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COMPACTNESS IN $C_s(T)$ AND APPLICATIONS

Richard HAYDON (*)

1. - INTRODUCTION.

In this paper I look at some properties of compact subsets of $C_s(T)$ which have applications to the "more interesting" space $C_c(T)$. A little light is cast on the difficult problem of when $C_c(T)$ may be a Kelley space, the concept of infra-k_R-space is examined, and lastly I offer two generalizations of a theorem of BUCHWALTER concerning the repletion υT.

The notations throughout are "standard Lyon". The algebra $C(T)$ of all continuous real-valued functions ont the completely regular space T may be endowed with the topology either of simple, compact or bounded convergence on T and is then denoted by $C_s(T)$, $C_c(T)$ or $C_b(T)$, respectively.

2. - ON KELLEY SPACES $C_c(T)$.

The characterization of $M(T)$ as the space $C_c(\theta T)' = M_c(\theta T)$, of all measures of compact support on the c-repletion θT, enables one to deduce ([BI] and [H_1]) that $C_c(\theta T)$ is always a Kelley space ([B_1]) and that, when T is a k_R-space, $C_c(T)$ is Kelley if and only if T is c-replete. Put into an attractively symmetric form :

$$C_c(T) \text{ is a complete Kelley space} \Longleftrightarrow T \text{ is a c-replete } k_R\text{-space.}$$

One can, however, say more, namely that, when T is a k_R-space, $C_c(\theta T)$ is the *Kelleyfié* $\bar{k}\, C_c(T)$ ([B_1]) of $C_c(T)$.

But what can we say if we do not assume T to be a k_R-space ? We can note first that the property used to prove the above results is not the full strength

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of being a k_R-space, but only that the compact discs of $C_c(T)$ should be equicontinuous. H. BUCHWALTER has introduced the definition of a property intermediate between these :

(2.1) DEFINITION. - T *is said to be an* infra-k_R-space *if every* precompact *subset of* $C_c(T)$ *is equicontinuous.*

If T is the space of (H_2), θT is infra-k_R and not k_R. Evidently, when T is an infra-k_R-space, θT is also infra-k_R and we have $C_c(\theta T) = \bar{k}\, C_c(T)$.

But this last equality does not hold for arbitrary T, as has been pointed out in (H_1). I want to consider here the problem posed at the end of that Note :

If $C_c(T)$ is Kelley, need T be c-replete ?

This question remains open still, but I am able to give some partial results and to show how it is linked to properties of compactness in $C_s(T)$.

(2.2) PROPOSITION. - *Let* T *be non-c-replete and suppose that* $C_c(T)$ *is a Kelley space. Then there is a compact disc in* $C_c(T)$ *that is not compact in* $C_s(\theta T)$.

Proof. - The continuous characters of the algebra $C_c(T)$ are the evaluations $\delta_t (t \in T)$. If $u \in \theta T \setminus T$, u is not continuous on $C_c(T)$ and, since $C_c(T)$ is Kelley, not continuous on some compact disc of $C_c(T)$. This disc is not compact in $C_s(\theta T)$.

I know of no example of a space T for which some compact subset of $C_c(T)$, even of $C_s(T)$, fails to be compact in $C_s(\theta T)$. Propositions (2.4) and (2.8) suggest that such a space (if one exists !) would be difficult to construct.

Let us denote by $\mathcal{R}(T)$ the set of all closures in T of K_σ subsets of T and consider the property :

(A) Every function $\psi \in \mathbb{R}^T$ *which coincides on each* $C \in \mathcal{R}$ *with a suitable* $f \in C(T)$ *is itself in* $C(T)$.

This property was introduced by J.D. PRYCE who proved :

(2.3) THEOREME ([P], Theorem 2.4). - *If* T *has property (A) then every relatively countably compact (rcc) subset of* $C_s(T)$ *is relatively compact (rc) in* $C_s(T)$.

(2.4) PROPOSITION. - *When* θT *has property (A) the compact subsets of* $C_s(T)$ *are compact in* $C_s(\theta T)$.

