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THREE EXAMPLES IN THE THEORY OF SPACES OF CONTINUOUS FUNCTIONS

Richard HAYDON (*)

1. - INTRODUCTION.

This article is an expanded version of the note (H), giving the same examples in rather greater detail. The first example is of a completely regular space X in which every bounded subset is finite but such that the repletion υX contains an infinite compact subset. This enables us to construct a barrelled lcs E whose associated bornological space $\tilde{E}$ is not barrelled. The second example is of a pseudocompact space Y in which every compact subset is finite ; $C_c(Y)$ is then quasibarrelled but not barrelled.

The spaces X and Y were constructed in answer to questions posed by H. BUCHWALTER during the Summer School on topological vector spaces, Brussels, September 1972. The last paragraph is devoted to a question posed by BUCHWALTER in (B) concerning the bidual T'' of the completely regular space T.

For notations and terminology I follow (B) and (BS). In particular, I say that a locally convex space is quasibarrelled (not infrabarrelled) if every bornivorous barrel is a neighbourhood of zero. The bornological space $\tilde{E}$ associated with the lcs E has the same underlying vector space as E and, as neighbourhoods of zero, all discs that are bornivorous in E.

Two of constructions that follow involve the use of ordinals. When Ω is an ordinal, $[0,\Omega]$ is the (compact) space of all ordinals $\xi\leq\Omega$, endowed with the order topology. I denote by $\omega=\omega_0$ the first infinite ordinal, and by ω_n the first ordinal of greater cardinality than ω_{n-1}. An ordinal ξ is called *isolated* if $\xi = \eta+1$ for some η (ξ is then an isolated point of $[0,\xi]$) ; otherwise ξ is a *limit ordinal*. A limit ordinal ξ is said to be *cofinal with* ω if there is an

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increasing sequence (ζ_n) with $\zeta_n<\xi$ for all n and $\zeta_n \uparrow \xi$.

2. - ON THE BORNOLOGICAL SPACE ASSOCIATED WITH A BARRELLED SPACE.

Let W be the product space $[0,\omega_2] \times [0,\omega]$ and define the subspace X to consist of all (ξ,n) with $n<\omega$ and ξ an ordinal $<\omega_2$, not cofinal with ω, together with the additional point (ω_2,ω). Every point x of X, other than (ω_2,ω) is a P-point (that is, every G_δ containing x is a neighbourhood of x). If B is a bounded subset of X, then, for every $\xi<\omega_2$, $B\cap([0,\xi[\times [0,\omega[)$ is a bounded subset of the open and closed subspace $X\cap([0,\xi[\times [0,\omega[)$. This is a P-space and so every bounded subset of it is finite ([GJ], 4K.3). It is an immediate deduction that every bounded subset of X is finite.

I shall now show that the repletion υX is equal to the subspace $Z = X\cup(\{\omega_2\} \times [0,\omega[)$ of W. Z is replete since it is the intersection of a family of F_σ subsets of the compact space W. Moreover, X is dense in Z. So it is enough to prove that every f in C(X) has an extension g that is continuous on Z. By a standard argument, there exists, for each $n<\omega$, an ordinal $\xi_n<\omega_2$ such that $f(\eta,n) = f(\xi_n,n)$ whenever $(\eta,n)\in X$ and $\eta\geq\xi_n$. Let us put $\xi = \operatorname{Sup} \xi_n<\omega_2$ and define $g(\omega_2,n) = f(\xi+1,n)$ for all $n<\omega$. It remains only to prove that $g(\omega_2,n) \to f(\omega_2,\omega)$ as $n \to \omega$. Given $\varepsilon>0$, there are η_0 and m_0 such that $|f(\omega_2,\omega) - f(\eta,n)| \leq \varepsilon$ whenever $(\eta,n)\in X$, $\eta>\eta_0$ and $n\geq m_0$. But this means that $|f(\omega_2,\omega) - g(\omega_2,n)| \leq \varepsilon$ whenever $n\geq m_0$.

Let us put $E = C_c(X)$, which is barrelled because X is a μ-space (corollaire 8 of [B]). Now $C_c(X) = C_s(X)$ and, by théorème 5 of [BS], we know that $\widetilde{C_s(X)} = C_s(\upsilon X)$. We now see, however, that, since υX contains an infinite compact subset $\{\omega_2\} \times [0,\omega]$, $C_s(\upsilon X) \neq C_c(\upsilon X)$ and $\tilde{E} = C_s(\upsilon X)$ is not barrelled (théorème 3 of [BS]).

3. - A PSEUDOCOMPACT SPACE IN WHICH EVERY COMPACT SUBSET IS FINITE.

This example resulted from a conversation between the author and Jim SIMONS.

