PUBLICATIONS DU DÉPARTEMENT DE MATHÉMATIQUES DE LYON

MANABU HARADA

Note on Categories of Indecomposable Modules

Publications du Département de Mathématiques de Lyon, 1972, tome 9, fascicule 4
, p. 11-25

<http://www.numdam.org/item?id=PDML_1972__9_4_11_0>

Publications du
Département de
Mathématiques
Lyon 1972 t.9-4

NOTE ON CATEGORIES OF INDECOMPOSABLE MODULES

by Manabu HARADA

Let R be a ring with identity and M a unitary right R-module wich is a directsum of indecomposable, injective modules. E. Matlis [13] posed the following question : for any direct summand L of M, is L also a directsum of indecomposable injective modules ? Recently, U.S. Kahlon [9] and K. Yamagata [16] studied this problem under an assumption that the singular submodule of L is equal to zero.

In this short note, we shall show that if the singular submodule of L is equal to zero, then the affirmative answer of Matlis' problem is an immediate consequence from [6] and [10]. Especially, in the section 4, we shall give simpler proofs of generalized Kahlon' results [9]. In sections 2 and 3, we shall give some supplementary results of [7] and [8] .

I. DEFINITIONS

Let R be a ring with identity. We assume that all modules in this note are unitary right R-modules. Let M be an R-module. If $\mathrm{End}_R(M) = S_M$ is a local ring (the Jacobson radical is a unique maximal ideal among left an right ideals), M is called *completely indecomposable*.

Let $\mathcal{A}$ be the induced full sub-category from all completely indecomposable modules M_α in the category of right R-modules $\mathcal{M}_R$, namely every object in $\mathcal{A}$ is a direct sum of some family of $\{M_\alpha\}$ (see [6], § 3). Let M^1 , M^2 be objects in $\mathcal{A}$ and $M = \Sigma_{I^\alpha} \oplus M_\alpha^i$; $M_\alpha^i \in \{M_\alpha\}$. We put $[M^1, M^2] \cap \mathfrak{J}' = \{f| \in \mathrm{Hom}_R(M^1, M^2),\ p_\beta f i_\beta^1 : M_\beta^1 \to M_\beta^2$

is non-isomorphic for all $\beta \in I^1$, $\beta' \in I^2$, where $i^1_{\beta'}: M^1_{\beta'} \to M^1$ is the injection and $p^2_\beta: M^2 \to M^2_\beta$ is the projection}. Then $\mathcal{J}'$ is an ideal in $\mathcal{A}$ and $\mathcal{A}/\mathcal{J}'$ is a completely reducible C_3-abelian category by [6], Theorem 7. If $M^1 = M^2$, we denote $[M^1, M^1] \cap \mathcal{J}'$ by $\mathcal{J}'$.

Let A and f be an object and a morphism in $\mathcal{A}$, respectively. By $\bar{A}$ and $\bar{f}$ we denote the residue classes of A and f in $\mathcal{A}/\mathcal{J}'$. Let $A \supset B$ be in $\mathcal{A}$ and i the inclusion of B to A. If $\bar{i}$ is isomorphic in $\mathcal{A}/\mathcal{J}'$, we say B is a *dense submodule* of A (see [7] ,p. 310-311). We assume $A = C \oplus D$ as R-modules and let e the projection of A onto C. By $\bar{C}$ we denote Im $\bar{e}$ in $\mathcal{A}/\mathcal{J}'$, even though C is not in $\mathcal{A}$. Next, we assume that $A \supset B$ are in $\mathcal{M}_R$ and $B = \sum_K \oplus T_\alpha$ as R-modules. If $\sum_{K'} \oplus T_{\alpha'}$ is a direct summand of A for any finite subset K' of K, then we say that B is a *finitely direct summand* of A (with respect to the decomposition $\sum_K \oplus T_\alpha$). It is clear that every directsum of injective modules is a finitely direct summand of its extension module.

We summarize here definitions of the exchange property given in [6], [7], [8] and [10].

Let $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ be sets of completely indecomposable modules. We put $M = \sum_I \oplus M_\alpha$ we recall Condition II given in [6] , §3.