Proof. - When (f_n) is a sequence in $C(T)$ and $u \in \upsilon T$, there exists $t \in T$ such that $f_n(t) = f_n^{\upsilon}(u)$ for every integer n. It follows at once from this that the rcc subsets of $C_s(T)$ and $C_s(\upsilon T)$ (hence also of $C_s(\theta T)$) are the same. Thus an rc subset of $C_s(T)$ is rcc in $C_s(\theta T)$ and, by the theorem of PRYCE, rc in $C_s(\theta T)$. If a subset is compact in $C_s(T)$ it is rc and closed, hence compact, in $C_s(\theta T)$.

We can note that θT satisfies (A) if θT is k_R or if there is a dense K_σ subset of θT, in particular if T is pseudocompact or has a dense B_σ (σ-bounded) subset.

Write $R'(T)$ for the set of all closures in T of B_σ subsets of T. PRYCE's theorem allows the generalization below.

(2.5) PROPOSITION. - *Let* T *be a completely regular space that satisfies :*
(A') Every $\psi \in \mathbb{R}^T$ *which coincides on each* $C \in R'$ *with a suitable* $f \in C(T)$ *is itself in* $C(T)$.
Then every rcc subset of $C_s(T)$ *is rc in* $C_s(T)$.

Proof. - Suppose first that T satisfies (A'). I shall show that the bidual T'' satisfies (A). Recall that the bidual of T is defined ((B_2)) as the space T'' of all continuous characters of the algebra $C_b(T)$, embedded as a subspace of θT.

Let ψ be a real-valued function on T'' and suppose that for all $C \in R(T'')$ there is an $f \in C(T'')$ with $f|C = \psi|C$. Now if B is a bounded subset of T, $\overline{B}$, taken in T'', is compact, so that the T'' closure $\overline{D}$ of any $D \in R'(T)$ is in $R(T'')$. Thus, for every such D, there is a $g \in C(T)$ such that $g|D = \psi|D$. Applying (A'), we see that $\psi|T \in C(T)$. Let us denote by ϕ the continuous extension of $\psi|T$ to T''. It will be enough to prove that $\phi=\psi$. If B is bounded in T, $\overline{B}$ is compact in T'' ; ϕ and ψ are both continuous on $\overline{B}$ and coincide on B. Hence ϕ and ψ coincide on $\overline{B}$. But by proposition 2 of (B_2) we know that $T'' = \bigcup\{\overline{B}$; B bounded in T$\}$ and we can deduce that ϕ and ψ coincide on T''.

If now A is rcc in $C_s(T)$, A is rcc in $C_s(T'')$ by the same reasoning as was used in proposition (2.4). A is therefore rc in $C_s(T'')$ and so certainly rc in

$C_s(T)$.

(2.6) DEFINITION. - *A space* X *is said to be* angelic *([P], p. 534) if*
(i) rcc $\Longrightarrow$ rc for the subsets of X, *and*
(ii) every element of the closure of an rc subset A *of* X *is the limit of some sequence in* A.

If $T \in R'(T)$, we know already by the first part that (i) is satisfied. PRYCE showed that $C_s(T)$ is angelic if $T \in R(T)$ ([P], theorem 2.5). Therefore $C_s(T'')$ is angelic. If A is rc in $C_s(T)$ (and hence also in $C_s(T''))$ and $f \in \overline{A}$ (the closure being the same in the two topologies), there is a sequence in A that converges to f in $C_s(T'')$, and which converges to f, a fortiori, in $C_s(T)$. Then :

(2.7) PROPOSITION. - *If* $T \in R'(T)$ *(particularly if* T *is pseudocompact),* $C_s(T)$ *is angelic.*

(2.8) PROPOSITION. - *Let* T *be a (completely regular) space in which all closed and discrete subspaces are* C^∞*-embedded (particularly if* T *is normal or countably compact) and that satisfies :*

(B) For every $u \in \theta T \setminus T$ *there is a base* $\mathcal{U}$ *of neighbourhoods of* u *in* θT *such that, whenever* $\mathcal{V} \subset \mathcal{U}$ *and the cardinality of* $\mathcal{V}$ *is strictly less than that of* $\mathcal{U}$*, then* $T \cap (\bigcap \mathcal{V})$ *is nonempty.*

Then the compact subsets of $C_s(T)$ *are compact in* $C_s(\theta T)$.

