injective, K is always a locally direct summand of M. This property is useful to consider the problem of Matlis [29] , which is the property III) in case of injective modules.

Those concepts are mutually related in the following theorem (Theorems 3.1.2 and 3.2.5) : Let M and $\{M_\alpha\}_I$ be as above. Then the following statements are equivalent.

1) M *satisfies the take out property of any subset* I' *of* I *and for any* $\{N_\beta\}_J$ .

2) *Every direct summand of* M *has the exchange property in* M.

3) $\{M_\alpha\}_J$ *is a locally semi-*T*-nilpotent system.*

4) *Every locally direct summand of* M *is a direct summand of* M.

5) $\underline{J}' \cap \mathrm{End}_R(M)$ *is equal to the Jacobson radical* J *of* $\mathrm{End}_R(M)$.

6) $\mathrm{End}_R(M)/J$ *is a regular ring in the sense of Von Neumann and every idempotents in* $\mathrm{End}_R(M)/J$ *are lifted to* $\mathrm{End}_R(M)$.

We study the propery III in Chapters 3 and 4 and give a special answer for it, even though it is not complete, (Theorem 3.2.7), (cf.[6,7,17, 18,24,38]).

In 1960 H. Bass [2] defined (semi-) perfect rings as a generalization of semi-primary rings and E. Mares [28] further generalized them to (semi-) perfect modules in 1963. In Chapter 5 we shall prove the following theorem (Theorem 5.2.1) ; let $\{P_\alpha\}_I$ be a set of projective modules and $P = \sum_I \oplus P_\alpha$ . Then $J(P)$ *is small in* $P$ *if and only if* $J(P_\alpha)$ *is small in* $P_\alpha$ *for all* $\alpha \in I$ *and* $\{P_\alpha\}_I$ *is a locally semi-*T*-nilpotent system.* Using this theorem and Mares' results, we shall study structures of (semi-) perfect modules.

In Chapter 6 we shall study injective modules. Let $\{E_\alpha\}_I$ be a set of injective modules and $\underline{B}$ the induced category from $\{E_\alpha\}$ . First we shall prove that $\underline{B}/\underline{J}$ *is an abelian Grothendieck and spectral category,* where $\underline{J}$ is the radical of $\underline{B}$ (Theorem 6.2.1). We shall study decompositions of injective modules by making use of this theorem (cf. [10, 29, 31]). Finally we shall consider the Matlis'problem (cf. [9, 12, 25, 38,40,41]). Relating to it, we shall give the following theorem (Theorem 6.5.3) ; Let $\{E_\alpha\}_I$ be a set of injective and indecomposable modules, $E = \sum_I \oplus E_\alpha$ and $\underline{A}'$ the induced category from the all completely indecomposable modules. Then the following statements are equivalent.

1) $\{E_\alpha\}_I$ *is a locally semi-*T*-nilpotent system.*

2) *Every module* M *in* $\underline{A}'$ *which is an extension of* E *contains* E *as a direct summand.*

3) *Every module* M *in* $\underline{A}'$ *which is an essential extension of* E *coincides with* E.

4) *For any monomorphism* f *in* $End_R(E)$ Im f *is a direct summand of* E.

This lecture note gives some applications of the theory of category to the theory of modules, however conversely we can apply some concepts in this note to special categories and define *semi-perfect* or *semi-artinian Grothendieck categories,* which preserve many properties of semi-perfect or semi-artinian rings (see [22]).

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# CHAPTER 1. A PRINCIPAL THEOREM

We shall assume the reader has some knowledge about elementary definitions and properties of modules and categories. We refer to [11,30] for them.

## 1.1. IDEALS.

We always study additive categories $\underline{A}$ and so we shall assume that categories in this note are additive, unless otherwise stated. We shall use the following notations :

$\underline{M}_R$ ; the category of all right R-modules, where R is a ring with identity.

$\underline{A}_m$ ; the class of all morphisms in $\underline{A}$.

For $\alpha,\beta$ in $\underline{A}_m$ " $\alpha\beta$ is defined" implies codomain of $\beta$ = domain of $\alpha$ and "$\alpha\pm\beta$ is defined" implies domain of $\alpha$ = domain of $\beta$ and codomain of $\alpha$ = codomain of $\beta$.

We shall define ideals in an additive category $\underline{A}$.

DEFINITION.- Let $\underline{C}$ be a subclass of $\underline{A}_m$. If $\underline{C}$ satisfies the following conditions, $\underline{C}$ is called a *left ideal* of $\underline{A}$.

1. For any $\alpha\in\underline{A}_m$ and $\beta\in\underline{C}$ if $\alpha\beta$ is defined, $\alpha\beta\in\underline{C}$.
2. For any $\gamma,\delta\in\underline{C}$, if $\gamma\pm\delta$ is defined, $\gamma\pm\delta\in C$, (cf. [5]).

We can define similarly right or two-sided ideals in $\underline{A}$. Let $\underline{C}$ be a two-sided ideal in $\underline{A}$. If $[A,A]\cap\underline{C}$ is the Jacobson radical of $[A,A]$ for all $A\in\underline{A}$, $\underline{C}$ iscalled the *Jacobson radical* of $\underline{A}$, (if $\underline{A}$ has finite co-products, the Jacobson radical is uniquely determined, (see [16,27])).

The following notion is essential in this note.

DEFINITION.—Let $\underline{A}$ be an additive category and $\underline{C}$ a two-sided ideal in $\underline{A}$. We define a *factor category* $\underline{A}/\underline{C}$ of $\underline{A}$ with respect to $\underline{C}$ as follows :

1 The objects in $\underline{A}/\underline{C}$ coincide with those in $\underline{A}$ (for A in $\underline{A}$, $\bar{A}$ means that $\bar{A}$ is considered in $\underline{A}/\underline{C}$).

2 For $\bar{A},\bar{B} \in \underline{A}/\underline{C}$, $[\bar{A},\bar{B}] = [A,B]/[A,B] \cap \underline{C}$ (for $f \in [A,B]$, $\bar{f}$ means the residue class of f in $[A,B]/[A,B] \cap \underline{C}$).

*Remarks* 1.—It is clear $\underline{A}/\underline{C}$ is also an additive category. In general even if $\underline{A}$ is abelian, $\underline{A}/\underline{C}$ is not abelian. If we want to use structures of factor categories, we should find good ideals $\underline{C}$ such that $\underline{A}/\underline{C}$ become good categories.

2. Let $A = \sum_{i=1}^{n} \oplus A_i$ in $\underline{A}$. Then there exists inclusions $i_k$ and projections $p_k$ such that $1_A = \sum i_k p_k$; $p_k i_k = 1_{A_k}$ and $i_j p_k = 0$ if $j \neq k$. Those relations are preserved in $\underline{A}/\underline{C}$ , i.e. $\bar{1}_A = \sum \bar{i}_k \bar{p}_k$ , $\bar{p}_k \bar{i}_k = \bar{1}_{A_k}$ and $\bar{i}_j \bar{p}_k = 0$ if $j \neq k$. Hence, $\bar{A} = \sum \oplus \bar{A}_i$ in $\underline{A}/\underline{C}$. This is not true for infinite coproducts.

3. If A,B are isomorphic each other in $\underline{A}$, then there exist morphisms $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow A$ such that $\alpha\beta = 1_B$ and $\beta\alpha = 1_A$. Hence, $\bar{A},\bar{B}$ are isomorphic each other in $\underline{A}/\underline{C}$. However the converse is not true, in general. If $\underline{C}$ is the Jacobson radical, the converse is also true. Because, if $\bar{A},\bar{B}$ are isomorphic, there exist $\alpha' : A \longrightarrow B$, $\beta' : B \longrightarrow A$

such that $\bar{\alpha}'\bar{\beta}' = \bar{1}_B$ and $\bar{\beta}'\bar{\alpha}' = \bar{1}_A$. Hence, $1_A-\beta'\alpha'$ is in the radical of $[A,A]$. Therefore, $\beta'\alpha'$ is a unit in $[A,A]$. Similarly, $\alpha'\beta'$ is a unit in $[B,B]$. Hence, $\alpha', \beta'$ are isomorphisms.

PROPOSITION 1.1.1. - *Let* $\underline{A},\underline{B}$ *be additive categories and* $T : \underline{A} \longrightarrow \underline{B}$ *an additive covariant functor. Then* $\underline{C} = \{\alpha \mid \epsilon \underline{A}_m, T\alpha=0\}$ *) *is a two-sided ideal in* $\underline{A}$ *and* $T = \bar{T}\circ\psi$ *, where,* $\psi : \underline{A} \longrightarrow \underline{A}/\underline{C}$ *is a natural functor and* $\bar{T} : \underline{A}/\underline{C} \longrightarrow \underline{B}$ *is naturally induced from* T.

1.2. ABELIAN CATEGORIES.

Let $\underline{A}$ be an additive category. There are many equivalent definitions for $\underline{A}$ to be *abelian*. We shall take the following :

i For any two objects A,B in $\underline{A}$ the coproduct $A \oplus B$ of A and B is defined and belongs to $\underline{A}$.

ii $\underline{A}$ contains a zero object (so does an additive category).

iii For each morphism f in $\underline{A}$, Ker f and Coker f exist in $\underline{A}$.

iv (normal) For each monomorphism f in A, f is a kernel of some morphism in $\underline{A}$.

iv' (conormal) For each epimorphism f in $\underline{A}$, f is a cokernel of some morphism in $\underline{A}$.

In this section, we shall rewrite the above definition of an abelian category by virtue of another terminologies, which are very familiar to the ring theory.

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*) In general, it is not a set, but we shall use the same notation as the set. We always use such notations.

Let $\underline{A}$ be an additive category and S a subclass of $\underline{A}_m$. We put $(S:\alpha)_r = \{\beta | \in \underline{A}_m$, $\alpha\beta$ is defined and $\alpha\beta \in S\}$ *), $(S:\alpha)_l = \{\beta | \in \underline{A}_m$, $\beta\alpha$ is defined and $\beta\alpha \in S\}$. If $(0:\alpha)_r \neq 0$ for some $\alpha \in \underline{A}_m$, $\alpha$ is called a left *zero-divisor*. Similarly, we define a right zero-divisor. From the definitions, we know that $\alpha$ is monomorphic (epimorphic) if only only if $\alpha$ is not left (right) zero-divisor. Let $C \xrightarrow{\alpha'} A \xrightarrow{\alpha} B$ be a sequence. Then $\alpha'$ is the kernel of $\alpha$ if and only if $(0:\alpha')_r=0$ and $(0;\alpha)_r=\alpha'\underline{A}_m$, where $\alpha'\underline{A}_m = \{\alpha\,\gamma | \gamma \in \underline{A}_m$, $\alpha\,\gamma$ is defined$\}$. $\alpha$ is the cokernel of $\alpha'$ if and only if $(0:\alpha)_l=0$ and $(0;\alpha')_l=\underline{A}_m\alpha$.

PROPOSITION 1.2.1.–*Let* $\underline{A}$ *be an additive category with finite co-products. Then* A *is abelian if and only if* $\underline{A}$ *satisfies the following conditions*

*1 For each* $\alpha \in \underline{A}_m$, *there exists* $\beta \in \underline{A}_m$ *such that* $(0:\beta)_r=0$ *and* $(0:\alpha)_r=\beta\underline{A}_m$.

*2 For each* $\alpha \in \underline{A}_m$ *there exists* $\beta'$ *such that* $(0:\beta')_l=0$ *and* $(0:\alpha)_l = \underline{A}_m\beta'$.

*3 For each* $\gamma \in \underline{A}_m$ *such that* $(0:\gamma)_r=0$, $(0:(0:\gamma)_l)_r = \gamma\underline{A}_m$.

*4 For each* $\gamma' \in \underline{A}_m$ *such that* $(0:\gamma')_l=0$, $(0:(0:\gamma')_r)_l=\underline{A}_m\gamma'$.

*Proof*. - By the assumption $\underline{A}_m$ satisfies i, ii in the above definition and iii corresponds to 1,2 from the above remark. We assume $\underline{A}$ is abelian Let $\gamma$ be as in 3. Then there exists a cokernel $\beta$ of $\gamma$ ; $0 \to A \xrightarrow{\gamma} B \xrightarrow{\beta} C \to 0$ exact. Then $\gamma = \mathrm{Ker}\beta$ and $\beta = \mathrm{Coker}\,\gamma$ . Hence, $(0:\beta)_r = \gamma A_m$ and $(0:\gamma)_l=\underline{A}_m\beta$ from the above remark. Therefore, $(0:(0:\gamma)_l)_r=(0:\underline{A}_m\beta)_r=(0:\beta)_r=\gamma\underline{A}_m$.

4 is dual to 3. Conversely, we assume $\underline{A}_m$ satisfies $1 \sim 4$. We know from the remark that 1,2 guarantee the existence of kernel and cokernel for any $\alpha \in \underline{A}_m$. Let $\gamma: A \to B$ be monomorphic. Then there exists $\beta \in \underline{A}_m$ such that $\beta$ is epimorphic and $(0:\gamma)_1 = \underline{A}_m\beta$ from 2. Furthermore, $(0:(0:\gamma)_1)_r = (0:\underline{A}_m\beta)_r = (0:\beta)_r = \gamma\, \underline{A}_m$ by 3. Hence, $\gamma = \mathrm{Ker}\, \beta$ and we have iv. iv' is dual to iv. Therefore, $\underline{A}$ is abelian.

## 1.3. AMENABLE CATEGORIES.

We shall define some special categories which we shall use later.

DEFINITION. Let $\underline{A}$ be an additive category. $\underline{A}$ is called *regular* if $[A,A]$ is a regular ring in the sense of Von Neumann for all $A \in \underline{A}$. $\underline{A}$ is called *amenable* if $\underline{A}$ has finite co-products and for any idempotent e in $[A,A]$ splits, i.e. $A = \mathrm{Im}\, e \oplus \mathrm{Ker}\, e$ for all $A \in \underline{A}$, (see [11]). $\underline{A}$ is called *spectral* if all $f \in \underline{A}_m$ splits (see [13]).

PROPOSITION 1.3.1. - *Let $\underline{A}$ be an additive, amenable and regular category. Then $\underline{A}$ is abelian.*

*Proof.* - Since $\underline{A}$ is amenable, $\underline{A}$ satisfies the assumption in (1.2.1). We shall show $\underline{A}$ satisfies $1 \sim 4$ in (1.2.1). Let $\alpha: A \to B$ be monomorphic. Put $\alpha' = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} : A \oplus B \to A \oplus B$. Since $\underline{A}$ is regular, there exists $x = (x_{ij}) \in [A\oplus B, A\oplus B]$ such that $\alpha' x \alpha' = \alpha'$. Hence, $\alpha = \alpha x_{12} \alpha$ . Put $e = x_{12}\alpha$ , then $e = e^2$ and $\alpha e = \alpha$. Hence, $\underline{A}_m\alpha = \underline{A}_m e$. Since $\underline{A}$ is amenable,

$e = i_e e'$ , where $e' : A \to \mathrm{Im}\ e$ is epimorphic and $i_e : \mathrm{Im}\ e \to A$ is the inclusion. Thus, we have $(0:\alpha)_r = (0:A_m\alpha)_r = (0:\underline{A}_m e)_r = (0:e)_r = (1_A - e)\underline{A}_m \subseteq i_{(1-e)}\underline{A}_m \subseteq (0:\alpha)_r$. Hence $(0:\alpha)_r = i_{(1-e)}\underline{A}_m$ and $(0:(0:\alpha)_r)_1 = (0:i_{(1-e)})_1 = \underline{A}_m e = \underline{A}_m\ \alpha$, which gives 2 and 4 in (1.2.1). From the duality we obtain 1 and 3. Therefore, $\underline{A}$ is abelian.

We can easily see from the above proof that Im e = Im $\alpha$. Thus, we have

COROLLARY 1.3.2 [35] . *Let* $\underline{A}$ *be an additive and amenable category, Then* A *is (abelian) spectral if and only if* $\underline{A}$ *is (abelian) regular.*

1.4 A principal theorem on indecomposable modules

Let R be a ring with identity. We consider always unitary right R-modules M. If $\mathrm{End}_R(M)$ is a local ring (i.e. its radical is a unique max maximal left or right ideal), M is called *completely indecomposable* module (briefly c.inde.). It is clear that c.inde. module is indecomposable as a directsum, however the converse is not true. We note that the radical is equal to the set of all non-isomorphisms in $\mathrm{End}_R(M)$ if M is c.inde. by the following.

LEMMA. 1.4.1. - *Let* $M_i$ , i = 1.2.3 *be* (c.) *inde. and* $f_i : M_i \to M_{i+1}$ R-*homomorphisms for* i = 1,2. *if* $f_2 f_1$ *is isomorphic,* $f_i$ *are isomorphic.*

*Proof.* - Since $f_2 f_1$ is isomorphic, $f_1$ is monomorphic and $f_2$ is epimorphic. Furthermore, $M_2 = \mathrm{Im}\ f_1 \oplus \mathrm{Ker}\ f_2$. Hence, $\mathrm{Ker}\ f_2 = 0$ and $\mathrm{Im}\ f_1 = M_2$.

Let $\{M_\alpha\}_I$ , $\{N_\beta\}_J$ be sets of modules and put $M = \sum_I \oplus M_\alpha$ and $N = \sum_J \oplus N_\beta$. We shall describe $\mathrm{Hom}_R(M,N)$ as the set of matrices. Let $\alpha_{ij} : M_j \to N_i$ be R-homomorphisms. If I and J are finite, $\mathrm{Hom}_R(M,N) = \{(J \times I)$ matrices $(\alpha_{ij})\}$. We assume I and J are infinite. Let m be an element in $M_1$ and $f \in \mathrm{Hom}_R(M,N)$.

Then $f(m) = \sum_{i=1}^{n} n_{\beta i}$ ; $n_{\beta i} \in N_{\beta i}$. From this remark, we can define a summable set of homomorphisms $\{\alpha_{j1}\}_j$ as follows : for any m in $M_1$ $\alpha_{J1}(m) = 0$ for almost all $j \in J$. In this case $\sum_J \alpha_{J1}$ has a meaning and

$\alpha_{J1} : M_1 \to N$ is an R-homomorphism. A matrix $(\alpha_{ij})$ is called *column summable* if $\{\alpha_{ji}\}_j$ is summable for all $i \in I$. Then it is clear that $\mathrm{Hom}_R(M,N)$ is isomorphic to the modules of all column summable matrices with entries $\alpha_{ij}$.

Let $T = \sum_K \oplus T_\delta$ be another module and $f \in \mathrm{Hom}_R(M,N)$, $g \in \mathrm{Hom}_R(N,T)$. We assume $f = (\alpha_{ij})$ and $g = (\beta_{pq})$ as above. Then we can easily show that $gf = (\beta_{pq})\ (\alpha_{ij})$. Thus, if M=N=T, End(M) is isomorphic to the ring of all column summable matrices $(\alpha_{ij})$.

Now, we shall assume that all $M_\alpha$ , $N_\beta$ and $T_\gamma$ are c.inde. . We define a subset.

$J'^{(\beta,\alpha)} = \{(\alpha_{ij}) \mid \in \mathrm{Hom}_R(M,N)$ *and no one of* $\alpha_{ij}$ *is isomorphic*$\}$, ($J'^{(\beta,\alpha)}$ *may depend on decompositions* M *and* N).

LEMMA 1.4.2. - *Let* $M = \sum_I \oplus M_\alpha$ , $N = \sum_J \oplus N_\sigma$ *and* $T = \sum_K \oplus T_\rho$ *and all* $M_\alpha$ , $N_\sigma$ *and* $T_\rho$ *c.inde.. Then* $\mathrm{Hom}_R(N,T)J'^{(\sigma,\alpha)} \subsetneq J'^{(\beta,\alpha)}$ , $J'^{(\rho,\sigma)}$. $\mathrm{Hom}_R(M,N) \subsetneq J'^{(\rho,\alpha)}$ .

*Proof.* - Let $f = (a_{ij}) \in J'^{(\sigma,\alpha)}$, $h = (b_{jk}) \in \mathrm{Hom}_R(N,T)$ and $hf = (x_{ts})$, where $x_{ts} = \sum_K b_{tk}a_{ks}$. If $M_s \not\approx T_t$, $x_{ts}$ is not isomorphic. We suppose $M_s \approx T_t$. Let $m \neq 0$ be in $M_s$. Since $(a_{ij})$ is column summable, there exists a finite subset $J'$ of $J$ such that $a_{ks}(m) = 0$ if $k \in J-J'$. Put $x_{ts} = \sum_{K_i \in J'} b_{tk_i}a_{k_i s} + \sum_{J-J'} b_{tk}a_{ks}$. Then neither the latter nor former term is isomorphic by the definition of $J'$ and (1.4.1). Thus $x_{ts}$ is not isomorphic by the remark before (1.4.1) and the fact $M_s \approx T_t$. Hence, $hf \in J'^{(\rho,\alpha)}$. Similarly we have the last part.

PROPOSITION. 1.4.3 [1] *The above module* $J'^{(\sigma,\alpha)}$ *does not depend on decompositions of* M *and* N. *Especially, if* M=N, J' *is a two-sided ideal in* $\mathrm{End}_R(M)$.

*Proof.* - Let $M = \sum_I \oplus M_\alpha$ and $N = \sum_J \oplus N_\sigma = \sum_{J'} \oplus N'_{\sigma'}$. Put $T = N = \sum_{J'} \oplus N'_{\sigma'}$ in (1.4.2). Then for any $f \in J'^{(\sigma,\alpha)}$, $f = 1_N f \in J'^{(\sigma,\alpha)}$. Therefore, $J'^{(\sigma,\alpha)} \subseteq J'^{(\sigma',\alpha)}$. Similarly, we obtain $J'^{(\sigma',\alpha)} \subseteq J'^{(\sigma,\alpha)}$ and hence $J'^{(\sigma',\alpha)} = J'^{(\sigma,\alpha)}$.

From (1.4.3) we denote $J'^{(\sigma,\alpha)}$ by J'.

We shall give here elementary properties of a ring.

LEMMA 1.4.4. - *Let* R *be a ring and* e,f *idempotents such that* $eR \overset{\phi_1}{\approx} fR$ *and* $(1-e)R \overset{\phi_2}{\approx} (1-f)R$. *Then there exists a regular element* a *in* R *such that* $f = a^{-1}ea$.

