NOUVELLES ANNALES DE MATHÉMATIQUES

WORONTZOFF

Sur les fonctions symétriques

Nouvelles annales de mathématiques 3e série, tome 12 (1893), p. 116-122

<http://www.numdam.org/item?id=NAM_1893_3_12__116_1>

SUR LES FONCTIONS SYMÉTRIQUES;

PAR M. WORONTZOFF.

1. En désignant respectivement par $x_1, x_2, \ldots, x_n$ et par $x'_1, x'_2, \ldots, x'_m$ les racines des équations

$$F(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0,$$

$$f(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x + b_m = 0 \qquad (m \leq n),$$

posons

$$s_q = x_1^q + x_2^q \ldots + x_n^q, \qquad s'_q = x_1'^q + x_2'^q + \ldots + x_m'^q,$$

$$A_{q-r} = -\frac{a_0}{q}\,\frac{ds_q}{da_r} = \sum_{k=1}^{k=n} a_0 \frac{x_k^{n+q-r-1}}{F(x_k)}$$

$$= \sum_{q-r} x_1^{q_1} x_2^{q_2} \ldots x_n^{q_n}, \qquad (r \leqq q),$$

où la somme $\sum\limits_{q-r} x_1^{q_1} x_2^{q_2} \ldots x_n^{q_n}$ se rapporte à toutes les solutions entières positives de l'équation

$$q_1 + q_2 + \ldots + q_n = q - r.$$

Comme

$$(1) \quad -q a_q = a_0 s_q + a_1 s_{q-1} + \ldots + a_{q-2} s_2 + a_{q-1} s_1 \quad (q > 0),$$

$$(2) \quad a_0 A_q + a_1 A_{q-1} + \ldots + a_{q-1} A_1 + a_q A_0 = 0 \qquad (A_0 = 1),$$

$$(3) \quad q A_q = A_0 s_q + A_1 s_{q-1} + \ldots + A_{q-1} s_2 + A_{q-2} s_1 \quad (^1),$$

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(1) En posant, pour abréger,

$$= \sum_{k=1}^{k=k} x_k^q, \qquad A_q^{(k)} = \sum_q x_1^{q_1} x_2^{q_2} \ldots x_k^{q_k} = \sum_{h=0}^{h=q} A_{q-h}^{(k-1)} x_k^h, \qquad u_q^{(k)} = \sum_{h=1}^{h=q} s_h^{(k)} A_{h-q}^{(k)},$$

admettons que, pour toutes les valeurs entières et positives de q, $u_q^{(k)} = q A_q^{(k)}$: on obtient alors, pour toutes les valeurs entières positives de q,

$$u_q^{(k+1)} = \sum_{h=1}^{h=q} s_h^{(k+1)} (A_0^{(k)} x_{k+1}^{q-h} + A_1^{(k)} x_{k+1}^{q-h-1} + \ldots + A_{q-h-1}^{(k)} x_{k+1} + A_{q-h}^{(k)})$$

$$= q \sum_{h=0}^{k=q} A_{q-h}^{(k)} x_{k+1}^h = q A_q^{(k+1)}.$$

Comme $u_q^{(1)} = q A_q^{(1)}$ ou $x_1^q + x_1^{q-1} x_1 + \ldots + x_1 x_1^{q-1} = q x_1^q$, q étant un entier positif quelconque, on a, pour toutes les valeurs entières positives de q,

$$u_q^{(2)} = q A_q^{(2)}, \qquad \ldots, \qquad u_q^{(n)} = q A_q^{(n)}.$$

on trouve

$$(4)\quad \begin{cases} s_q = q\sum(-1)^i \dfrac{(i-1)!\, a_0^{-i} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}}{\alpha_1!\,\alpha_2!\ldots\alpha_n!} \\ \quad = q\sum(-1)^{i-1} \dfrac{(i-1)!\, \mathrm{A}_0^{-i} \mathrm{A}_1^{\alpha_1} \mathrm{A}_2^{\alpha_2} \ldots \mathrm{A}_n^{\alpha_n}}{\alpha_1!\,\alpha_2!\ldots\alpha_n!}, \end{cases}$$