Proof. - It is enough to show that every character $u \in \theta T$ is continuous on each compact $A \subset C_s(T)$. Suppose then that u is not continuous on such an A ; there is a net (f_α) in A such that $f_\alpha \to f$ in $C_s(T)$ while $f_\alpha^\theta(u) \to \ell \neq f^\theta(u)$. We can assume that the f_α are uniformly bounded by 1, that $f_\alpha \to 0$ in $C_s(T)$ and that $f_\alpha^\theta(u) = 1$ for all α.

Let $\mathcal{U}$ be a base of neighbourhoods of u in θT with the property of (B). Then if $B \subset C(T)$, $\mathcal{V} \subset \mathcal{U}$ and card B, card $\mathcal{V}$ are strictly less than card $\mathcal{U}$, there exists $t \in T$ such that $t \in \bigcap \mathcal{V}$ and $f(t) = f^\theta(u)$ for every $f \in B$. Let us denote by Ω the first ordinal of cardinality card $\mathcal{U}$ and index $\mathcal{U}$ as $(U_\xi)_{\xi<\Omega}$. I shall define, by transfinite induction, families (x_ξ) in T and (g_ξ) in A with the properties :

(a) $g_\xi(x_\zeta) \leq 1/2$ $(\xi \geq \zeta)$,
(b) $g_\xi(x_\zeta) = 1 = g_\xi^\theta(u)$ $(\xi < \zeta)$,
(c) $x_\xi \to u$ in θT.

Let $x_\circ$ be an arbitrary point of T and choose $\alpha_\circ$ such that $f_{\alpha_\circ}(x_\circ) \leq 1/2$. Put $g_\circ = f_{\alpha_\circ}$. Suppose that x_ξ and g_ξ have been defined for all ξ less than some $\eta < \Omega$ and that (a) and (b) are satisfied. Since the cardinality of $[0,\eta[$ is less than card $\mathcal{U}$, there exists $x_\eta \in T \cap (\bigcap_{\xi<\eta} U_\xi)$ such that
$g_\xi(x_\eta) = g_\xi^\theta(u) = 1$ $(\xi < \eta)$.

Let us now choose, for each finite subset S of $[0,\eta]$, an α_S such that $f_{\alpha_S}(x_\xi) \leq 1/2$ $(\xi \in S)$. Let g_η be a cluster point of the net (f_{α_S}), directed by the upward filtering set of finite subsets of $[0,\eta]$. Then we have
$g_\eta(x_\xi) \leq 1/2$ $(\xi \leq \eta)$ and
$g_\eta^\theta(u) = 1$ (because there is $t \in T$ with $g_\eta(t) = g_\eta^\theta(u)$ and $f_{\alpha_S}(t) = f_{\alpha_S}^\theta(u)$ for every finite set $S \subset [0,\eta]$).

Since, by construction, each x_η is in $\bigcap_{\xi<\eta} U_\xi$, we see that $x_\eta \to u$ in θT. I shall now show that $\{x_\eta \,;\, \eta < \Omega\}$ is a closed discrete subspace of T. If not, there is $\zeta < \Omega$ such that $\{x_\eta \,;\, \eta < \zeta\}$ has an accumulation point x in T. Choose to be the least such ordinal ; then x is in the closure of $\{x_\eta \,;\, \xi < \eta < \zeta\}$ for each $\xi < \zeta$. Hence $g_\xi(x) = 1$ for every $\xi < \zeta$. Let g be a cluster point of the net $(g_\xi)_{\xi<\zeta}$. Then
$g(x) = 1$, but
$g(x_\eta) \leq 1/2$ $(\eta < \zeta)$, since
$g_\xi(x_\eta) \leq 1/2$ $(\eta < \xi < \zeta)$. This contradicts the continuity of g at x.

Since $\{x_\xi \,;\, \xi < \Omega\}$ is a closed discrete subspace of the space T, there is a continuous function $f \in C(T)$ with $f(x_\xi) = 0$ (ξ an isolated ordinal)
$f(x_\xi) = 1$ (ξ a limit ordinal).
But such an f can have no extension that is continuous on θT, and this contradiction ends the proof.

