Résumé des principales formules de la théorie des fonctions elliptiques

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RÉSUMÉ DES PRINCIPALES FORMULES DE LA THÉORIE DES FONCTIONS ELLIPTIQUES (1).

FONCTIONS DE WEIERSTRASS.

Développements en produits et séries infinis.

$$\sin u = u \prod{}' \left(1 - \frac{u}{m\pi}\right) e^{\frac{u}{m\pi}},$$

$$\cot u = \frac{1}{u} + \sum{}' \left(\frac{1}{u - m\pi} + \frac{1}{m\pi}\right),$$

$$\frac{1}{\sin^2 u} = \frac{1}{u^2} + \sum{}' \frac{1}{(u - n\pi)^2}.$$

$$(w = 2m\omega + 2n\omega'),$$

$$\sigma u = u \prod{}' \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}},$$

$$\zeta u = \frac{1}{u} + \sum{}' \left(\frac{1}{u - w} + \frac{1}{w} + \frac{u}{w^2}\right),$$

$$p u = \frac{1}{u^2} + \sum{}' \left[\frac{1}{(u - w)^2} - \frac{1}{w^2}\right],$$

$$-\frac{1}{2} p' u = \sum \frac{1}{(u - w)^3}.$$

(1) Les candidats à l'Agrégation de Mathématiques seront autorisés à se servir, pour les compositions écrites, de ce Tableau qu'ils trouveront à la librairie Gauthier-Villars.

Développements en séries entières.

$$\sigma u = u + \star - \frac{g_2}{2^4.3.5}u^5 - \frac{g_3}{2^3.3.5.7}u^7 - \frac{g_2^2}{2^9.3^2.5.7}u^9 - \ldots,$$

$$\zeta u = \frac{1}{u} + \star - \frac{g_2}{2^2.3.5}u^3 - \frac{g_3}{2^2.5.7}u^5 - \frac{g_2^2}{2^4.3.5^2.7}u^7 - \ldots,$$

$$\wp u = \frac{1}{u^2} + \star + \frac{g_2}{2^2.5}u^2 + \frac{g_3}{2^2.7}u^4 + \frac{g_2^2}{2^4.3.5^2}u^6 + \frac{3g_2g_3}{2^4.5.7.11}u^8 + \ldots.$$

$$g_2 = 60\sum{}'\frac{1}{w^4}, \qquad g_3 = 140\sum{}'\frac{1}{w^6}.$$

Relations entre $\wp u$ et ses dérivées.

$$\wp'^2 u = 4\wp^3 u - g_2\wp u - g_3 = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3),$$

$$\wp'' u = 6\wp^2 u - \tfrac{1}{2}g_2,$$

$$\wp''' u = 12\wp u\wp' u.$$

Homogénéité.

$$\sigma(\mu u \mid \mu\omega, \mu\omega') = \mu\sigma(u \mid \omega, \omega'),$$

$$\zeta(\mu u \mid \mu\omega, \mu\omega') = \frac{1}{\mu}\zeta(u \mid \omega, \omega'),$$

$$\wp(\mu u \mid \mu\omega, \mu\omega') = \frac{1}{\mu^2}\wp(u \mid \omega, \omega').$$

$$g_2(\mu\omega, \mu\omega') = \frac{1}{\mu^4}g_2(\omega, \omega'),$$

$$g_3(\mu\omega, \mu\omega') = \frac{1}{\mu^6}g_3(\omega, \omega').$$

Dégénérescence.

1° $\omega' = \infty$:

$$\wp u = -\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 + \left(\frac{\pi}{2\omega}\right)^2\frac{1}{\sin^2\left(\frac{\pi u}{2\omega}\right)},$$

$$\left(\frac{\pi}{2\omega}\right)^2 = \frac{9g_3}{2g_2}, \qquad e_1 = \frac{3g_3}{g_2}, \qquad e_2 = e_3 = -\frac{3g_3}{2g_2};$$

$$g_2^3 - 27g_3^2 = 0.$$

$$\zeta u = \frac{\pi}{2\omega}\operatorname{cotang}\frac{\pi u}{2\omega} + \frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 u,$$

$$\sigma u = e^{\frac{1}{6}\left(\frac{\pi u}{2\omega}\right)^2}\frac{2\omega}{\pi}\sin\frac{\pi u}{2\omega}.$$