$$(5)\quad \begin{cases} \dfrac{a_q}{a_0} = \sum \dfrac{s_1^{\sigma_1} s_2^{\sigma_2} \ldots s_q^{\sigma_q}}{(-1)^{\sigma_1}(-2)^{\sigma_2}\ldots(-q)^{\sigma_q}\sigma_1!\,\sigma_2!\ldots\sigma_q!} \\ \quad = \sum(-1)^i \dfrac{i!\,\mathrm{A}_0^{-i} \mathrm{A}_1^{\alpha_1} \mathrm{A}_2^{\alpha_2} \ldots \mathrm{A}_n^{\alpha_n}}{\alpha_1!\,\alpha_2!\ldots\alpha_n!}, \end{cases}$$

$$(6)\quad \begin{cases} \mathrm{A}_q = \sum(-1)^i \dfrac{i!\, a_0^{-i} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}}{\alpha_1!\,\alpha_2!\ldots a_n!} \\ \quad = \sum \dfrac{s_1^{\sigma_1} s_2^{\sigma_2} \ldots s_q^{\sigma_q}}{1^{\sigma_1} 2^{\sigma_2}\ldots q^{\sigma_q}\,\sigma_1!\,\sigma_2!\ldots\sigma_q!}, \end{cases}$$

où $\sigma_1, \sigma_2, \ldots, \sigma_q, \alpha_1, \alpha_2, \ldots, \alpha_n$ sont les solutions entières positives des équations

$$\sigma_1 + 2\sigma_2 + \ldots + q\sigma_q = q, \qquad \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = q,$$

et où $i = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

En vertu des relations

$$\frac{dx_k}{da_h} = -\frac{x_k^{n-h}}{\mathrm{F}'(x_k)}, \qquad \frac{da_k}{ds_r} = -\frac{a_{k-r}}{r}, \qquad \frac{d\mathrm{A}_k}{ds_r} = \frac{\mathrm{A}_{k-r}}{r}$$
$$(k \geqq r > 0),$$

on obtient aussi

$$(7)\quad s_q = \frac{1}{(q-1)}\left(a_1\frac{ds_{q-1}}{da_0} + 2a_2\frac{ds_{q-1}}{da_1} + 3a_3\frac{ds_{q-1}}{da_2} + \ldots + na_n\frac{ds_{q-1}}{da_{n-1}}\right)$$

$$(8)\quad \begin{cases} a_q = \dfrac{1}{q}\Big[-s_1 a_{q-1} + s_2\dfrac{da_{q-1}}{ds_1} + 2s_3\dfrac{da_{q-1}}{ds_2} \\ \qquad + 3s_4\dfrac{da_{q-1}}{ds_3} + \ldots + (q-1)s_q\dfrac{da_{q-1}}{ds_{q-1}}\Big], \end{cases}$$

$$(9)\quad \begin{cases} \mathrm{A}_q = \dfrac{1}{q}\Big[s_1\mathrm{A}_{q-1} + s_2\dfrac{d\mathrm{A}_{q-1}}{ds_1} + 2s_3\dfrac{d\mathrm{A}_{q-1}}{ds_2} \\ \qquad + 3s_4\dfrac{d\mathrm{A}_{q-1}}{ds_3} + \ldots + (q-1)s_q\dfrac{d\mathrm{A}_{q-1}}{ds_{q-1}}\Big]. \end{cases}$$

(*Nouvelles Annales*, p. 328; 1891.)

2. Prenons l'équation

$$
(10)\quad \left\{
\begin{aligned}
\Phi(a_n+y) &= [y+\mathrm{F}(x'_1)][y+\mathrm{F}(x'_2)]\dots[y+\mathrm{F}(x'_m)] \\
&= y^m + \frac{\mathrm{D}^{m-1}a_n\Phi(a_n)}{1.2\dots(m-1)}y^{m-1}+\dots \\
&\quad + \mathrm{D}a_n\Phi(a_n)y+\Phi(a_n) \\
&= (a_n+y)^m + \mathrm{C}_1(a_n+y)^{m-1} \\
&\quad + \mathrm{C}_2(a_n+y)^{m-2}+\dots+\mathrm{C}_{m-1}(a_n+y)+\mathrm{C}_m = 0,
\end{aligned}
\right.
$$

