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QUOTIENT OVERRINGS

By Hans H. STORRER

§ 1. Let R be an associative ring with unit element and let Q be its maximal right quotient ring. (See [10] for details. Insofar torsion theories are concerned our basic references are [11] and [14]). A *right overring* S of R is a ring such that $R \subseteq S \subseteq Q$. Left overrings are defined similarly.

Overrings of commutative integral domains have been studied rather extensively. We quote the following result for further reference :

1.1. PROPOSITION. - *Let* R *be a commutative integral domain.*

Then the following conditions are equivalent :

(a) R *is a Prüfer domain (i.e. every finitely generated ideal is invertible).*

(b) every overring is integrally closed,

(c) every overring is a flat R*-module.*

Proof. For an account in book form as well as generalizations to commutative rings with zero-divisors, see [12, p. 132-134, 237].

Thus it would seem useful to have means of describing the overrings of R in terms of the ring R itself. This can obviously be done for quotient overrings. By a *right quotient overring* S of R we mean a right overring of R, which is the quotient ring $R_{\mathcal{F}}$ of R relative to some idempotent filter (topology) $\mathcal{F}$ of right ideals. Since $Q = R_{\mathcal{D}}$, where $\mathcal{D}$ is the idempotent filter of dense right ideals [10, p. 96] , S is a right quotient overring if and only if $S = R_{\mathcal{F}} = \{q \in Q \mid q^{-1}R \in \mathcal{F}\}$ where $\mathcal{F}$ is an idempotent filter contained in $\mathcal{D}$. Notice, that $\mathcal{F}$ is not uniquely determined by S in general.

The following result, due to Lambek [11, p. 39] characterizes the quotient overrings :

1.2. LEMMA. - *The right overring* S *is a right quotient overring of* R *if and only if for every* $s \in S$ *and for every* $q \in Q \setminus S$ *there exists an* $r \in R$ *such that* $sr \in S$, $qr \notin S$.

1.3. LEMMA. - *Let* S *be a right quotient overring of* R, *let* $s_k \in S$ $(k = 1,\ldots,n)$ *and* $q \in Q \setminus S$. *Then there exists an* $r \in R$ *such that* $s_k r \in R$ $(k = 1,\ldots,n)$, $qr \notin S$.

Proof. By 1.2. there is an $r_1 \in R$ such that $s_1 r_1 \in R$, $qr_1 \notin S$. Next, there exists an $r_2 \in R$ such that $(s_2 r_1) r_2 \in R$, $(qr_1) r_2 \notin S$. Continuing in this way, we obtain $r = r_1 r_2 \ldots r_n$ with the desired property.

We will now restrict our attention to rings having the property, that every right overring is a right quotient overring. Such a ring will be called a *right* L-*ring*. The class of L-rings includes a large variety of rings :

1.4. EXAMPLES :

(a) A right rationally complete ring [4,10] is trivially a right L-ring, since R = Q. As particular examples, we mention the right self-injective and the commutative Artinian rings.

(b) The ring of 2×2 triangular matrices over a field (Q is then the full matrix ring) is a (left and right) L-ring, since there are no proper overrings.

(c) Let the following conditions be satisfied :

(i) The inclusion map $R \to Q$ is a ring epimorphism (i.e. the canonical map $Q \otimes_R Q \to Q$ is an isomorphism).

(ii) Q is left and right R-flat.

(iii) Every overring S is flat as a left R-module.

Then R is a right L-ring.

Proof. By (ii) and (iii) the map $i \otimes i : S \otimes_R S \to Q \otimes_R S \to Q \otimes_R Q$ induced by the inclusion $i : S \to Q$ is injective.

Thus the map in the top row of the commutative diagram

$$\begin{tikzcd} S \otimes_R S \arrow[r] \arrow[d, "i \otimes i"'] & S \arrow[d, "i"] \\ Q \otimes_R Q \arrow[r, "\cong"] & Q \end{tikzcd}$$

is injective and hence bijective.

Its follows, that every overring S is a right flat epimorphic extension, and these are well known to be right quotient overrings (see e.g. [14]).

The conditions (i), (ii) and (iii) are satisfied in the following cases (d) and (e).

(d) R is a Prüfer domain (use 1.1.).