2° $\omega = \infty$, $\omega' = \infty$:

$$pu = \frac{1}{u^2}, \qquad \zeta u = \frac{1}{u}, \qquad \sigma u = u,$$
$$e_1 = e_2 = e_3 = 0, \qquad g_2 = 0, \qquad g_3 = 0.$$

Périodicité et formules d'addition.

$$\begin{aligned}
&p(u + 2\omega) = pu,\\
&p(u + 2\omega') = pu,\\
&\zeta(u + 2\omega) = \zeta u + 2\eta, \qquad \eta = \zeta\omega,\\
&\zeta(u + 2\omega') = \zeta u + 2\eta', \qquad \eta' = \zeta\omega',\\
&\sigma(u + 2\omega) = -e^{2\eta(u+\omega)}\sigma u,\\
&\sigma(u + 2\omega') = -e^{2\eta'(u+\omega')}\sigma u,\\
&\sigma(u + 2m\omega + 2n\omega') = (-1)^{mn+m+n}e^{(2m\eta+2n\eta')(u+m\omega+n\omega')}\sigma u;\\
&\eta\omega' - \omega\eta' = \tfrac{1}{2}\pi i,
\end{aligned}$$

le coefficient de i dans le rapport $\frac{\omega'}{\omega}$ étant supposé positif.

$$pu - pv = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u\,\sigma^2 v},$$

$$\frac{p'u}{pu - pv} = \zeta(u+v) + \zeta(u-v) - 2\zeta u,$$

$$\frac{-p'v}{pu - pv} = \zeta(u+v) - \zeta(u-v) - 2\zeta v,$$

$$\frac{1}{2}\,\frac{p'u - p'v}{pu - pv} = \zeta(u+v) - \zeta u - \zeta v,$$

$$pu - p(u+v) = \frac{1}{2}\,\frac{\partial}{\partial u}\left(\frac{p'u - p'v}{pu - pv}\right),$$

$$pu + pv + p(u+v) = \frac{1}{4}\left(\frac{p'u - p'v}{pu - pv}\right)^2;$$

$$p(u+\omega) - e_1 = \frac{(e_1 - e_2)(e_1 - e_3)}{pu - e_1},$$

$$p(u+\omega+\omega') - e_2 = \frac{(e_2 - e_1)(e_2 - e_3)}{pu - e_2},$$

$$p(u+\omega') - e_3 = \frac{(e_3 - e_1)(e_3 - e_2)}{pu - e_3}.$$

Racines e_1, e_2, e_3. — Fonctions σ_1, σ_2, σ_3.

$$e_1 = pω, \qquad e_2 = p(ω + ω'), \qquad e_3 = pω',$$

$$p'u = -2\frac{\sigma(u-ω)\,\sigma(u-ω')\,\sigma(u+ω+ω')}{\sigma ω\,\sigma ω'\,\sigma(ω+ω')\,\sigma^3 u},$$

$$\sigma_1 u = e^{\eta u}\frac{\sigma(ω-u)}{\sigma ω},$$

$$\sigma_2 u = e^{(\eta+\eta')u}\frac{\sigma(ω+ω'-u)}{\sigma(ω+ω')},$$

$$\sigma_3 u = e^{\eta' u}\frac{\sigma(ω'-u)}{\sigma ω'};$$

$$pu - e_1 = \left(\frac{\sigma_1 u}{\sigma u}\right)^2, \qquad pu - e_2 = \left(\frac{\sigma_2 u}{\sigma u}\right)^2, \qquad pu - e_3 = \left(\frac{\sigma_3 u}{\sigma u}\right)^2;$$

$$p'u = -\frac{2\sigma_1 u\,\sigma_2 u\,\sigma_3 u}{\sigma^3 u}.$$

Les fonctions σ_1, σ_2, σ_3 sont paires.