*Proof.* - $R = eR\oplus(1-e)R = fR\oplus(1-f)R$. Therefore, $\phi = \phi_1 + \phi_2 \in End_R(R)=R_1$, say $\phi = a_1$. Then it is clear that $a_1e_1 = f_1a_1$, ($R_1$ means the set of the left multiplications of elements in R).

We shall later make use of the following.

COROLLARY 1.4.5. - *Let* P *be a vector space over a division ring* $\Delta$, *say* $P = \sum_I \oplus u_\sigma\Delta$ . *Let* $S = End_\Delta(P)$ *and* e *an idempotent in* S. *Then there exist a subset* J *of* I *and a regular element* a *in* S *such that for the projection* $f: P\to\sum_J \oplus v_\gamma\Delta$ $e = a^{-1}fa$.

*Proof.* - Let $eP = \sum_J \oplus v_\gamma\Delta$ and we may assume $P = \sum_J \oplus u_\rho\Delta \oplus \sum_{I-J} \oplus u_\sigma\Delta$ . Since $eS \approx Hom_\Delta(P,eP) \approx Hom_\Delta(P,fP)$, we have the corollary by (1.4.4).

Now, we shall enter into a main part of this section. Let $\{M_\alpha\}_I$ be a set of c.inde. Modules. By $\underline{A}(\underline{A}_f)$ we shall denote the full sub-additive category in $\underline{M}_R$ , whose objects consist of all kinds of (finite) direct sums $\sum_K \oplus T_\gamma$ such that $T_\gamma$' s are isomorphic to some $M_\beta$ in $\{M_\alpha\}_I$. We call $\underline{A}$ $(\underline{A}_f)$ the (finitely) *induced category* from $\{M_\alpha\}_I$, (we shall use the same terminology even if $\{M_\alpha\}$ are not c.inde.).

DEFINITION . - Let $\underline{B}$ be an additive category. If $\underline{B}$ satisfies the following properties, $\underline{B}$ is called a *Grothendieck category*.

1 $\underline{B}$ is abelian.

2 $\underline{B}$ has any co-products.

3 Let $B\in\underline{B}$ and $\{B_\alpha\}$, C sub-objects of B such that $\{B_\alpha\}$ is a directed set. Then

$$(\cup B_\alpha) \cap C = \bigcup (B_\alpha \cap C).$$

(This corresponds to a fact that functor $\varinjlim$ is exact (see [30], Ch. 3)).

4 $\underline{B}$ has a generator, (this implies $\underline{B}$ is complete (see [14] )).

Definition. Let $\underline{B}$ be as above. If every object in $\underline{B}$ is *artinian (noetherian)* with respect to sub-objects, $\underline{B}$ is called *artinian (noetherian)*. If every object in $\underline{B}$ is a co-product of minimal objects, B is called *completely reducible*. If the Jacobson radical of $\underline{B}$ is zero, $\underline{B}$ is called *semi-simple*.

LEMMA 1.4.6. - *Let* A *be a semi-simple category with finite co-products. If* $\alpha \neq 0 \in [M,N]$, *there exist* $\beta$, $\beta' \in [N,M]$ *such that* $\beta\alpha \neq 0$ *and* $\alpha\beta' \neq 0$.

*Proof.* - Put $P = M \oplus N$, $S = [P,P]$ and $\alpha^* = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$. If $[N,M]\alpha = 0$, $S\alpha^*$ is nilpotent, which is a contradiction. Similarly, we have $\alpha[N,M] \neq 0$.

COROLLARY 1.4.7. - *Let* $\underline{A}$ *be as above. If* $[M,M]$ *is a division ring,* M *is a minimal object.*

*Proof.* - Let $M \supsetneq N$. Then $[M,N] = 0$. Hence, $[N,M] = 0$ by (1.4.6) and so the inclusion map : $N \to M$ is zero.

From now on, by $[M,N]$ we shall denote $\mathrm{Hom}_R(M,N)$ for R-modules M,N.

THEOREM 1.4.8 (Principal theorem) [17] .-*Let* $\{M_\alpha\}_I$ *be a set of* c. inde. *modules and* $\underline{A}$,$\underline{A}_f$ *the induced category and finitely induced category, respectively. Let* $\underline{J}'$ *be the ideal in* $\underline{A}$ *defined before (1.4.2). Then* $\underline{A}/\underline{J}'$ ($\underline{A}_f/\underline{J}'$) *is a Grothendieck and completely reducible (completely) category.*

*Proof.* - We put $\bar{\underline{A}} = \underline{A}/\underline{J}'$ $(\bar{\underline{A}}_f = \underline{A}_f/\underline{J}')$. From the definition of co-product and (1.4.3) we can easily show $\overline{\sum_J \oplus M_\gamma} = \sum_J \oplus \bar{M}_\gamma$. Put $\bar{S}_M = [M,M]/[M,M]\cap\underline{J}'$ for an object $M = \sum_J \oplus M_\gamma$ in $\underline{A}$. Then $\bar{S}_M = \{(\bar{a}_{\sigma\tau})$, column finite $\}$, since $\bar{a}_{\sigma\tau} = 0$ for almost all $\sigma$. We rearrange M as follows : $M = \sum_\alpha \sum_{I_\alpha \ni \beta} \oplus M_{\alpha\beta}$ ; $M_{\alpha\rho} \approx M_{\alpha\rho'}$, and $M_{\alpha\rho} \not\approx M_{\alpha'\rho'}$, if $\alpha \neq \alpha'$ . Then $[\bar{M},\bar{M}] = \prod_\alpha [\sum_{I_\alpha} \oplus \bar{M}_{\alpha\beta}, \sum_{I_\alpha} \oplus \bar{M}_{\alpha\beta}]$, $[\sum_{I_\alpha} \oplus \bar{M}_{\alpha\beta}, \sum_{I_\alpha} \oplus \bar{M}_{\alpha\beta}] \approx \{(x_{\alpha\beta}) |$ column finite and $x_{\alpha\beta} \in [\bar{M}_\alpha, \bar{M}_\alpha] = \Delta_\alpha$, which is a division ring$\}$. Therefore, $\bar{\underline{A}}$ and $\bar{\underline{A}}_f$ are regular and semi-simple. Next, we shall show that they are amenable. Put $\bar{S}_\alpha = [\sum_{I_\alpha} \oplus \bar{M}_{\alpha\beta}, \sum_{I_\alpha} \oplus \bar{M}_{\alpha\beta}]$, then $\bar{S}_M = \prod_\alpha \bar{S}_\alpha$. Let $\bar{e}$ be an idempotent in $\bar{S}_M = \prod_\alpha \bar{S}_\alpha$; $\bar{e} = \prod \bar{e}_\alpha$ , $\bar{e}_\alpha \in \bar{S}_\alpha$ , $\bar{e}_\alpha^2 = \bar{e}_\alpha$ . Then there exist a regular element $\bar{a}_\alpha \in \bar{S}_\alpha$ and a projection $f_\alpha : \sum_{I_\alpha} \oplus M_{\alpha\beta} \to \sum_{J_\alpha} \oplus M_{\alpha\beta}$, in $\underline{M}_R$ such that $\bar{e}_\alpha = \bar{a}_\alpha^{-1} \bar{f}_\alpha \bar{a}_\alpha$ by (1.4.5), (note $\bar{S}_\alpha$ may be regarded as the endomorphism ring of a vector space). Since $f_\alpha$ is the projection in $\underline{M}_R$ , $\bar{f}_\alpha$ splits in $\bar{\underline{A}}$. Hence, so does $\bar{e}_\alpha$ since $\bar{a}_\alpha$ is regular, and $\bar{e}_\alpha : \bar{M}_\alpha \xrightarrow{\bar{f}\bar{a}_\alpha} \operatorname{Im} \bar{f}_\alpha \xrightarrow{\bar{a}_\alpha^{-1}\ \bar{i}\bar{f}_\alpha} \bar{M}_\alpha$.

Therefore, so does $\bar{e}$, which implies that $\bar{\underline{A}}$ $(\bar{\underline{A}}_f)$ is amenable. Thus, $\bar{\underline{A}}$ $(\bar{\underline{A}}_f)$ is abelian and spectral by (1.3.2). On the other hand, $\bar{M}_\alpha$ is a minimal object by (1.4.7). Hence, $\bar{\underline{A}}$ is completely reducible. Finally we shall show that $\bar{\underline{A}}$ satisfies the condition 3) in the definition of Grothendieck categories. Let $\{\bar{A}_\alpha\}_K$ be a directed set of subobjects in an object $\bar{F}$ and

$\bar{B}$ a subobject in $\bar{F}$. Put $\bar{C} = \bigcup_K(\bar{A}_\alpha \cap \bar{B})$, then $\bar{B} = \bar{C} \oplus \bar{B}_o$, since $\underline{A}$ is spectral. $(\bigcup_K \bar{A}_\alpha) \cap \bar{B} = (\bigcup_K \bar{A}_\alpha) \cap (\bar{C} \cup \bar{B}_o) = \bar{C} \cup ((\bigcup_K \bar{A}_\alpha \cap \bar{B}_o)$, since $\bar{C} \subsetneq \bigcup_K \bar{A}_\alpha$. We assume $(\bigcup_K \bar{A}_\alpha) \cap \bar{B}_o = \bar{D} \neq 0$. From an exact sequence : $\sum_K \oplus \bar{A}_\alpha \xrightarrow{\bar{f}} \bigcup_K \bar{A}_\alpha \longrightarrow 0$ we obtain a monomorphism $\bar{g}:\bar{D} \longrightarrow \Sigma \oplus \bar{A}_\alpha$ such that $\bar{f}\bar{g} = 1_{\bar{D}}$ , because $\bar{\underline{A}}$ is spectral. Let $\bar{D}_o$ be a minimal sub-object in $\bar{D}$. Then $\bar{g}|\bar{D}_o$ is a column finite matrix from the first part. Hence, Im $(\bar{g}|\bar{D}_o) \subsetneq \sum_1^n \oplus \bar{A}_{\alpha_i}$ and so $\bar{D}_o \subsetneq \bigcup_{i=1}^n \bar{A}_{\alpha_i} \subsetneq \bar{A}_\beta$ for some $\beta \in K$ such that $\beta \geq \alpha_i$. Thus, $\bar{D}_o \subsetneq \bar{A}_\beta \cap \bar{B} \subseteq \bar{C}$ and $\bar{D}_o \subsetneq \bar{B}_o$ , which is a contradiction. Therefore, $(\bigcup_K \bar{A}_\alpha) \cap \bar{B} = \bigcup_K(\bar{A}_\alpha \cap \bar{B})$.

## CHAPTER 2. THE THEOREM OF KRULL-REMAK-SCHMIDT-AZUMAYA.

In this chapter we shall prove the titled theorem as an application of (1.4.8).

2.1. Azumaya' theorem :

Let $\{M_\alpha\}_I$ be a set of c.inde. modules and $M = \sum_I \oplus M_\alpha$.

LEMMA 2.1.1 [1] .-*Let* M *and* $\{M_\alpha\}_I$ *be as above and* $S_M = [M, \ ]$ . *Let* a *be any element in* $S_M$. *Then for any finite subset* $\{M_{\alpha i}\}_{i=1}^n$ *of* $\{M_\alpha\}_I$ , *there exists a set* $\{M_i\}_{i=1}^n$ *of direct sumand of* M *such that* $M = \sum_{i=1}^n \oplus M_i \sum_{\alpha \neq \{\alpha_i\}} \oplus M_\alpha$ *and* $M_{\alpha i}$ *is isomorphic to* $M_i$ *via* a *or* (1-a) *for each* i.

*Proof.* - Let $e_1$ be the projection of M to $M_{\alpha 1}$ . Then $e_1 a | M_{\alpha 1}$ and $e_1(1-a)e_1 | M_{\alpha 1}$ are in $[M_{\alpha 1} , M_{\alpha 1}]$ and $1_{M_{\alpha 1}} = (e_1 a e_1 + e_1(1-a)e_1) | M_{\alpha 1}$ . Since $M_{\alpha 1}$ is c.inde., either $e_1 a e_1 | M_{\alpha 1}$ or $e_1(1-a)e_1 | M_{\alpha 1}$ is isomorphic : $M_{\alpha 1} \xrightarrow{b} b(M_{\alpha 1}) \xrightarrow{e_1} M_{\alpha 1}$ , where b = a or (1-a). Hence, M = $b(M_{\alpha 1}) \oplus \mathrm{Ker}\ e_1 = b(M_{\alpha 1}) \oplus \sum_{\alpha \neq \alpha_1} \oplus M_\alpha$ . Repeating this argument on the last decomposition, we obtain (2.1.1).

LEMMA 2.1.2 [1] ..*Let* J' *be the ideal in § 1.4. Then* J' *does not contain non-zero idempotents.*

*Proof.* - Let e be a non-zero idempotent in $S_M$. Then there exists a finite subset $\{M_{\alpha i}\}_{i=1}^{n}$ of $\{M_\alpha\}_I$ such that $eM \cap \sum_{i=1}^{n} \oplus M_{\alpha i} \neq 0$. We apply (2.1.1) to e and $\{M_{\alpha i}\}_{i=1}^{n}$. Then we can find a direct summand $\sum_{i=1}^{n} \oplus M_i$ of M such that $M_i = b_i(M_{\alpha i})$, where $b_i$ = e or (1-e). It is impossible that all $b_i$ are equal to (1-e). Hence, $e_i e e_{\alpha i}$ is isomorphic for some i, where $e_{\alpha_i} : M \longrightarrow M_{\alpha_i}$, $e_i : M \longrightarrow M_i$ are projections. Therefore, $e \notin J'$ by (1.4.3).

LEMMA 2.1.3.- *Let* $M = \sum_{i=1}^{n} \oplus N_i$ *and* $N_i$ c.inde.. *Then* J' *is the Jacobson radical of* $S_M$.

*Proof.* - Let $x = (x_{ij})$ be in J'. Then we note that $1-x_{ii}$ is regular in $S_{M_i}$ and that a sum of non isomorphisms of $S_{M_i}$ is not isomorphic. By the above remark and (1.4.2) we can find regular matrices P,Q in $S_M$ such that $P(1-X)Q = 1_M$. Hence, X is quasi-regular.

We shall consider a similar lemma in a case of infinite sum in the next section.

Now we can prove the Krull-Remak-Schmidt-Azumaya' theorem.

THEOREM 2.1.4 [1, 7,17]. - *Let* $\{M_\alpha\}_I$ , $\{N_\beta\}_J$ *be sets of* c.inde. *modules such that* $M = \sum_I \oplus M_\alpha = \sum_J \oplus N_\beta$ *. Then*

*I) There exists a one-to-one mapping* $\phi$ *of* I *onto* J *such that*

$M_\alpha \approx N_{\phi(\alpha)}$ *for all* $\alpha \in I$ *and hence,* $|I| = |J|$, *where* $|I|$ *is the cardinal of* $I$.

*II) For any finite subset* $I'$ *of* $I$, *there exists a one-to-one mapping* $\psi$ *of* $I'$ *into* $J$ *such that* $M_i \approx N_{\psi(i)}$ *for all* $i \in I'$ *and*

$$M = \sum_{i \in J'} \oplus N_{\psi(i)} \oplus \sum_{I-I'} \oplus M_{\alpha'} .$$

*II') For any finite subset* $I'$ *of* $J'$, *there exists a one-to-one* mapping $\psi'$ *of* $I'$ *into* $J$ *such that* $M_i \approx N_{\psi'(i)}$ *for all* $i \in I'$ *and*

$$M = \sum_{I'} \oplus M_i \oplus \sum_{J-\psi'(I')} \oplus N_{\beta'} .$$

*III) Let* $M'$ *be a direct summand of* $M$, *then* $M'$ *is isomorphic to some* $\sum_{i=1}^{n} \oplus M_{\alpha i}$ *or for any* $m < \infty$ $M'$ *contains a direct summand, which is isomorphic to some* $\sum_{i=1}^{m} \oplus M_{\alpha i}$.

*Proof.* - I) Let $\underline{A}$ be the induced category from $\{M_\alpha, N_\beta\}_{(I,J)}$ and $\underline{J}'$ the ideal in $\underline{A}$ defined in 6.1.4. Then $\underline{A}/\underline{J}' = \bar{\underline{A}}$ is a Grothendieck and completely reducible category by (1.4.8). Furthermore, we know from its proof that $\bar{M} = \sum_I \oplus \bar{M}_\alpha = \sum_J \oplus \bar{N}_\beta$. Since $\bar{M}_\alpha$ and $\bar{N}_\beta$ are minimal objects, there exists a one-to-one mapping $\phi$ of $I$ onto $J$ such that $\bar{M}_\alpha \approx \bar{N}_{\phi(\alpha)}$, (note that we may use the similar argument in $\bar{\underline{A}}$ to the ring theory, since $\bar{\underline{A}}$ is a good category). On the other hand, $S_{M_\alpha} \cap \underline{J}'$ is equal to the Jacobson radical. Hence, $\bar{M}_\alpha \approx \bar{N}_{\phi(\alpha)}$ implies $M_\alpha \approx N_{\phi(\alpha)}$ as R-modules by the remark 3 in § 1.1.

II) Put $M_o = \sum_{I'} \oplus M_{\alpha i}$ and let $p : M \to M_o$ be the projection. Then $\overline{M} =$ $= \mathrm{Ker}\ \bar{p} \oplus \sum_{I'} \oplus \overline{N}_{\psi(\alpha_i)}$, since $\bar{\underline{A}}$ is completely reducible, where $\overline{M}_{\alpha_i} \approx \overline{N}_{\psi(\alpha i)}$. It is clear $\mathrm{Im}\ \bar{p} = \sum_{I'} \oplus \bar{M}_{\alpha i}$. Put $N_0 = \sum_{I'} \oplus N_{\psi(\alpha_i)}$ and let $i : N_0 \to M$ be the inclusion. Then $\overline{pi}$ is isomorphic in $\bar{\underline{A}}$. Since $N_0 \in \underline{A}_f$, $\underline{J}' \cap [N_0, N_0]$ is equal to the radical of $[N_0, N_0]$ by (2.1.3). Hence, pi is isomorphic in $\underline{M}_R$ by the remark 3 in § 1.1. Therefore, $M = N_0 \oplus \mathrm{Ker}\ p$ in $\underline{M}_R$ and so $M = N_0 \oplus \sum_{I-I'} \oplus M_{\alpha'}$. It is clear that $M_{\alpha_i} \approx N_{\psi(\alpha_i)}$ in $\underline{M}_R$.

II') The following argument is dual to that in the above. Put $M_0{}' = \sum_{I'} \oplus M_{\alpha_i}$. Since $\bar{\underline{A}}$ is completely reducible,

$$\bar{M} = \bar{M}_0{}' \oplus \sum_{J-\psi'(I')} \oplus N_{\beta'} \text{ , where } \psi' : I' \to J \text{ and } \overline{M}_{\alpha'_i} \approx \overline{N}_{\psi'(\alpha'_i)} \quad \ldots(*).$$

Let p' be the projection of M to $N_0{}' = \sum_{I'} \oplus N_{\psi'(\alpha'_i)}$. It is clear that $\mathrm{Ker}\ \bar{p}' = \sum_{J-\psi'(I')} \oplus N_{\beta'}$, $\mathrm{Im}\ \bar{p} = \sum_{I'} \oplus \overline{N}_{\psi'(\alpha'_i)}$ and $\bar{p}|\bar{M}_0{}'$ is isomorphic by (*). Let $i' : M_0{}' \to M$ be the inclusion, then $\overline{p'i}$ is isomorphic. Since $M_0{}'$ is in $\underline{A}_f$, p'i is isomorphic in $\underline{M}_R$. Therefore, $M = M_0{}' \oplus \mathrm{Ker}\ p'$ in $\underline{M}_R$ and $M = M_0{}' \oplus \sum_{J-\psi'(I')} \oplus N_{\beta}{}'$.

III) Let e be a projection of M to M'. Since $\bar{\underline{A}}$ is completely reducible, $\mathrm{Im}\ \bar{e} = \sum_{I'} \oplus \overline{M'_\alpha}$, where $M'_\alpha$ are isomorphic to some $M_\beta$ in $\{M_\alpha\}_I$. Put $M_0 = \sum_{I'} \oplus M'_{\alpha'_i} \oplus\!> M_0{}' = \sum_{i=1}^{t} \oplus M'_{\alpha'_i}$ in $\underline{M}_R$. Then from the definition of $\underline{A}$, we have the following R-homomorphisms : $i : M_0' \to M_0 \overset{i'}{\to} M$ and $p: M \overset{e'}{\longrightarrow} M_0 \longrightarrow M_0'$ such that $\bar{i}$ is the inclusion $\bar{M}_0' \to \bar{M}$, $\bar{p}: \bar{M} \longrightarrow \bar{M}_0'$ is the projection and $\bar{i}'\bar{e}' = \bar{e}$. Since $M_0' \in \underline{A}_f$ and $\overline{pei}$ is isomorphic in $\bar{\underline{A}}$, so is pei in $\underline{M}_R$ ;

$$M_0' \overset{i}{\longrightarrow} M \overset{e}{\longrightarrow} M \overset{f}{\longrightarrow} M_0' \quad \ldots \; (**).$$

Hence, Im e in $\underline{M}_R$ = M' contains Im ei , which is a direct summand of M and isomorphic to $\sum_{i=1}^{n} \oplus M'_{\alpha_i'}$ . If I' is infinite, the above argument gives the last part in III). We assume I' is finite. In this case, we can take $M_0' = M_0$. Hence, M' = Im e contains Im ei as R-direct summand from (**). On the other hand, Im $\bar{e}$ = Im $\overline{ei}$ and hence, M' is equal to Im ei by (2.1.2) , which is isomorphic to $\sum_{i=1}^{t<\infty} \oplus M'_{\alpha_i'}$ .

REMARK 1. In the above proof, we used only an assumption "I' is finite" to obtain that $\underline{J}' \cap [M_0, M_0]$ is equal to the radical of $[M_0, M_0]$ for some module $M_0$. Hence, if we can show the above property with another assumption, the proofs given above are still valid. We shall make use of this fact in Chapter 3.

## 2.2 SEMI-T-NILPOTENT SYSTEM.

We shall give, in this section, a new concept which is a generalization of T-nilpotency defined by H. Bass [2] .

