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Functional topology and abstract variational theory

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FUNCTIONAL TOPOLOGY

AND

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INTRODUCTION

Abstract basis of the theory. — We are concerned with the existence of equilibria in the large, stable or unstable. The elements may be points, curves, or general configurations. These elements are regarded as points of an abstract metric space M on which there is given a real function F such that $0 \leqq F \leqq 1$. The theory is based on two hypotheses, termed respectively the *F-accessibility* of M and the *upper-reducibility* of F. The critical points of F are topologically defined and our two hypotheses lead to the existence of critical points in a way similar to that in which the compactness of M and lower semi-continuity of F lead to the absolute minimum of F. In the minimum theory compactness and lower semi-continuity imply that any o-dimensional homology class contains a cycle at a minimum level; this is a way of saying that F assumes an absolute minimum on each connected subset of M. Let H be a k-dimensional homology class. The F-accessibility of M implies that the numbers b such that the subset $F \leqq b$ of M contains a k-cycle of H have a minimum s (termed a cycle limit). The upper-reducibility of F then implies that this cycle limit s is assumed by F at some topological critical point.

We note however a vital difference between the minimum theory and the critical point theory.

In the minimum theory it is sufficient to show that the greatest lower bound of F *is assumed at some point by* F *while in the general theory one must show that the cycle limit is not only assumed by* F *at some point but is assumed at some critical point of* F.

The compactness of M and the lower semi-continuity of F imply the F-accessibility of M but are by no means implied by F-accessibility. The cycles used are Vietoris cycles, otherwise the F-accessibility of M would not be implied by compactness and lower semi-continuity. Upper-reducibility and lower semi-continuity are independent properties. A continuous function possesses both of them; but functions exist which are lower semi-continuous without being upper-reducible, and conversely. The functionals of an ordinary positive definite variational problem are upper-reducible. Lower semi-continuity is quite inadequate for the general critical point theory, and must be replaced by upper-reducibility or some related property.

Abstract critical point theory can be applied to the theory of functions of a finite number of variables, for example to harmonic functions of two or three variables (1) (Kiang [1, 2, 3]), to the study of equilibria of floating bodies, to countless geometric problems such as determining normals from a point to a manifold (M [2], p, 4o3), to problems in celestial mechanics (Birkhoff [1, 2]). The most extended applications up to the present time have been in the calculus of variations in the large (M [5, 2]). In this memoir we apply our theory to an abstractly formulated problem in the calculus of variations in the large. We are concerned with « homotopic » extremals which join two fixed points.

In this application we start with an abstract metric space Σ with a symmetric distance function. On Σ we suppose there is given a secondary metric with a distance function in general not symmetric. (*Cf.* Menger [4].) In terms of this secondary metric a length J is

(1) References will be found at the end of the memoir. References to Morse will be indicated by the letter M.

defined. Restricting ourselves to curves η which join two fixed points a and b we set

$$F(\eta) = \frac{J(\eta)}{1 + J(\eta)},$$

when $J(\eta)$ is finite. Otherwise we set $F(\eta) = 1$. Two curves joining a to b are assigned the usual Fréchet distance. The space M here becomes the space of curves η joining a to b, with two curves with a null distance regarded as the same point of M.

There are two principal hypotheses on this function F and space M. The first is that the set of points on Σ at a bounded secondary distance from any fixed point of Σ is compact. This is called *finite* J-*compactness* of Σ. The second is that Σ is *locally* J-*convex* in the sense that points of Σ which are sufficiently near together can be joined by a unique minimizing arc (more precisely defined in § 14). These two hypotheses insure that the space M is F-accessible and that F is upper-reducible. The general theory is thus applicable to the variational problem.

The theory of critical points of functions goes back at least to Kronecker [1]. Poincaré [1] recognized the relation of such a theory to problems in differential equations in the large. The work of Hilbert and Tonelli [1] on the absolute minimum and the concepts of Fréchet [1] and Menger [1, 2, 3, 4] furnish a partial background for the abstract theory. Lusternik [1] and Schnirelmann added interesting ideas. The contributions and applications of Birkhoff [1, 2] have been most significant.

Our bibliography is not meant to be complete but merely to list recent papers used by the author or papers which may be of particular historical interest to the reader. In § 4 and 5 we shall have occasion to refer to hitherto unpublished proofs of an important theorem and a lemma by R. Baer and E. Čech respectively. A more extended bibliography (1) is given in the author's Colloquium Lectures on « The calculus of variations in the large » (M [5]).