(e) R is a hereditary noetherian prime ring (all notions are two-sided), cf. [9].

(f) Every right overring S of a right L-ring R is a right L-ring. This follows from 1.2. and the fact, that the maximal right quotient rings of R and S coincide.

(g) A left L-ring need not be a right L-ring :

Let Q be the ring consisting of all 4×4 matrices over a field, having the following form.

$$\begin{pmatrix} a & 0 & 0 & 0 \\ b & c_1 & x & 0 \\ d & y & c_2 & 0 \\ e & z & f & a \end{pmatrix}$$

The matrices with $x = 0$ form a subring S of Q and those with $x = y = z = 0$ and $c_1 = c_2$ form a subring R of S.

Using the methods of [17] , one finds, that R is left rationally complete, thus a left L-ring, and that Q is the maximal right quotient ring of R.

Let now x and y be non-zero and consider

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in S \qquad \xi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in Q\setminus S$$

The $\eta\rho\in R$ implies $\xi\rho\in S$ for all $\rho\in R$, thus R is not a right L-ring.

We now wish to show, that the property of being a right L-ring is Morita invariant. We shall use the following description of Morita equivalence [3, p. 47].

1.5. PROPOSITION. - *Two rings* R *and* R' *are Morita equivalent if and only if* $R' \cong eR_ne$ *, where* e *is an idempotent of the* $n \times n$ *matrix ring* R_n *such that* $R_neR_n = R_n$.

1.6. PROPOSITION. - *Let* Q *be the maximal right quotient ring of* R. *Then*

(a) Q_n *is the maximal right quotient ring of* R_n,

(b) if $e\in R$ *is an idempotent such that* $ReR = R$, *then* eQe *is the maximal right of* eRe.

(c) the property of being right rationally complete is a Morita invariant property of rings.

Proof. (a) is due to Utumi [18, (2.3)].
(b) and (c) : We first show, that $eRe \subsetneq eQe$ is a rational extension [4] of right eRe-modules. Let $epe, eqe \in eQe$, $epe \neq 0$. Then since $R \subsetneq Q$ is a rational extension of right R-modules, there exists an $r \in R$ such that $eper \neq 0$, $eqer \in R$. Since $ReR = R$, we have $eperRe \neq 0$, $eqerRe \subseteq R$. Thus there exists an $r_1 \in R$ such that $(epe)(err_1e) \neq 0$, $(eqe)(err_1e) \in eRe$. To prove (b) it therefore remains to show, that eQe is right rationally complete, and this will follow, if we prove (c).
Now if eRe is right rationally complete, then so is R. Indeed $R \subsetneq Q$ rational implies $eRe \subsetneq eQe$ rational, hence $eRe = eQe$ and since $ReR = R$, this implies $R = Q$. If $R' \cong eR_ne$ as in 1.5. and if R is not rationally complete, then neither is R_n nor R'. This implies (c) and completes the proof of (b).

1.7. LEMMA. - *If* R_n *is a right* L*-ring for some* n*, then so is* R.

Proof. Let $R \subsetneq S \subsetneq Q$, $s \in S$, $q \in Q \setminus S$, consider the $n \times n$ diagonal matrices $\mathrm{diag}(s,\ldots,s)$ and $\mathrm{diag}(q,\ldots,q)$, and apply 1.2 to $R_n \subsetneq S_n \subsetneq Q_n$.

1.8 LEMMA. - *Let* e *be an idempotent of* R *such that* $ReR = R$. *If* eRe *is a right* L*-ring, then so is* R.

Proof. Let $R \subsetneq S \subsetneq Q$, $s \in S$, $q \in Q \setminus S$. Since $ReR = R$, there exist elements $a_k, b_k \in R$ $(k = 1,\ldots,n)$ with $\sum a_k e b_k = 1$. For the same reason, there exist $u, v \in R$ such that $euqve \notin eSe$ (for $eRqRe \subseteq S$ implies $q \in S$). Consider now the elements $eb_ksve \in eSe$ $(k = 1,\ldots,n)$ and $euqve \in eQe \setminus eSe$. Since eQe is the maximal right quotient ring of eRe by 1.6 , eSe is a right overring of eRe and hence a quotient overring. By 1.3 there exists an $ere \in eRe$ with $(eb_ksve)(ere) \in eRe$ for $k = 1,\ldots,n$

$(euqve)(ere) \notin eSe$. This implies $q(vere) \notin S$, and $\Sigma\, a_k e b_k svere = s(vere) \in R$. Thus R is a right L-ring by 1.2.