Let $\{M_\alpha\}_I$ be a set of modules (not necessarily c.inde.). Let $\underline{A}$ be the induced category from $\{M_\alpha\}$ and $\underline{C}$ an ideal in $\underline{A}$. Take any countably infinite subset $\{M_{\alpha_i}\}$ of $\{M_\alpha\}$ and a set of morphisms $\{f_i : M_{\alpha_i} \to M_{\alpha_{i+1}}$,

$f_i \in \underline{C}\}$. If for any such sets and any element m in $M_{\alpha_1}$, there exists a natural number n (depending on the sets and m) such that $f_n f_{n-1} \dots f_1(m) = 0$, $\{M_\alpha\}_I$ is called a *locally semi-T-nilpotent system* **with respect to** $\underline{C}$. Let $\{M_i\}^\infty$ be a countable set of modules $M_i$ such that $M_i$ are isomorphic to some ones in $\{M_\alpha\}$. If any such set and any set of morphisms $f_i$ satisfy the above, we say $\{M_\alpha\}$ a *locally T-nilpotent system*, ([17,28]). If I is finite, we understand by the definition that $\{M_\alpha\}_I$ is a locally semi-T-nilpotent system. If the above n does not depend on any element m in $M_{\alpha_1}$, we omit the word "locally". If every $M_\alpha$ is finitely generated, we have this situation.

In this section, we give a principal lemma (2.2.3), which we shall frequently use later.

Let $M = \sum_I \oplus M_\alpha$ and describe End(M) = $S_M$ by the ring of the column summable matrices. We may assume I is well ordered. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ (or $\alpha_1 > \alpha_2 \cdots > \alpha_n$) be in I and $b_{\alpha_i \alpha_{i-1}} \in [M_{\alpha_{i-1}}, M_{\alpha_i}]$. Then by $b(\alpha_n, \alpha_{n-1}, \dots \alpha_1)$ we denote $b_{\alpha_n\alpha_{n-1}} b_{\alpha_{n-1}\alpha_{n-2}} \dots b_{\alpha_2\alpha_1}$ for the sake of simplicity.

LEMMA 2.2.1 (Konig graph theorem). - *Let* M, $\{M_\alpha\}_I$ *and* $\underline{C}$ *as above. Let* $f = (b_{\sigma\tau})$ *be in* $S_M \cap \underline{C}$. *Put* $F_\tau = \{b(\alpha_n, \alpha_{n-1}, \dots, \alpha_2, \alpha_1 = \tau)$, *for any* $n \geq 2\}$. *We assume* $\{M_\alpha\}_I$ *is locally semi-T-nilpotent system with respect to* $\underline{C}$. *Then for any element* $x_\tau$ *in* $M_\tau$, $b(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)(X_\tau) = 0$ *for almost all* b *in* $F_\tau$.

*Proof.* - Since $(b_{\sigma\tau})$ is column summable, there exists a finite subset $T_1$ of I such that $b_{\sigma\tau}(x_\tau) = 0$ for all $\sigma \in I-T_1$. Let $\beta$ be in $T_1$. Then the subset $T_2 = \{\gamma | b(\gamma,\beta,\tau)(x_\tau) \neq 0\}$ of I is also finite. On the other hand, $\{M_\alpha\}_I$ is locally semi-T-nilpotent and $b_{\sigma\tau} \in \underline{C}$, since $\underline{C}$ is an ideal. Hence, (2.2.1) is clear from Konig graph theorem.

REMARK 2. Let $b(\alpha_n, \alpha_{n-1} \ldots, \alpha_1)$ be as above. Then for $\tau < \sigma$ $\sum_{\alpha_i} b(\sigma, \alpha_\tau, \ldots, \alpha_2, \tau)$ is an element in $[M_\tau, M_\sigma]$.

LEMMA 2.2.2. - *Let* $\{M_\alpha\}_{I}$, M *and* $\underline{C}$ *be as above. We assume* $\{M_\alpha\}_I$ *is locally semi-*T*-nilpotent with respect to* $\underline{C}$. *Let* $(b_{\sigma\tau})$ *be in* $S_M \cap \underline{C}$ *such that* $b_{\sigma\tau} = 0$ *if* $\sigma \leqslant \tau$ . *Then* $(b_{\sigma\tau})$ *is quasi-regular, (cf. [33,36]).*

*Proof.* - Put $B = (b_{\sigma\tau})$. Then each entry of the column of $B^n$ consists of some elements in $F_\tau$ . Hence, $\sum_1^\infty B^n$ has a meaning and is an element in $S_M$ by (2.2.1). Put $A = \sum_1^\infty B^n$. Then $(-A)B-B = -A$. Hence, B is quasi-regular.

LEMMA 2.2.3 [19] (principal lemma).*Let* $\{M_\alpha\}_I$ *be a set of modules and* $\underline{C}$ *an ideal in the induced category from* $\{M_\alpha\}$. *By* $S_\alpha$ *we denote* $End(M_\alpha)$ . *Suppose*

*1)* $\underline{C} \cap S_\alpha \subseteq J(S_\alpha)$ *for* $\alpha \in J$.

*2) If* $\{a_\alpha\}_{I'}$ *is a set of morphisms in* $\underline{C} \cap [M_\sigma, M_\tau]$ *such that* $\{a_\alpha\}_{I'}$ *is summable, then* $\sum_{I'} a_\sigma \in \underline{C} \cap [M_\alpha, M_\tau]$.

*3)* $\{M_\alpha\}_I$ *is a locally semi-*T*-nilpotent system with respect to* $\underline{C}$.
*Then* $\underline{C} \cap S_M \subsetneq J(S_M)$ .

*Proof.* - Let $A' = (a'_{\sigma\tau})$ be in $\underline{C} \cap S_M$ and put $A = (a_{\sigma\tau}) = E - A'$, where E is the identity matix. We shall show that A is regular in $S_M$ by the similar argument to (2.1.3). Since $Aa'_{\sigma\sigma}$ is in $J(S_\sigma)$ by 1, $a_{\sigma\sigma}$ is regular in $S_\sigma$ . Put $b_{\sigma 1} = a_{\sigma 1}a_{11}^{-1}$ for $\sigma > 1$, then $\{b_{\sigma 1}\}_\sigma$ is summable and $b_{\sigma 1} \in \underline{C}$.
We shall define $b_{\sigma\tau}$ for $\sigma > \tau$ with the following properties :

i) $\{b_{\sigma\tau}\}_\sigma$ is summable and $b_{\sigma\tau} \in \underline{C}$.

ii) $b_{\sigma\tau} = -y_{\sigma\tau}\, y_{\tau\tau}^{-1}$ , where

$y_{\sigma\tau} = a_{\sigma\tau} + \sum_{\tau > \alpha_t} b(\sigma, \alpha_t, \alpha_{t-1}, \dots, \alpha_1) a_{\alpha_1 \tau} \cdots$ (*), (cf. Remark 2).

We defined $\{b_{\sigma_1}\}$ with i) and ii). We suppose we have defined $\{b_{\sigma\rho}\}$ for $\rho < \tau$. Then since every terms in (*) are defined, we can define $y_{\sigma\tau}$ by (*). Since $\sum_{\tau > \alpha_t} b(\tau, \alpha_t, \dots, \alpha_1)\, a_{\alpha_1 \tau} \in \underline{C} \cap S_\tau \subsetneq J(S_\tau)$ by (2.2.1) and 1.2 , $y_{\tau\tau}$ is regular in $S_\tau$ . Hence, we can define $b_{\sigma\tau}$ by ii). It is clear from (2.2.1) and 2 that $\{b_{\sigma\tau}\}$ is summable and $b_{\sigma\tau} \in \underline{C}$ . Now, we define $C = (c_{\sigma\tau})$ by setting $c_{\sigma\sigma} = 1_\sigma$ , $c_{\sigma\tau} = 0$ for $\sigma < \tau$ and
$c_{\sigma\tau} = \sum_{\alpha_i} b(\sigma\ \alpha_\tau, \dots, \alpha_2\ \tau)$ ($\in \underline{C} \cap [M_\tau, M_\sigma]$) for $\sigma > \tau$. Then C is column summable and hence, $C \in S_M$. Put $D = CA = (d_{\sigma\tau})$. First we shall show $d_{\sigma\tau} = 0$ for $\sigma > \tau$ .

$$d_{\sigma\tau} = \sum_\rho c_{\sigma\rho} a_{\rho\tau} = a_{\sigma\tau} + \sum_{\rho < \sigma} c_{\sigma\rho} a_{\rho\tau} = a_{\sigma\tau} + \sum_{\rho < \sigma} \sum_{\alpha_i} b(\sigma\ \alpha_\tau, \dots, \alpha_2, \rho).$$

$$a_{\rho\tau} = a_{\sigma\tau} + \sum_{\tau>\alpha_t} b(\sigma,\alpha_\tau,\ldots,\alpha_1)a_{\alpha_1\tau} + b_{\sigma\tau}\Big(\sum_{\alpha'_i} b(\tau\ \alpha'_{t'},\ldots,\alpha'_1)a_{\alpha'_1\tau}$$

$$+a_{\tau\tau}\Big)+ \sum_{\sigma>\alpha''_{t''}>\tau} b_{\sigma\alpha''_{t''}}\Big(\sum_{\alpha''_i} b(\alpha''_{t''},\ldots,\alpha''_1)a_{\alpha''_1\tau}+a_{\alpha''_1\tau}\Big)=y_{\sigma\tau}+b_{\sigma\tau}y_{\tau\tau}+\sum_{\sigma>\alpha''_{t''}>\tau} b_{\sigma\alpha''_{t''}}\, d_{\alpha''_{t''}\tau}$$

$$= \sum_{\sigma>\alpha''_{t''}>\tau} b_{\sigma\alpha''_{t''}} d_{\alpha''_{t''}\tau} \quad \ldots (**).$$

It is clear from $(**)$ $d_{\tau+1\tau} = 0$ for all $\tau$. If we use the transfinite induction on $\sigma,\tau$, we can show $d_{\sigma\tau} = 0$ if $\sigma>\tau$ from $(**)$. Futhermore, $d_{\sigma\sigma} = \sum b(\sigma,\alpha_t,\ldots,\alpha_1)a_{\alpha_1\sigma} + a_{\sigma\sigma}$ is regular in $S_\sigma$. Put $C_1 = \mathrm{diag}(d_{11}^{-1},\ldots,d_{\sigma\sigma}^{-1},\ldots)$ and $K = E-C_1CA=E-C_1D$. Then the entries of K, which are in the diagonal or under the diagronal, are all zero and the entries of upper the diagonal belong to $\subseteq$ by ii) and 2. Hence, K is quasi-regular by 3 and (2.2.2), (which is a case of $\alpha_1>\alpha_2>\ldots>\alpha_n$). Therefore, $C_1CA$ is regular in $S_M$. Again using (2.2.2), we know C is regular in $S_M$. Thus so is A. Therefore, $C\cap S_M \subseteq J(S_M)$.

REMARK 3. - In the introduction we defined" take out property" of a module M, which is the property II) in (2.1.4) without the assumption of the finiteness of I'. In that definition, we assumed that any kinds of decompositions of M should have the take out property. Now we fix a decompesition of M : $M = \sum_I \oplus M_\alpha$ , $M_\alpha$ are c.inde.. We shall note that if this decomposition has the take out property for any another decompositions $M = \sum_J \oplus N_\beta$ , then so do any kinds of decompositions of $M{:}M=\sum_K \oplus M'_{\alpha'}$. Because, let $M = \sum_I \oplus M_\alpha = \sum_K \oplus M'_{\alpha'} = \sum_J \oplus N_\beta$. Then there exist a one-to-mapping $\phi$ of

K onto I and a set of isomorphisms $f_{\alpha'}: M'_{\alpha'} \to M_{\phi(\alpha)}$ . Put $F = \Sigma f_{\alpha'} \in S_{M'}$ which is isomorphic. Hence, $M = \sum_{I} \oplus M_{\alpha} = \sum_{J} \oplus F(N_{\gamma})$. If we apply the take out property for those decompositions, we obtain

$M = \sum_{I'} \oplus F(N_{\psi(\alpha)}) \oplus \sum_{I-I'} \oplus M_{\alpha}$ . Therefore, $M = F^{-1}(M) = \sum_{\alpha \in K'} \oplus N_{\psi(\alpha)} \oplus \sum_{K-K'} \oplus M'_{\alpha'}$.

# CHAPTER 3. SEMI-T-NILPOTENCY AND THE RADICAL

We have defined a (locally) semi-T-nilpotency for a set of modules $\{M_\alpha\}_I$ in Chapter 2. In this chapter we study some relations between a semi-T-nilpotency of a set of c. inde. modules $\{M_\alpha\}_I$ and the radical of End(M), where $M = \sum_I \oplus M_\alpha$.

## 3.1. EXCHANGE PROPERTY.

We shall define, in this section, the exchange property of a direct summand of a modules, which is slightly weaker than the usual one (cf.[4]).

DEFINITION. Let N be an R-module and N a direct summand of M. We say N has the *α-exchange property in* M if for any decomposition of M:$M=\sum_I \oplus T_\gamma$ with $|I| \leqslant \alpha$, there exists always a new decomposition $M = N \oplus \sum_I \oplus T_\gamma'$, such that $T_\gamma' \subsetneq T_\gamma$, (and hence, $T_\gamma'$ is a direct summand of $T_\gamma$ for all $\gamma \in I$). If N has the *α-exchange property for* any α, we say N has the *exchange property in* M. If in the above, N has the α-exchange property whenever all $T_\gamma$ are c.inde., we say N has the α-exchange property with respect to c.inde. modules.

REMARKS 1. It is clear from the definition that M has always the exchange property in M.

2. Suppose $M = \sum_{i=1}^{n} \oplus N_i$. If $N_1, N_2$ have the α-exchange property in M, then so does $N_1 \oplus N_2$ by [4]. However, the converse is not true.

Furthermore, even if neither $N_1$ nor $N_2$ has the $\alpha$-exchange property in M, it is possible that $N_1 \oplus N_2$ so does.

LEMMA 3.1.1. - *Let* $\{M_\alpha\}_I$ *be a set of (c.inde.) modules and* $M = \sum_I \oplus M_\alpha$ . *Suppose* M *satisfies the take out property for any subset* I' *with* $|I'| \leq \chi_0$ . *Then* $\{M_\alpha\}_I$ *is a locally semi-*T*-nilpotent system (with respect to* $\underline{J}'$*).*

*Proof.* - Let $\{M_i\}_1^\infty$ be a subset of $\{M_\alpha\}_I$ and $\{f_i : M_i \to M_{i+1}\}$ a set of given morphisms. First we shall show that some of $\{f_i\}$ is not monomorphic. Put $M_i' = \{m_i + f_i(m_i) | m_i \in M_i\} \subseteq M_i \oplus M_{i+1} <\oplus\ M$ and $M_o = \sum_{I-I_o} \oplus M_\gamma$ , where $I_o = (1,2,\dots,n\dots)$. Then it is clear that

$$M = M_1 \oplus M_2' \oplus M_3 \oplus M_4' \oplus \dots \oplus M_o = M_1' \oplus M_2 \oplus M_3' \oplus M_4 \oplus \dots \oplus M_o . \quad \dots(*)$$

We assume that all $f_i$ are monomorphic and use the take out property for the above decomposition. We take a subset $I' = (2,4,\dots,2n,\dots)$. Then we obtain from the take out property that

$$M = M_1' \oplus M_3' \oplus \dots \oplus M_o \oplus \psi_2(M_2) \oplus \psi_4(M_4) \oplus \dots \oplus \psi_{2n}(M_{2n}) \oplus \dots\dots (**)$$

where $\psi_{2n}(M_{2n})$ is equal to one of modules in the first decomposition except modules in $M_o$ . From the above assumptions no one of $\{f_i\}$ is epimorphic. Hence, every $M_{2n}'$ has to be equal to some $\psi_{2m}(M_{2m})$. Therefore, $\sum_{I'} \oplus \psi_{2n}(M_{2n}) \supseteq \sum_{I'} \oplus M_{2m}'$. We shall show $\sum_{I'} \oplus \psi_{2n}(M_{2n}) = \sum_{I'} \oplus M_{2m}'$. If $\sum_{I'} \oplus \psi_{2n}(M_{2n}) \neq \sum_{I'} \oplus M_{2m}'$ , we had some 2i such that $\psi_{2i}(M_{2i})$ is equal to some $M_{2k+1}$. First we assume that we had $\psi_{2n}(M_{2n}) = M_{2i+1}$ and $\psi_{2m}(M_{2m}) = M_{2j+1}$

for $i < j$. Then since $M_{2k}'$ is equal to some $\psi_{2p}(M_{2p})$, $M_{2i+1}+M_{2i+1}'+M_{2i+2}'+$ $\dots+M_{2j}'+M_{2j+1}$ is a direct sum from (**). We shall denote $f_p f_{p-1}\dots f_q$ by $\theta(p,q)$ for $p>q$. Let $x \neq 0$ be in $M_{2i+1}$, then

$$\begin{aligned}
x = x+f_{2i+1}(x) &\in M_{2i+1}' \\
-f_{2i+1}(x)-f_{2i+2}f_{2i+1}(x) &\in M_{2i+2}' \\
&\dots\dots \\
&\dots\dots \qquad (***) \\
\pm(\theta(2j-1,2i+1)(x)+\theta(2j,2i+1)(x)) &\in M_{2j}' \\
\mp\theta(2j,2i+1)(x) &\in M_{2j+1},
\end{aligned}$$

which is a contradition to the above. Therefore, if $\Sigma \oplus\psi_{2n}(M_{2n}) \neq \Sigma \oplus M_{2n}'$, we should have only one $\psi_{2k}(M_{2k})$ which is equal to some $M_{2i+1}'$. Thus,

$$M = \sum_{p=1}^{2i} \oplus M_p' \oplus M_{2i+1} \oplus M_{2i+1}' \oplus \sum_{k>2i+1} \oplus M_k' \oplus M_o = \sum_{q=1}^{2i+1} \oplus M_q \oplus \operatorname{Im} f_{2i+1} \oplus \sum_{k>2i+1} \oplus M_k' \oplus M_o.$$

Since $f_{2i+1}$ is not epimorphic, we can show by the same argument to (***) that $M_{2i+2} \not< M$. Therefore, some of $\{f_i\}$ has to be non-monomorphic. From those arguments, we may assume there are infinite many of non-monomorphisms $f_j$ among $\{f_i\}$. Let $f_{i_1}, f_{i_2}, \dots, f_{i_n}, \dots$ be such a set. Put $\theta(i_{k+1}-1, i_k) = g_k$. Then all $g_k$ are non-monomorphic. In order to show that $\{f_i\}$ is a locally semi-T-nilpotent system, it is sufficient to show that so is $\{g_k\}$. We put $M_k{*} = M_{i_k}$. Let $x \neq 0 \in \operatorname{Ker} g_i$, then $x \in M_i^* \cap M_i^{*}{}'$. When we use the above argument for $\{M_k{*}\}$, we know from (**) that $\psi_{2n}(M_{2n}{*})$ is not equal to any $M_{2m+1}^*$. Therefore, $\psi_{2n}(M_{2n}^*)$ is equal to some $M_{2m}^{*\prime}$ and $M = M_1^{*\prime} \oplus M_2^{*\prime} \oplus \ldots \oplus M_o$ (it is possible that some $M_{2m}^{*\prime}$ may not appear in this decomposition). Take $x \neq 0 \in M_1^*$ and use the formular $(***)$, then we know that there exists some t such that $\theta(t,1)(x) = 0$. Therefore, $\{f_i\}$ is a locally semi-T-nilpotent system.

We shall later make use of the following lemma and we can prove it by the similar argument to the above and so we shall leave a proof to the reader.

LEMMA 3.1.1'.-*Let* $\{M_\alpha\}_I$ *and* $\{N_\beta\}_J$ *be sets of c.inde. modules. Put* $T = \sum_I \oplus M_\alpha \oplus \sum_J \oplus N_\beta$ . *We assume that* $\sum_J \oplus N_\beta$ *has the* $\chi_o$*-exchange property in* T. *Then for any countable subsets* $\{M_i\}$ *and* $\{N_i\}$ *of* $\{M_\alpha\}_I$ *and* $\{N_\beta\}_J$, *respectively and for any non-isomorphisms* $f_i: M_i \to N_i$, $g_i: N_i \to M_{i+1}$; *and for any* $x \in M_1$, *there exists* m *such that* $g_m f_m \cdots g_1 f_1(x) = 0$.

The following main theorem gives us an answer in a case where we drop the assumption of finiteness in Azumaya' theorem (2.1.4).

THEOREM 3.1.2 [19,24] (MAIN THEOREM). - *Let* $\{M_\alpha\}_I$ *be a set of c.inde. modules and* $M = \sum_I \oplus M_\alpha$ . *Then the following statements are equivalent.*

*1)* M *satisfies the take out property for any subset* I' *and any other decompositions (cf. 2 Remark 3 in Chapter 3).*

*2) Every direct summand of* M *has the exchange property in* M.

*3) Every direct summand of* M *has the exchange property in* M *with respect to c.inde. modules.*

*4)* $\{M_\alpha\}_I$ *is a locally semi-*T*-nilpotent system with respect to* $\underline{J}'$ *defined in §1.4.*

*5)* $\underline{J}' \cap \mathrm{End}(M)$ *is equal to the Jacobson radical of* End(M).

*Proof.* - 1) $\rightarrow$ 4) It is clear from (3.1.1).

4) $\rightarrow$ 5) Since $S_M/\underline{J}' \cap S_M$ is semi-simple by (1.4.8), $S_M \cap \underline{J}' \supseteq J(S_M)$, where $S_M = \mathrm{End}(M)$. We shall prove the converse inclusion from (2.2.3). The first condition in (2.2.3) is clear for $S_\alpha$. Let $\{a_i\}$ be a set of element in $\underline{J}' \cap [M_\sigma, M_\tau]$ such that $\{a_i\}$ is summable. Put $a = \Sigma a_i$. If $M_\sigma \not\approx M_\tau$, then $a \in \underline{J}' \cap [M_\sigma, M_\tau]$. If $M_\sigma \approx M_\tau$, we can show by the same argument in the proof of (1.4.2) that a is not isomorphic. Hence, $a \in \underline{J}' \cap [M_\sigma, M_\tau]$, which is the second condition in (2.2.3). The third one is equal to 4). Hence, $\underline{J}' \cap S_M \subseteq J(S_M)$ by (2.2.3).