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(1) The following book will appear shortly: *Seifert und Threlfall, Variationsrechnung im Grossen* (Theorie von Marston Morse. Teubner, Berlin). This book is highly recommended. The authors begin with two axioms similar to our acces sibility hypothesis, but referring to singular cycles. These axioms are satisfied when the critical values cluster at most at infinity and when the critical points

PART I.

CRITICAL LIMITS.

1. The space M and its topology. — Let M be a space of elements $p, q, r, \ldots$ in which a number pq is assigned to each ordered pair of points such that

$$\text{I. } pp = 0, \qquad \text{II. } pq \neq 0 \text{ if } p \neq q, \qquad \text{III. } pr \leqq pq + rq.$$

Upon setting $r = p$ in III we see that $pq \geqq 0$ and upon setting $p = q$ that $qr = rq$. The space M is termed a metric space. The distance qr is termed symmetric since $qr = rq$. The elements $p, q, r, \ldots$ are termed « points » and pq the « distance » from p to q. Neighborhoods, limit points, sets relatively open or closed can now be defined in the usual way (Hausdorff [1]). In particular if e is a positive number the e-neighborhood A_e of a point set A shall consist of all points p with a distance from A less than e. A set $B \subset M$ (read B on M) will be said to be compact if every infinite sequence in B contains a subsequence which converges to a point in B.

We shall use Vietoris cycles (Vietoris [1]). Singular cycles (Lefschetz [1]) taken in the classical sense are inadequate in a number of ways. This deficiency may be illustrated as follows. Let V be a compact subset of M and V_e the e neighborhood of V. Let u be an arbitrary singular k-cycle not on V. Corresponding to each positive e suppose that u is homologous to a cycle on V_e. It is not always true that u is homologous to a cycle on V as examples would show. The corresponding theorem for Vietoris cycles however is true as we shall see.

We proceed with a systematic outline of the Vietoris theory generalized and modified to meet our needs.

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are isolated. In this way the most important cases are treated in the simplest way. To obtain greater generality Vietoris cycles seem to be useful. In fact the present author has shown in Morse, *Sur le calcul des variations* (*Bulletin de la Société mathématique de France*, 1939), that the accessibility hypothesis is not in general satisfied when ordinary cycles are used, even when F is of class C^n on a regular analytic manifold and when the critical values are finite in number.

Let A be a set of $k+1$ points of M. We term A a *vertex k-cell* of M, as distinguished from an algebraic k-cell now to be defined. For $k>0$ the orderings of A will be divided into two classes, any ordering of one class being obtained from any other of the same class by an even permutation. The vertex k-cell A taken with one of these classes of orderings will be termed a positively oriented k-cell and taken with the other class a negatively oriented k-cell. An oriented k-cell α_k may be represented by a succession

$$\eta A_0 \dots A_k \tag{1.1}$$

of its vertices preceded by η, where $\eta = 1$ or -1 according an the ordering $A_0 \dots A_k$ belongs to α_k or not. We shall say that each oriented $(k-1)$-cell $\eta(-1)^i A_0 \dots A_{i-1} A_{i+1} \dots A_k$ is *positively related* to α_k. We shall say that α_k admits the norm e if the distances between the vertices of α_k are less than e.

The cell α_k will be regarded as null if at least two of its vertices are coincident. Let δ be an element in an arbitrary field Δ (van der Waerden [1]). By a k-chain of norm e is meant a symbolic sum u of the form $\delta_i a_i$, $i=1, \dots, m$, in which δ_i is in Δ and a_i is an oriented k- cell of norm e (We understand that a repeated subscript or superscript is to be summed). The chain u will be termed *reduced* if none of the cells a_i are null, if each is positively oriented and no two cells a_i are identical.

An arbitrary chain will be *reduced* as follows. Let a be an arbitrary positively oriented k-cell and b the corresponding negatively oriented k-cell. Any term of the form δb in the chain u will be replaced by $-\delta a$. All terms involving a will then be summed. Finally all terms involving null cells or coefficients will be dropped. The resulting reduced chain v will be regarded as formally equal to the original chain u. In particular if v is null we regard u as null.

By the sum of two chains $\delta_i a_i$ and $\delta'_j a'_j$ is meant the chain $\delta_i a_i + \delta'_j a'_j$. The expression $\delta[\delta_i x_i]$ shall mean the chain $(\delta\delta_i)x_i$. We understand that a chain u equals a chain v if $u-v$ reduces to the null chain. It is readily seen that k-chains of norm e form an additive abelian operator group (van der Waerden [1], p. 132).