$$\frac{d}{du}\frac{\sigma u}{\sigma_\lambda u} = \frac{\sigma_\mu u}{\sigma_\lambda u}\frac{\sigma_\nu u}{\sigma_\lambda u}, \qquad \frac{d}{du}\frac{\sigma_\mu u}{\sigma_\nu u} = -(e_\mu - e_\nu)\frac{\sigma_\lambda u}{\sigma_\nu u}\frac{\sigma u}{\sigma_\nu u},$$

$$\frac{d}{du}\frac{\sigma_\lambda u}{\sigma u} = -\frac{\sigma_\mu u}{\sigma u}\frac{\sigma_\nu u}{\sigma u}.$$

Valeurs réelles de pu quand $ω$ et $\frac{ω'}{i}$ sont réelles.

Considérons le rectangle de sommets o, $ω$, $ω + ω'$, $ω'$. Quand l'argument u décrit le contour de ce rectangle dans le sens o, $ω$, $ω + ω'$, $ω'$, o, la fonction pu diminue constamment de $+\infty$ à $-\infty$:

1° Quand u va de o à $ω$, pu est réel et décroît de ∞ à e_1 ; $p'u$ est négatif.

2° Quand u va de $ω$ à $ω + ω'$, pu décroît de e_1 à e_2, $p'u$ est purement imaginaire positive.

3° La variable u allant de $ω + ω'$ à $ω'$, pu décroît de e_2 à e_3, $p'u$ est réelle et positive.

4° Enfin u revenant de ω' à o, pu décroît de e_3 à $-\infty$; $p'u$ est purement imaginaire négative.

En tout point pris dans le rectangle, pu est imaginaire.

FONCTIONS DE JACOBI.

Séries trigonométriques.

$$q = e^{-\pi\frac{K'}{K}}, \qquad v = \frac{\pi u}{2K},$$

$$H(u) = 2\sqrt[4]{q}\sin v - 2\sqrt[4]{q^9}\sin 3v + 2\sqrt[4]{q^{25}}\sin 5v - \ldots,$$

$$H_1(u) = 2\sqrt[4]{q}\cos v + 2\sqrt[4]{q^9}\cos 3v + 2\sqrt[4]{q^{25}}\cos 5v + \ldots,$$

$$\Theta(u) = 1 - 2q\cos 2v + 2q^4\cos 4v - 2q^9\cos 6v + \ldots,$$

$$\Theta_1(u) = 1 + 2q\cos 2v + 2q^4\cos 4v + 2q^9\cos 6v + \ldots.$$

$\lambda = e^{-\frac{i\pi}{4K}(2u+iK')}$,	*Zéros de* $H(u)$	$2mK + 2niK'$,
$H_1(u) = H(u+K)$,	» $H_1(u)$	$(2m+1)K + 2niK'$,
$\Theta(u) = \frac{1}{i\lambda}H(u+iK')$,	» $\Theta(u)$	$2mK + (2n+1)iK'$,
$\Theta_1(u) = \frac{1}{\lambda}H(u+K+iK')$,	» $\Theta_1(u)$	$(2m+1)K + (2n+1)iK'$.

Produits infinis.

$$v = \frac{\pi u}{2K}, \qquad A = (1-q^2)(1-q^4)(1-q^6)\ldots,$$

$$H(u) = A\,2\sqrt[4]{q}\sin v(1-2q^2\cos 2v+q^4)(1-2q^4\cos 2v+q^8)\ldots;$$

$$H_1(u) = A\,2\sqrt[4]{q}\cos v(1+2q^2\cos 2v+q^4)(1+2q^4\cos 2v+q^8)\ldots,$$

$$\Theta(u) = A(1-2q\cos 2v+q^2)(1-2q^3\cos 2v+q^6)\ldots,$$

$$\Theta_1(u) = A(1+2q\cos 2v+q^2)(1+2q^3\cos 2v+q^6)\ldots.$$

Addition d'une demi-période ou d'une période.