Let us choose, for each infinite subset A of the set $\mathbb{N}$ of natural numbers, one element of $\beta\mathbb{N}\setminus\mathbb{N}$ that is adherent to A. Call this element u_A and form the space $Y = \mathbb{N}\cup\{u_A \; ; A\subset \mathbb{N}, \text{ A infinite}\}$, topologised as a subspace of $\beta\mathbb{N}$. Then the cardinality of Y is c = card $\mathbb{R}$, since there are c infinite subsets of $\mathbb{N}$. Every infinite compact subset of $\beta\mathbb{N}$, on the other hand, has cardinality 2^c (p. 133 of [GJ]) so that the compact subsets of Y are finite.

To show that Y is pseudocompact, suppose, if possible, that $f \in C(Y)$ is unbounded. Since $\mathbb{N}$ is dense in Y, there is a sequence (n_k) in $\mathbb{N}$ with $|f(n_k)| \geq k$. Put $A = \{n_k \ ; \ k \in \mathbb{N}\}$ and note that u_A is in the $\beta\mathbb{N}$ closure of $\{n_k \ ; \ k \geq m\}$ for every m, which implies that $|f(u_A)| \geq m$ for every m, and this is absurd.

Since the compact sets of Y are finite, $C_c(Y) = C_s(Y)$, which is quasi-barrelled (théorème 2 of [BS]), while $C_c(Y)$ is not barrelled since Y is not a μ-space.

4. - ON THE BIDUAL T''.

Recall that the bidual of the completely regular space T is defined in [B] to be the space of continuous characters of the algebra $C_b(T)$, topologised as a subspace T'' of βT. The space μT is obtained by a transfinite iteration of the bidual operation, and the following problem occurs ([B], problème 2) :

Find a space T such that T'' is not a μ-space.

Again in [B] a subset P of a space R is said to be *distinguished* in R if every bounded subset of R is contained in the R-closure of some bounded subset of P. Proposition 4 of [B] shows that if T is distinguished in T'' then $T''=\mu T$, but the question of the converse implication has remained open. The following construction, reminiscent in some ways of that of §2, settles both the above questions.

Let us define some subspaces of the product

$Q = [0,\omega_1] \times [0,\omega]$:

$R = Q \setminus \{(\xi,\omega) \ ; \ \xi \text{ is a countable limit ordinal}\}$,

$S = R \setminus \{(\omega_1,\omega)\}$,

$T = R \setminus (\{\omega_1\} \times [0,\omega])$,

$U = R \setminus (\{\omega_1\} \times [0,\omega[)$.

Let $\mathcal{B}$ denote the collection of all subsets of T which are of the form

(1) $B = (([0,\omega_1[\times \{n_1,\dots,n_k\}) \cup (\{\xi_1,\dots,\xi_k\} \times [0,\omega]))$, where $n_1,\dots,n_k<\omega$ and $\xi_1,\dots,\xi_k$ are isolated ordinals $<\omega_1$.

I assert that $\mathcal{B}$ is a base for the bounded subsets of T.

Certainly each $B\in\mathcal{B}$ is bounded ; indeed each B is a pseudocompact topological space. Suppose now that C is a subset of T not contained in any member of $\mathcal{B}$. I shall show C is not bounded.

There are sequences (m_k) and (ξ_k) that increase (not necessarily both strictly) to ω and to some countable limit ordinal ξ, respectively, and such that (ξ_k,m_k) is in C for each integer k. Choose sequences (n_k), (η_k) increasing strictly to ω and ξ, respectively, and such that $n_k<m_k$, $\xi_k<\eta_k$ for all k. Then the sets $D_k = T\cap(]\eta_k, \xi]\times [n_k,\omega])$ form a locally finite family of open and closed subsets of T. So $f = \sum_{k=0}^{\infty} \chi_{D_k}$ is in $C(T)$. But $f(\xi_k,m_k) \geqslant k$ for each k, so that f is not bounded on C.

By an argument similar to one used in §2, we see that $\upsilon T=R$. Now the bidual T" is equal to $\bigcup\{\overline{B}^{\upsilon} ; B\in\mathcal{B}\}$ where $\overline{B}^{\upsilon}$ denotes the closure of B taken in υT ([B], proposition 2). Now for each B of the form (1), $\overline{B}^{\upsilon} = (([0,\omega_1] \times \{n_1,\dots,n_k\})\cup(\{\xi_1,\dots,\xi_k\} \times [0,\omega]))$, so that T" can be identified as the space S. But S is not a μ-space, since $\{\omega_1\} \times [0,\omega[$ is a non-compact bounded subset of S. Indeed $S'' = R = \mu T$.

Finally we can show in just the same way that $\{B\cup\{(\omega_1,\omega)\} ; B\in\mathcal{B}\}$ is a base for the bounded subsets of U and that $U'' = R = \mu U$. The compact subset $\{\omega_1\} \times [0,\omega]$ of U" is not contained in the U" closure of any bounded subset of U. So U is not distinguished in U".

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