5) $\rightarrow$ 1) Let $M' = \sum_{I'} \oplus M_\gamma$ and e the projection of M to M'. It is clear by (1.4.3) that $(\underline{J}' \cap S_M) \cap S_{M'} = \underline{J}' \cap S_{M'}$. On the other hand, it is well known that $eS_Me = S_{M'}$ and $J(S_{M'}) = eJ(S_M)e$. Hence, $J(S_{M'}) = \underline{J}' \cap S_{M'}$, which guarantees 1) by Remark 1 in § 2.1.

2) $\rightarrow$ 3) It is clear from the definition.

3) $\rightarrow$ 1) 3) implies 4) by (3.1.1) and hence, implies 1).

1) $\rightarrow$ 2) In order to show this, we need the following proposition. If we use it, the proof is clear.

PROPOSITION 3.1.3. - *Let* $\{M_\alpha\}$ *and* M *be as in (3.1.2). Then the following statements are qeuivalent.*

*1) The property III in the introduction ; every direct summand of* M *is a direct sum of c.inde. modules* $M'_\alpha$ *such that* $M'_\alpha$ *are isomorphic to some* $M_\gamma$ *in* $\{M_\alpha\}_I$, *is true.*

*2) For any idempotents* e,f *in* $S_M$ *we have*

$eS_M \approx fS_M$ *if and only if* $eS_M/e(\underline{J}'\cap S_M) \approx fS_M/f(\underline{J}'\cap S_M)$.

*Proof.* - 1) $\rightarrow$ 2) Put $\bar{S}_M = S_M/\underline{J}'\cap S_M$, $eM = \Sigma \oplus M'_\alpha$, and $fM = \Sigma \oplus M''_{\alpha''}$. We Assume $\bar{e}\bar{S}_M \approx \bar{f}\bar{S}_M$. Then Im $\bar{e} \approx$ Im $\bar{f}$ in $\bar{\underline{A}}$, where $\underline{A}$ is the category in (1.4.8). Hence, since Im $\bar{e} = \Sigma \oplus \bar{M}'_\alpha$ and Im $\bar{f} = \Sigma \oplus \bar{M}''_{\alpha''}$, $M'_{\alpha'}$ is isomorphic to some $M''_{\alpha''}$ and vice versa by (1.4.8). Therefore, $eM \approx fM$, which implies $eS_M \approx fS_M$.

2) $\rightarrow$ 1) Let M' be a direct summand of M and e the projection. We showed in the proof of (2.1.4) that there exists an idempotent f in $S_M$ such that $fM = \sum_{I'} \oplus M_\delta$ ; $I' \subseteq I$ and $\bar{e}\bar{S}_M \approx \overline{fS}_M$. Hence, $eS_M \approx fS_M$ implies $eM \approx fM$.

COROLLARY 3.1.4 [7]. - *Let* $\{M_\alpha\}$ *be as in (3.1.2). If one of the conditions in (3.1.2) is satisfied, then the property III is true for* M.

REMARKS 1. We can replace 2) and 3) in (3.1.2) by the $\chi_0$-exchange property by virtue of (3.1.1).

2. Let Z be the ring of integers and p a prime. Then $\{Z/p^i\}_{i=1}^\infty$ is not a semi-T-nilpotent system. Hence, $M = \sum_{i=1}^\infty \oplus Z/p^i$ does not satisfy any statements in (3.1.2). However, M satisfies the property III (see § 4.2).

3. Let $\{M_\alpha\}_I$ be a set of indecomposable modules with finite composition lengthes which do not exceed a fixed natural number n. Then $\{M_\alpha\}_I$ is a T-nilpotent system with respect to $\underline{J}'$ (see [17]).

4. Let K be a field and R the ring of lower tri-angular matrices with infinite degree. Put $M = \sum_i \oplus e_{ii}R$, where $e_{ii}$ are matrix units in R. Then $\{e_{ii}R\}$ is not a semi-T-nilpotent system, but M satisfies the property III (see § 4.2).

5. Let R be the ring of upper tri-angular matrices. Then $\{e_{ii}R\}$ is a T-nilpotent system.

3.2. DENSE SUBMODULES.

In this section we shall give a special answer to the property III. Let $\{M_\alpha\}_I$ be a set of c.inde.modules and $M = \sum_I \oplus M_\alpha$ . By $\underline{A}$ we denote the induced category from $\{M_\alpha\}_I$. Let $\underline{J}'$ be the ideal in $\underline{A}$ defined in § 1.4. We denote $\underline{A}/\underline{J}'$ by $\bar{\underline{A}}$.

DEFINITION. – Let M and N be in $\underline{A}$ such that N is a submodule in M, $i: N \to M$ inclusion. If $\bar{i}$ is isomorphic in $\bar{\underline{A}}$, i.e. $\bar{N} = \bar{M}$; N is called a *dense submodule* in M, (note that if N is a submodule of M which is a direct sum of c.inde. modules and $\bar{i}$ is isomorphic in $\bar{\underline{C}}$, then $N \in \underline{A}$, where $\underline{C}$ is the induced category from all c.inde.modules).

NOTATION.–Let e be an idempotent in $S_M$=End(M). Then $M = eM \oplus (1-e)M$ in $\underline{M}_R$. We do not know whether $eM \in \underline{A}$ or not, however we shall denote Im $\bar{e}$ in $\bar{\underline{A}}$ by $\overline{eM}$ for the sake of conveniency. It is clear that if $eM \in \underline{A}$, Im $\bar{e} = \overline{eM}$ in $\bar{\underline{A}}$. We note that even if f(M) is in $\underline{A}$ for some $f \in S_M$; Im $\bar{f}$ is not equal to $\overline{f(M)}$ in general.

PROPOSITION 3.2.1. - *Every dense submodule of* M *is isomorphic to* M.

*Proof.* - Since $\overline{M} = \sum_I \oplus \overline{M}_\alpha = \overline{N} = \Sigma \oplus \overline{N}_\gamma$ , $M \approx N$ as R-modules by (1.4.8), where $N_\gamma$'s are c.inde. modules.

PROPOSITION 3.2.2. - *Let* M *and* P *in* $\underline{A}$ *and* $\overline{M} \supseteq \overline{P}$ *in* $\underline{A}$. *Then there exists a submodule* $P_o$ *in* M *which satisfies the followings :*

*1)* $P_o$ *is in* $\underline{A}$ *i.e.* $P_o = \sum_J \oplus M'_\alpha{}'$.

*2) For any finite subset* J' *of* J $\sum_{J'} \oplus M'_\alpha$, *is a direct summand of* M. *If* $\{M'_\alpha\}_J$ *is a locally semi-*T*-nilpotent system with respect to* $\underline{J}$'*, then* $P_o$ *is a direct summand of* M.

*3)* $P_o \approx P$ *as* R*-modules.*

*Furthermore, if* e *is an idempotent in* $S_M$ *and* $\overline{P} = \text{Im } \overline{e}$, *then we can find such* $P_o$ *in* Im e *in* $\underline{M}_R$.

*Proof.* - Since $\overline{\underline{A}}$ is completely reducible by (1.4.8), there exist R-homomorphisms i: P $\rightarrow$ M and p:M $\rightarrow$ P such that $\overline{pi} = 1_{\overline{P}}$. Let $P = \sum_K \oplus P_\gamma$ ; $P_\gamma$ are c.inde.. For a subset K' of K we denote the injection : $P_{K'} = \sum_{K'} \oplus P_\gamma \rightarrow P$ and the projection : $P \rightarrow P_{K'}$ by $i_{K'}$ and $p_{K'}$, respectively : $P_{K'} \underset{p_{K'}}{\overset{i_{K'}}{\rightleftarrows}} P \underset{p}{\overset{i}{\rightleftarrows}} M$. Then $\overline{p_{K'}pii_{K'}} = \overline{1}_{P_{K'}}$ If either K' is finite or $\{P_\gamma\}_{K'}$ is semi-T-nilpotent, $S_{P_{K'}} \cap \underline{J}' = J(S_{P_{K'}})$ by (3.1.2). Hence, $p_{K'}pii_{K'}$ is R-isomorphic. Therefore, $ii_{K'}$ is monomorphic in $\underline{M}_R$ for every finite subset K' of K, which means i is monomorphic in $\underline{M}_R$. Put $P_o$ = Im i in $\underline{M}_R$. Then $P_o$ satisfies 1)∼3). Suppose Im e = P. Then M = P ⊕ (1-e)M and hence, pei = pi. Put $P_o$ = Im ei in $\underline{M}_R$. From the above argument, we know that $P_o$ satisfies the all requirment in(3.2.2).

REMARK 6. Let $N = \sum_{i=1}^{n} \oplus M_i$ be a submodule of M via the inclusion $i_N$. Then we know by the above proof that $\bar{i}_N$ is monomorphic in $\bar{\underline{A}}$ if and only if N is a direct summand of M.

LEMMA 3.2.3 [1]. - *Let* M *and* $\underline{J}'$ *be as above. Then for any* $f \in \underline{J}' \cap S_{M'}$ $1_M - f$ *is monomorphic.*

*Proof.* - Suppose Ker (1-f) ≠ 0. Then there exists a finite subset I' of I such that Ker $(1-f) \cap \sum_{I'} \oplus M_\alpha \neq 0$. By (2.1.1) we obtain a set of direct summands $\{M'_{\phi(\alpha')}\}_{I'}$ such that $M = \sum_{I'} \oplus M'_{\phi(\alpha')} \oplus \sum_{I-I'} \oplus N_\alpha$ and $M_{\alpha'} \cong M_{\phi(\alpha')}$ for each $\alpha' \in I'$ via either f or (1-f). However, f is in $\underline{J}'$ and hence, we must obtain those isomorphisms by (1-f), which is a contradiction. Therefore, Ker (1-f) = 0.

We shall give criteria for submodules to be dense.

THEOREM 3.2.4. - *Let* $\{M_\alpha\}_I$ *be a set of c.inde. modules,* $\underline{A}$ *the induced category from* $\{M_\alpha\}_I$ *and* $\underline{J}'$ *the usual ideal in* $\underline{A}$. *Let* N *be in* $\underline{A}$. *i.e.* $N = \sum_J \oplus N_\gamma$ *and a submodule of* M *via the inclusion* $i_N : N \to M$. *Then the followings are equivalent.*

*1)* N *is a dense submodule of* M.

*2)* $\bar{i}_N$ *is monomorphic in* $\underline{A}/\underline{J}'$ and *for any direct summand* P *of* M, *there exists a finite subset* J' *of* J *such that* $P \cap N_{J'} \neq 0$ or $P \oplus N_{J'}$ *is not a direct summand of* M, *where* $N_{J'} = \sum_{J'} \oplus N_{\gamma'}$.

*3)* $\bar{i}_N$ *is monomorphic and* N *contains* Im (1-f) *in* $\underline{M}_R$ *for some* $f \in \underline{J}'$. *Hence,* Im (1-f) *is a dense submodule in* M *for all* $f \in \underline{J}'$. *Furthermore, the above* $N_{J''}$ *is a direct summand of* M *if either* J" *is finite or* $\{N_\gamma\}_{J''}$ *is a semi-*T*-nilpotent system.*

*Proof.* - 1) $\Rightarrow$ 2) Since P contains a direct summand of M which is c.inde. by (2.1.4), we may assume P is c.inde.. Furthermore, since $\underline{A}$ is a Grothendieck category and $\bar{P}$ is minimal in $\bar{\underline{A}}$, $\bar{P} \subseteq \sum_{J'} \oplus \bar{N}_{\gamma'} = \bar{N}_{J'}$ for some finite subset J' of J. Suppose $P \cap N_{J'} = 0$ and $P \oplus N_{J'}$ is a direct summand of M ; $M = P \oplus N_{J'} \oplus M_o$. Let $i : P \oplus N_{J'} \to M$ be the inclusion. Then $\mathrm{Im}\, \bar{i} = \bar{P} \oplus \bar{N}_{J'}$ , which is a contradiction. Hence, $P \oplus N_{J'}$ is not a direct summand of M.

2) $\Rightarrow$ 1) We assume that $\bar{i}_N$ is monomorphic and $\bar{M} \neq \bar{N}$. Then there exists a minimal object $\bar{M}_\alpha$ such that $\bar{M}_\alpha \cap \bar{N} = 0$. Hence, for any finite subset J' of J $\bar{M}_\alpha \cap \bar{N}_{J'} = \bar{0}$. Therefore, $M_\alpha \oplus N_{J'}$ is a direct summand of M by Remark 6 (take first a formal direct sum $M_\alpha \oplus N_{J'}$ and consider a natural mapping from $M \oplus N_{J'}$ to $M_\alpha \cup N_{J'} \subseteq M$).

1) $\Rightarrow$ 3) Since $\bar{i}_N$ is isomorphic, there exists an R-homomorphism $j \in [M,N]$ such that $\bar{i}_N \bar{j} = \bar{1}_M$. Then $f = 1 - i_N j \in \underline{J}'$ and Im (1-f) in $\underline{M}_R \subseteq \mathrm{Im}\, i_N = N$.

3) $\Rightarrow$ 1) Since 1-f is monomorphic by (3.2.3), Im (1-f) in $\underline{M}_R$ = N' is in $\underline{A}$. Put $1-f: M \xrightarrow{(1-f')'} N' \xrightarrow{i} M$. Then $\overline{1}_M = \overline{1-f} = \overline{i}(\overline{1-f})'$. Hence, $\overline{i}$ is isomorphic in $\underline{A}$, since (1-f)' is isomorphic in $\underline{M}_R$. Therefore, Im (1-f)' is a dense submodule in M. Since $\overline{i}_N$ is monomorphic and $N \supseteq$ Im (1-f), N is also dense.

The remaining part is clear from Remark 6 and (3.1.2).

REMARK 7. - In general, we have many dense submodules P in $M = \sum_{i=1} \oplus M_i$ , for instance such as $P \cap \sum_{i=1}^{n} \oplus M_i = 0$ for some $n < \infty$ or $P \cap M_i \neq 0$ for all i (see [18] ).

In the above we showed that if J' is a finite set, then $N_{J'}$ is a direct summand of M for a dense submodule N. We generalize this property as follows :

DEFINITION.-Let $A \supset B$ be R-modules and $B = \sum_J \oplus B_\gamma$ . If for any finite subset J' of J, $\sum_{J'} \oplus B_\gamma$ is a direct summand of A, we call B a *locally direct summand* of A (with respect to the decomposition $B = \sum_J \oplus B_\gamma$).

We note that if all $B_\gamma$ are injective, B is always a locally direct summand of A. We shall use this fact in Chapter 6. In general $B = \sum_I \oplus B_\gamma$ is a locally direct summand of $\prod_I B_\gamma$ .

THEOREM 3.2.5. - *Let* $\{M_\alpha\}_I$ *be a set of c.inde.modules and* $M = \sum_I \oplus M_\alpha$. *Then the following statements are equivalent.*

*1)* $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent system with respect to* $\underline{J}'$

*2) Every dense submodules coincide with* M.

*3) Every locally direct summand* M' *of* M *with* **re**spect *to* $M' = \sum_K \oplus T_\alpha$

*with any cardinal* $|K|$ *is a direct summand of* M.

*4) 3) is true for decomposition with* $|K| \leq \chi_0$.

*5) 4) is true whenever all* $T_\alpha$ *are c.inde. modules.*

*6)* $S_M/J(S_M)$ *is a regular ring in the sense of Von Neumann and every idempotent in* $S_M/J(S_M)$ *is lifted to* $S_M$.

*Proof.* - 1) $\Rightarrow$ 2) Every dense submodule N of M is a direct summand of M by the last part of (3.2.4). Hence, N = M by (2.1.2).

2) $\Rightarrow$ 1) Since Im (1-f) is dense in M for $f \in \underline{J}' \cap S_M$, 1-f is regular by 2). Hence, $\underline{J}' \cap S_M \subsetneq J(S_M)$, , which implies 1) from (3.1.2).

1) $\Rightarrow$ 3) Every direct summand of M is a direct sum of c.inde.modules by (3.1.4). The assumption of locally direct summand implies that $\sum_K \oplus \bar{T}_\alpha$ is a subobject of $\bar{M}$ via $\bar{i}_{M'}$ , where $i_{M'}$ : M' $\to$ M inclusion. Hence, M' is a direct summand by Remark 1 in § 2.1 and (3.1.2).

3) $\Rightarrow$ 4) $\Rightarrow$ 5) They are clear.

5) $\Rightarrow$ 1) We shall recall the proof of (3.1.1). Let $\{M_i\}_1^\infty$ be a countable subset of $\{M_\alpha\}$ and $\{f_i : M_i \to M_{i+1}\}$ a given set of morphisms in $\underline{J}'$. We defined the submodule $M' = M_1' \oplus M_2' \oplus \dots$ in M. Since $M_1' \oplus \dots \oplus M'_n \oplus M_{n+1} = \sum_{i=1}^{n+1} \oplus M_i$ for any n, M' is a locally direct summand of M. Hence, M' is a direct summand of M and hence, so is in $M_o = \sum_{i=1}^{\infty} \oplus M_i$. Since M' = im (1-f) in $\underline{M}_R$, M' is a dense submodule of $M_o$, where $f = \sum_{i=1}^{\infty} -e_{i\,i+1} f_i$ ; $e_{ij}$'s are matrix units in $S_M$ . Hence, $M = M_o$. If we use the formula (***) in the proof of (3.1.1), then we know that $\{f_i\}_1^\infty$ is a locally semi-T-nilpotent system.

1) $\Rightarrow$ 6) Since $\underline{J}' \cap S_M = J(S_M)$ by (3.1.2) and $\underline{A}/\underline{J}'$ is a regular ring by (1.4.8), so is $S_M/J(S_M)$. Let $f \in S_M$ such that $\bar{f}^2 = \bar{f}$. Then there exists a direct summand $M_1$ of M such that $\bar{M}_1 = \text{im}\ \bar{f}$ by (3.2.2). Let e be the projection of M to $M_1$. Since $\text{Im}\ \bar{f} = \text{Im}\ \bar{e}$, $\text{Im}\ (\overline{1-f}) \approx \text{Im}\ (\overline{1-e})$ in $\bar{\underline{A}}$. Hence, there exists a regular element $\bar{a}$ in $\bar{S}_M$ such that $\bar{f} = \bar{a}^{-1}\bar{e}\bar{a}$ by (1.4.4). Since $\underline{J}' \cap S_M = J(S_M)$, a is regular in $S_M$ and hence, $a^{-1}ea$ is a idempotent.

6) $\Rightarrow$ 1) $\underline{J}' \cap S_M \supseteq J(S_M)$ by (1.4.8). Since $S_M/J(S_M)$ is regular, $(J' \cap S_M)/J(S_M)$ contains a non-zero idempotent if $\underline{J}' \cap S_M/J(S_M)$. Then this idempotent is lifted to $S_M$ by 6) and hence it is in $\underline{J}' \cap S_M$, which contradicts (2.1.2).

COROLLARY 3.2.6. - *Let* R *be a local ring with* T*-nilpotent radical* J(R) *and* S *the ring of column finite matrices over* R *with any degree. Then every idempotent in* S/J(S) *is lifted to* S.

*Proof.* - Put $M = \sum_I \oplus R$, then $S \approx \text{End}_R(M)$.

The following theorem is some generalization of (3.2.4) and is a special answer to the property III.

THEOREM 3.2.7. - *Let* $\{M_\alpha\}_I$ , M *and* $\underline{A}$ *be as in (3.2.4). Let* $M = \sum_J \oplus N_\gamma$, *where* $N_\gamma$ *may not be in* $\underline{A}$. *Then there exists a set of submodules* $\{P_\gamma\}_J$ *of* M *as follows :*

*1)* $N_\gamma \supseteq P_\gamma$ *and* $P_\gamma \in \underline{A}$.

*2)* $\sum \oplus P_\gamma$ *is a dense submodule in* M.

*Proof.* - Let $\Pi_\gamma$ be the projection of M to $N_\gamma$ (note that $\Pi_\gamma$ is regarded as an element in $[M,M]$ . It is clear that $\{\Pi_\gamma\}$ is a summable set and $1_M = \sum_J \Pi_\gamma$ . Let $M_1$ be an element in $\{M_\alpha\}$. For any non-zero element $m_1$ in $M_1$ we have $\Pi_\gamma(m_1) = 0$ for all $\gamma \in J-J'$, where J' is a finite subset of J. Hence, $\Pi_\gamma|M_1 \in \underline{J}'$ for all $\gamma \in J-J'$. We shall express $\Pi_\gamma$ as matrices $(x^\gamma_{\alpha\beta})$ in §1.4. Since $\{\Pi_\gamma\}_J$ is summable, so is $\{x^\gamma_{\alpha\beta}\}_J$ for any $\alpha$ $\beta$. It is clear $\Pi_\gamma|M_1 = (x^\gamma_{\alpha 1})_{\alpha \in I}$. Therefore, $\sum_{J-J'} \Pi_\gamma|M_1 \in \underline{J}'$ (see the proof of (1.4.2)). Then $\bar{M}_1 = \mathrm{Im}\ \bar{1}_M/\bar{M}_1 \subsetneq \mathrm{Im}(\sum_{J'} \bar{\Pi}_\gamma|\bar{M}_1 + (\overline{\sum_{J-J'} \Pi_\gamma|M_1}) = \mathrm{Im}(\Sigma \bar{\Pi}_\gamma|\bar{M}_1) \subsetneq \sum_{J'} \mathrm{Im}\ \bar{\Pi}_\gamma$. Hence, $\bar{M} = \sum \mathrm{Im}\ \bar{\Pi}_\gamma$ . On the other hand, there exists a set $\{P_\gamma\}_J$ of a submodule in $N_\gamma$ such that $P_\gamma \in \underline{A}$ and $\bar{P}_\gamma = \mathrm{Im}\ \bar{\Pi}_\gamma$. It is clear that $\bigcup_{\delta \in K} \mathrm{Im}\ \bar{\Pi}_\delta = \sum_{\delta \in K} \oplus\ \mathrm{Im}\ \bar{\Pi}_\delta$ for any finite subset K of J, and so

$$\bar{M} = \sum_J \oplus\ \mathrm{Im}\ \bar{\Pi}_\delta = \sum_J \oplus\ \bar{P}_\gamma .$$

We shall call such $P_\gamma$ *a dense submodule in* $N_\gamma$.

The following proposition shows that dense submodules in $N_\gamma$ are maximal submodules in $N_\gamma$ up to isomorphism in some senses.

PROPOSITION 3.2.8. - *Let* M *be as above and* N *a direct summand of* M. *Let* N' *be a dense submodule in* N *and* T *a submodue of* N *and in* $\underline{A}$. *If* T *is a locally direct summand of* N, T *is isomorphic to a direct summand of* N'. *Every countably generated* R-*submodule of* N *is isomorphic to some submodule of* N'.