We shall now define the boundary operator β. If a is an oriented k-cell with $k>0$ and e is the unit element in Δ, ea will be a k-chain and βea shall be the $(k-1)$-chain $\Sigma_i eb_i$ where b_i is an arbitrary

$(k-1)$-cell positively related to a. More generally we set $\beta\delta_i a_i = \delta_i \beta e a_i$.

If $k = 0$ we understand that $\beta ea = 0$. If u is an arbitrary k-chain one sees that $\beta\beta u = 0$. *Cf.* Seifert, Threlfall [1], p. 60. We term βu the boundary chain of u. One finds that $\beta\delta u = \delta\beta u$ while for two k-chains u and v, $\beta(u+v) = \beta u + \beta v$.

The preceding k-chains and k-cells are termed *algebraic* k-chains to distinguish them from Vietoris chains to be defined later. The term algebraic will be omitted when it is clear from the context that the chain is algebraic. In particular this will be the case whenever the norm e of the chain is mentioned.

Let B and C be compact subsets of M such that $B \subset C$. An algebraic k-chain u on C will be termed a cycle mod B on C if βu is on B. We term u e-homologous to 0 mod B on C and write $u \sim_e 0$ mod B on C if there exists a $(k+1)$-chain z of norm e on C such that $\beta z = u + v$ where v is a k-chain on B. If $B = 0$, then $v = 0$, and the phrase mod B is omitted.

Let $u = (u_n)$ be a sequence of algebraic k-cycles u_n, $n = 0, 1, \ldots$ mod B on C with norms e_n respectively. If the numbers e_n tend to zero as n becomes infinite, and if for each integer n there exist « connecting » homologies of the form

$$(1.2) \qquad u_n \sim_{e_n} u_{n+1} \qquad (\text{mod B on C}),$$

u is termed a (Vietoris) k-cycle mod B on C and C a *carrier* of u. We write $u \sim 0$ mod B on C if corresponding to each positive number e there exists an integer N so large that $u_n \sim_e 0$ mod B on C whenever $n > N$. The set C is termed a *carrier* of the homology $u \sim 0$ mod B on C. Vietoris k-cycles u and v are termed homologous, $u \sim v$ mod B on C, if $u - v \sim 0$ mod B on C. If (u_n) is a k-cycle mod B on C, any infinite subsequence (v_n) of (u_n) defines a k-cycle $v \sim u$ mod B on C.

The algebraic k-cycles u_n will be termed the *components* of $u = (u_n)$. If u and v are Vietoris k-cycles mod B on C, the algebraic k-cycles $u_n + v_n$ are the components of a Vietoris k-cycle mod B on C which we denote by $u + v$. Similarly the algebraic k-cycles δu_n are the components of a Vietoris k-cycle which we denote by δu.

In the remainder of this memoir the term k-cycle mod B on C shall mean a Vietoris k-cycle mod B on C unless otherwise stated.

We shall have occasion to refer to a k-cycle v mod B' on C' where B' and C' are not necessarily compact. We shall understand thereby that v is a k-cycle mod B on C where B and C are compact subsets of B' and C' respectively and $B \subset C$. Homologies mod B' on C' are similarly defined.

The set of k-cycles homologous to a given k-cycle u is termed the *homology class* defined by u. If u is not homologous to zero, u is termed *non-bounding*. The set of all k-cycles forms a group G_k of which the bounding k-cycles form a subgroup H_k. The group G_k mod H_k is composed of homology classes and is termed the *homology group*. These groups are operator groups; that is if u is an element of one of these groups and $\delta \subset \Delta$, then δu is likewise an element of the group. When $\delta \neq 0$, δ possesses an inverse with respect to division, and the presence of δu in an operator group implies the presence of u in the group. Moreover when $\delta \neq 0$, the relations $\delta u = 0$ and $u = 0$ are equivalent, as well as the homologies $\delta u \sim 0$ and $u \sim 0$.

A sequence of algebraic k-chains w_n on a compact subset x of M, with norms e_n tending to zero will be termed a *formal k-chain* w. We term x a *carrier* of w. The sum $u + v$ of two formal k-chains u and v shall be the formal k-chain whose components are $u_n + v_n$ while δu shall be the formal k-chain whose components are δu_n. Formal k-chains make up an additive abelian operator group. If w is formal k-chain with components w_n, the set of algebraic $(k-1)$-cycles βw_n defines a formal chain which we denote by βw. The formal chain βw will not in general be a Vietoris $(k-1)$-cycle because it will lack the necessary « connecting homologies ».