II (Take out). *For any subset I' of I and any other decomposition $M = \sum_J \oplus N_\beta$, there exists a subset $\{N_{\phi(\gamma)}\}_{\gamma \in I'}$, of $\{N_\beta\}_J$ such that $M_\gamma \approx N_{\phi(\gamma)}$ for all $\gamma \in I'$ and $M = \sum_{I'} \oplus N_{\phi(\gamma)} \oplus \sum_{\alpha \in I - I'} \oplus M_\alpha$.*

II' (Put in). *For the same assumption as above, there exists a subset $\{N_{\psi(\gamma)}\}_{\gamma \in I'}$ such that $M_\gamma \approx N_{\psi(\gamma)}$ for all $\gamma \in I'$ and $M = \sum_{\alpha \in I'} \oplus M_\alpha \oplus \sum_{\beta \in J - \psi(I')} \oplus N_\beta$ where ϕ and ψ are one-to-one mappings of I' into J.*

If we replace the subset I' by $I-I'$, then II and II' are equivalent by Azumaya's theorem [1]. Furthermore, Azumaya [1] showed that II and II' are satisfied for any finite subset I' of I.

We remark that if a given decomposition $M = \sum_I \oplus M_\alpha$ satisfies II or II', then any decomposition $M = \sum_J \oplus N_\beta$ does the same property. Because, let $M = \sum_K \oplus T_\delta$ be another decomposition with T_δ indecomposable. Then there exists an automorphism σ of M such that $\sigma(N_\beta) = M_{\pi(\beta)}$ by Azumaya's theorem, where π is a one-to-one mapping of J to I. We apply II or II' for the decompositions $M = \sum \oplus M_{\pi(\beta)} = \sum_K \oplus \sigma(T_\delta)$. Then we have $M = \sum_{\gamma \in J'} \oplus \sigma(T_{\phi(\gamma)}) \oplus \sum_{\beta \in J-J'} \oplus M_{\pi(\beta)}$ or $M = \sum_{\beta \in J'} \oplus M_{\pi(\beta)} \oplus \sum_{\delta \in K-\psi(J')} \oplus \sigma(T_\delta)$. Hence, $M = \sigma^{-1}(M) = \sum_{J'} \oplus T_{\phi(\gamma)} \oplus \sum_{\beta \in J-J'} \oplus N_\beta$ or $M = \sum_{J'} \oplus N_\beta \oplus \sum_{\delta \in K-\psi(J')} \oplus T_\delta$

We note II and II' are independent for fixed two decompositions $M = \sum_I \oplus M_\alpha = \sum_J \oplus N_\beta$ and a given subset I' of I. For example, we assume there exist non-isomorphic monomorphisms f_i of M_i to M_{i+1} for all $i \in K \subseteq I$. We put $M_i' = \{m_i + f_i(m_i) \mid \in M_I \oplus M_{i+1},\ m_i \in M_i\}$. Then $M = M_1' \oplus M_2 \oplus M_3' \oplus \ldots \oplus M_0 = M_1 \oplus M_2' \oplus M_3 \oplus M_4' \oplus \ldots \oplus M_0$, where $M_0 = \sum_{\alpha \in I-K} \oplus M_\alpha$. It is clear that $N' = M_1' \oplus M_3' \oplus \ldots \oplus M_0$ has the property II for the second decomposition in the above. However, by the proof of [6], Lemma 9 we know that if N had the property II', then $\{f_i\}$ would be a locally semi-T-nilpotent system (see (see [7], § 1 for the definition). Similarly, $N = M_1 \oplus M_3 \oplus \ldots \oplus M_0$ has the property II' for the first decomposition, however if N had the property II, then $\{f_i\}$ would be a locally semi-T-nilpotent system.

We say that a direct summand T of M has the *exchange property in M* if for any decomposition $M = \sum_J \oplus U_\delta$ (U_δ are not necessarily indecomposable),

$M = T \oplus \Sigma \oplus U'_\delta$ and $U_\delta \supseteq U'_\delta$ for all δ J. Especially, if T has the above property, whenever all U_δ are indecomposable , we say T has the *exchange property in M for indecomposable modules*. We refer the reader for terminologies to [6] and [7].

2. DIRECT SUMMANDS

First we recall somme of main theorems in [7] and [10].