où

$$
\mathrm{C}_r = \left[\frac{\mathrm{D}^{m-r}a_n\Phi(a_n)}{1.2\dots(m-r)}\right] a_n = 0;
$$

alors, d'après la formule (5), on a

$$
(11)\quad \left\{
\begin{aligned}
\Phi(a_n) &= \mathrm{F}(x'_1)\,\mathrm{F}(x'_2)\dots\mathrm{F}(x'_m) \\
&= \sum \frac{\mathrm{S}_1^{\sigma_1}\mathrm{S}_2^{\sigma_2}\dots\mathrm{S}_m^{\sigma_m}}{(-1)^{\sigma_1}(-2)^{\sigma_2}\dots(-m)^{\sigma_m}\sigma_1!\,\sigma_2!\dots\sigma_m!}
\end{aligned}
\right.
$$

$$(\sigma_1+2\sigma_2+\dots+m\sigma_m=m)$$

et aussi

$$
(12)\quad \left\{
\begin{aligned}
\Phi(a_n) &= \mathrm{F}(x'_1)\,\mathrm{F}(x'_2)\dots\mathrm{F}(x'_m) \\
&= a_n^m+\mathrm{C}_1a_n^{m-1}+\mathrm{C}_2a_n^{m-2}+\dots+\mathrm{C}_{m-1}a_n+\mathrm{C}_m,
\end{aligned}
\right.
$$

$$
(13)\quad \left\{
\begin{aligned}
\mathrm{C}_r &= \frac{\Phi^{(m-r)}(0)}{1.2\dots(m-r)} \\
&= \sum \frac{\mathrm{S}'^{\sigma_1}_1\mathrm{S}'^{\sigma_2}_2\dots\mathrm{S}'^{\sigma_r}_r}{(-1)^{\sigma_1}(-2)^{\sigma_2}\dots(-r)^{\sigma_r}\sigma_1!\,\sigma_2!\dots\sigma_r!}
\end{aligned}
\right.
$$

$$(\sigma_1+2\sigma_2+\dots+r\sigma_r=r)$$

ou

$$
(14)\quad \left\{
\begin{aligned}
\mathrm{S}_q &= (-1)^q\sum_{k=1}^{k=m}[\mathrm{F}(x'_k)]^q \\
&= (-1)^q\sum\frac{q!}{\alpha_0!\,\alpha_1!\dots\alpha_n!}a_0^{\alpha_0}a_1^{\alpha_1}\dots a_n^{\alpha_n}\mathrm{S}'_{\alpha_{n-1}+2\alpha_{n-2}+\dots+n\alpha_0}
\end{aligned}
\right.
$$

$$(\alpha_0+\dots+\alpha_n=q),$$

$$
(15)\quad \left\{
\begin{aligned}
\mathrm{S}'_q &= (-1)^q\sum_{k=1}^{k=m}[\mathrm{F}(x'_k)-a_n]^q \\
&= (-1)^q\sum\frac{q!}{\alpha_0!\,\alpha_1!\dots\alpha_{n-1}!}a_0^{\alpha_0}a_1^{\alpha_1}\dots a_{n-1}^{\alpha_{n-1}}\mathrm{S}'_{\alpha_{n-1}+2\alpha_{n-2}+\dots+n\alpha_0}
\end{aligned}
\right.
$$

$$(\alpha_0+\dots+\alpha_{n-1}=q),$$

$$(16)\quad S'_q = \sum_{k=1}^{k=m} x_k'^q = q\sum (-1)^i (i-1)!\, \frac{b_0^{-i} b_1^{\beta_1} b_2^{\beta_2} \ldots b_m^{\beta_m}}{\beta_1!\,\beta_2!\ldots\beta_m!}$$
$$(\beta_1 + 2\beta_2 + \ldots + m\beta_m = q,\quad i = \beta_1 + \beta_2 + \ldots \beta_m).$$

En faisant

$$m = n-1,\qquad b_0 = na_0,\qquad b_1 = (n-1)a_1,\qquad \ldots,$$
$$b_{n-2} = 2a_{n-2},\qquad b_{n-1} = a_{n-1},$$

nous pouvons appliquer les formules précédentes au calcul du discriminant de l'équation $F(x) = 0$,

$$(-1)^{n\left(\frac{n-1}{2}\right)} \frac{n^n}{a_0^{n-1}} F(x'_1) F(x'_2) \ldots F(x'_{n-1}) = \pi(x_1 - x_2)^2.$$