In the special case where w is a k-cycle mod B on C, βw is a cycle on C. For the homologies « connecting » the components w_n of w imply the existence of an algebraic $(k+1)$-chain z_n on C and a k-chain v_n on B such that $\beta z_n = w_{n+1} - w_n + v_n$, where the norms involved tend to zero as n becomes infinite.

Applying the operator β to both sides of this relation we find that $0 = \beta w_{n+1} - \beta w_n + \beta v_n$, implying homologies connecting the components of βw.

Deformation chains and operators. — Corresponding to any map or deformation of a set $A \subset M$ onto a set B, and corresponding to any algebraic k-chain on A we shall now define an algebraic $(k+1)$-chain D u termed a deformation chain of u.

Let Z be an $(n+1)$-dimensional euclidean prism whose bases are closed n-simplices A and A^*. Let a be A or any simplex on the boundary of A and let λ_a be the lateral face of Z with base a, or if $a = A$ the prism Z. Corresponding to a let P_a be an arbitrary inner point of λ_a. One can subdivide the prisms λ_a in the order of their dimensions into the simplices determined by P_a and the simplices on the boundary of λ_a (supposing that the lateral faces of λ_a have already been subdivided).

Corresponding to each k-simplex w there exist two oppositely oriented algebraic k-cells whose vertices are the vertices of w. We say that these algebraic k-cells are *associated* with w. Let u be an algebraic k-cell associated with one of the k-simplices a of A and let fu be the algebraic k-cell obtained by replacing the vertices of u by corresponding vertices of A^*. It is possible to associate algebraic $(k+1)$-cells with the respective simplices of the subdivided lateral face λ_a in such a manner that the sum Du of these algebraic $(k+1)$-cells is a $(k+1)$-chain whose boundary βDu consists of algebraic k-cells associated with simplices on the boundary of λ_a. If we impose the condition that the chain βDu contain the term u, Du is uniquely determined. If we set $D0 = 0$, we find that $\beta Du = u - fu - D\beta u$. (*Cf.* Seifert, Threlfall [1], § 29).

More generally let the vertices of an algebraic n-cell x on M be mapped onto a set of points on M. Suppose each vertex p on x is thereby replaced by a point fp, and x is replaced by an algebraic n-cell fx. Let the vertices of the preceding n-simplex A be mapped in a one-to-one way onto the vertices of x. Let the vertices of A^* be mapped onto those of fx so that the vertices of A and A^* on the same lateral edge of Z are mapped onto vertices p and fp of x and fx respectively. Under the map of the vertices of A onto vertices of x each k-simplex a of A *determines* a vertex k-cell $\alpha(a)$ of x. Correspon-to α let $Q_{\alpha(a)}$ be an arbitrary vertex of α. We map P_a onto $Q_{\alpha(a)}$.

With the vertices of A and A^* so mapped onto the vertices of x and fx respectively and with the points P_a mapped onto the points $Q_{\alpha(a)}$ each vertex of the subdivision of Z has a unique image on M. Let u be an algebraic k-cell of vertices of A and let v be the corresponding algebraic k-cell of x. The vertices of Du (for each u) will now be replaced by their images on M determining thereby an algebraic $(k+1)$-chain which we denote by Dv. In particulary $D\beta v$ will thereby be defined. It folows that

$$\beta D v = v - fv - D\beta v, \tag{1.3}$$

where fv denotes the image of v in the mapping of x onto fx. We term Dv a *deformation chain* belonging to the map of x onto fx and the points Q_α.

Let T be a mapping of a point set S onto a set fS in which an algebraic j-cell x on S is replaced by an algebraic j-cell fx on fS. Corresponding to each vertex j-cell α on S let a point Q_α of α be uniquely determined. Corresponding to the mapping T, the points Q_α, and any algebraic k-cell v on S, let Dv be a deformation chain defined as in the preceding paragraph. With Dv so defined (for each v and each k) (1.3) will hold. Let u_i be a finite set of algebraic k-cells on S and set $u = \delta_i u_i$. We define Du by the relation $Du = \delta_i Du_i$.
It folows that

$$\beta D \delta_i u_i = \beta \delta_i D u_i = \delta_i \beta D u_i.$$

Upon using (1.3) we see that

$$\beta D u = \delta_i u_i - \delta_i f u - \delta_i D\beta u_i = u - fu - D\beta u. \tag{1.4}$$

We term D a *deformation operator* belonging to T. D is uniquely determined by T and S and by the choice of the preceding points Q_α corresponding to the respective vertex k-cells α on S. If the points Q_α were not uniquely chosen the relations (1.4) would not hold in general. The choice of Q_α must certainly be independent of the $(k+1)$-cells ou whose boundaries α lies.