THEOREM 1 ([7],[10]). - *Let M be a direct sum of a family of completely indecomposable modules* $\{M_\alpha\}_I$. *Then the following statements are equivalent.*

1) M satisfies the property of "take out"

2) Every direct summand of M has the exchange property in M

3) Every direct summand of M has the exchange property in M for indecomposable modules

4) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent system*

5) $\mathcal{J}'$ *is the Jacobson radical* J *of* $S = \mathrm{End}_R(M)$

6) Every finitely direct summand M' of M such that $M' = \sum_K \oplus T_\alpha$ *is a direct summand of M for any* $[K]$ *and any family* $\{T_\alpha\}$

6') 6) is valid for any K with $K = \aleph_0$, *and*

7) 6') is valid whenever all T_α *are completely indecomposable,*

8) S/ is a regular ring (and self injective as a one sided module) and every idempotents in $S/\mathcal{J}$ *are lifted to S, where* $|K|$ *is the cardinal number of K.*

Proof. 2)$\longleftrightarrow$ 4) is proved by [8] , Corollary to Proposition 1. We note 1)$\longrightarrow$ 4) in § 1. 2)$\longrightarrow$ 3) $\longrightarrow$ 1) is clear. 4)$\longleftrightarrow$5) is proved by [10],Theorem. 6)$\longrightarrow$ 6') $\longrightarrow$ 7) is trivial. 7)$\longrightarrow$4). Let $\{M_{\alpha i}\}_1^{\infty}$ be any countable sub-family of $\{M_\alpha\}_I$ and let $\{f_i\}_1^{\infty}$ be a family of non-isomorphisms $f_i : M_i = M_{\alpha i} \to M_{i+1} = M_{\alpha i+1}$. Put $M' = \sum_1^{\infty} \oplus M_i'$, where $M_i' = \{m_i + f_i(m_i) | \in M,\ m_i \in M_i\}$. Since $\sum_1^{n} \oplus M_i' \oplus M_{n+1} = \sum_1^{n+1} \oplus M_i$, M' is a finitely direct summand of M. Hence , M' is direct summand of $\sum_1^{\infty} \oplus M_i\ (=M_0) \subseteq M$ by 7). We know from [8], Theorem 2 that M' is a dense submodule of M_0. Therefore, $M' = M_0$ by [6], lemma 7, which means that $\{M_i\}_1^{\infty}$ is a locally semi-T-nilpotent (cf. [6], the proof of Lemma 9). 4)$\longrightarrow$6). Let $M' = \sum_K \oplus T_\alpha$ be a finitely direct summand of M. We may assume from 2) that all T_α are indecomposable. Now, we consider the above modules in $\mathcal{A}/\mathcal{J}'$. Let i be the inclusion of M' to M. Since M' is a finitely direct summand of M and $\mathcal{A}/\mathcal{J}'$ is a C_3-abelian, $\bar{i}$ is the inclusion of $\bar{M}'$ into $\bar{M}$ and $\bar{M}' = \sum_K \oplus \bar{T}_\alpha$. Then $\bar{M}'$ is a coretract of $\bar{M}$ by [6], Theorem 7. $\{T_\alpha\}_K$ is a locally semi-T-nilpotent system. Therefore, i is a coretract of M by 5) (cf. [7], the proof of Proposition 2). 5)$\longleftrightarrow$8) It is clear from [6], Lemma 7 and 13 Corollary to Lemma 6.

Remark 1. In the above proof of 4)$\longrightarrow$6) we only make use of a fact that $\{T_\alpha\}_K$ is a locally semi-T-nilpotent system.

Next, we study a general type of Matlis' problem. The following theorem combines [10] and [14].

THEOREM 2. - *Let M be a directsum of completely indecomposable modules M_α ; $M_\alpha = \sum_I \oplus M_\alpha$ and $\{M_\beta\}_J$ the sub-family of countably generated R-modules M_β of $\{M_\alpha\}_I$. We assume $\{M_\gamma\}_{I-J}$ is a locally semi-T-nilpotent system. Then every direct summand of M is in $\mathcal{A}$.*