*Proof.* - We leave the proof to the reader (cf. (4.2.1)).

# CHAPTER 4. THE EXCHANGE PROPERTY

Let $\{M_\alpha\}_I$ be a set of c.inde. modules and $M = \sum_I \oplus M_\alpha$ as before. In chapter 3 we have considered a case where every direct summand of M has the exchange property in M. We shall concentrate, in this chapter , in a direct summand of M which has the exchange property in M.

## 4.1. SEMI-T-NILPOTENCY AND THE EXCHANGE PROPERTY.

Let M be as above, $\underline{A}$ the induced category from $\{M_\alpha\}_I$ and $\bar{\underline{A}} = \underline{A}/\underline{J}'$ as before. It is clear that if a direct summand N of M has the exchange property in M, then $N \in \underline{A}$.

PROPOSITION 4.1.1.- *Let* $M = N_1 \oplus N_2$. *If either* $N_1$ *is a finitely generated* R-*module or its dense submodule is a direct sum of c.inde. modules* $\{M'_{\alpha'}\}_J$ *such that* $\{M'_{\alpha'}\}_J$ *is a locally semi*-T-*nilpotent system, then* $N_1 \in \underline{A}$.

*Proof.* - If $N_1$ is finitely generated, $N_1$ is contained is some $\sum_{i=1}^n \oplus M_{\alpha_i} \angle \oplus M$. Hence, $N_1$ is a direct summand of $\sum_{i=1}^n \oplus M_{\alpha_i}$ . Therefore, $N_1 \in \underline{A}$ by (2.1.4), III. If a dense submodule N' of N is of form in the assumption, then $N_1 = N'$ by (3.2.4).

The following proposition is true in a general case (see [4,38]), however we shall prove it by virtue of a structure of $\bar{\underline{A}}$.

PROPOSITION 4.1.2 [4,38]. - *Let* M *be as before. If* $M = N_1 \oplus N_2$ *and* $N_1 = \sum_{i=1}^{n} \oplus M_{\alpha_i}$, $M_{\alpha_i}$ *'s are c.inde, then* $N_1$ *has the exchange property in* M.

*Proof*. - Let $M = \sum_{I'} \oplus Q_\alpha$ be any decomposition. Then each $Q_\alpha$ contains a dense submodule $P_\alpha$, $P_\alpha = \sum_{J_\alpha \ni j} \oplus P_{\alpha_{j'}}$, $P_{\alpha_{j'}}$ s are c.inde.. Then $\bar{M} = \bar{N}_1 \oplus \bar{N}_2 = \sum_{I'} \oplus \bar{Q}_\alpha = \sum_{I'} \sum_{J_\alpha} \oplus \bar{P}_{\alpha_j}$ (see Notation in § 3.2). Since $\bar{N}_1 = \sum_{i=1}^{n} \oplus \bar{M}_{\alpha_i}$, $\bar{N}_1$ is contained in a co-product of finite many of $\bar{P}_{\alpha_j}$, say $\sum_{i=1}^{m} \sum_{J'_i} \oplus \bar{P}_{ij'}$, where $J'_i$ is a finite subset of $J_i$. Hence, $P = \sum_{i=1}^{m} \sum_{J'}^{i=1} \oplus P_{ij}$ contains a direct summand $N_1'$ such that $N_{1'} \approx N_1$ in $\underline{M}_R$ and $\bar{N}_1' = \bar{N}_1$ by (3.2.2). Since $N_{1'}$, $P \in \underline{A}_f$, $P = N_{1'} \oplus \sum_i \sum_{J_i''} \oplus P_{ij}$ ; $J_i'' \subsetneq J_i'$ by (3.1.2) and (2.1.3). Furthermore, P is a direct summand of M by (3.2.2). Since $Q_i \supset \sum_{J_i'} \oplus P_{ij'}$, $Q_i = \sum_{J'_i} \oplus P_{ij} \oplus Q_i'$. Hence, $M = N_1' \oplus \sum_i \sum_{J_i''} \oplus P_{ij} \oplus \sum_{i=1}^{m} \oplus Q_i' \oplus \sum_{\alpha \neq i} \oplus Q_\alpha$. Let p be the projection of M to $N_1'$ in the above decomposition. Since $\bar{N}_1' = \bar{N}_1$, $\bar{p}|\bar{N}_1 = \bar{p}|\bar{N}_1'$ and $\overline{pi_{N_1}} = \bar{i}_{N_1}$. Since $N_1 \in \underline{A}_f$, $pi_{N_1}$ is isomorphic in $\underline{M}_R$ by (2.1.3). Hence, $M = N_1 \oplus \text{Ker } p = N_1 \oplus \sum_{i=1}^{m} (\sum_{J_i''} \oplus P_{ij} \oplus Q_i') \oplus \sum_{\alpha \neq i} \oplus Q_\alpha$.

We note that if I is finite, we may regard $\{M_\alpha\}_I$ as a locally semi-T-nilpotent system, (see § 3.2).

THEOREM 4.1.3. - *Let* $\{M_\alpha\}_I$ *be a set of c.inde.modules and* $M = \sum_I \oplus M_\alpha = N_1 \oplus N_2$. *Suppose* $N_1 = \sum_{I'} \oplus M'_{\alpha'}$ ; $M'_{\alpha'}$ *are c.inde. If* $\{M'_{\alpha'}\}_{I'}$ *is a locally semi-T-nilpotent system,* $N_1$ *and* $N_2$ *have the exchange property in* M.

*Proof.* - First, we shall show that $N_2$ has the exchange property in M. Let $M = \sum_J \oplus Q_\alpha$ be any decomposition. By (3.2.7) each $Q_\alpha$ contains a dense submodule $P_\alpha = \sum_{T_\alpha} \oplus P_{\alpha i}$ . Since $\bar{A}$ is a completely reducible and Grothendieck category by (1.4.8), we have

$$\bar{M} = \bar{N}_2 \oplus \sum_{J \ni \alpha} \sum_{T_\alpha'} \oplus \bar{P}_{\alpha i'} \text{ , where } T'_\alpha \subsetneq T_\alpha \quad \ldots \quad 1) .$$

It is clear that $\bar{N}_1 \approx \sum_J \sum_{T_\alpha'} \oplus \bar{P}_{\alpha i'}$ . Hence , $\{P_{\alpha i'}\}_{J\ T'_\alpha}$ is a locally semi-T-nilpotent system by the assumption. Put $p_{N_1}$ be the projection of M to $N_1$ with Ker $p_{N_1} = N_2$. From 1) we know that $\bar{p}_{N_1} | \sum_J \sum_{T_{\alpha'}} \oplus \bar{P}_{\alpha i'}$ is isomorphic in $\bar{A}$. Hence, $p_{N_1} | \sum_J \sum_{T'_\alpha} \oplus P_{\alpha i'}$ is isomorphic in $\underline{M}_R$ by (3.1.2). Therefore, $M = \sum_J \sum_{T'_\alpha} P_{\alpha i'} \oplus \text{Ker } p_{N_1} = \sum_J \sum_{T_\alpha} \oplus P_{\alpha i'} \oplus N_2$. Since $\sum_{T_{\alpha'}} \oplus P_{\alpha i'} \subsetneq Q_\alpha$ , $N_2$ has the exchange property in M. Next, we shall show that $N_1$ has the exchange property in M. From the similar argument to 1) we have a dense submodule $P_\alpha = P'_\alpha \oplus P''_\alpha$ in $Q_\alpha$ such that

$$\bar{M} = \bar{N}_1 \oplus \sum_J \oplus \bar{P}'_\alpha \quad \ldots \quad 2) \text{ and}$$

$$\bar{N}_1 \approx \sum_J \oplus \bar{P}''_\alpha \quad \ldots \quad 3).$$

Since $\bar{M} = \sum_J \oplus \bar{P}'_\alpha \oplus \sum_J \oplus \bar{P}''_\alpha$, there exists $p \in [M, \Sigma \oplus P''_\alpha]$ in $\underline{M}_R$ such that

$\mathrm{Ker}\ \bar{p}$ in $\underline{\bar{A}} = \sum_J \oplus \bar{P}'_\alpha$ and $\bar{p}|\bar{M}$ is the projection of $\bar{M}$ to $\Sigma \oplus \bar{P}''_\alpha$ ... 4).

From 3) and (3.1.2) we obtain $M = \sum_J \oplus P''_\alpha \oplus \mathrm{Ker}\ p$ and hence, $Q_\alpha =$

$P''_\alpha \oplus (\mathrm{Ker}\ p \cap Q_\alpha)$. Then $M = \sum_J \oplus Q_\alpha = \sum_J \oplus P''_\alpha \oplus \sum_J \oplus(\mathrm{Ker}\ p \cap Q_\alpha) =$

$\sum_J \oplus P''_\alpha \oplus \mathrm{Ker}\ p$. Hence,

$$\mathrm{Ker}\ p = \sum_J \oplus(\mathrm{Ker}\ p \cap Q_\alpha) \quad \ldots \quad 5).$$

From 2) and 4) $\mathrm{Ker}\ \bar{p} \cap \bar{N}_1 = 0$ and $\bar{p}(\bar{M}) = \bar{p}(\bar{N}_1) = \Sigma \oplus \bar{P}''_\alpha$. On the other hand, from 3) we know that $p|N_1$ is isomorphic in $\underline{M}_R$. Hence, $M = N_1 \oplus \mathrm{Ker}\ p =$ $N_1 \oplus \Sigma \oplus (\mathrm{Ker}\ p \cap Q_\alpha)$ by 5).

The following theorem is a generalization of (3.1.2),2) and 5).

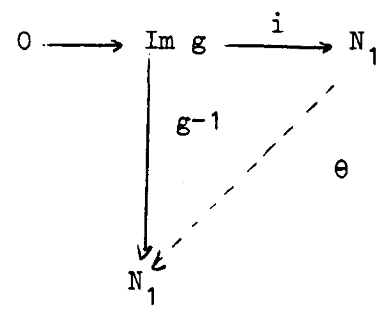
THEOREM 4.1.4. - *Let* $M$ *and* $\{M_\alpha\}_I$ *be as in (4.1.3) and* $M = N_1 \oplus N_2$. *Let* $f$ *be the projection of* $M$ *to* $N_1$. *Then* $fJ'f = f\underline{J}f$ *if and only if every direct summand of* $N_1$ *has the exchange property in* $M$. *In that case* $N_2$ *also has the exchange property in* $M$, *where* $J' = \underline{J}' \cap S_M$ *and* $\underline{J} = J(S_M)$.

*Proof.* - We assume $f\underline{J}'f = f\underline{J}f$. Since $\underline{A}$ is completely reducible, there exists a subset $K$ of $I$ such that $\mathrm{Im}\ \bar{f} \approx \sum_K \oplus \bar{M}_\alpha = \bar{M}_K$. Let $e$ be a projection of $M$ to $M_K$. Then $fS_M/fJ' \approx eS_M/eJ'$. Hence, there exist $a \in eS_Mf$, $b \in fS_Me$ such that $ba \equiv f$ and $ab \equiv e$ (mod $J'$). Put $f-ba = n \in J'$. Then $n = fnf \in fJ'f =$ $fJf$, which is equal to the radical $S_{N_1} = \mathrm{End}(N_1)$. Therefore, $ba$ is an automorphism in $S_{N_1}$ : $N_1 = fM \xrightarrow{a} eM \xrightarrow{b} N_1$. Then $eM = a(fM) \oplus \mathrm{Ker}\ b$ in $\underline{M}_R$.

On the other hand, since $ab \equiv e \pmod{J'}$, $\bar{b}|\overline{eM} \to \bar{M}$ is monomorphic (note $eM \in \underline{A}$). By considering a dense submodule of Ker b, we know $\overline{\text{Ker } b} = \bar{0}$ in $\bar{\underline{A}}$. Therefore, Ker b = 0 by (2.1.2) and $eM \approx fM$ in $\underline{M}_R$. Since $fJ'f=fJf$, $\{M_\alpha\}_K$ is a locally semi-T-nilpotent sytem by (3.1.2). Hence, every direct summand of $N_1 = fM(\in \underline{A})$ has the exchange property in M by (4.1.3). Conversely, we assume that every direct summand $N_1'$ of $N_1$ has the exchange property in M. Then $N = \sum_K \oplus T_\gamma$ ; $T_\gamma$ are c.inde. and $N_1'$ has the exchange property in $N_1$. Hence, $\{T_\gamma\}_K$ is a semi-T-nilpotent by (3.1.1). Therefore, $fJ'f = fJf$ by (1.4.3) and (3.1.2). The remaining part is clear from (4.1.3).

COROLLARY 4.1.5. - *Let* M *and* $N_i$ *be as in (4.1.3). We suppose that for every monomorphism* g *in* $S_{N_1}$ Im g *is a direct summand of* $N_1$ *i.e.* $gS_{N_1} = eS_{N_1}$ *and* $e = e^2$. *Then every direct summand of* $N_1$ *has the exchange property in* M. *Especially, if* $N_1$ *is quasi-injective,* $N_1$ *satisfies the condition.*

*Proof.* - Let f be the projection of M to $N_1$ and $e \in fJ'f$. Then (1-a) is monomorphic by (3.2.3). Futhermore, $(1-a)|N_2 = 1_{N_2}$ and Im (1-a) = Im $((1-a)|N_1) \oplus N_2$. From the assumption, Im $((1-a)|N_1)$ is a direct summand of $N_1$ and hence, Im (1-a) is a direct summand of M. On the other hand, Im(1-a) is a dense submodule of M by (3.2.4). Therefore, Im (1-a) = M and so Im $((1-a)|N_1) = N_1$. Hence, a is quasi-regular in $S_{N_1}$ and $fJ'f \subsetneq fJf$. It is clear $fJf \subsetneq fJ'f$, since $J \subsetneq J'$. Now we assume $N_1$ is quasi-injective and g is a monomorphism in $S_{N_1}$. Then we have a commutative diagram :




Since $g^{-1}$ is epimorphic, $N_1$ = im g ⊕ Ker θ.

Faith and Walker [9] proved the above corollary and Warfield [39] did in a more general case, where $N_1$ is injective. Fuchs [12] generalized [39] in a case of quasi-injective modules. Kahlon [25] and Ymagata [40] studied the corollary when all $M_\alpha$ are injective.

As we see above, the locally semi-T-nilpotency of a submodule N guarantees the exchange property in M (more strongly for all direct summands of N). However, the converse is not true, for example M itself has the exchange property in M, but its direct summands do not. Of course this is a special example.

Let Z be the ring of integers and $p_i$ primes. Put $M = \sum_{i=1}^{\infty} \oplus Z/p_1^i \oplus \sum_{i=1}^{\infty} \oplus Z/p_2^i$, $(p_1 \neq p_2)$. Since $N_1 = \sum_{i=1}^{\infty} \oplus Z/p_1^i$ is the set of all $p_1$-primary $N_1$ has the exchange property in M, but $\{Z/p_1^i\}_1^{\infty}$ is not semi-T-nilpotent. This example is similar to the first case. Let $M = \sum \oplus Z/p^i = N_1 \oplus N_2$. Then $N_1$ has the exchange property in M if and only if either $N_1$ or $N_2$ is isomorphic to a finite direct sum of $\{Z/p^i\}$, (see (4.1.7)). Hence, in this case either $N_1$ or $N_2$ must have the property of semi-T-nilpotency.

In the following we shall study those situations (I do not know whether the concepts of the exchange property and semi-T-nilpotency are equivalent, except special cases).

Let $M = \sum_I \oplus M_\alpha$ be as before and $M = N_1 \oplus N_2$. We noted that if $N_1$ has the exchange property in M, then $N_i \in \underline{A}$.

PROPOSITION 4.1.6. -*Let* M, $N_i$ *be as above. We assume that* $N_i = \sum_\gamma \sum_{J(i)_\gamma \ni \beta} \oplus M(i)_{\gamma\beta}$, *where* $M(i)_{\gamma\beta}$ *are c.inde. and* $M(1)_{\gamma\beta} \approx M(1)_{\gamma\beta'} \approx M(2)_{\gamma\beta''} \approx M(2)_{\gamma\beta'''}$ *and* $M(i)_{\gamma\beta} \not\approx M(j)_{\gamma'\beta'}$, *if* $\gamma \neq \gamma$ *.Furthermore, we assume that if* $0 \leq |J(2)_\gamma| < \infty$, $|J(1)_\gamma| \leq |J(2)_\gamma|$. *Then* $N_1$ *has the* ($\chi_o$-)*exchange property in* M *if and only if* $\{M(1)_{\gamma\beta}\}$ *is a locally semi-*T*-nilpotent system with respect to* $\underline{J}$!

*Proof*. - "If" part is clear from (4.1.3). Conversely, let $\{M(1)_{\gamma_i\beta_i}\}_{i=1}^{\infty}$ be a subset of $\{M(1)_{\gamma\beta}\}$ and $\{f_i | \overline{M(1)_{\gamma_i\beta_i}} \to M(1)_{\gamma_{i+1}\beta_{i+1}}$ and $f_i \in \underline{J}\}$ .

From the assumption, we may assume that $J(2)_{\gamma_i} \neq \emptyset$ for all i and that if $|J(1)_{\gamma_i}| = \infty$ , $|J(2)_{\gamma_i}| = \infty$. In order to show that $\{f_i\}$ is a semi-T-nilpotent, we may change $f_{2i}$ by suitable $g_{2i}: M(2)_{\gamma_i\beta_i} \to M(1)_{\gamma_{i+1}\beta_{i+1}} \in J'$ from the above assumption. Then we obtain the proposition from (3.1.1).

If $|J(2)_\gamma| = \infty$ for all $\gamma$, the assumption is satisfied.

PROPOSITION 4.1.7. - *Let* $\{M_i\}_1^\infty$ *be a set of c.inde. modules such that* $M_i$ *is monomorphic but not isomorphic to* $M_{i+1}$ *for all* i.

1) *Let* $M = \sum_1^\infty \oplus M_i = N_1 \oplus N_2$. *Then* $N_1$ *has the* ($\chi_o$-)*exchange*

*property in* M *if and only if* $N_1$ *or* $N_2$ *is a direct sum of c.inde. modules* $\{M_{i'}\}_J$ *which is locally semi-*T*-nilpotent (in this case* $N_1$ *or* $N_2$ *is a finite direct sum).*

*2) Furthermore, we assume that any of* $M_i$ *is itself a locally* T*-nilpotent system and* $M = \sum_I \oplus T_\alpha$ ; $T_\alpha$ *is isomorphic to some* $M_i$. *Then we have the same statement as in 1).*

*Proof.* - 1) If $N_1$ and $N_2$ are infinite directsums of c.inde. modules, it contradicts (3.1.1'). We can prove it similarly to 1) and (4.1.6) and we leave it to the reader.

Contrary to the assumption in (4.1.7) we have

PROPOSITION 4.1.8. - *Let* $M = \sum_I \oplus M_\alpha$ *and* $M_\alpha$ *isomorphic to a fixed c.inde. module* $M_1$ *for all* α. *Let* $M = N_1 \oplus N_2$. *Then* $N_1$ *has the exchange property in* M *if and only if* $N_1$ *is a direct sum of c.inde.modules* $\{M_{\alpha'}\}_J$ *which is a locally semi-*T*-nilpotent system.*

We leave the proof to the reader.

4.2. THE PROPERTY III.

We shall study the property III in the introduction, namely let $M = \sum_I \oplus M_\alpha$ be in $\underline{A}$, then every direct summand of M is in $\underline{A}$. Whether the property III is true for any M in $\underline{A}$ or not is still an open problem. If $\{M_\alpha\}$ is a locally semi-T-nilpotent system, this property is true by (3.1.2).

We shall give the combined result (4.2.5) of [38] and [24] .

LEMMA 4.2.1. - *Let* $M = \sum_I \oplus M_\alpha = N_1 \oplus N_2$ *be as before. For any* $x$ *in* $N_1$ *there exists a direct summand* $N_0$ *of* $N_1$ *such that* $x \in N_0$ *and* $N_0 \in \underline{A}$.

*Proof.* - It is clear that there exists a finite subset J of I such that $x \in M_J = \sum_J \oplus M_\alpha$. Since $M_J$ has the exchange property in M by (4.1.2), $M = M_J \oplus N_1' \oplus N_2'$, where $N_i' \subseteq N_i$. Put $N_i'' = N_i \cap (M_J \oplus N_j')$ $(i \neq j)$. Then $x \in N_1''$ and $M = \sum_{i=1}^{2} \oplus (N_i' \oplus N_i'')$. Hence, $M_J \approx \sum_{i=1}^{2} \oplus N_i''$ and so $N_i'' \in \underline{A}$ by (2.1.4).

COROLLARY 4.2.2. - *Let* $M = N_1 \oplus N_2$ *be as above. If* $N_1$ *is countably generated,* $N_1 \in \underline{A}$.

*Proof.* - We can prove it by an induction from (4.2.1).

LEMMA 4.2.3 [26]. - *Let* M *be a direct sum of countably generated* R-*modules. Then every direct summand of* M *is also a direct sum of countably generated* R-*modules.*

See [26] or [34] for the proof.

LEMMA 4.2.4 [4,38.] - *Let* $M = \sum_I \oplus M_\alpha$ *and let all* $M_\alpha$ *be countably generated and c.inde.modules. Then the property III is true for* M.

*Proof.* - It is clear form (4.2.2) and (4.2.3).

THEOREM 4.2.5. - *Let* $\{M_\alpha\}_J, \{M^*_\beta\}_K$ *be sets of c.inde.modules such that* $\{M_\alpha\}_J$ *is a semi-T-nilpotent system with respect to* $\underline{J}'$ *and* $\sum_K \oplus M_\beta$ *satisfies the property III for any direct summand of it. Then* $M = \sum_J \oplus M_\alpha \oplus \sum_K \oplus M_\beta$ *satisfies the property III for any direct summand of* M.

*Proof.* - Let $M = N_1 \oplus N_2$. Since $\sum_J \oplus M_\alpha = N_o$ has the exchange property in M by (4.1.3), $M = M_o \oplus N_1' \oplus N_2'$, where $N_i = N_i' \oplus N_i''$. Hence, $M/M_o \approx N_1' \oplus N_2' \approx \sum_K \oplus M_\beta^*$. Therefore, $N_i' \in \underline{A}$ from the assumption. On the other hand, $N_1'' \oplus N_2'' \approx M_o$ and hence, $N_i'' \in \underline{A}$ by (3.1.4).