1.9 PROPOSITION. - *The property of being a right* L-*ring is a Morita invariant property of rings.*

Proof. As before, we show, that the negation of the property under consideration is Morita invariant. If $R' \cong eR_n e$ as in 1.5 and if R is not a right L-ring, then the result follows form 1.7 and 1.8.

Another straightforward result is

1.10. PROPOSITION. - *A finite product of rings as a right* L-*ring if and only if every factor is a right* L-*ring.*

Proof. One uses the fact, that the quotient ring of a product is the product of the quotient rings [10 p. 100] and 1.2.

§ 2. In this section, R always denotes a commutative integral domain with quotient field Q. We consider the following types of overrings, each more general than the preceding one.

(QR) $S = R_\Sigma$, the ring of fractions relative to a multiplicatively closed subset Σ of R. ($R_\Sigma = \Sigma^{-1}R$).

(F) S is a flat epimorphic extension of R.

(QQR) S is the intersection of rings of type (QR), i.e.

$$S = \bigcap R_{\Sigma_\lambda}$$

(L) $S = R_{\mathcal{G}}$, a quotient overring.

(GQR) $S = R_{\mathcal{S}}$, $\mathcal{S}$ a multiplicatively closed set of ideals or R.

2.1. REMARKS.

(a) Rings of type (GQR) - generalized quotient rings - are defined as follows : $\mathcal{S}$ is a set of ideals of R such that $K,L \in \mathcal{S}$ implies $KL \in \mathcal{S}$. $R_{\mathcal{S}}$ is the set of all $q \in Q$ such that there is a $K \in \mathcal{S}$ with $qK \subseteq R$.

(See [2] and [7] for results and further literature on overrings of this type).

(b) The implication (QR) $\Rightarrow$ (F) is well -known.

(c) S is a flat epimorphic extension of R if and only if S is a flat overring (using the argument of 1.4. c.). It is know, that S is a flat overring if and only if S is the intersection of localizations $R_{P \cap R}$, where P runs through the prime ideals of S. [12, p. 91]. Thus (F) $\Rightarrow$ (QQR). (See [2] and [5] for results concerning overrings of type (QQR)).

(d) (QQR) $\Rightarrow$ (L) : Every ring of fractions R_{Σ_λ} is a also a quotient overring $R_{\mathcal{F}_\lambda}$ and $\cap R_{\mathcal{F}_\lambda} = R_{\mathcal{F}}$, where $\mathcal{F} = \cap \mathcal{F}_\lambda$.

(e) (L) $\Rightarrow$ (GQR) since every idempotent filter is multiplicatively closed.

(f) As we shall see below, none of the above implication is reversible.

2.2 LEMMA. - *If* R *is noetherian, then the classes* (QQR), (L) *and* (GQR) *coïncide.*

Proof. One has to show, that an overring of type (GQR) is of type (L), and that one type (L) is of type (QQR). Let $\mathcal{S}$ be any multiplicatively closed set of ideals of R, then $\mathcal{F}$, the set of all ideals containing an ideal from $\mathcal{S}$ is closed under intersection and is multiplicative. Since R is noetherian, a well-know result (see e.g. [19, 1.22]) shows, that $\mathcal{F}$ is an idempotent filter. Since $R_{\mathcal{F}} = R_{\mathcal{S}}$, the classes (GQR) and (L) coïncide.

If now $\mathcal{F}$ is an arbitrary idempotent filter, then there exists an injective module E such that $\mathcal{F} = \{ I \mid \mathrm{Hom}_R(R/I,E) = 0 \}$. Write E as a sum of indecomposable injectives $E_\lambda = E(R/P_\lambda)$ (injective hulls), then $\mathcal{F} = \cap \mathcal{F}_\lambda$ and $R_{\mathcal{F}} = \cap R_{\mathcal{F}_\lambda}$. Since $R_{\mathcal{F}_\lambda}$ is just the localization R_P, R is of type (QQR).