Proof. Let $M = N_1 \oplus N_2$ and $K = I-J$. Each N_i contains a dense submodule $T_i = \sum_{L_i} \oplus M'_{\gamma i}$ such that $M \approx T_1 \oplus T_2$ by [6], Theorem 1 for $i = 1,2$, where $M'_{\gamma i}$ is isomorphic to some M_α ; $\alpha \in I$. We divide L_i into two partitions $L_i = J_i \cup K_i$ such that for $\gamma_i \in J_i$ (resp. K_i) $M'_{\gamma i}$ is isomorphic to some $M_\alpha ; \alpha \in J$ (resp. K) . Since $\{M'_{\gamma\, i}\}_{K_i}$ is locally semi-T-nilpotent, $T'_i = \sum_{K_i} \oplus M'_{\gamma i}$ is a direct summand of N_i by Remark 1 and [7] , proposition 2, say $N_i = T'_i \oplus N'_i$ Furthermore, $T'_1 \oplus T'_2$ has the exchange property in M by [7], Theorem 2. Hence, $N'_1 \oplus N'_2 \approx \sum_{J'} \oplus M_\beta \oplus \sum_{K'} \oplus M_\gamma$. We consider those modules in $\mathscr{A}/\mathscr{J}'$. Then $\bar{M} = \bar{T}'_1 \oplus \bar{T}'_2 \oplus \sum_{J_1} \oplus \bar{M}'_{\gamma i} \oplus \sum_{J_2} \oplus \bar{M}'_{\gamma i}$ .On the other hand, $\bar{M} = \bar{N}_1 \oplus \bar{N}_2$ $= \bar{T}'_1 \oplus \bar{N}'_1 \oplus \bar{T}'_2 \oplus \bar{N}'_2 = \bar{T}'_1 \oplus \bar{T}'_2 \oplus \sum_{J'} \oplus \bar{M}'_\beta \oplus \sum_{K'} \oplus \bar{M}'_\gamma$, where $M'_\delta \approx M_\delta$. Hence $\sum_{J_1} \oplus \bar{M}'_{\gamma i} \oplus \sum_{J_2} \oplus \bar{M}'_{\gamma i} \approx \sum_{J'} \oplus \bar{M}'_\beta \oplus \sum_{K'} \oplus \bar{M}'_\gamma$. Since all $M'_{\gamma i}$ in the left side are countably generated, $K' = \emptyset$ by [6], Theorem 7. Therefore, N'_1 is in $\mathscr{A}$ by [14] or [7], Proposition 3. We have completed the proof.

In Theorem 2 if N_1 is injective, NN_1 is in $\mathscr{A}$ by [4], [9] or [15] without any assumption. Similarly

PROPOSITION 1. - *Let M be in $\mathscr{A}$ and N a direct summand of M. If N is projective, N is in $\mathscr{A}$.*

Proof. By [11], Theorem 1, N is a directsum of countably generated R-submodules P_α . Furthermore, P_α is in $\mathscr{A}$ by [7], Proposition 3. Hence, N is in $\mathscr{A}$.

The following corollary was given with an assumption that $J(P)$ is small in P by [12], Theorem 5.5 and [7], Proposition 5.

COROLLARY. - *Let M be in $\mathcal{A}$ and R-projective. Then every direct summand P of M is in $\mathcal{A}$.*

It is also clear.

We give a property of dense submodules.

PROPOSITION 2. - *Let M be in $\mathcal{A}$ and N a direct summand of M. Then there exists a submodule N' of satisfying the following properties :*
1) N' is in $\mathcal{A}$, 2) N' is a finitely direct summand of N and 3) If $T = \sum_J \oplus T_\alpha$ is finitely direct summand of N, T is isomorphic to a direct summand of N', where T_α's are indecomposable.
Especially, every countable generated R-submodule of N is isomorphic to a submodule of N'. Every submodule N'' of N satisfying 1) and 2) is isomorphic to a direct summand of any dense submodule of N.

Proof. Let N' be a dense submodule of N. Then N' satisfies 1) and 2) by [7], Proposition 2. Let e and e_α be projections of M to N and T_α with respect to given decompositions. Then $ee_\alpha = e_\alpha$ for all $\alpha \in J$. We consider those modules in $\mathcal{A}/\mathfrak{J}$. Since T is a finitely direct summand of $N (\subseteq M)$, $\bar{T} = \sum_J \oplus \bar{T}_\alpha \subsetneq Im\, \bar{e} = \bar{N}'$. Hence, T is isomorphic to a direct summand of N' by [6], Theorem 7. We easily see that for two finitely generated submodules $T_1 \supseteq T_2$ in N, we can find a direct summand T_i' of N such that $T_i' \supseteq T_i$, $T_1' \supseteq T_2'$ and T_i' is a finite directsum of indecomposable modules for $i = 1,2$ (cf. [7], the proof of Proposition 3). Since $T_1' \supseteq T_2'$, we can find a monomorphism of T_1 to N' which is an extension of a given monomorphism of T_2 to N' by [6], Theorem 7.