Proposition (2.8) applies in particular to the non-c-replete P-space of ([GJ], 9.L). In this case there exists, for every $\psi \in \mathbb{R}^{\theta T}$ and every $C \in \mathcal{R}(\theta T)$, a function $f \in C(\theta T)$ with $f|C = \psi|C$; a situation very different from that considered in proposition (2.4).

For the last result in this paragraph, we return to the methods of propositions (2.4) and (2.5).

(2.9) PROPOSITION. - *A compact subset of* $C_s(T)$ *remains compact in* $C_s(\mu T)$.

Proof. - By the characterization of μT as the space obtained by transfinite iteration of the bidual operation ([B2], théorème 2), it is enough to prove that a compact subset A of $C_s(T)$ is compact in $C_s(T'')$. Such an A is countably compact in $C_s(T'')$ and hence, for each bounded $B \subset T$, $A|\overline{B}^{\upsilon}$ is countably compact in $C_s(\overline{B}^{\upsilon})$. But countable compactness and compactness coincide in this space, since $\overline{B}^{\upsilon}$ is compact. Thus, for all characters u in $\overline{B}^{\upsilon}$, $u|A$ is continuous for the topology of pointwise convergence on B, and we deduce that $u|A$ is $C_s(T)$-continuous for every $u \in T''$.

(2.10) COROLLARY. - *If* $C_c(T)$ *is a Kelley space then* T *is a* μ*-space, i.e.* $C_c(T)$ *cannot be Kelley without being barrelled.*

3. - INFRA-k_R-SPACES.

The space T of (H_2) has given us an example of a complete lcs $E = C_c(T)$, the Kelleyfié of which, $F = \overline{k}E = C_c(\theta T)$, is not quasi-complete. F is, however, a p-*semi-reflexive* space ([DJ]), that is to say, every precompact subset is relatively compact. In this example θT happens to be an infra-k_R-space, but it would seem, a priori, that the property "every precompact set is relatively compact" was a good deal weaker than the infra-k_R-property, "every precompact set is equicontinuous". But it turns out that this is not the case.

(3.1) THEOREM. - T *is an infra-*k_R*-space if and only if every precompact subset of* $C_c(T)$ *is relatively compact in* $C_s(T)$.

(3.2) COROLLARY. - T *is an infra-*k_R*-space if and only if* $C_c(T)$ *is* p*-semi-reflexive.*

We shall need a definition and two preliminary results.

(3.3) <u>DEFINITION</u>. - *Let us say that a subset* H *of* C(T) *is* closed under lattice operations *(or, more simply,* lattice-closed*) if* $f \vee g \in H$ *and* $f \wedge g \in H$ *whenever* $f, g \in H$. *If* $H \subset C(T)$, *define the* lattice-closed hull ΛH *of* H *to be the smallest lattice closed set that contains* H.

(3.4) <u>LEMMA</u>. - *For a subset* H *of* C(T) *the following are equivalent :*

(a) H *is precompact in* $C_c(T)$ *;*

(a') for every compact $K \subset T$, $H|K$ *is bounded and equicontinuous in* C(K) *(i.e.* $H|K \in \mathcal{H}(K)$) *;*

(b) ΛH *is precompact in* $C_c(T)$ *;*

(b') for every compact $K \subset T$, $\Lambda H|K \in \mathcal{H}(K)$.

Proof. - The equivalences (a)<=>(a') and (b)<=>(b') are consequences of ASCOLI's theorem. (a') is equivalent to (b') since the lattice-closed hull of an equicontinuous set is equicontinuous.

(3.5) <u>PROPOSITION</u>. - *A lattice-closed, relatively compact subset of* $C_s(T)$ *is equicontinuous.*

Proof. - Let H be such a set and suppose, if possible, that H is not equicontinuous at some $t \in T$. We can assume that, for some $\varepsilon > 0$, there are, for each neighbourhood U of t, a function $h_U \in H$ and a point $t_U \in U$ such that

$$h_U(t_U) \geq h_U(t) + \varepsilon.$$

Now the set $\{h(t) \; ; \; h \in H\}$ is bounded in $\mathbb{R}$ and there exists a subnet of $(h_U(t))$ convergent to some $\alpha \in \mathbb{R}$. That is to say that there is a base $\mathcal{U}$ of neighbourhoods of t such that $h_U(t) \to \alpha$ as U decreases through $\mathcal{U}$. We can suppose that $|h_U(t) - \alpha| \leq \varepsilon/3$ $(U \in \mathcal{U})$, so that $h_U(t) \leq \alpha + \varepsilon/3$ and $h_U(t_U) \geq \alpha + 2\varepsilon/3$ for all $U \in \mathcal{U}$.