COROLLARY 4.2.6. - *Let* $M = \sum_I \oplus M_\alpha$ *and* $M_\alpha$ *c.inde..Let* $\{M_\beta\}_K$ *be the subset of* $\{M_\alpha\}$ *which consists of all countably generated* R*-modules. If* $\{M_\gamma\}_{I-K}$ *is a locally semi-*T*-nilpotent system with respect to* $\underline{\mathfrak{J}}'$, *then the property III is true for* M.

*Proof.* - It is clear form (4.2.4) and (4.2.5).

Finally, we add here a corollary to (4.2.4).

Corollary 4.2.7. - *Let* M, $N_i$ *be as in (4.1.3). If* $N_1$ *is* R*-projective,* $N_1 \in \underline{A}$. *Especially, if* M *is* R*-projective, the property III is true for* M.

*Proof.* - Every R-projective module is a directsum of countably generated R-modules by (4.2.3) and hence, $N_1 \in \underline{A}$ by (4.2.4).

## CHAPTER 5. SEMI-PERFECT MODULES

H. Bass [2] defined semi-perfect or perfect rings as a generalization of semi-primary rings in 1960. Later E. Mares [28] succeeded to generalize them to modules in 1963.

In this chapter we shall give many interesting properties of semi-perfect modules given by [19, 28] . We always assume that a ring R contains the identity and modules are right R-modules and unitary.

### 5.1. Semi-perfect modules

Let $M \supseteq N$ be R-modules. If any submodule T of M with property : $M = T+N$, always coincides with M, N is called *small in* M.

LEMMA 5.1.1. - *Let* $A \subseteq B \subseteq M \subseteq N$ *be* R-*modules. Then*

*1) If* B *is small in* M, *then* A *is small in* N.

*2) Let* $\{A_i\}_1^n$ *be a finite set of small submodules in* M, *then* $\sum_{i=1}^{n} A_i$ *is also small in* M.

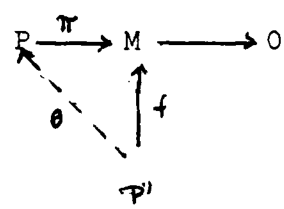
*3) Let* f *be a homomorphism of* M *to* M'. *If* A *is small in* M, f(A) *is small in* M'.

*Proof.* - It is clear from the definition.

DEFINITION. - Let $P \xrightarrow{\pi} M \longrightarrow 0$ be an exact sequence of R-modules. If P is R-projective and $\mathrm{Ker}\,\pi$ is samll in P, we say P is a *projective cover* of M. We shall denote it by $(P, \pi)$ and P by P(M), respectively.

LEMMA 5.1.2. - *Projective covers* $(P,\pi)$ *of* M *are unique up to isomorphism if they exist. If* $P' \longrightarrow M \longrightarrow 0$ *is an exact sequence with* P' *projective, then* $(P,\pi)$ *is naturally imbedded in* P' *as a direct summand.*

*Proof.* - From a diagram ;

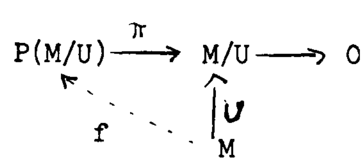


we have θ and $P = \operatorname{Im}\theta + \operatorname{Ker}\pi$, since P' is projective and f is surjective. Hence, $P = \operatorname{Im}\theta$, which implies $P' = P_0 \oplus \operatorname{Ker}\theta$, since P is projective. The first part is clear from the last.

DEFINITION. - Let P be an R-module. If P is R-projective and every factor modules of P have projective covers, we call P *semi-perfect*. If every direct sum of copies of P are semi-perfect, we call P *perfect*.

LEMMA 5.1.3 [2,28]. - *Let* M *be semi-perfect and* U *a submodule of* M. *Let* $\nu_U : M \to M/U$ *be the natural epimorphism. Then there exist projective submodules* P *and* V *of* M *and of* U, ***respectively such*** *that* $M = P \oplus V$, $\nu_U | P \to M/U$ *is a projective cover and* $U \cap P$ *is small in* P.

*Proof.* - Take a diagram ;



Then $M = P \oplus \text{Ker } f$ by (5.1.2), where $P \overset{f}{\approx} P(M/U)$ and $P \cap U$ is small in P. It is clear $\text{Ker } f \subseteq U$.

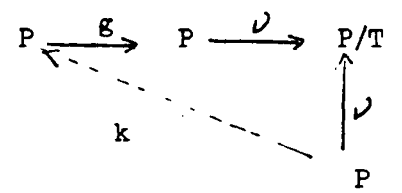
COROLLARY 5.1.4. - *Let* M *be semi-pefect. Then for any submodule* U *of* M, U *is small in* M *or there exists a non-zero direct summand* V *of* M *such that* $U \supseteq V$.

*Proof.* - If U is not small in M, $U \supsetneq U \cap P$ by (5.1.1) and (5.1.3). Hence, $P \not\subseteq U$ and so $V \neq 0$.

LEMMA 5.1.5 [37]. - *Let* P *be* R-*projective and* $S_P = \text{End}(P)$.

*Then* $J(S) = \{ f \mid \in S, \text{ Im } f$ *is small in* $P \}$ .

*Proof.* - Denote the set of right side in (5.1.6) by J'(S). It is clear from (5.1.1) that J'(S) is a two-sided ideal in S. For any $f \in S$ we have $P = \text{Im } f + \text{Im } (1-f)$. Hence, if $f \in J'(S)$, $P = \text{Im } (1-f)$. Since P is projective, $P = \text{Ker } (1-f) \oplus P'$. Put $K = \text{Ker } (1-f)$. Then $K = f(K) \subseteq f(P)$, which is small in P. Hence, $P = P'$ and $K = 0$. Therefore, $J'(S) \subseteq J(S)$. Conversely, let $g \in J(S)$. We shall show that g(P) is small in P. Let $P = T + g(P)$ for some $T \subsetneq P$ and consider a diagramm ;

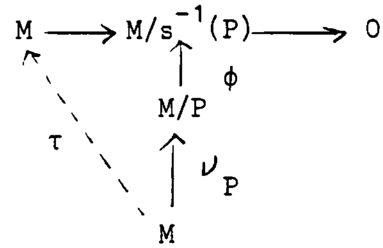



($\nu g$ is surjective).

Then $(1-gk)=0$ and hence, $\nu = 0$, since $gk \in J(S)$. Therefore, $P = T$.

PROPOSITION 5.1.6. - *Let* M *be a semi-perfect module. Then* S/J(S) *is a regular ring in the sense of Von Neumann, where* $S = End_R(M)$, *(cf. [23, 28]).*

*Proof.* - Let $s \in S$. Then there exists a submodule P of M such that $M = Im\ s + P$ and $P \cap Im\ s$ is small in M by (5.1.3). We define an R-homomorphism $\phi : M/P \to M/s^{-1}(P)$ by setting $\phi(s(m)+P) = m+s^{-1}(P)$, which is clearly well defined. Now consider a diagram ;




Then $ts(m)-m \in s^{-1}(P)$ and hence $s(ts(m)-m) \in P \cap Im\ s$. Therefore, $s-sts \in J'(S) = J(S)$ by (5.1.5).

For any R-module A we put $J(A) = \cap$(Maximal submodules in A) or $J(A) = A$ is there exist non maximal submodules. If A = R, J(R) is the Jacobson radical of R. We note that every small submodule in A is contained in J(A) and that $f(J(A)) \subseteq J(B)$ for any R-homomorphism f of A to B.

From now on, we shall denote $Hom_R(A,B)$ by [A,B] and $End_R(A)$ by $S_A$.

PROPOSITION 5.1.7. - *Let* P *be* R-*projective. Then* J(P) *is small in* P *if and only if* $J(S_P) = [P,J(P)]$ . *In this case* $S_P/J(S_P) \approx End_R(M/J(P))$ *as rings.*

*Proof*. - From the above remark we always have $J(S_P) \subseteq [P,J(P)]$ by (5.1.6) for projective P. If $J(P)$ is small in P, $[P,J(P)] \subseteq J(S_P)$ by (5.1.5). Conversely, suppose $[P,J(P)] = J(S_P)$ and $P = J(P)+N$ for some submodule N. Then we consider a diagram ;

$$\begin{array}{ccccc} J(P) & \longrightarrow & J(P)/N \cap J(P) & \longrightarrow & 0 \\ & \nwarrow & \wr\wr & & \\ & & P/N & & \\ & h & \uparrow \nu & & \\ & & P & & \end{array}$$

From it we obtain $J(P) = h(P)+N \cap J(P)$ and hence, $P = N+J(P) = N+h(P)$. Since $h \in [P,J(P)] = J(S_P)$, $h(P)$ is small in P. Therefore, $P = N$ and we have shown that $J(P)$ is small in P. Since P is projective, we have an exact sequence ; $0 \to [P,J(P)] \to S_P \to [P,P/J(P)] \to 0$. It is clear that $[P,P/J(P)] = [P/J(P),P/J(P)]$ by the above remark.

LEMMA 5.1.8.- *Let* $\{A_\alpha\}_I$ *be a set of* R*-modules such that* $[A_\alpha,J(A_\alpha)] \subseteq J(S_{A_\alpha})$ *for all* $\alpha \in I$. *Put* $A = \sum_I \oplus A_\alpha$ . *If* Ker $(1-f) \neq 0$ *for some* $f \in S_A$, *then* Im $f \neq J(\text{Im } f)$.

*Proof*. - Put B = Im f and suppose $B = J(B)$. Since $J(B) \subseteq J(A)$, $f \in [A,J(A)]$. Ker $(1-f) \neq 0$ implies that there exists a subset $\{1,2,\dots,n\}$ such that $(\sum_{i=1}^{n} \oplus A_i) \cap \text{Ker } (1-f) \neq 0$. The following argument is analogous to the proof of (2.1.1.). Let $e_1$ be the projective of A to $A_1$. Since $f \in [A,J(A)]$ , $e_1 f e_1 | A_1 \in [A_1,J(A_1)] \subseteq J(S_{A_1})$ by the assumption. Hence,

$e_1(1-f)e_1|A_1$ is an automorphism of $A_1$ : $A_1 \xrightarrow{(1-f)e_1} A \xrightarrow{e_1} A_1$ and so $A = (1-f)(A_1) \oplus \text{Ker } e_1 = (1-f)(A_1) \oplus \sum_{\alpha \neq 1} \oplus A_\alpha$ and $A_1 \overset{1-f}{\approx} (1-f)(A_1)$

Now, we repeat the same argument on the latest decomposition and on $A_2$. Then we have $A = (1-f)(A_1) \oplus (1-f)(A_2) \oplus \sum'_{\alpha=1,2} \oplus A_\alpha$. Finally, we have that $(1-f)|(\sum_1^n \oplus A_i)$ is isomorphic from this argument, which is a contradiction.

Hence, $B \neq J(B)$.

COROLLARY 5.1.9 [2]. - *If* P *is* R-*projective*, $P \neq J(P)$ *and* $J(P) = PJ(R)$.

*Proof.* - It is clear $J(R) = [R, J(R)]$ and P is a direct summand of copies of R. Hence, $P \neq J(P)$ from (5.1.8). The last part is also clear.

We note that (5.1.9) shows that P contains a maximal submodule.

COROLLARY 5.1.10. - *If* M *is semi-perfect*, J(M) *is small in* M.

*Proof.* - By (5.1.4) either J(M) is small in M or J(M) contains a non-zero submodule V such that $M = V \oplus V'$. If we had the latter, then $J(M) = J(V) \oplus J(V')$ and $J(V) = J(M) \cap V = V$. Hence, $V = 0$ by (5.1.9).

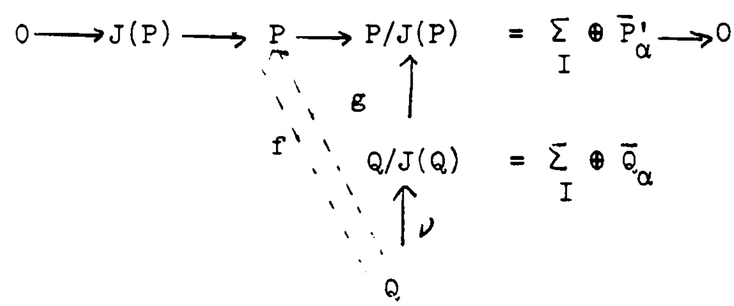
PROPOSITION 5.1.11. - *Let* M *be semi-perfect. Then* M/J(M) *is a semi-simple module.*

*Proof.* - Put $\bar{M} = M/J(M)$ and $\bar{U} = U/J(M)$ for a submodule $U \supseteq J(M)$. By (5.1.3) there exist submodules P,V in M such that $M = P \oplus V$, $V \subseteq U$ and $U \cap P$ is small in M. Then $U \cap P \subseteq J(M)$. On the other hand, $(P + J(M)) \cap U = (P \cap U) + J(M) = J(M)$. Hence, $\bar{M} = \bar{P} \oplus \bar{U}$. Therefore, M is semi-simple (since

R contains the identity or $J(M) \neq M$).

LEMMA 5.1.12. - *Let* P *be an* R*-projective module such that* J(P) *is small in* P. *Suppose that* P/J(P) *is a direct sum of submodules* $\{\bar{P}'_\alpha\}_I$ *as* R/J(R)*-modules and that for each* $\alpha \in I$, *there exists a projective module* $Q_\alpha/J(Q_\alpha) \approx \bar{P}'_\alpha$ . *Then the above decomposition of* $\bar{P}$ *is lifted to* P.

*Proof.* - Put $Q = \sum_I \oplus Q_\alpha$ . Since $P \to P/J(P)$ is a projective cover of P/J(P), $P/J(P) \approx Q/J(Q)$ and Q is projective, $Q = P \oplus Q'$ by (5.1.2) :



Then $Q = P + J(Q) = P \oplus J(Q')$ and hence, $Q' = 0$.

COROLLARY 5.1.13. - *Let* M *be semi-perfect and* $M/J(M) = \sum \oplus \bar{M}'_\alpha$ .

*Then there exists a decomposition of* M : $M = \sum_I \oplus M_\alpha$ *which induces the above. Especially* M *is a direct sum of c.inde. modules.*

*Proof.* - We know from the proof of (5.1.11) that M satisfies the condition in (5.1.12). Hence, we obtain the first part from (5.1.12). Since M/J(M) is semi-simple by (5.1.11), $M = \sum_J \oplus M''_\beta$ , where $M''_\beta / J(M''_\beta)$ are minimal by the first part. Since $End(M''_\beta/J(M''_\beta)) = End(M''_\beta)/J(EndM''_\beta)$ by (5.1.7), $M''_\beta$ is c.inde..

From this corollary we can apply the results in the previous chapters to semi-perfect modules.

THEOREM 5.1.4 [28]. - *Let* M *be semi-perfect. Then we obtain*

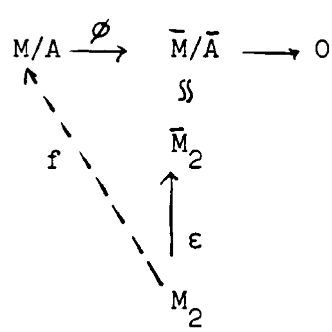
*1)* J(M) *is small in* M.

*2)* M/J(M) *is semi-simple.*

*3) Every decomposition of* M/J(M) *such as* $M/J(M) = M_1' \oplus M_2'$ *is lifted to* M.

*Conversely, if a projective module* M *satisfies 1)* ∼ *3), then* M *is semi-perfect.*

*Proof.* - We have shown the first half. We assume a projective module M satisfies 1) ∼ 3). Let A be a submodule of M and put M = M/J(M) and $\bar{A}$ (A+J(M))/J(M). From 2) and 3) there exist submodules $M_1, M_2$ such that $M = M_1 \oplus M_2$ and $\bar{M}_1 = \bar{A}$. Then we have a diagram ;




Ker ϕ = (A+J(M))/A is small in M/A by 1) and (5.1.1). Hence, f is surjective. On the other hand, Ker f ⊊ Ker ε = $J(M_2)$, which is small in $M_2$ by 1). Therefore, $(M_2, f) = P(M/A)$.

## 5.2. SEMI-T-NILPOTENCY AND SEMI-PERFECTION

We have shown by (5.1.3) that every semi-perfect modules are directsums of c.inde.projective modules. In this section, we shall consider the converse case.

THEOREM 5.2.1. - *Let* $\{P_\alpha\}_I$ *be a set of projective modules* $P_\alpha$ *and* $P = \sum_I \oplus P_\alpha$. *Then* $J(P)$ *is small in* $P$ *if and only if* $J(P_\alpha)$ *is small in* $P_\alpha$ *for all* $\alpha \in I$ *and* $\{P_\alpha\}_I$ *is a locally semi-*T*-nilpotent system with respect to the radical* $\underline{J}$ *(of the induced category from* $\{P_\alpha\}_I$*).*

*Proof.* - If $J(P)$ is small in $P$ then $J(P_\alpha)$ is small in $P_\alpha$ by (5.1.1). Let $\{P_i\}_1^\infty$ be a subset of $\{P_\alpha\}_I$ and $\{f_i : P_i \to P_{i+1}$ and $f_i \in \underline{J}\}$. Put $P_i' = \{p_i + f_i(p_i) \mid \in P_i \oplus P_{i+1} < \oplus P, p_i \in P_i\}$. Since $J(P_i) \oplus J(P_{i+1})$ is small in $P_i \oplus P_{i+1}$, $f_i(p_i) \in J(P_{i+1})$ by (5.1.7). Then $P = \sum_1^\infty + P_i' + \sum_{\gamma \neq (i)} + P_\gamma + J(P)$. Since $J(P)$ is small in $P$, $P = \sum_{i=1}^\infty \oplus P_i' \oplus \sum_{\gamma \neq (i)} \oplus P_\gamma$. Hence, $\{P_\gamma\}_{I}$ is a locally semi-T-nilpotent system from (***) in the proof of (3.1.1). Conversely, we assume that $J(P_\alpha)$ is small in $P_\alpha$ for all $\alpha \in I$ and $\{P_\alpha\}_I$ is locally semi-T-nilpotent. Then $[P_\alpha, J(P_\alpha)] = J(S_{P_\alpha})$ by (5.1.7). We shall put $\underline{C} \cap [P_\alpha, P_\beta] = [P_\alpha, J(P_\beta)]$ in (2.2.3). Then $\underline{C}$ satisfies all conditions in (2.2.3). Hence, $[P, J(P)] \subseteq J(S_P)$, which implies that $J(P)$ is small in $P$ by (5.1.7).

COROLLARY 5.2.2. - *Let* $\{P_\alpha\}_I$ *and* P *be as above. Then* P *is (semi-)perfect if only if* $P_\alpha$ *is (semi-)perfect and* $\{P_\alpha\}_I$ *is a locally (semi-)*T-*nilpotent system with respect to* $\underline{J}$.

*Proof.* - We assume that P is semi-perfect. Then each $P_\alpha$ is semi-perfect and J(P) is small in P by (5.1.14). Hence, $\{P_\alpha\}_I$ is locally semi-T-nilpotent. If P is perfect, consider any co-products of copies of P, then the above argument shows that $\{P_\alpha\}_I$ is locally T-nilpotent. Conversely, we assume that each $P_\alpha$ is semi-perfect. Then by (5.1.11) and (5.1.13) P/J(P) is a semi-simple module and $P = \sum_J \oplus P'_\beta$, where $P'_\beta$ are c.inde.. Since $\{P_\alpha\}_I$ is a locally semi-T-nilpotent system with respect to $\underline{J}$, so is $\{P'_\beta\}_J$. Furthermore, $\underline{J}' \cap [P'_\beta, P'_\beta] = \underline{J} \cap [P'_\beta, P'_\beta]$, (see § 1.4 for the definition of $\underline{J}'$). Hence, every idempotent in $S_P/J(S_P)$ is lifted to $S_P$ by (3.2.5). J(P) is small in P by (5.2.1). Therefore, P is semi-perfect by (5.1.14). If $\{P_\alpha\}_I$ is locally T-nilpotent, we can use the above argument on any co-products of copies of P. Hence, P is perfect.

COROLLARY 5.2.3 [33,36]. - *Let* S *be any ring with radical* J(S) *and* $(S)_I$ *the ring of column finite matrices over* S *with any degree* I. *Then* $J((S_I)) = (J(S))_I$ *if and only if* J(S) *is right* T-*nilpotent.*

*Proof.* - Put $M = \sum_I \oplus S$, then $[M,M] = (S)_I$ and $[M,J(M)] = (J(S))_I$. Hence, $(J(S))_I = J((S)_I)$ if and only if J(M) is small in M by (5.1.7) and hence if and only if J(S) is right T-nilpotent by (5.2.1).

THEOREM 5.2.4. - *Let* P *be an indecomposable and projective modules.*

*Then* P *is semi-perfect if and only if* P *is c.inde.* .

*Proof.* - If P is semi-perfect, P is c.inde. by (5.1.13). The converse is a special case of the following theorem.

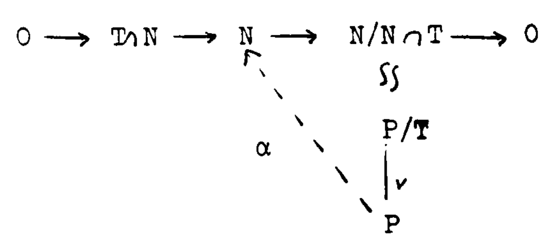
THEOREM 5.2.4'. - *Let* P *be projective, then we have the following equivalent statements.*

*1)* $S_P$ *is a local ring.*

*2) Every proper submodule of* P *is small in* P.

*3)* P *is semi-perfect and indecomposable.*

*Proof.* - 1) → 2) Since $S_P$ is local, P is c.inde. and hence, $J(S_P)$ consist of all non-isomorphisms in $S_P$. Let N be a proper submodule of P and P = T+N for some submodule T in P. Then we have a diagram ;




Since N ≠ P, $\alpha \in J(S_P)$ and N= T∩N+Im α. Hence, P = T+Im α. Since Im α is small in P by (5.1.5), P=T.

2) → 1) Let $f \neq 0 \in S_P$ be a non-isomorphism. If Im f = P, P = $P_o \oplus$Ker f, since P is projective. Hence, Ker f = 0 by 2), which contradicts the assumption. Therefore, Im f ≠ P. Let g be another non-isomorphism in $S_P$.