Hence, every countably generated R-submodule of N is isomorphic to a submodule of N' by the standard argument. Let N'' be a submodule satisfying 1) and 2). Then $\bar{N}'' \subseteq \bar{N}'$. Hence, the last statement is clear from [6], theorem 7.

3. EXCHANGE PROPERTY

It seems to the author that the difficulty of the exchange property in M comes from the following facts. Let M be in $\mathscr{A}$ and $M = N_1 \oplus N_2 \oplus N_3$ as R-modules. It is well known from [2] that if N_1 and N_2 have the exchange property in M, then so is $N_1 \oplus N_2$, however the converse is not true. Furthermore, even if neither N_1 nor N_2 has the exchange property in M, it is possible that $N_1 \oplus N_2$ does.

We note that if a direct summand L of M has the exchange property, then L is in $\mathscr{A}$. The following theorem is a slight generalization of some parts in Theorem 1.

THEOREM 3. - *Let M be in $\mathscr{A}$ and $M = N_1 \oplus N_2$. Let f be the projection of M Onto N_1. Then $f\mathcal{J}'f = f\mathcal{J}f$ if and only if every direct summand of N_1 has the exchange property in M. In the case N_2 also has the exchange property in M, where $\mathcal{J}'$ is the ideal defined in §1 and $\mathcal{J}$ is the Jacobson radical of $S = \mathrm{End}_R(M)$.*

Proof. "Only if" . Let $M = \sum_I \oplus M_\alpha$ and M'_α ' s are completely indecomposable. We can find a subset J of I such that $\bar{M}_J = \sum_J \oplus \bar{M}_\alpha \approx \mathrm{Im}\, \bar{f}$ in $\mathscr{A}/\mathcal{J}'$ by [6], Theorem 7. Let e be the projection of M to M_J . Then $fS/f\mathcal{J}' \approx eS/e\mathcal{J}'$.

Hence, there exist $a \in eSf$, $b \in fSe$ such that $ba \equiv f$ (mod $\mathfrak{J}'$). Put $f\text{-}ba = n \in \mathfrak{J}'$, then $n \in f\mathfrak{J}'f = f\mathfrak{J}f$, which tis the radical of $S_{N_1} = \mathrm{End}_R(N_1)$. Hence, ba is an automorphism of fS as an S-module. Therefore, $eS = f_1S \oplus f_2S$ and $f_1S \overset{a}{\approx} fS$, $f_2S = \mathrm{Ker}\ b$ and $f_i^2 = f_i$. Since b induces $eS/e\mathfrak{J}' \approx fS/f\mathfrak{J}'$, $f_2S = f_2\mathfrak{J}' \subsetneq \mathfrak{J}'$. Hence, $f_2 = 0$ by [1], Theorem 1 or [6], Lemma 7 and $eS \approx fS$, which implies $N_1 \approx M_J$. Therefore, $\{M_\alpha\}_J$ is a locally semi-T-nilpotent system by Theorem 1. Thus, we have poved "only if" from [8], Corollary to Theorem 2. "if". $N_1 = \sum_K \oplus M'_\gamma$ and $\{M'_\gamma\}$ is a locally semi-T-nilpotent system by [8], Corollary to Theorem 2. Hence, $f\mathfrak{J}'f = f\mathfrak{J}f$ by Theorem 1 and [6], Lemma 5. The remaining part is clear from [8], Theorem 2.

COROLLARY . - *Let M and N_1 be as above. If for every monomorphism g in S_{N_1}. Im g is a direct summand of N (i.e. $gS_{N_1} = eS_{N_1}$, $e^2 = e$), then N_2 and every direct summand of N_1 have the exchange property in M. eEspecially, if N_1 is quasi-injective, N_i has the exchange property in M for $i = 1,2$, (cf. [4]).*

Proof. Let $M = N_1 \oplus N_2$ and f be the projection of M to N_1. We take any element a in $f\mathfrak{J}'f$. Then Ker $(1-a) = 0$ by [1], Theorem 2 and Im $(1-a) =$ Im$((1-a)|N_1) \oplus N_2$. Since Im $((1-a)|N_1)$ is a direct summand of N_1 by the assumption, Im$(1-a)$ is a direct summand of M. On the other hand, Im $(1-a)$ is a dense submodule of M by [7], Theorem 2 and hence, $M =$ Im $(1-a)$. Therefore, $f\text{-}a$ is an automorphism of N_1 ,which implies $f\mathfrak{J}'f = f\mathfrak{J}f$.Hence, N_i has the exchange property by the theorem. The remaining part of the corollary is immediate from the above.