$$\lambda = e^{-\frac{i\pi}{4K}(2u+iK')}, \qquad \mu = e^{-\frac{i\pi}{K}(u+iK')},$$

$$\begin{array}{rlrl}
H(u+K) = & H_1(u), & H(u+iK') = & i\lambda\,\Theta(u),\\
\Theta(u+K) = & \Theta_1(u), & \Theta(u+iK') = & i\lambda H(u),\\
H_1(u+K) = & -H(u), & H_1(u+iK') = & \lambda\Theta_1(u),\\
\Theta_1(u+K) = & \Theta(u), & \Theta_1(u+iK') = & \lambda H_1(u),\\
H(u+K+iK') = & \lambda\Theta_1(u), & H(u+2iK') = & -\mu H(u),\\
\Theta(u+K+iK') = & \lambda H_1(u), & \Theta(u+2iK') = & -\mu\,\Theta(u),\\
H_1(u+K+iK') = & -i\lambda\,\Theta(u), & H_1(u+2iK') = & \mu H_1(u),\\
\Theta_1(u+K+iK') = & i\lambda H(u), & \Theta_1(u+2iK') = & \mu\Theta_1(u).
\end{array}$$

Dans tout ce qui suit, nous supposerons K et K′ particularisés de telle façon que

$$\sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + \ldots.$$

Relations entre les σ et les fonctions de Jacobi.

$$\sigma(u|K, iK') = \frac{H(u)}{H'(0)}e^{\frac{1}{2}\frac{\eta}{K}u^2},$$

$$\sigma_1(u|K, iK') = \frac{\sigma(K+u)}{\sigma K}e^{-\eta u} = \frac{H_1(u)}{H_1(0)}e^{\frac{1}{2}\frac{\eta}{K}u^2},$$

$$\sigma_2(u|K, iK') = \frac{\sigma(K+iK'+u)}{\sigma(K+iK')}e^{-(\eta+\eta')u} = \frac{\Theta_1(u)}{\Theta_1(0)}e^{\frac{1}{2}\frac{\eta}{K}u^2},$$

$$\sigma_3(u|K, iK') = \frac{\sigma(iK'+u)}{\sigma iK'}e^{-\eta' u} = \frac{\Theta(u)}{\Theta(0)}e^{\frac{1}{2}\frac{\eta}{K}u^2}.$$

Ces formules donnent les fonctions σ construites avec les périodes spéciales K et iK'. D'après les formules d'homogénéité, les fonctions σ construites avec deux périodes ω et ω′, assujetties à la seule condition

$$\frac{\omega'}{\omega} = \frac{iK'}{K},$$

s'obtiennent immédiatement :

$$\sigma(u|\omega, \omega') = \frac{\omega}{K}\,\frac{H\left(\frac{K}{\omega}u\right)}{H'(0)}e^{\frac{1}{2}\frac{\eta}{\omega}u^2},$$

$$\sigma_1(u|\omega,\omega') = \frac{H_1\left(\frac{K}{\omega}u\right)}{H_1(0)}\,e^{\frac{1}{2}\frac{\eta}{\omega}u^2},$$

$$\sigma_2(u|\omega,\omega') = \frac{\Theta_1\left(\frac{K}{\omega}u\right)}{\Theta_1(0)}\,e^{\frac{1}{2}\frac{\eta}{\omega}u^2},$$

$$\sigma_3(u|\omega,\omega') = \frac{\Theta\left(\frac{K}{\omega}u\right)}{\Theta(0)}\,e^{\frac{1}{2}\frac{\eta}{\omega}u^2}.$$

FONCTIONS $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$

$$\operatorname{sn} u = \frac{1}{\sqrt{k}}\frac{H(u)}{\Theta(u)},$$

$$\operatorname{cn} u = \sqrt{\frac{k'}{k}}\frac{H_1(u)}{\Theta(u)}, \qquad \sqrt{k} = \frac{H(K)}{\Theta(K)} = \frac{H_1(0)}{\Theta_1(0)},$$

$$\operatorname{dn} u = \sqrt{k'}\frac{\Theta_1(u)}{\Theta(u)}, \qquad \sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)}.$$

Addition d'une demi-période ou d'une période.