Now let us define, for each finite subset $F = \{U_1, \ldots, U_n\}$ of $\mathcal{U}$, $g_F = h_{U_1} \vee \cdots \vee h_{U_n}$ and note that $g_F(t) \leq \alpha + \varepsilon/3$ for all F, and $g_F(t_U) \geq \alpha + 2\varepsilon/3$ whenever $U \in F$.

Each g_F is in H and so there is a subnet of (g_F) convergent in $C_s(T)$ to some g (in fact, to $g = \operatorname{Sup}_{U \in \mathcal{U}} h_U$) and we see that $g(t) \leq \alpha + \varepsilon/3$ while $g(t_U) \geq \alpha + 2\varepsilon/3$ $(U \in \mathcal{U})$. This contradicts the continuity of g at t.

Proof of theorem (3.1). - The necessity of the condition comes from the fact that a pointwise bounded equicontinuous subset of $C(T)$ is relatively compact in $C_s(T)$.

Suppose now that the condition is satisfied and that H is a precompact subset of $C_c(T)$. By lemma (3.4), ΛH is precompact in $C_c(T)$, and hence relatively compact in $C_s(T)$. But now, by proposition (3.5), we deduce that ΛH is equicontinuous.

4. - TWO GENERALIZATIONS OF A THEOREM OF BUCHWALTER.

H. BUCHWALTER has shown that, if υT is a k_R-space, then necessarily $\upsilon T = \theta T$. There follow two generalizations of this result.

(4.1) LEMMA. - *If* $H \in \mathcal{H}(T)$ *and card* H *is non-measurable, then the metrizable space* T_H *is replete and* $H^\upsilon \in \mathcal{H}(\upsilon T)$.

Proof. - Recall that T_H is defined to be the Hausdorff quotient of T endowed with the pseudometric $d(s,t) = \sup_{h \in H} |h(s)-h(t)|$. There is an injection $T_H \to \mathbb{R}^H$ so that card $T_H \leq c^{\text{card } H}$. Now if m,n are non-measurable cardinals, so is m^n ([I], p. 128) and it follows that T_H is replete.

H factors through the quotient mapping $\pi_H : T \to T_H$, as $H = H_1 \circ \pi_H$ where $H_1 \in \mathcal{H}(T_H)$. Since T_H is replete, π_H extends to $\pi_H^\upsilon : \upsilon T \to T_H$ and $H^\upsilon = H_1 \circ \pi_H^\upsilon \in \mathcal{H}(\upsilon T)$.

(4.2) THEOREM. - *Let* T *be a completely regular space and suppose either :*

(a) υT *has property* (A), *or*

(b) υT *is an infra-*k_R*-space.*

Then $\upsilon T = \theta T$.

Proof :

(a) Let $H \in \mathcal{H}(T)$. H is relatively compact in $C_s(T)$ and hence relatively countably compact in $C_s(\upsilon T)$. By the theorem of PRYCE, H is relatively compact in $C_s(\upsilon T)$. We can deduce that the topologies of $C_s(T)$ and of $C_s(\upsilon T)$ coincide on H and hence that, for any $u \in \upsilon T$, $u|H$ is continuous for the topology of simple convergence on T. But this is exactly the condition for a character u to be in θT.

(b) Again suppose $H \in \mathcal{H}(T)$. As above, it will be enough to show that H^{υ} is relatively compact in $C_s(\upsilon T)$ and hence enough to show that H^{υ} is precompact in $C_c(\upsilon T)$. This will be true provided that J^{υ} is precompact in $C_c(\upsilon T)$ for each countable $J \subset H$. But, by lemma (4.1), we know that each J^{υ} is even in $\mathcal{H}(\upsilon T)$.

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