We shall need the fact that the operators β and f are commutative. That is whenever u is an algebraic k-chain on S,

$$\beta f u = f \beta u. \tag{1.5}$$

It is clear that (1.5) is true when u is a cell. It follows that (1.5) holds as stated.

Let θ be a continuous deformation on M of a set of points A with t the time in the deformation and $0 \leqq t \leqq 1$. Let $t_0 < t_1 \ldots < t_n$ be a set of values of t such that $t_0 = 0$ and $t_n = 1$. Under θ an arbitrary point p_0 of A will be replaced at the times $t_1, \ldots, t_n$ by points $p_1 \ldots,$

p_n respectively. Let T_i denote the mapping of the points p_i on their correspondents p_{i+1}. Let D_i be a deformation operator « belonging » to the mapping T_i. Let z be an arbitrary algebraic k-chain on A and z_i its image under θ at the time t_i. Then for i not summed,

$$\beta D_i z_i = z_i - f_i z_i - D_i \beta z_i \qquad (i = 0, 1, \ldots, n-1), \tag{1.6}$$

where $f_i z_i$ denotes the image of z_i under T_i.

We define Δz as the sum $D_i z_i$ ($i = 0, 1, \ldots, n-1$) understanding that this definition holds for each dimension. Observing that $z_i = f_{i-1} z_{i-1}$ (i not summed) and making use of (1.5) we find that (for i not summed),

$$\beta z_i = \beta f_{i-1} z_{i-1} = f_{i-1} \beta z_{i-1} \qquad (i > 0). \tag{1.7}$$

It follows from (1.6) and (1.7) that

$$\beta \Delta z = z - fz - \Delta \beta z, \tag{1.8}$$

where fz is the final image of z under θ. We term Δ a *deformation operator* belonging to θ and z.

Let z be an algebraic k-chain on A whose images under θ for $0 \leqq t \leqq 1$ admit the norm e. The operator Δ can be so chosen that Δz has the norm e. For z also admits a norm $\delta < e$ provided δ differs sufficiently little from e. Let $\eta = e - \delta$. Upon subdividing the time interval $(0, 1)$ sufficiently finely by the times t_i the preceding mappings will displace the vertices involved a distance less that $\eta/2$ so that the cells of Δz will have norms $\delta + \eta/2 + \eta/2 = e$.

2. F-accessibility. — Let $F(p)$ be a real single-valued function of the point p on M. We suppose that the values of F lie between 0 and 1 inclusive. The functionals of the calculus of variations can be reduced to the form $F(p)$ by a simple transformation as we shall see. By the set $F \leqq b$ will be meant the subset of points of M at which $F(p) \leqq b$.

Let U be an homology class with elements which are non-bounding k-cycles u. If u is on $F \leqq b$, b will be called a *cycle bound* of u and of U. The greatest lower bound of the cycle bounds of U will be called the *cycle limit* $s(u)$ of U and of the elements u of U. If U is the class of bounding k-cycles, $s(u)$ will not be defined.

As pointed out in the introduction we shall make two principal assumptions, namely the assumption of F-accessibility of M and upper-reducibility of F.

Under the hypothesis of F-*accessibility any non-bounding k-cycle which is homologous to zero* mod $F \leqq c + e$ *for each positive e is homologous to a k-cycle on* $F \leqq c$.

If c is a cycle limit of the homology class U, F-accessibility implies that there exists a k-cycle v of U on $F \leqq c$. A non-bounding k-cycle v which lies on the set $F \leqq s(v)$ will be termed *canonical*. Under the hypothesis of F accessibility there is at least one canonical k-cycle in each non-null homology class. In § 5 we shall see that F-accessibility is implied if each set $F \leqq c$ for which $c < 1$ is compact. These conditions for F-accessibility while sufficient are by no means necessary. We shall see that F-accessibility implies that each cycle limit is a cap limit (§ 3) while the hypothesis of upper reducibility will imply (§ 8) that each cap limit is assumed by F at some critical point.