Thus the classes (L) and (QQR) coïncide as well.

Using different methods, the equality of the classes (GQR) and (QQR) can be extended to certain non-noetherian domains (see [8] and a remark in [7]).

We shall say, that R is a QR-domain if every overring of R is of type (QR) and similarly for the other types. Thus one has the implications :

QR-domain $\Rightarrow$ F-domain $\Rightarrow$ QQR-domain $\Rightarrow$ L-domain $\Rightarrow$ GQR-domain.

By 1.1, the F-domains are just the Prüfer domains.

For noetherian domains, most of the classes coincide :

2.3 PROPOSITION. - *Let* R *be a noetherian domain.*

(a) R *is a* QR-*domain if and only if* R *is a Dedekind domain such that a power of every ideal is principal.*

(b) R *is an* F-*domain if and only if* R *is a Dedekind domain.*

(c) R *is a* GQR-*domain if and only if* R *is a Dedekind domain.*

Proof. For (a) see [6], and for (c) see [7]. (b) follows from the remark above.

We now quote four counter-examples, which show, that none of the implications above is reversible. This also validates the claim made in 2.1. f.

(1) A QR-domain, which is not an F-domain : It suffices to take a Dedekind domain, which does not satisfy 2.3. a. For an example, see e.g. [6].

(2) In [5] there is an example of a QQR-domain, which is not a Prüfer domain hence not an F-domain.

(3) An L-domain, which is not a QQR-domain : In [2.5. 11], an integral domain R (called D in the refrrence quoted) is constructed whith the following properties : R has only one overring S different from R and Q.R has only one non-zero prime ideal M and $M^2 = M$. Furthermore $S = R_{\mathscr{S}}$, where $\mathscr{S}$ is the set consisting of M alone. Since $M^2 = M$, however, $\mathscr{F} = \{M, R\}$ is a idempotent filter and $S = R_{\mathscr{S}} = R_{\mathscr{F}}$.

Thus every overring of R is of type (L), i.e. R is an L-domain. On the other hand, it is not hard to see, that every overring of type (QR) is an intersection of localizations at prime ideals (in fact $R_\Sigma = \cap R_{P_\lambda}$, where P_λ are the ideals of R maximal with respect to not meeting Σ), and therefore every overring of type (QQR) is also an intersection of such localizations. It follows, that S is not of type (QQR) ; R is not a QQR-domain.

(4) In [7,2.9] , there is an example of a GQR-domain having an overring (in fact a localization) which is not a GQR-domain. Since, by 1.4.f, an overring of an L-domain is again an L-domain, we obtain an example of a GQR-domain, which is not an L-domain.

2.4. PROPOSITION. - *For an integrally closed domain, the following conditions are equivalent :*

(a) R *is a Prüfer domain,*

(b) R *is a* QQR-*domain,*

(c) R *is an* L-*domain,*

(d) R *is a* GQR-*domain.*

Proof. The implications (a) ⇒ (b) ⇒ (c) ⇒ (d) have already been noted. (d) ⇒ (a) is in [7,2.4] . A different proof (c) ⇒ (a) is also available : By [19,2.7] any quotient overring of an integrally closed domain is integrally closed, and 1.1 may be applied. (In fact, this proof also works for (GQR)).

Since an overring of an L-domain is an L-domain 1.4.f., we obtain

2.5. COROLLARY . - *The integral closure of an* L-*domain is a Prüfer domain.*

An analoguous result holds for QQR-domains [5,1.7] and is conjectured for GQR-domains [7].

2.6. PROPOSITION. - R *is an* L-*domain if and only of* R_M *is an* L-*domain ofor every maximal ideal* M *of* R.

Proof. If R is an L-domain, then so is its overring R_M. To prove the converse, we first show, that the assumption implies, that every local overring S of R is a quotient overring. Let N be the maximal ideal of S, then $P = N \cap R$ is a prime ideal of R and if M denotes a maximal ideal of R containing P, then $R_M \subsetneq R_P \subseteq S$. By assumption, S is a quotient overring of R_M, hence of R, according to the lemma below. If T is now an arbitrary overring, then T is equal to the intersection of its localizations at the maximal ideals [1, chap. II, §3, n° 3], and since each localization is a quotient overring, so is T. Thus R is an L-domain.