3. Posons

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = F_n(x),$$
$$a_0 x^{n-1} + a_1 x^{n-2} + \ldots + a_{n-2} x + a_{n-1} = F_{n-1}(x),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$F_n(x'_1) F_n(x'_2) \ldots F_n(x'_m) = \Phi_n(a_n),$$
$$F_{n-1}(x'_1) F_{n-1}(x'_2) \ldots F_{n-1}(x'_m) = \Phi_{n-1}(a_{n-1}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Si l'on connait $\Phi_{n-1}(a_{n-1})$, la valeur de $\Phi_n(a_n)$ s'en déduit immédiatement par de simples différentiations. En effet, on a

$$\Phi_n(a_n) = a_n^m + C_1 a_n^{m-1} + \ldots + C_{m-1} a_n + C_m$$

où

$$C_m = \Phi_n(0) = (-1)^m \frac{b_m}{b_0} \Phi_{n-1}(a_{n-1}).$$

En vertu de la relation

$$b_0 \frac{d\Phi_n(a_n)}{da_{n-m}} + b_1 \frac{d\Phi_n(a_n)}{da_{n-m+1}} + \ldots$$
$$+ b_{m-1} \frac{d\Phi_n(a_n)}{da_{n-1}} + b_m \frac{d\Phi_n(a_n)}{da_n} = 0$$

ou

$$\frac{d\Phi_n(a_n)}{da_n} = -\frac{1}{b_m}\left[b_0\,\frac{d\Phi_n(a_n)}{da_{n-m}} + b_1\,\frac{d\Phi_n(a_n)}{da_{n-m+1}} + \ldots + b_{m-1}\,\frac{d\Phi_n(a_n)}{da_{n-1}}\right],$$

ou, plus généralement,

$$\frac{d^r\Phi_n(a_n)}{da_n^r} = -\frac{1}{b_m}\left[b_0\,\frac{d}{da_{n-m}}\,\frac{d^{r-1}\Phi_n(a_n)}{da_n^{r-1}} + b_1\,\frac{d}{da_{n-m+1}}\,\frac{d^{r-1}\Phi_n(a_n)}{da_n^{r-1}} + \ldots + b_{m-1}\,\frac{d}{da_{n-1}}\,\frac{d^{r-1}\Phi_n(a_n)}{da_n^{r-1}}\right],$$

on trouve

$$\begin{aligned}
C_{m-1} &= \left[\frac{d\Phi_n(a_n)}{da_n}\right]_{a_n=0} \\
&= -\frac{1}{b_m}\left(b_0\,\frac{dC_m}{da_{n-m}} + b_1\,\frac{dC_m}{da_{n-m+1}} + \ldots + b_{m-1}\,\frac{dC_m}{da_{n-1}}\right), \\
C_{m-2} &= \frac{1}{1.2}\left[\frac{d^2\Phi_n(a_n)}{da_n^2}\right]_{a_n=0} \\
&= -\frac{1}{2b_m}\left(b_0\,\frac{dC_{m-1}}{da_{n-m}} + b_1\,\frac{dC_{m-1}}{da_{n-m+1}} + \ldots + b_{m-1}\,\frac{dC_{m-1}}{da_{n-1}}\right), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
C_{m-r} &= \frac{1}{1.2\ldots r}\left[\frac{d^r\Phi_n(a_n)}{da_n^r}\right]_{a_n=0} \\
&= -\frac{1}{rb_m}\left(b_0\,\frac{dC_{m-r+1}}{da_{n-m}} + b_1\,\frac{dC_{m-r+1}}{da_{n-m+1}} + \ldots + b_{m-1}\,\frac{dC_{m-r+1}}{da_{n-1}}\right), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
C_1 &= \frac{1}{1.2\ldots(m-1)}\left[\frac{d^{m-1}\Phi_n(a_n)}{da_n^{m-1}}\right]_{a_n=0} \\
&= -\frac{1}{(m-1)b_m}\left(b_0\,\frac{dC_2}{da_{n-m}} + b_1\,\frac{dC_2}{da_{n-m+1}} + \ldots + b_{m-1}\,\frac{dC_2}{da_{n-1}}\right).
\end{aligned}$$