The rôle of a Vietoris cycle in relation to accessibility is shown by the following example. Let the space M consist of the closure in the xy-plane of the set of points $x = \sin 1/y$ where $0 < y \leqq 1$. Let $F = y$ on M. Let p be a point on M at which $F > 0$. The point p can be regarded as the components of a Vietoris 0-cycle u. One sees that $s(u) = 0$ and that there is a canonical 0-cycle in the homology class of u, for example a Vietoris 0-cycle whose components are identical with the origin. If however one regards p as a singular cycle there is no singular cycle on $F = 0$ homologous to p and the hypothesis of F-accessibility fails.

3. **The rank conditions.** — Bounding k-cycles u possess no cycle limits $s(u)$. Let G be the group of all k-cycles. With some but not all of the elements u of G we have thus associated a number $s(u)$. We term $s(u)$ the *rank* of u. The ranks of k-cycles satisfy the following three conditions :

I. *If u has a rank and* $\delta \neq 0$, $s(u) = s(\delta u)$. II. *If* u, v, *and* $u + v$ *have ranks*, $s(u + v) \leqq \max[s(u), s(v)]$. III. *If u and v have unequal ranks*, $s(u + v)$ *exists*.

To verify I we observe that the homologies $u \sim v$ and $\delta u \sim \delta v$ are equivalent if $\delta \neq 0$. I follows directly. To verify II let $\sigma = \max[s(u), s(v)]$ and let e be an arbitrary positive constant. There are cycles u' and v' respectively in the homology classes of u and v and on $F \leqq \sigma + e$. But $u' + v'$ is in the homology class of $u + v$ and on $F \leqq \sigma + e$. Rank condition II follows. To establish III one has merely to show that $u + v$ is non-bounding. If $u + v$ were bounding, u and $-v$ would be in the same homology class so that $s(u) = s(-v) = s(v)$, contrary to hypothesis.

k-caps. We shall now define a new set of ranks termed cap limits. Cap limits also satisfy the rank conditions.

We begin with several definitions. A point set A will be said to be *definitely below* a (written d-below a) if A lies on $F < a - e$ for some positive e. The phrase d-mod$F < a$ shall be understood to mean mod some compact set d-below a. If u is a k-cycle on $F \leqq a$ d-mod $F < a$, an homology

$$u \sim 0 \qquad (\text{on } F \leqq a \ d\text{-mod } F < a) \tag{3.1}$$

will be called an *a-homology*. A k-cycle u on $F \leqq a$ d-mod $F < a$ not a-homologous to zero will be called a k-cap with *cap limit a*. We write $a = a(u)$. We note that $a(u)$ is uniquely determined by the formal k-chain w whose components are the components of u. In fact $a(u)$ is the greatest lower bound c of numbers b such that w is on $F \leqq b$. For the case $a(u) > c$ is impossible since u would then satisfy an $a(u)$-homology. The case $a(u) < c$ is equally impossible since it would imply that w is on $F \leqq a(u) < c$.

The k-cap limits $a(u)$ satisfy the three rank conditions provided $s(u)$ is replaced by $a(u)$ and the group G of k-cycles is replaced by the group of formal k-chains. That I is satisfied follows as for the ranks $a(u)$. That II is satisfied follows from the fact that $a(u)$ is the greatest lower bound of numbers b such that the components of u are on $F \leqq b$. Turning to III suppose that $a(u) < a(v)$. Then u is $a(v)$-homologous to zero. Hence $u + v$ is not $a(v)$ homologous to zero. Since $u + v$ is on $F \leqq a(v)$, $a(v)$ is a cap limit of $u + v$ and III is proved. The k-cap limits also satisfy a fourth rank condition as follows.

IV. *If* $u_1, \ldots, u_m$ and, $v_1 \ldots, v_n$ *have ranks at most* a_0 *while the* *sums* $u = \Sigma u_i$ *and* $v = \Sigma v_j$ *have no rank and* $u + v$ *has a rank, then* $a(u+v) < a_0$.

The cycle limits $s(u)$ satisfy IV vacuously with $s(u)$ replacing $a(u)$.

Recall that a non-bounding k-cycle u is termed canonical if u is on $F \leqq s(u)$ where $s(u)$ is the cycle limit of u. With this understood a first connection between cap limits and cycle limits is as follows.