In Theorem 4 below, we shall show the converse of Corollary in a special case.

Remark 2. [6], Proposition 10 and [9], Theorem I are special cases of Corollary I.

It is shown in [8], Pemark in p. 52 that the exchange property does not imply the locally semi-T-nilpotency. In a special case we have

PROPOSITION 3. *Let* $\{M_i\}^{\infty}_1$ *be a set of completely indecomposable modules such that* M_i *is monomorphic, but not isomorphic to* M_{i+1} *(cf. [6], p. 340 and [8], Corollary 3). 1) Let* $M = \sum_1^{\infty} \oplus M_i = N_1 \oplus N_2$. *Then* N_1 *has the exchange property in* M *if and only if either* N_1 *or* N_2 *is a directsum of indecomposable modules* $\{M_i'\}$ *which is a semi-T-nilpotent system, (in this case, a finite directsum of* M_i'*). 2) We further assume that each* M_i *itself is a locally T-nilpotent system and* $M = \Sigma \oplus M'_{\alpha}$; $M'_{\alpha} \approx M_i$ *for some* i *and* $M = N_1 \oplus N_2$. *Then we have the same statement in 1).*

Proof. 1) "If part" is clear from [8], Theorem 2. We assume that N_1 has the exchange property. Then N_i is in $\mathcal{A}$: say $N_i = \sum_{K \in J^i} \oplus T^i_k$, where $T^i_k \approx M_m$ for some m. if J^i vere infjnite for $i = 1, 2$, we vould have a contradiction from the assumption and [8], Lemma 2. 2) We can prove it similarly to 1).

PROPOSITION 4. - *Let* $M = \sum_I \oplus M_{\alpha}$ *and* M_{α} *be isomorphic to a completely indecomposable module* M_1 *for all* $\alpha \in I$. *Let* $M = N_1 \oplus N_2$. *Then* N_1 *has the exchange property in* M *if and only if* M *itself is a locally T-nilpotent system or either* N_1 *or* N_2 *is isomorphic to a finite directsum of* M_1.

Proof. It is clear from [8], Lemma 2.

COROLLARY. - *Let P be a completely indecomposable and projective module and $M = \sum_I \oplus P_\alpha$; $P_\alpha \approx P$. Let $M = N_1 \oplus N_2$. Then N_1 has the exchange property in M if and only if either N_1 or N_2 is semi-perfect or equivalentely $J(N_i)$ is small in N_I for $i = 1$ or 2.*

Proof. It is clear from Proposition 4 and [7], Theorem 7.

4. MODULES WITH ZERO SINGULAR SUBMODULES.

In this section, we study Matlis' problem and give simpler proofs of slightly generalized results of [9], Theorems 2 and 3.

Let N be an R-module. We denote the singular submodule of N by $Z(N)$, namely $Z(N) = \{n | \in N,\ (o:n) \text{ is large in } R\}$. The following lemma is well known and essential in this section.

LEMMA. - *Let $\{N_\alpha\}_I$ be a set of indecomposable injective modules. If $Z(N_{\alpha'}) = 0$ any homomorphism of N_α to $N_{\alpha'}$ is either isomorphic or zero. Especially, if $Z(N_\alpha) = 0$ for all $\alpha \in I$, $\{N_\alpha\}$ is a locally T-nilpotent system.*

Let M be in $\mathfrak{A}$ and $M = N_1 \oplus N_2$. Then we note that N_1 always contains direct summands which are finite directsums of indecomposable modules (cf. [6], Corollary 1 in p. 334).

PROPOSITION 5. - *Let M be in $\mathfrak{A}$ and $M = N_1 \oplus N_2$. We assume that for any decomposable direct summand T_1, T_2 of N_1 non-zero elements in $\mathrm{Hom}_R(T_1, T_2)$ are always isomorphic. Then N_i is in $\mathfrak{A}$ for $i = 1, 2$.*

Proof. Let $\sum_J \oplus T_\beta$ be a dense submodule of N_1, where T_β 's are indecomposable. Every T_β is a direct summand of N_1 by [6], Proposition 2. Hence $\{T_\beta\}_J$ is a locally T-nilpotent system by the assumption. Therefore, $N_1 = \sum_J \oplus T_\beta$ and N_2 is in $\mathcal{A}$ by Remark 1, [7], Proposition 2 and [8], Theorem 2.