$$\operatorname{sn}(u+K) = \frac{\operatorname{cn} u}{\operatorname{dn} u}, \qquad \operatorname{sn}(u+iK') = \frac{1}{k\operatorname{sn} u},$$

$$\operatorname{cn}(u+K) = -\frac{k'\operatorname{sn} u}{\operatorname{dn} u}, \qquad \operatorname{cn}(u+iK') = -i\frac{\operatorname{dn} u}{k\operatorname{sn} u},$$

$$\operatorname{dn}(u+K) = \frac{k'}{\operatorname{dn} u}, \qquad \operatorname{dn}(u+iK') = -i\frac{\operatorname{cn} u}{\operatorname{sn} u},$$

$$\operatorname{sn}(u+K+iK') = \frac{\operatorname{dn} u}{k\operatorname{cn} u},$$

$$\operatorname{cn}(u+K+iK') = \frac{-ik'}{k\operatorname{cn} u}.$$

$$\operatorname{dn}(u+K+iK') = ik'\frac{\operatorname{sn} u}{\operatorname{cn} u},$$

$$\operatorname{sn}(u+2K) = -\operatorname{sn} u, \qquad \operatorname{sn}(u+2iK') = \operatorname{sn} u,$$

$$\operatorname{cn}(u+2K) = -\operatorname{cn} u, \qquad \operatorname{cn}(u+2iK') = -\operatorname{cn} u,$$

$$\operatorname{dn}(u+2K) = \operatorname{dn} u, \qquad \operatorname{dn}(u+2iK') = -\operatorname{dn} u.$$

Argument purement imaginaire.
Relation entre $\wp u$ *et* sn u.

$$\operatorname{sn}(iu \mid K, iK') = i\,\frac{\operatorname{sn}(u \mid K', iK)}{\operatorname{cn}(u \mid K', iK)}, \qquad \wp(u \mid K, iK') = -\frac{1+k^2}{3} + \frac{1}{\operatorname{sn}^2 u};$$

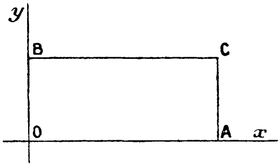
$$\operatorname{cn}(iu \mid K, iK') = \frac{1}{\operatorname{cn}(u \mid K', iK)}, \qquad \left\{ \begin{aligned} &\wp(u \mid \omega, \omega') = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2(u\sqrt{e_1 - e_3})}; \\ &\omega\sqrt{e_1 - e_3} = K, \qquad \omega'\sqrt{e_1 - e_3} = iK'. \end{aligned} \right.$$

$$\operatorname{dn}(iu \mid K, iK') = \frac{\operatorname{dn}(u \mid K', iK)}{\operatorname{cn}(u \mid K', iK)},$$

Valeurs réelles de sn u, cn u, dn u *quand* K *et* K' *sont réels.*

$$OA = K, \qquad OB = K' \ (fig.\ 1),$$

Fig. 1.



u	O	A	C	B
sn u	0	1	$\frac{1}{k}$	∞

u	A	O	B
cn u	0	1	∞

u	C	A	O	B
dn u	0	k'	1	∞

Formules d'addition.

$$\operatorname{cn}\alpha = \operatorname{cn} u \operatorname{cn}(u - \alpha) + \operatorname{sn} u \operatorname{sn}(u - \alpha) \operatorname{dn}\alpha.$$

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \qquad k^2 + k'^2 = 1,$$
$$k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1,$$

$$\operatorname{sn}(u + v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

$$\operatorname{cn}(u+v)=\frac{\operatorname{cn}u\operatorname{cn}v-\operatorname{sn}u\operatorname{sn}v\operatorname{dn}u\operatorname{dn}v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v},$$

$$\operatorname{dn}(u+v)=\frac{\operatorname{dn}u\operatorname{dn}v-k^2\operatorname{sn}u\operatorname{sn}v\operatorname{cn}u\operatorname{cn}v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v}.$$

Dérivées.