THEOREM 3.1. — *Under the hypothesis of* F-*accessibility* (1) *a canonical non-bounding* k-*cycle* u *with cycle limit* $s(u)$ *is a* k-*cap with cap limit* $s(u)$.

If u were not a k-cap with cap limit $s(u)$, there would exist some constant b less than $s(u)$ such that u would be homologous to zero mod $F \leqq b$. Under the hypothesis of F-accessibility u would then be homologous to a k-cycle on $F \leqq b$, contrary to the definition of the cycle limit $s(u)$. The proof of the theorem is complete.

4. The rank theory. — Abstracting the relations of § 3 we suppose that we have an additive abelian operator group G with coefficients in the field Δ. With certain of the elements u of G we associate a rank $\rho(u)$ in a simply ordered set $[\rho]$. The rank $\rho(0)$ shall not be defined. In referring to the rank conditions I to IV of § 3 we shall understand that $s(u)$ and $a(u)$ are replaced by $\rho(u)$.

The elements of G with rank (with 0 added) in general will not form a group. We shall nevertheless be able to establish various theorems which have immediate bearing on the existence and enumeration of critical points and limits. We shall be concerned with subgroups of G. We suppose throughout that these subgroups g are operator subgroups, that is if u is in g and δ is in Δ, δu is in g. Our isomorphisms shall be operator isomorphisms, that is, if u corresponds to v, δu corresponds to δv. We shall be concerned with various properties of subsets of elements of G, for example the property of having rank. A property A will be termed an *operator property* if whenever u has the property A and $\delta \neq 0$, δu has the property A. By

(1) This is the only place in § 3 and § 4 where the hypothesis of F-accessibility is used.

a subgroup g of G with operator property A is meant an operator subgroup of G every element of which with the possible exception of o has the property A. The group g will be termed *maximal* if it is a proper subgroup of no subgroup of G with property A. The following example shows that there may be several maximal groups with a given property A.

Let G be a group generated by three elements a, b, c with coefficients in the field of integers mod 2. Suppose that $a, b. a+b, c$ are the only elements of G with the property A. Then a and b together generate a maximal group with property A, as does c.

LEMMA 4.1. — *If g is a maximal subgroup of* G *with operator property* A *and v is an element of* G *with property* A. *there exists an element z in g and an element w which is null or fails to have the property* A, *such that $v = z + w$.*

By virtue of the definition of g as a maximal subgroup of G with property A there exists a $\delta \neq 0$ and an element v_1 in g together with an element v_2 of G which is null or fails to have the property A, such that $\delta v = v_1 + v_2$. But there exists a $\delta' \neq 0$, such that $\delta'\delta = 1$. Upon setting $\delta' v_1 = z$ and $\delta' v_2 = w$, we have $v = z_1 + w$. Since v_1 has the property A and A is an operator property, $\delta' v_1$ has the property A. Similarly $\delta' v_2$ fails to have the property A or is null, and the proof of the lemma is complete.

The *dimension* of the group G is the cardinal number μ of a maximal linearly independent subset of elements of G with coefficients in Δ. Two such groups with the same dimension are operator isomorphic, This fact is easily proved in case μ is finite, and we shall use the fact in no other case.

In the author's earlier work critical points were counted in terms of « type numbers ». These type numbers were dimensions of groups composed of cycles (« type groups », § 9) associated with the respective critical sets. When these dimensions are finite the earlier theory is relatively adequate. But in the general case it is necessary to investigate the « type groups » more closely. The method will be that of comparison of groups by means of isomorphisms. The isomorphisms admitted will be restricted in nature by F. Otherwise the only invariants would be the dimensions.

We begin with a well-known lemma in group theory.

LEMMA 4.2. — *If* H *is an operator subgroup of* G *and* R = G mod H *while* m *is a maximal group of elements of* G *not in* H, *then* R *and* m *are isomorphic in such a manner that an element of* m *corresponds to the coset of* R *to which it belongs.*

It is clear that each element of m is in at least one coset of R and that different elements u and v are in different cosets. Moreover there is an element of m in each coset of R. Otherwise let z be a coset which contains no element of m and let u be an element of z. If v is an element of m, $u - v$ is not in H since u is not in H. Hence $\delta u - \delta v$ is not in H unless $\delta = 0$. Thus u and m generate a group of elements not in H, with m as a proper subgroup, contrary to the nature of m. We conclude that there is an element of m in each coset z of R. The lemma follows readily.