Then $P \supsetneqq \mathrm{Im}\, f + \mathrm{Im}\, g \supseteq \mathrm{Im}(f+g)$. Hence, the set of non-isomorphisms in $S_P$ is the two-sided ideal, which means that $S_P$ is local.

2) → 3) It is clear.

3) → 2) Let T be a proper submodule of P and P' = P(P/T). Since P is projective, $P = P' \oplus P''$ by (5.1.2). Hence, P' = P and T is small in P.

REMARK. - If P is semi-perfect and indecomposable, J(P) is a unique maximal submodule of P by (5.2.4'),2). Hence, $P \approx eR$ for some idempotent e, since P is cyclic. Thus, *there exist semi-perfect modules if and only if* R *contains a local idempotent* e, *i.e.* eRe *is a local ring.*

COROLLARY 5.2.5. - *Let* P *be a semi-perfect. Then there exist maximal ones among perfect direct summands of* P *and those modules are isomorphic each other.*

*Proof.* - Let $P = \sum_I \oplus P_\alpha$ and $P_\alpha$ c.inde.. Let $\underline{S}$ be the set of subset $\{P_\gamma\}_J$ of $\{P_\alpha\}_I$ such that $\{P_\gamma\}_J$ is locally T-nilpotent. We can find a maximal one in $\underline{S}$ by Zorn's lamma, say $\{P_\gamma\}_J$, since $\{P_\alpha\}_I$ is semi-T-nilpotent. Put $P_o = \sum_J \oplus P_\gamma$, then $P_o$ is a desired perfect summand of P by (5.2.3). Let $P = \sum_J \oplus P_\gamma \oplus \sum_K \oplus P_\delta = \sum_{J'} \oplus P'_\gamma \oplus \sum_{K'} \oplus P'_\delta$, where $\sum_J \oplus P_\gamma$ and $\sum_{J'} \oplus P'_\gamma$ are maximal perfect submodules. Then $P_\gamma$ and $P'_\gamma$ are themselves T-nilpotent, respectively. Hence, if $P_\gamma$ is isomorphic to some $P'_{\delta''}$ in $\{P'_{\delta'}\}_K$, $\{\{P'_{\gamma'}\}_{J'}, P_\gamma\}$ is locally T-nilpotent. Which contradicts to the maximality of $\sum_{J'} \oplus P'_{\gamma'}$. Therefore, $\sum_J \oplus P_\gamma \approx \sum_{J'} \oplus P'_{\gamma'}$, by (2.1.4).

PROPOSITION 5.2.6. - *Let* P *be semi-perfect and* $P_o$ *a projective sub-module in* P. *Then* $P_o$ *is a direct summand of* P *if and only if* $J(P) \cap P_o = J(P_o)$.

*Proof.* - Suppose $J(P_o) = J(P) \cap P_o$, then $P_o/J(P_o) \subseteq P/J(P)$. By (5.1.13) there exists a direct summand $P_1$ of P such that $P_1/J(P_1) \oplus P_o/J(P_o) = P/J(P)$. On the other hand, the formal directsum $P_1 \oplus P_o$ is isomorphic to P by (5.1.12). Hence, $J(P_o)$ is small in $P_o$. Consider a diagram ;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J(P_o) & \longrightarrow & P_o & \xrightarrow{\nu i} & P/(P_1+J(P)) & \longrightarrow & 0 \text{ (exact)} \\ & & & & & {\scriptstyle g}\nwarrow & \uparrow{\scriptstyle \nu} & & \\ & & & & & & P & & \end{array}$$

where i is the inclusion. Then $(1_{P_o} - gi)(P_o) \subseteq J(P_o)$ and hence, $(1_{P_o} - gi) \in J(S_{P_o})$ by (5.1.5). Therefore, gi is isomorphic in $S_{P_o}$, which means that $P_o$ is direct summand of P. The converse is clear.

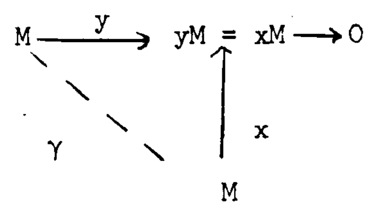
## 5.3. PROJECTIVE ARTINIAN MODULES.

Let M be an R-module. If for every series $M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ of submodules $M_i$ of M there exists n such that $M_n = M_{n+t}$ for all t, we call M *artinian*. Let T be a subset of $S_M$. We put $TM = \{f(m) \mid f \in T \text{ and } m \in M\}$.

LEMMA 5.3.1. - *Let* M *be artinian and projective. If* $AM = A^2 M \neq 0$ *for a right ideal* A *in* $S_M$, *Then* A *contains a non-zero idempotent.*

*Proof.* - Since M is artinian, there exists a minimal submodule $N = A'M$ with respect to properties $N' = A''M = A''^2M \neq 0$ for a right ideal $A'' \subseteq A$.

Then $A'$ is not nilpotent. Hence, there exists $x$ in $A'$ such that $xA' \neq 0$. Again from the assumption we can find a minimal one among submodules $x'M, (x' \in A)$ and $x'M \neq 0$, say $xM$, $(x \in A)$. Since $xA'A'M = xA'M \neq 0$, there exists $y \in xA'$ such that $yA' \neq 0$. Then $yM \subseteq xA'M \subseteq xM$. Hence, $yM = xM$ by the minimality of $xM$. Now, consider a diagram ;




Then $x = yr = xa$, where $a \in A$ . Hence, $x = xa = xa^2 = \dots$ . Therefore, $a$ is not nilpotent and $x(a-a^2) = 0$. Put $n = a^2-a$. If $n = 0$, $a$ is a non-zero idempotent. Suppose $n \neq 0$. Put $A^* = \{z \mid \in A', xz = 0\}$ , then $A' \supsetneq A^* \ni n$. We consider a series ; $A^*M \supseteq A^{*2}M \supseteq \dots \supseteq A^{*n}M \supseteq \dots$ . Since $M$ is artinian, $A^{*n}M = A^{*n+1}$ for some $n$. Since $A'M \supseteq A^*M$ and $A'M$ is the minimal one, $A'M = A^{*n}M$ or $A^{*n}M = 0$. On the other hand, $xA' \neq 0$ and $xA^* = 0$ and hence, $A^{*n} = 0$, which implies that $n$ is nilpotent. Next, put $a_1 = a+n-2an$, then all $a,n$ and $a_1$ commute each other, since they are generated by $a$. Hence, $(-n+2an)$ is also nilpotent and $a_1$ is not nilpotent. Furthermore, $a_1{}^2-a_1 = n^2(4n-3)$. Repeating this argument we get non-nilpotent elements $a_i \in A'$ such that $(a_i-a_i{}^2) = n^{2^i}z_i$, $z_i \in S_M$. Since $n$ is nilpotent, we have a non-zero idempotent $a_t$ in $A'$.

COROLLARY 5.3.2. - *Let* $M$ *be as above. Then* $S_M$ *is a semi-primary ring.*

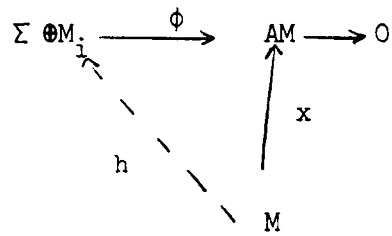
*Proof.* - Since M is artinian, M is a finite directsum of indecomposable, projective module $M_i$. First we assume $M = M_1$. For any right ideal A in $S_M$, $A^nM = A^{n+1}M$ for some n. If $A^n \neq 0$, A contains a non-zero idempotent e by (5.3.1). Since M is indecomposable, e=1. Therefore, $S_M$ is a local ring with nilpotent radical. Next, we may assume $M = \sum_{i=1}^{n} \sum_{j=1}^{s_i} \oplus M_{ij}$, where $M_{ij}$'s are indecomposable and $M_{ij} \approx M_{ij'}$, $M_{ij} \not\approx M_{i'j}$, if $i \neq i'$. Then $S_M = \{(s_{ij}) | s_{ij} \in S_{ij} = [\sum_{k=1}^{s_j} \oplus M_{jk'}, \sum_{k'=1}^{s_i} \oplus M_{ik'}]\}$. Since $M_{ij}$ is c.inde. from the above,

$$J(S_M) = \begin{pmatrix} J(S_{11}) & S_{12} & \cdots & S_{in} \\ S_{21} & J(S_{22}) & \cdots & S_{2n} \\ & & \cdots\cdots & \\ S_{n1} & & \cdots\cdots & J(S_{nn}) \end{pmatrix}$$

by (2.1.3). Furthermore. All $J(S_{ii})$ are nilpotent and hence, $J(S_M)$ is nilpotent and $S_M/J(S_M) \approx \sum_{i=1}^{n} \oplus S_{ii}/J(S_{ii})$. It is clear that $S_{ii}/J(S_{ii})$ is the ring of matrices over a division ring $S_{M_{i1}}/J(S_{M_{i1}})$.

LEMMA 5.3.3. - *Let* M *be* R*-projective and* A *a finitely generated right ideal in* $S_M$. *Then* $A = [M, AM]$. *Furthermore, if* M *is* R*-finitely generated,* $A' = [M, A'M]$ *for any right ideal* A' *in* $S_M$.

*Proof.* - Let $A = \sum_{i=1}^{n} a_i S_M$. Then we shall consider a diagram ;




where $M_i \approx M$ for all i, $\phi=(a_1,a_2,\dots,a_n)$ and x is any element in $[M,AM]$.

We shall denote h by $\begin{pmatrix} h_1 \\ h_2 \\ h_n \end{pmatrix}$. Then $x = \sum_{i=1}^{n} a_i h_i$ is in A. Hence,

$[M,AM] \subseteq A$. It is clear $A \subseteq [M,AM]$. If M is finitely generated, we replace $\sum_{i=1}^{n} \oplus M_i$ by $\sum_{a \in A} \oplus M_a$ in the above, then $h(M) \subseteq \sum_{i=1}^{t} \oplus M_{a_i}$. Hence, we can make use of the same argument.

THEOREM 5.3.4. - *Let* M *be* R-*projective and artinian. Then* M *is a perfect* R-*finitely generated module and* $S_M$ *is right artinian.*

*Proof.* - It is clear from the proof of (5.3.2) that $M = \sum_{i=1}^{n} \oplus M_i$, where $M_i$' s are c.inde.. Furthermore, since $S_M$ is semi-primary by (5.3.2), $M_i$ is a (locally) T-nilpotent system with respect to $\underline{J}$. Therefore, M is perfect by (5.2.2) and (5.2.4) and $M_i$ is cyclic. Furthermore, (5.3.3) gives a lattice monomorphism of the set of right ideals in $S_M$ into the set of submodules of M. Hence, $S_M$ is right artinian.

# CHAPTER 6. INJECTIVE MODULES

In this chapter we assume that the reader knows elementary properties of injective modules and we refer to [8] for them.

We mainly study some application of (1.3.2) to injective modules and hence, we shall consider directsums of indecomposable and injective modules. We reproduce [10, 25, 29, 31, 40] by virtue of factor categories and study the Matlis'problem in § 6.5.

## 6.1. ENDOMORPHISM RINGS OF INJECTIVE MODULES.

In this section we shall recall some properties of the endomorphism rings of injective modules, which we make use of later. If the reader is not familiar to them, consult [8].

As a dual of the concept "small", we shall define the concept "large". Let $M \supseteq N$ be R-modules. If for any son-zero submodule T of M, $N \cap T \neq 0$, we say N is *large* submodule in M or M is an *essential extension* of N. We denote it by $M \overset{\prime}{\supset} N$.

As a dual of (5.1.6) we have

LEMMA 6.1.1. *Let* E *be injective and* $S_E = \mathrm{End}(E)$. *Then* $J(S_E) = \{f| \in S_E, \mathrm{Ker}\ f \overset{\prime}{\subsetneq} E\}$ *and* $S_E/J(S_E)$ *is a regular ring.*

As a dual of (5.1.14).

LEMMA 6.1.2. *Let* E *and* $S_E$ *be as above. Then a finite set of mutually orthogonal idempotents in* $S_E/J(S_E)$ *is lifted to* $S_E$.

As a dual of projective cover, we define an injective envelope (injective hull) E of R-modules M as follows ; E is injective and M is large in E. Contrary to projective covers , every modules have injective hulls and every injective hulls are isomorphic (dual to (5.1.2)). Hence by E(M) we shall denote an injective hull of M.

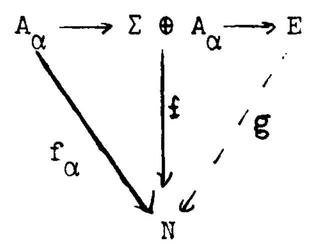
## 6.2. CATEGORIES OF INJECTIVE MODULES.

We shall give here an application of (1.3.2) to injective modules. Let M be an R-injective module. We shall define a full sub-additive category $\underline{C}(M)$ in $\underline{M}_R$ as follows (cf. the induced category in § 1.4) ; the objects in $\underline{C}(M)$ consist of all direct summands of any products $\prod_I M_\alpha$ ; $M_\alpha \approx M$. If M is an injective and cogenerator in $\underline{M}_R$ , then $\underline{C}(M)$ is the category of all injective modules. We also call $\underline{C}(M)$ the *category of injective modules induced from* M. Let $\underline{J}$ be the radical of C(M) (see §1.1. for the definition).

THEOREM 6.2.1 [17, 39] . - *Let* M *be and* R-*injective module and* $\underline{C}(M)$ *the category of injective modules induced from* M *and* $\underline{J}$ *the radical of* $\underline{C}(M)$. *Then* $\underline{C}(M)/\underline{J}$ is a Grothendieck and spectral category.

*Proof*. - We shall denote $\underline{C}(M)/\underline{J}$ by $\bar{\underline{C}}(M)$. Then $\bar{\underline{C}}(M)$ has a finite coproducts from the definition and Remark 2 in § 1.1, and $\bar{\underline{C}}(M)$ is a regular category from (6.1.1). Furthermore, (6.1.2) shows that $\bar{\underline{C}}(M)$ is amenable. Hence, $\bar{\underline{C}}(M)$ is an abelian spectral category by (1.3.2). We shall show that $\bar{\underline{C}}(M)$

has finite co-products. Let $\{\bar{A}_\alpha\}_I$ be a set of objects in $\bar{\underline{C}}(M)$. Since $A_\alpha < \oplus \prod_{I_\alpha} M$, $\sum_I \oplus A_\alpha < \oplus_\alpha \prod_{I_\alpha} M$ and $E = E(\sum_I \oplus A_\alpha)$ is an object in $\underline{C}(M)$, since $\prod \prod M$ is injective. We show $\bar{E} = \sum \oplus \bar{A}_\alpha$ . Let N be any object in $\underline{C}(M)$ and $\{\bar{f}_\alpha : \bar{A}_\alpha \to \bar{N}\}$ a set of morphisms, where $f_\alpha : A_\alpha \to N$ is a representative. Then there exists $f : \Sigma \oplus A_\alpha \to N$ in $\underline{M}_R$ such that




Since N is injective, there exists $g : E \to N$ which commutes the above diagram. We can easily show from (6.1.1) that g does not depend on a choice of representative $f_\alpha$ and that $\bar{g}$ is uniquely determined,(cf. the proof of (3.2.7)). Also we can similarly show that for a given $\bar{g}:\bar{E} \to \bar{N}$, there exists a unique set of $\bar{f}_\alpha : \bar{A}_\alpha \to \bar{N}$ such that $\bar{g} = \pi \bar{f}_\alpha$ Hence, $\bar{E} = \Sigma \oplus \bar{A}_\alpha$. Next we shall show that $\bar{\underline{C}}(M)$ has a generator. Let $\underline{S}$ be the set of right ideals K in R such that $E_K = E(R/K) \in \underline{C}(M)$. Put $\bar{U} = \sum_{K \in \underline{S}} \oplus \bar{E}_K$. Let T be an object in $\underline{C}(M)$ and $t \neq 0 \in T$. Then $T \supseteq tR \approx R/(0:t)_r$ and $E(R/(0;t)_r) \in \underline{S}$, since T is an injective and in $\underline{C}(M)$. Therefore, $E_{(0;t)_r}$ is isomorphic to a direct summand of T, which implies $[\bar{U},\bar{T}] \neq 0$. Finally, we shall show similarly to the proof of (1.4.8) that $(\bigcup_K \bar{A}_\alpha) \cap \bar{B} = \bigcup_K (\bar{A}_\alpha \cap \bar{B})$

for a subobject $\bar{B}$ and a directed set of subobjects $\{\bar{A}_\alpha\}_K$ in a given object $\bar{F}$. Put $\bar{C} = \bigcup_K (\bar{A}_\alpha \cap \bar{B})$, then $\bar{B} = \bar{C} \oplus \bar{B}_o$ and $(\bigcup_K \bar{A}_\alpha) \cap \bar{B} = \bar{C} \cup ((\bigcup_K \bar{A}_\alpha) \cap \bar{B}_o)$.

We put $\bar{D} = (\bigcup_K \bar{A}_\alpha) \cap \bar{B}_o$ and assume $\bar{D} \neq 0$. From an exact sequence

$\sum_K \oplus \bar{A}_\alpha \to \bigcup_K \bar{A}_\alpha \to 0$, we obtain a monomorphism $\bar{g} : \bar{D} \to \sum_K \oplus \bar{A}_\alpha$ such that $\bar{f}\bar{g} = 1_{\bar{D}}$. We note that g is R-monomorphic, since $\underline{J}$ is the Jacobson radical and that $\sum_K \oplus \bar{A}_\alpha = \overline{E(\sum \oplus A_\alpha)}$. Put D' = Im g in $\underline{M}_R$. Then $\bar{D}' = \text{Im}\ \bar{g}$. Since $D' \neq 0$, $D' \cap \sum_K \oplus A_\alpha \neq 0$ in $\underline{M}_R$. Let $x \neq 0$ be an element in $D' \cap \sum_K \oplus A_\alpha$ and let E(xR), $E_1(xR)$ be injective hulls of xR in D' and $\sum_{i=1}^{n} \oplus A_{\alpha_i}$, respectively, where $x \in \sum_{i=1}^{n} \oplus A_{\alpha_i}$. Then $\overline{E(xR)} = \overline{E_1(xR)} \subseteq \sum_1^n \oplus \bar{A}_{\alpha_i}$ from Remark 2 below. Hence, $\bar{E}(g^{-1}(x)R) \subseteq \bigcup \bar{A}_{\alpha_i} \subseteq \bar{A}_\beta$ for some β such that $\beta \geqslant \alpha_i$ and $\bar{E}(g^{-1}(x)R) \subseteq \bar{D}$, which is a contradiction.

REMARKS 1. We noted in the proof of (1.4.8) that $\overline{\sum_I \oplus M_\alpha} = \sum_I \oplus \bar{M}_\alpha$ in the factor category of c.inde.modules. However, in $\underline{C}(M)$ $\sum_I \oplus A_\alpha$ is not, in general, an object in $\underline{C}(M)$ and $\sum_I \oplus \bar{A}_\alpha$ means $\bar{E}(\sum_I \oplus A_\alpha)$.

2. Let E, E' be injective and f : E → E'. We shall find Ker f and Im f in $\underline{C}(M)$. Let K = Ker f in $\underline{M}_R$ and E" = E(K) in E. Then $E = E'' \oplus E_1$. We define $f' \in [E,E']$ by setting $f' = (0, f \mid E_1)$. Then $\text{Ker}\ (f-f') = K \oplus E_1 \subseteq E$. Hence, $\bar{f} = \bar{f}'$. Therefore, Ker $\bar{f}$ = Ker $\bar{f}'$ = $\bar{E}''$ and Im $\bar{f}$ = Im $\bar{f}'$ = $\overline{f(E_1)}$. This argument shows that Ker $\bar{f}$ (Im $\bar{f}$) does not depend on a choice of injective hulls of K in E and that we can give direct proofs of many

results in the following without factor category. However, if we use the factor category, the proofs are simple and natural in some sense.

3. If $\underline{J} \neq 0$, $\overline{\pi A_\alpha} \neq \pi \overline{A}_\alpha$ for $A_\alpha \in \underline{C}(M)$ in general.

4. Instead of injective modules, we can consider the full sub-additive category $\underline{P}$ of projective modules in $\underline{M}_R$. However, in this case $\underline{P}/\underline{J}$ is not spectral. We know that $\underline{P}/\underline{J}$ is spectral and Grothendieck category if and only if R is right perfect ring (see [19]).

For any R-module M we put $Z(M) = \{ m \mid \in M, (0:m)_r \subseteq' R\}$ . It is clear that Z(M) is an R-submodule of M and we call Z(M) the *singular submodule of* M.

LEMMA 6.2.2. - *Let* M *be an injective module with* $Z(M) = 0$, *then* $J(S_M) = 0$.

*Proof*. - Let $f \in J(S_M)$. Then Ker $f \subseteq' M$ and so Z(M/Ker f) = M/Ker f. On the other hand, M/Ker f is isomorphic to a submodule of M. Hence, M = Ker f.

PROPOSITION 6.2.3. - *Let* M *be an injective* R-*module with* $Z(M) = 0$. *Then* $\underline{C}(M)$ *is a spectral and Grothendieck category with generator* M. *For any morphism* f *in* $\underline{C}(M)$, Ker f(Im f) *in* $\underline{C}(M)$ *is equal to* Ker f (Im f) *in* $\underline{M}_R$.

*Proof*. - From (6.2.2) we obtain $\underline{J} = 0$. Hence, $\underline{C}(M)$ is a spectral and Grothendieck category. Furthermore, since M is a cogenerator in $\underline{C}(M)$, M is a generator. The remaining part is clear from Remark 2.

COROLLARY 6.2.4. - *Let* N *be an* R*-module with* $Z(N) = 0$ *and* $Q_1, Q_2$ *injective submodules in* N. *Then* $Q_1+Q_2$ *and* $Q_1 \cap Q_2$ *are injective.*

*Proof.* - Let $E = E(N)$ and consider $\underline{C}(E)$. Then $Q_i \in \underline{C}(E)$ and $Q_1+Q_2$ and $Q_1$ $Q_2$ are an image and a kernel in $\underline{M}_R$ of morphisms in $\underline{C}(E)$, respectively. Hence , they are injective in $\underline{M}_R$ by (6.2.3).

LEMMA 6.2.5. - *Let* $\underline{B}$ *be a full sub-additive category in* $\underline{M}_R$. *Suppose* $\underline{B}$ *contains a generator (cogenerator) in* $\underline{M}_R$. *Then every monomorphism (epimorphism) in* $\underline{B}$ *is monomorphic (epimorphic) in* $\underline{M}_R$.