En désignant respectivement par Δ_n et Δ_{n-1} les discriminants des équations $F_n(x) = 0$, $F_{n-1}(x) = 0$, et en faisant

$$m = n-1, \qquad b_0 = na_0, \qquad b_1 = (n-1)a_1, \qquad \ldots, \qquad b_{n-2} = 2a_{n-2}, \qquad b_{n-1} = a_{n-1},$$

on obtient

$$\begin{aligned}
\Delta_n &= \Pi(x_1-x_2)^2\\
&= (-1)^{n\left(\frac{n-1}{2}\right)}\frac{n^n}{a_0^{n-1}}\,\mathrm{F}(x'_1)\,\mathrm{F}(x'_2)\ldots\mathrm{F}(x'_{n-1})\\
&= (-1)^{n\left(\frac{n-1}{2}\right)}\frac{n^n}{a_0^{n-1}}\,(a_n^{n-1}+\mathrm{C}_1a_n^{n-2}+\mathrm{C}_2a_n^{n-3}+\ldots+\mathrm{C}_{n-2}a_n+\mathrm{C}_{n-1})\\
&= \mathrm{C}'_0a_n^{n-1}+\mathrm{C}'_1a_n^{n-2}+\mathrm{C}'_2a_n^{n-3}+\ldots+\mathrm{C}'_{n-2}a_n+\mathrm{C}'_{n-1},
\end{aligned}$$

où

$$\mathrm{C}'_{n-1}=(\Delta_n)_{a_n=0}-\frac{a_{n-1}^2}{a_0^2}\Delta_{n-1},$$

et, en vertu de la relation

$$\sum_{k=1}^{k=n}(n-k+1)a_{k-1}\frac{d\Delta_n}{\partial a_k}-\sum_{k=1}^{k=n}\frac{d\Delta_n}{\partial x_k}=0$$

où

$$\frac{d^r\Delta_n}{\partial a_n^r}=-\frac{1}{a_{n-1}}\sum_{k=1}^{k=n-1}(n-k+1)a_{k-1}\frac{d}{\partial a_k}\,\frac{d^{r-1}\Delta_n}{\partial a_n^{r-1}},$$

$$\begin{aligned}
\mathrm{C}'_{n-2} &= \left(\frac{d\Delta_n}{\partial a_n}\right)_{a_n=0}\\
&= -\frac{1}{a_{n-1}}\left[na_0\frac{d\mathrm{C}'_{n-1}}{\partial a_1}+(n-1)a_1\frac{d\mathrm{C}'_{n-1}}{\partial a_2}+\ldots+2a_{n-2}\frac{d\mathrm{C}'_{n-1}}{\partial a_{n-1}}\right],\\
\mathrm{C}'_{n-3} &= \frac{1}{1.2}\left(\frac{d^2\Delta_n}{\partial a_n^2}\right)_{a_n=0}\\
&= -\frac{1}{2a_{n-1}}\left[na_0\frac{d\mathrm{C}'_{n-2}}{\partial a_1}+(n-1)a_1\frac{d\mathrm{C}'_{n-2}}{\partial a_2}+\ldots+2a_{n-2}\frac{d\mathrm{C}'_{n-2}}{\partial a_{n-1}}\right],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\mathrm{C}'_1 &= \frac{1}{1.2\ldots(n-2)}\left(\frac{d^{n-2}\Delta_n}{\partial a_n^{n-2}}\right)_{a_n=0}\\
&= -\frac{1}{(n-2)a_{n-1}}\left[na_0\frac{d\mathrm{C}'_2}{\partial a_1}+(n-1)a_1\frac{d\mathrm{C}'_2}{\partial a_2}+\ldots+2a_{n-2}\frac{d\mathrm{C}'_2}{\partial a_{n-1}}\right],\\
\mathrm{C}'_0 &= (-1)^{n\left(\frac{n-1}{2}\right)}\frac{n^n}{a_0^{n-1}}.
\end{aligned}$$

(J.-A. SERRET, *Cours d'Algèbre supérieure*, t. I, p. 461; 1885)