COROLLARY. - ([16], Theorem 4). *Let $\{M_\alpha\}_I$ be a family of indecomposable injective modules and $M = \sum_I \oplus M_\alpha$. If $M = N_1 \oplus N_2$ and $Z(N_1) = 0$, then N_i is in $\mathcal{A}$ for $i = 1,2$ (cf. [9], Theorem 3).*

Proof. It is clear from Lemma and Proposition 5.

THEOREM 4. - *Let $\{E_\alpha\}_I$ be a family of injective and indecomposable modules and $E = \sum_I \oplus E_\alpha$. Then the following statements are equivalent.*

1) $\{E_\alpha\}_I$ is a locally semi-T-nilpotent system

2) Any extension module M, in $\mathcal{A}$, of E contains E as a direct summand

3) There are no proper and essential extensions, in $\mathcal{A}$, of E and

4) $\mathrm{Im}\, g$ is a direct summand of E for any monomorphism g in S_E.

Proof. 4) $\longrightarrow$ 1) is proved by Theorem 1 and Corollary 1 to Theorem 3. 1) $\longrightarrow$ 4) by the assumption and Theorem 1 $\mathrm{Im}\, g = \sum_K \oplus E_\beta$; $E'_\beta \approx E_\alpha$ for some α. Since E'_β is injective, $\mathrm{Im}\, g$ is a finitely direct summand of M. On the other hand, $\{E'_\beta\}_K$ is a locally semi-T-nilpotent system by Theorem 1. Hence, $\mathrm{Im}\, g$ is a direct summand of M from Remark 1. 1) $\longrightarrow$ 2) is clear from Theorem 1. 2) $\longrightarrow$ 3) is trivial. 3) $\longrightarrow$ 1) We assume that $\{E_\alpha\}_I$ is not locally semi-T-nilpotent.

Then we have a sub-family $\{E_i\}_1^{\infty}$ of $\{E_\alpha\}_I$ such that there exist non-isomorphisms $f_i : E_i \longrightarrow E_{i+1}$ and an element x in E with a property : $f_n f_{n-1} \cdots f_1(x) \neq 0$ for all n. We note Ker $f_i \neq 0$ for all i, since E_i is injective an indecomposable. Put $E_i' = \{x_i + f_i(x_i) | \in E_i \oplus E_{i+1}\} \subseteq \sum_1^{\infty} \oplus E_j$ and $E = \sum_1 \oplus E_j \oplus E_o$. Then $E_i \cap (\sum_j \oplus E_j') \supseteq \text{Ker } f_i \neq 0$. Hence, $\sum_j \oplus E_j' \oplus E_o$ is essential in E. It is clear $x \in E - (\sum_j \oplus E_j' \oplus E_o)$. Let E^* be an injective hull of E. Then we can extend an isomorphism ϕ of $\sum_j \oplus E_j' \oplus E_o$ onto E to a monomorphism of E^*. Hence $\phi(\sum_j \oplus E_j' \oplus E_o) = E \subsetneq \phi(E) = \sum_I \oplus \phi(E_\alpha) \in \mathcal{A}$.

COROLLARY 1. - *We assume further in Theorem 4 that all E_α are noetherian. Then we obtain all of 1) $\sim$ 4) in Theorem 4.*

Proof. Let E_1, E_2 and E_3 be injective, indecomposable and noetherian modules, and $f_i : E_i \longrightarrow E_{i+1}$ non-isomorphisme. Then Ker $f_i \neq 0$ and Im $f_1 \cap$ Ker $f_2 \neq 0$ if $f_1 \neq 0$, since E_2 is uniform. Hence, Ker $f_1 \subsetneq$ Ker $f_2 f_1$ if $f_1 \neq 0$. Therefore, $\{E_\alpha\}_I$ is a T-nilpotent system.

COROLLARY 2. *Let M be in $\mathcal{A}$ and L a submodule of M. We assume that* $Z(L) = 0$ *and L is a directsum of injective modules. Then L is a direct summand of M* (cf. [9] , Theorem 2).

Proof. Since every injective submodule of M is in $\mathcal{A}$ by Corollary to Theorem 3. L is a direct summand of M by Lemma and Theorem 4.