*Proof.* - Let U a generator in $\underline{M}_R$ , which is contained in $\underline{B}$ and $f:A \to B$ a monomorphism in $\underline{B}$. Put Ker $f = C$ in $\underline{M}_R$. If $C \neq 0$, there exists $g \neq 0 \in [U,C]$ in $\underline{M}_R$ such that $ig \neq 0$, where $i:C \to A$ is the inclusion. However, $ig \in [U,A] \in \underline{B}$ and $fig = 0$, which is a contradiction.

PROPOSITION 6.2.6. - *Let* M *be an* R*-injective module. We assume* M *is a generator and cogenerator in* $\underline{M}_R$, (e.g. R *is a* Q.F. *ring). Then* $\underline{C}(M)$ *is an abelian category if and only if* R *is a semi-simple artinian ring.*

*Proof.* - We assume $\underline{C}(M)$ is abelian. We shall show for any morphism f in $\underline{C}(M)$ that (Ker f in $\underline{C}(M)$) = (Ker f in $\underline{M}_R$). Let $f : N \xrightarrow{f'} \mathrm{Im}\, f \xrightarrow{i} N'$ be a decomposition of f in $\underline{C}(M)$. Since $\underline{C}(M)$ is abelian, f' is epimorphic in $\underline{C}(M)$ and i is monomorphic in $\underline{C}(M)$. Hence, so are they in $\underline{M}_R$ by (6.2.5). Hence, (Im f in $\underline{C}(M)$) = (Im f in $\underline{M}_R$). Put $K_1$ = (Ker f in $\underline{C}(M)$) and $K_2$ = (Ker f in $\underline{M}_R$ ).

It is clear $K_1 \subsetneq K_2$ by (6.2.5). On the other hand, $K_1$ is R-injective and hence, $N = K_1 \oplus N''$ in $\underline{M}_R$. Then $N'' \in \underline{C}(M)$ and $N'' \overset{f}{\underset{\approx}{}}{}^{\text{in}}\ \underline{M}_R$ (Im f in $\underline{M}_R$) from the above. Hence, $K_1 = K_2$. Let A be any R-module, then there exists an R-exact sequence ; $0 \to A \to \prod_{I_1} M \to \prod_{I_2} M$. Since $\prod M \in \underline{C}(M)$, A = (Ker f in $\underline{M}_R$) = (Ker f in $\underline{C}(M)$). Hence, A is injective. Therefore, R is semi-simple and artinian. The converse is clear.

## 6.3. DECOMPOSITIONS OF INJECTIVE MODULES.

This section is a reproduction of [29] by virtue of factor category and we shall give a condition under which every injective module is an injective hull of some direct sum of c.inde. modules, which is equivalent to a fact that $\underline{A}/\underline{J}$ is completely reducible, where $\underline{A}$ is the full sub-additive category of all injective modules in $\underline{M}_R$.

LEMMA 6.3.1. *Let $\underline{B}$ be a full sub-additive category in $\underline{M}_R$. We assume that every direct summand in $\underline{M}_R$ of an object in $\underline{B}$ belongs to $\underline{B}$. Then every finite co-product in $\underline{B}/\underline{J}$ is lifted to $\underline{M}_R$.*

*Proof.* - Let B, $B_1$ and $B_2$ be in $\underline{B}$ and $\bar{B} = \bar{B}_1 \oplus \bar{B}_2$ in $\underline{B}/\underline{J}$. Then there exist morphisms $i_k : B_K \to B$ and $p_k : B \to B_k$ such that $\bar{1}_B = \overline{i_1 p_1} + \overline{i_2 p_2}$ and $\overline{p_k i_k} = \bar{1}_{Bk}$. Since $\underline{J}$ is the radical, $p_k i_k$ is isomorphic in $\underline{M}_R$. Hence, $M = \text{Im } i_1 \oplus \text{Ker } p_1$ in $\underline{M}_R$. By the assumption Im $i_1$ and Ker $p_1 \in \underline{B}$ and it is clear that Ker $\bar{p}_1 = \bar{B}_2$ and $\bar{B} = \text{Im } \bar{i}_1 \oplus \text{Ker } \bar{p}_1 = \bar{B}_1 \oplus \bar{B}_2$.

COROLLARY 6.3.2. - *Let* M *be* R-*injective. Then an object* N *in* $\underline{C}(M)/\underline{J}$ *is minimal if and only if* N *is indecomposable.*

*Proof.* - It is clear from (6.2.1) and (6.3.1).

PROPOSITION 6.3.3. - *Let* R *be a left perfect ring and* M R-*injective as a right* R-*module. Then* $\underline{C}(M)/\underline{J}$ *is a completely reducible and Grothendieck category.*

*Proof.* - Since R is left perfect, every right R-module contains minimal submodules by [2] . Let N be in $\underline{C}(M)$ and S(N) the socle of N in $\underline{M}_R$, i.e. $S(N) = \sum \oplus I_\alpha$ and $I_\alpha$'s are minimal R-modules. We know from the assumption that $N \supsetneq' \sum \oplus I_\alpha$. Hence, $\bar{N} = \Sigma \oplus \overline{E(I_\alpha)}$ by Remark 1 and $\overline{E(I_\alpha)}$ is a minimal object in $\bar{\underline{C}}(M)$ by (6.3.2).

Let $\underline{A}$ be the full sub-additive category of all injective modules in $\underline{M}_R$. By $\bar{\underline{A}}$ we shall always denote $\underline{A}/\underline{J}$ in the follows. We know from (6.3.3) that if R is a left perfect ring, then $\underline{A}$ is completely reducible. We shall give a condition for $\bar{\underline{A}}$ to be completely reducible [29] .

DEFINITION.-Let K be a right ideal in R. K is called *reducible* if there exist right ideal $K_i$ in R such that $K = K_1 \cap K_2$ and $K_i \neq K$. If K is not reducible, we call K *irreducible.*

We shall denote E(R/K) by $E_K$.

LEMMA 6.3.4. - *Let* E *be* R-*injective. Then the following statements are equivalent.*

*1)* E *is indecomposable.*

*2)* E *is an essential extension of any submodule.*

*3)* $E = E_K$ *for some irreducible right ideal* K.

*Furthermore,* $E_{K'}$ *is indecomposable for a right ideal* K', *then* K' *is irreducible.*

*Proof.* - 1) $\Leftrightarrow$ 2) It is clear from the definition.

2) $\Leftrightarrow$ 3) Let $x = 0 \in E$. Then $E \supseteq xR \approx R/(0:x)_r$. If $(0:x)_r = K_1 \cap K_2$, $R/(0:x)_r \supseteq K_1/(0:x)_r \oplus K_2/(0:x)_r$. By 2) we have $K_1$ or $K_2 = (0:x)_r$. Hence, $(0:x)_r$ is irreducible. This proof shows the last part.

3) $\Rightarrow$ 1) Let $E_K = E_1 \oplus E_2$ and $p_i \colon E \to E_i$ the projections. Put $K_i = \mathrm{Ker}(p_i | R/K)$. Then $K = K_1 \cap K_2$. We may assume $K = K_1$ from 3). Then Ker $p_1 = 0$ since $E \supseteq R/K$. Hence, $E_2 = 0$.

THEOREM 6.3.5 [17,29,39] . - *Let* $\bar{\underline{A}}$ *be as above. Then* $\bar{\underline{A}}$ *is completely reducible if and only if for every right ideal* K, K *always has a decomposition as follows :* $K = K_1 \cap K_2$ *and* $K_1$ *is irreducible and* $R \supseteq K_2 \neq K$.

*Proof.* - If $E_K$ is completely reducible, $E_K = E_1 \oplus E_2$ by (6.3.1) and (6.3.2), where $E_1$ is indecomposable. Then we have $K = K_1 \cap K_2$ and $E_1$ contains an isomorphic image of $R/K_1$ from the proof of 3) $\Rightarrow$ 1) of (6.3.4). Hence, $K_1$ is irreducible from (6.3.4). Conversely, if $K = K_1 \cap K_2$ and $K_2 \neq K$, then we have a natural exact sequence : $0 \to R/K \xrightarrow{\phi} R/K_1 \oplus R/K_2$ and $\phi(K_2/K) \subseteq R/K_1$. Hence, $E(R/K) \supseteq E(R/K_1)$ since $E(R/K_1)$ is indecomposable. We knew already from the proof of (6.2.1) that every injective module E contains some $E_K$. Therefore, E contains a minimal object in $\bar{\underline{A}}$ and hence

$\bar{\underline{A}}$ is completely reducible, since $\bar{\underline{A}}$ is a spectral, Grothendieck category.

COROLLARY 6.3.6. - *We have the following equivalent statements*

*1)* R *is a right noetherian ring.*

*2) Every injective modules are a direct sum of c.inde.modules.*

*3) Any directsums of injective modules are also injective,*

([3, 29, 32]).

*Proof.* - 1) $\Leftrightarrow$ 3) See [3] or [8] .

1) $\Rightarrow$ 2) Since R is right noetherian, the condition of (6.3.5) is satisfied and so $\underline{A}$ is completely reducible. Hence, for any injective module E, $E = E(\Sigma \oplus Q_\alpha)$ by Remark 1 and (6.3.2), where $Q_\alpha$'s are indecomposable and injective. Since $\Sigma \oplus Q_\alpha$ is injective, $E = \Sigma \oplus Q_\alpha$.

2) $\Rightarrow$ 3) Let $\{E_\alpha\}_I$ be a set of indecomposable injective modules. We put $E = E(\Sigma \oplus E_\alpha)$. Then we have $E = \sum_J \oplus Q_\beta$ by 2), where $Q_\beta$'s are indecomposable and $\bar{E} = \sum_J \oplus \bar{Q}_\beta = \Sigma \oplus \bar{E}_\alpha$ . Hence, $|J| = |I|$ and $\bar{E}_\alpha$ is isomorphic to some $\bar{Q}_\beta$ and vice versa, since $\bar{E}_\alpha$ and $\bar{Q}_\beta$ are minimal in $\bar{\underline{A}}$. Therefore $\sum_I \oplus E_\alpha \approx \sum_J \oplus Q_\beta$ is injective.

Remark 5. The completely reducibility of $\bar{\underline{A}}$ does not guarantee that R is a right noetherian ((6.3.3)). Furthermore, $\bar{\underline{A}}$ is not completely reducible in general (see [17]).

## 6.4. GOLDIE DIMENSION.

A. Goldie [15] defined a dimension of modules as a generalization of noetherian modules. J. Fort [10] and Y. Miyashita [31] generalized it independently to an infinite case. We shall reproduce them as an application of (6.2.1).

DEFINITION.- Let M be an R-module. If M is always essential extension of any non-zero sub-modules, M is called *uniform*. Let N be an R-module. We consider the set $\underline{S}$ of sub-modules T of N such that $T = \sum_I \oplus K_\alpha$, where $K_\alpha$'s are uniform. Put $\dim N = \max_I(|I|)$ if it exists (we shall show in (6.4.3) that dim N exists for any N).

THEOREM 6.4.1 [10, 17, 31]. - *Let* E *be* R-*injectives. Then* dim E *exists and we have a decomposition* $E = E_1 \oplus E_2$ *such that* $\dim E = \dim E_1$, $\dim E_2 = 0$ *and* $E_1$ *is a minimal injective submodule of* E *among injective submodules* E' *of* E *with decompositions as above. Furthermore this decomposition is unique up to isomorphism.*

*Proof.* - We take the factor category $\underline{\bar{A}}$ in § 6.3. Then dim E = 0 if and only if the socle $S(\bar{E})$ of $\bar{E}$ in $\underline{\bar{A}}$ is zero. We assume $S(\bar{E}) \neq 0$ and $S(\bar{E}) = \sum \oplus \bar{E}_\alpha = \overline{E(\sum \oplus E_\alpha)}$, where $E_\alpha$'s are indecomposable injectives. Then $E = E(\sum \oplus E_\alpha) \oplus E_2$ and $\dim E_2 = 0$. Let $N = \sum_J \oplus N_\alpha$ be a submodule in E, where $N_\alpha$'s are uniform. Then $E(N) = E(\sum_J \oplus E(N_\alpha))$ and $\overline{E(N_\alpha)}$ is minimal in $\underline{\bar{A}}$. Hence, $\overline{E(N)} \subseteq S(\bar{E})$ and so $|J| \leq |I|$. Therefore, $\dim E = |I|$. Let E' be an injective submodule of E such that $E = E' \oplus E_2'$, $\dim E' = \dim E$ and $\dim E_2' = 0$. Then $\bar{E}'$ contains $S(\bar{E}) = \sum_I \oplus \bar{E}_\alpha$. Hence, $E_1$ is a minimal one among injectives with such a decomposition. Let $E = E_1 \oplus E_2 = E_1' \oplus E_2'$ such that $\dim E_1 = \dim E_1'$ and $\dim E_2 = \dim E_2' = 0$ and $E_1$, $E_1'$ are minimal in such decompositions. Then $\bar{E}_1 = \bar{E}_1' = S(\bar{E})$ and hence, $\bar{E}_2 \approx \bar{E}_2'$. Since $\underline{J}$ is the radical, $E_1 \approx E_1'$ and $E_2 \approx E_2'$ in $\underline{M}_R$ by Remark 3 in § 1.1.

LEMMA 6.4.2. - *Let* $M = \sum_I \oplus M_\alpha$ *in* $\underline{M}_R$ *and* $N$ *a submodule of* $M$. *Put* $N_\alpha = M_\alpha \cap N$ *and* $N' = \sum_I \oplus N_\alpha$. *Then* $M \overset{\prime}{\geq} N$ *if and only if* $M_\alpha \overset{\prime}{\geq} N_\alpha$ *for all* $\alpha \in I$, (further, $M \overset{\prime}{\geq} N'$).

*Proof*. - Suppose $M_\alpha \overset{\prime}{\geq} N_\alpha$ for all $\alpha$ . Let $m \neq 0 \in M$ ; $m = \Sigma m_{\alpha_i}$ $m_{\alpha_i} \neq 0 \in M_{\alpha_i}$. From the assumption, there exists $r \in R$ such that $mr = m_{\alpha_1} r + \sum_{i \geq 2} m_{\alpha_i} r$ and $m_{\alpha_1} r \neq 0 \in N_{\alpha_1}$. Repeating this, we obtain $mR \cap N' \neq 0$. Hence, $M \overset{\prime}{\geq} N'$. The converse is clear.

PROPOSITION 6.4.3. - *Let* $N$ *be an* $R$-*module. Then* $\dim N$ *exists and* $N$ *is an essential extension of a submodule* $N_1 \oplus N_2$ *such that* $\dim N_1 = \dim N = |I|$ *and* $\dim N_2 = 0$ *and* $N_1$ *is an essential extension of* $\sum_I \oplus T_\alpha$ , *where* $T_\alpha$*'s are uniform.*

*Proof*. - Put $E = E(N)$. Then $E = E_1 \oplus E_2$ as in (6.4.1). Put $N_i' = N \cap E_i$. Then $E_i \overset{\prime}{\geq} N_i'$ and $N \overset{\prime}{\geq} N_1' \oplus N_2'$ by (6.4.2). Hence, $\dim E_2 = \dim N_2' = 0$ and $E_1 = E(N_1')$. Let $E_1 = E(\sum_I \oplus E_\alpha)$, where $E_\alpha$'s are indecomposable. Put $E_\alpha \cap N_1' = N_\alpha$ and $N_1 = \sum_I \oplus N_\alpha$ . Then $N_\alpha$'s are uniform and $N_1' \overset{\prime}{\geq} N_1$ by (6.4.2). Suppose $N \geq T' = \sum_J \oplus T_\alpha$ , where $T_\alpha$'s are uniform. Then $\overline{E(T')} = \overline{E(\sum_J \oplus E(T_\alpha))} = \Sigma \oplus \overline{E(T_\alpha)} \subseteq \bar{E}_1$. Hence, $|J| \leq |I|$ and $\dim N = \dim E = |I|$.

COROLLARY 6.4.4. [9] . - *Let* $\{E_\alpha\}_I$ *be a set of injective modules and* $Q = \sum_I \oplus E_\alpha$ . *Let* P *be a submodule of* Q *such that* $P = \sum_J \oplus P_\beta$ ; $P_\beta$'*s are indecomposable injectives. Then* $|J| \leq |I|$.

## 6.5. THE PROPERTY III IN INJECTIVE MODULES.

In this section we shall study the property III in a case where every c.inde. modules are injective, which is called Matlis'problem [29]. We do not know a complete answer for this problem and we shall give here some affirmative answers given by [25] and [40].

From the proof of (6.2.2) we have

LEMMA 6.5.1. - *Let* $\{N_\alpha\}_I$ *be a set of indecomposable injectives. If* $Z(N_\alpha) = 0$ *for some* $\alpha$ , *every non-zero element in* $[N_\gamma, N_\alpha]$ *is isomorphic. Especially, if* $Z(N_\alpha) = 0$ *for all* $\alpha \in \Gamma$, $\{N_\alpha\}_I$ *is a* T-*nilpotent system with respect to* J!

THEOREM 6.5.2 [21, 25, 40] . - *Let* $\{N_\alpha\}_J$ *be a set of indecomposable injectives and* $N = \sum \oplus N_\alpha$ . *Suppose* $N = M_1 \oplus M_2$ *and* $Z(M_1) = 0$. *Then* $M_i$ *is a directsum of c.inde. injectives for* i = 1,2.

*Proof*. - $M_i$ contains a dense submodule $T_i$ by (3.2.7). Let $T_1 = \sum_I \oplus T_\alpha$; $T_\alpha$'s are c.inde.. Since $Z(M_1) = 0$, $Z(T_1) = 0$. Hence, $\{T_\alpha\}_I$ is a T-nilpotent system by (6.5.1). Therefore, we have the theorem from (3.2.2) and (4.1.3).

THEOREM 6.5.3. - *Let* $\{E_\alpha\}_I$ *be a set of indecomposable injective modules and* $E = \sum_I \oplus E_\alpha$ . *Then the followings are equivalent.*

*1)* $\{E_\alpha\}_I$ *is a locally semi-*T*-nilpotent system with respect to* $\underline{J}'$.

*2) Every module in* $\underline{C}$ *which is an extension of* E *contains* E *as a direct summand.*

*3) There are no proper and essential extension of* E *which are in* $\underline{C}$.

*4) For each monomorphism* g *in* $S_E = \mathrm{End}(E)$, Im g *is a direct summand of* E,

*where* $\underline{C}$ *is the category of all c.inde.modules.*

*Proof.* - 4) $\Rightarrow$ 1) It is proved by (4.1.5).

1) $\Rightarrow$ 4) Let g be a monomorphism in $S_E$. Then $\mathrm{Im}\ g = \sum \oplus g(E_\alpha)$ and $E_\alpha \simeq g(E_\alpha)$. Since $g(E_\alpha)$ are injective, Im g is a locally direct summand of E. Hence, Im g is a direct summand of E by 1) and (3.2.5).

1) $\Rightarrow$ 2) It is clear from the above proof.

2) $\Rightarrow$ 3) It is also clear.

3) $\Rightarrow$ 1) Suppose $\{E_\alpha\}_I$ is not a locally semi-T-nilpotent. Then there exist a subset $\{E_{\alpha_i}\}_1^\infty$ of $\{E_\alpha\}_I$ and a set of non-isomorphisms $f_i : E_{\alpha_i} \to E_{\alpha_{i+1}}$ such that for some element x in $E_{\alpha_1}$ $f_n f_{n-1} \cdots f_1(x) \neq 0$ for all n. We note Ker $f_i \neq 0$, since $E_{\alpha_i}$ are injective and indecomposable. Put

$E_i' = \{x_i + f_i(x_i) \mid x_i \in E_{\alpha_i}\} \subseteq \sum_1^\infty \oplus E_{\alpha_i} < \oplus E$. Put $E = \sum_{i=1}^\infty \oplus E_{\alpha_i} \oplus E_o$,

$E_{\alpha_i} \cap (\Sigma \oplus E'_j) \geq \mathrm{Ker}\, f_i \neq 0$. Hence, $\Sigma \oplus E'_j \oplus E_o \subsetneq E$ by (6.4.2). It is clear $x \notin (\Sigma \oplus E'_j \oplus E_o)$. Let $E^*$ be an injective hull of E. Since $(\Sigma \oplus E'_j \oplus E_o) \underset{t}{\approx} E$, we can extend this isomorphim t to a monomorphism $\phi$ of $E^*$ .Therefore, $\phi(\Sigma \oplus E'_j \oplus E_o) = E \subsetneq \phi(E) = \sum_I \oplus \Phi(E_\alpha) \subseteq \underline{C}$. which is a contradiction.

COROLLARY 6.5.4. - *Let* $\{E_\alpha\}$ *and* E *be as above. Furthermore, we assume that each* $E_\alpha$ *is noetherian. Then all statements in (6.5.3) are true.*

*Proof.* - Let $\{E_i\}_1^\infty$ be a set of injective and indecomposable modules and $f_i : E_i \to E_{i+1}$ non-isomorphisms. Then $\mathrm{Ker}\, f_i \neq 0$, $\mathrm{Im}\, f_1 \cap \mathrm{Ker}\, f_2 \neq 0$ if $f_i \neq 0$, Since $E_2$ is uniform. Hence, $\mathrm{Ker}\, f_1 \subsetneq \mathrm{Ker}\, f_2 f_1$, if $f_1 \neq 0$. Therefore, $\{E_\alpha\}_I$ is a T-nilpotent system form the assumption.

COROLLARY 6.5.5. *Let* M *be a module in* $\underline{C}$ *and* L *a submodule of* M. *Suppose* L *is a direct sum of injective modules and* Z(L) = 0. *Then* L *is a direct summand of* M (cf. [9,21,25] ).

*Proof.* - Since every injective module in M is in $\underline{C}$ by (4.1.5), the corollary is clear from (6.5.3).

Remark 6. Let $\{E_\alpha\}$ be as in (6.5.3). In general $\{E_\alpha\}_I$ is not semi-T-nilpotent. Hence $E = \sum_I \oplus E_\alpha$ is not quasi-injective. Furthermore, even if $E_\alpha$ are noetherian, E is not injective. If E is (quasi-)injective or Z(E)=0 $\{E_\alpha\}$ is semi-T-nilpotent. However, the converse is not true (see [42]).

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