Si l'on suppose, comme dans ce qui précède, K et K′ liées par la condition

$$\sqrt{\frac{2K}{\pi}}=1+2q+2q^4+2q^9+\ldots$$

on a

$$\frac{d(\operatorname{sn}u)}{du}=\operatorname{cn}u\operatorname{dn}u,$$

$$\frac{d(\operatorname{cn}u)}{du}=-\operatorname{sn}u\operatorname{dn}u,$$

$$\frac{d(\operatorname{dn}u)}{du}=-k^2\operatorname{sn}u\operatorname{cn}u.$$

Développements en séries entières.

$$\alpha=\frac{1}{2}\left(k+\frac{1}{k}\right),$$

$$\operatorname{sn}u=u-2k\alpha\frac{u^3}{1.2.3}+4k^2(\alpha^2+3)\frac{u^5}{1.2.3.4.5}$$
$$-8k^3(\alpha^3+33\alpha)\frac{u^7}{1.2.3.4.5.6.7}+\ldots,$$

$$\operatorname{cn}u=1-\frac{u^2}{1.2}+(1+4k^2)\frac{u^4}{1.2.3.4}-(1+44k^2+16k^4)\frac{u^6}{1.2.3.4.5.6}+\ldots,$$

$$\operatorname{dn}u=1-k^2\frac{u^2}{1.2}+k^2(4+k^2)\frac{u^4}{1.2.3.4}$$
$$-k^2(16+44k^2+k^4)\frac{u^6}{1.2.3.4.5.6}+\ldots.$$

Autre notation. Fonctions ϑ.

$$\tau=\frac{\omega'}{\omega},\qquad q=e^{\tau\pi i}.$$

$$\vartheta_1(v\,|\,\tau)=2q^{\frac{1}{4}}\sin v\pi-2q^{\frac{9}{4}}\sin 3v\pi+2q^{\frac{25}{4}}\sin 5v\pi-\ldots,$$

$$\vartheta_2(v\,|\,\tau)=2q^{\frac{1}{4}}\cos v\pi+2q^{\frac{9}{4}}\cos 3v\pi+2q^{\frac{25}{4}}\cos 5v\pi-\ldots,$$

$$\vartheta_3(v\,|\,\tau) = 1 + 2q\cos 2v\pi + 2q^4\cos 4v\pi + 2q^9\cos 6v\pi + \ldots,$$
$$\vartheta_0(v\,|\,\tau) = 1 - 2q\cos 2v\pi + 2q^4\cos 4v\pi - 2q^9\cos 6v\pi + \ldots.$$

$$\vartheta_0(v+\tfrac{1}{2}) = \vartheta_3(v), \qquad \vartheta_1(v+\tfrac{1}{2}) = \vartheta_2(v),$$
$$\vartheta_2(v+\tfrac{1}{2}) = -\vartheta_1(v), \quad \vartheta_3(v+\tfrac{1}{2}) = \vartheta_0(v).$$

$$\vartheta_0(v+\tfrac{1}{2}\tau) = iq^{-\frac{1}{4}}e^{-v\pi i}\vartheta_1(v), \quad \vartheta_1(v+\tfrac{1}{2}\tau) = iq^{-\frac{1}{4}}e^{-v\pi i}\vartheta_0(v),$$
$$\vartheta_2(v+\tfrac{1}{2}\tau) = q^{-\frac{1}{4}}e^{-v\pi i}\vartheta_3(v), \quad \vartheta_3(v+\tfrac{1}{2}\tau) = q^{-\frac{1}{4}}e^{-v\pi i}\vartheta_2(v).$$

$$e^{-\frac{1}{2}\frac{\eta}{\omega}u^2}\sigma(u) = 2\omega\frac{\vartheta_1(v)}{\vartheta_1'(0)},$$
$$e^{-\frac{1}{2}\frac{\eta}{\omega}u^2}\sigma_1(u) = \frac{\vartheta_2(v)}{\vartheta_2(0)}, \qquad v = \frac{u}{2\omega},$$
$$e^{-\frac{1}{2}\frac{\eta}{\omega}u^2}\sigma_2(u) = \frac{\vartheta_3(v)}{\vartheta_3(0)},$$
$$e^{-\frac{1}{2}\frac{\eta}{\omega}u^2}\sigma_3(u) = \frac{\vartheta_0(v)}{\vartheta_0(0)}.$$