Remark 3. Let $\{E_\alpha\}_I$ be a family of indecomposable, injective modules. In general, $\{E_\alpha\}_I$ is not locally semi-T-nilpotent and hence, $\sum_I \oplus E_\alpha$ is not quasi-injective. Furthermore, even if all E_α is not injective. If either $E = \sum_I \oplus E_\alpha$ is (quasi-)injective or $Z(E) = 0$, $\{E_\alpha\}_I$ is a locally semi-T-nilpotent system. However, the converse is not true as follows. Let K be a commutative, local Frobenius ring with $Z(K) \neq 0, M_{21} = \sum_1^\infty \oplus K_i$; $K_i \approx K$, $M_{32} = \mathrm{Hom}_K(M_{21}, K) = \pi K_i$ and $M_{31} = K$. Then

$$R = \begin{pmatrix} K & & 0 \\ M_{21} & K & \\ M_{31} & M_{32} & K \end{pmatrix} = T(K,K,K;M_{ij})$$

is a ring (cf. [5], p. 23). Put $e_1 = T(1,0,0;0)$ and $e_3 = T(0,0,1;0)$, then $\mathrm{Hom}_K(Re_1, K) \approx e_3R$ is R-injective. Since $e_3Re_3 = K$ is local, e_3R is indecomposable and e_3R itself is a T-nilpotent system. It is clear that $Z(e_3R) \supseteq (Z(K),0,0) \neq 0$. Put $S_i = (M_{31}, \prod_{j=i}^\infty K_j, K) \subseteq e_3R$. Then $(0:S_1)_R \subsetneqq (0:S_j)_R$ if $i<j$. Hence, $\sum_1^\infty \oplus e_3R$ is not injective by [3]. However, I do not know whether $\sum_1^\infty \oplus e_3R$ is quasi-injective or not. If we can construct a self-injective and perfect, but not Σ-injective ring S (or a self-injective and local, but not Σ-injective ring with $Z(S) = 0$), then $\sum_I \oplus S$ is not quasi-injective but $\{S\}_I$ is a T-nilpotent system.

REFERENCES.

[1] G. AZUMAYA, *Correction and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem,* Nagoya Math. J. 1 (1950), p. 117-124.

[2] P. CRAWLY and B. JONNSON , *Refinements for infinite direct decomposition of algebraic systems,* Pacific J. Math. 14 (1964), p. 797-855.

[3] C. FAITH, *Rings with ascending condition on annihilators*, Nagoya Math. J. 27 (1966), 179-191.

[4] L. FUCHS, *On quasi-injective modules*, Annali della Scuola Norm. Sup. Pisa, 23 (1969), p. 541-546.

[5] M. HARADA, *QF-3 and semi-primary PP-rings* II, Osaka J. Math. 3 (1966) p. 21-27.

[6] M. HARADA and Y. SAI, *On categories of indecomposable modules* I, ibid 7 (1970), p. 323-344.

[7] M. HARADA, *On aategories of indecomposable modules* II, ibid. 8 (1971) p. 309-321.

[8] M. HARADA, *Supplementary remarks on categories of indecomposable modules*, ibid. 9 (1972), p. 49-55.

[9] U.S. KAHLON, *Problem of Krull-Schmidt-Remak -Azumaya-Matlis*, J. Indian Math. Soc. 35 (1971), p. 255-261.

[10] H. KANBARA, *Note on Krull-Remak-Schmidt-Azumaya's theorem*, Osaka J. Math. 9 (1972).

[11] I. KAPLANSKY, *Projective modules*, Ann of Math. 68 (1958), p. 372-377.

[12] E. MARES, *Semi-perfect modules*, Math. Z. 83 (1963), p. 347-360.

[13] E. MATLIS, *Injective modules over noetherian rings*, Pacific J. Math. 8 (1958), p. 511-528.

[14] R.B. WARFIELD Jr, *A Krull-Schmidt theorem for infinite sums of modules*, Proc. Amer. Math. Soc. 22 (1969), p. 460-465.

[15] R.B. WARFIELD Jr, *Decomposition of injective modules*, Pacific J. Math. 31 (1969), p. 263-276.

[16] K. YAMAGATA, *Non-singular rings and Matlis' problem*, Sci. Rep. Tokyo Kyoiku Daigaku 11 (1972), p. 186-192.

Manabu HARADA
OSAKA CITY UNIVERSITY
Department of mathematics.