2.7. LEMMA. - *Let* $S \subseteq T$ *be overrings or* R, *let* $R \subseteq S$ *be a flat epimorphic extension and let* T *be a quotient overring of* S. *Then* T *is a quotient overring of* R.

Proof. Let $\mathcal{F}$ be an idempotent filter of S such that $T = S_{\mathcal{F}}$. Following [13] we define an idempotent filter $\mathcal{G}$ on R : $\mathcal{G}$ is the set of all ideals I of R such that $IS \in \mathcal{F}$. We claim, that $R_{\mathcal{G}} = S_{\mathcal{F}}$. If $q \in R_{\mathcal{G}}$, then $qG \subseteq R$ for some G in $\mathcal{G}$, but then $qGS \subseteq S$ and since $GS \in \mathcal{F}$, we have $q \in S_{\mathcal{F}}$.

Conversely, assume, that $x \in S_{\mathcal{F}}$. Then there is an $F \in \mathcal{F}$ such that $xF \subseteq S$, and $xy \in S$ for all $y \in F$. Since $R \subseteq S$ is a flat epimorphism, the ideals $(xy)^{-1}R$ and $y^{-1}R$ belong both to the filter $\mathcal{E}$, consisting of all ideals K of R with $KS = S$. Thus $((xy)^{-1}R \cap y^{-1}R)S = S$, and we can find $a_1,\dots,a_n \in R$, $s_1,\dots,s_n \in S$ such that $xya_k \in R$, $ya_k \in R$ for $k = 1,\dots,n$ and $\Sigma a_i s_i = 1$. Let J_y be the ideal of R generated by $ya_1,\dots,ya_n$ Then $y \in J_yS$ and if J denotes the sum of the ideals J_y for all $y \in F$, then $F \subseteq JS$, thus $J \in \mathcal{G}$. Since $xJ \subseteq R$, it follows, that $x \in R$.

§ 3. There is another type of overring, which is of some interest : We will say, that an overring S of R is an *epimorphic overring*, if the inclusion map $R \subseteq S$ is a ring epimorphism. (Notice, that a general epimorphic extension of R need not be an overring of R, take e.g. the epimorphism $\mathbb{Z} \to \mathbb{Q} \times \mathbb{Z}_2$).

In [16] , it has been shown, that the integral domain R is a Prüfer domain if and only if every overring is epimorphic. The existence of L-domains which are not Prüfer domains shows, that not every quotient overring is epimorphic. Conversely, not every epimorphic overring need be a quotient overring. We shall give a counter-example, in which, however, R is not an integral domain, but only a commutative semiprime ring.

Let X be the real numbers with the discrete topology, X^* its one-point (Alexandroff) compactification and R the ring of continuous real-valued functions on X^*. Q can be described as the product ring $\mathbb{R}^X$. Let S be the subring of Q consisting of all those functions $f : X \to \mathbb{R}$ which are constant $(f(\xi) = \gamma)$ except for at most countably many values of ξ. R is then t e subring of S consisting of all those functions in S satisfying the additional property, that for any $\varepsilon > 0$ $|f(\xi) - \gamma| \leqslant \varepsilon$ for all but finitely many values of ξ .

This example has already been used in [15, 11.6] and it was shown, that $R \subseteq S$ is an epimorphic extension (in fact the largest epimorphic extension of R which is an overring). It remains to show, that S is not a quotient overring.

Let Y be an uncountable subset of X whose complement is also uncountable. Let $q \in Q \setminus S$ be defined by $q(\xi) = 1$ if $\xi \in Y$ and $q(\xi) = 0$ otherwise. Further, let Y' be a countable subset of Y and let $s \in S$ be defined by $s(\xi) = 1$ if $\xi \in Y'$ and $s(\xi) = 0$ otherwise. Then one shows, that if $sr \in R$ for some $r \in R$, then the constant value γ which r takes on almost everywhere has to be 0, and from this it follows, that $qr \in R \subseteq S$. Thus 1.2 is not satisfied and S is an epimorphic overring which is not a quotient overring.

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