Proceeding with the rank theory we shall say that two elements u and v of G are in the same *ρ-class* or *rank class* if u and v have the same rank while $u - v$ has no rank or a lesser rank. In case ranks are identified with cycle limits, homologous non-bounding k-cycles are in the same rank class, but k-cycles in the same rank class are not necessarily homologous. In case ranks are identified with cap limits two k caps with cap limit a are in the same rank class if and only if their difference is a-homologous to zero. An isomorphism between two subgroups of G of elements with rank will be termed a *rank isomorphism* if corresponding non-null elements are in the same rank class. With this understood we shall prove the following theorem.

THEOREM 4.1. — *When the rank conditions* I *to* IV *are satisfied, any two maximal groups* m_σ *of elements of* G *with the same fixed rank* σ *are rank isomorphic.*

Let g_σ be the group generated by the set of elements of G which possess ranks at most σ. Let H_σ be the subset of elements of g_σ without rank or with rank less than σ. We continue by proving the following statement.

α. *The elements of* H_σ *form a group.*

Let u and v be arbitrary eléments of H_σ. Writing E for « exists » and $\sim$ E for « does not exist », we have four cases :

$$
\begin{array}{lll}
(1) & \rho(u) \ \ \mathrm{E}, & \rho(v) \ \ \mathrm{E}, \\
(2) & \rho(u) \ \ \mathrm{E}, & \rho(v) \sim \mathrm{E}, \\
(3) & \rho(u) \sim \mathrm{E}, & \rho(v) \ \ \mathrm{E}, \\
(4) & \rho(u) \sim \mathrm{E}, & \rho(v) \sim \mathrm{E}.
\end{array}
$$

We shall show that $u+v$ is in H_σ. If $\rho(u+v)$ does not exist, $u+v$ is in H_σ. We assume therefore that $\rho(u+v)$ exists and seek to prove that $\rho(u+v)<\sigma$. This will follow from rank conditions II, III, III, IV respectively in cases (1), (2), (3), (4). In case (1), $\rho(u)$ and $\rho(v)$ are less than σ since u and v are in H_σ. It follows from II that $\rho(u+v)<\sigma$. In case (2), $\rho(u)<\sigma$ and $\rho(u+v)$ must be less than σ, for otherwise $\rho(v)$ would exist. Case (3) is similar. The result in case (4) follows from IV, and the proof of α is complete.

To establish the theorem observe that m_σ is a maximal subgroup of elements of g_σ not in H_σ. By virtue of Lemma 4.2 m_σ and the group $\mathrm{R}_\sigma = g_\sigma \bmod \mathrm{H}_\sigma$ are isomorphic, with elements of m_σ corres ponding to cosets of R_σ which contain them. If m'_σ is a second maximal group of elements of G with rank σ we can use R_σ to establish an isomorphism between m_σ and m'_σ in which corresponding elements are in the same coset of R_σ and so in the same rank class or null.

The following theorem is a consequence of α.

THEOREM 4.2. — *When the rank conditions* I *to* IV *are satisfied, the property of elements with rank being in the same rank class is transitive.*

Suppose u, v, and w have a common rank σ while u and v as well as u and w are in the same rank class. Then $u-v$ and $u-w$ are in H_σ. It follows from α that $v-w$ belongs to H_σ so that v and w are in the same rank class, and Theorem 4.2 is true.

LEMME 4.3. — *If* $u_1, \ldots, u_m$ *are elements of* G *with ranks satisfying* I, II, *and* III *and such that*

$$\rho(u_1) > \rho(u_i) \qquad (i=2, \ldots, m), \tag{4.1}$$

then $\rho(u_1+\ldots+u_m)$ *exists and equals* $\rho(u_1)$.

Suppose first that $m=2$. By virtue of rank condition III $\rho(u_1+u_2)$

exists. From I we see that $\rho(-u_2) = \rho(u_2)$. Since $u_1 = (u_1 + u_2) - u_2$ we can infer from II and (4.1) that

$$\rho(u_1) \leqq \max[\rho(u_1 + u_2), \rho(u_2)] \leqq \rho(u_1 + u_2),$$
$$\rho(u_1 + u_2) \leqq \max[\rho(u_1), \rho(u_2)] \leqq \rho(u_1),$$

so that $\rho(u_1) = \rho(u_1 + u_2)$. The proof of the lemma can be completed by induction with respect to m.

LEMMA 4.4. — *If h is an operator subgroup of* G *with finite dimension r, and with ranks which satisfy* I, II, III, *the elements of h have at most r different ranks.*

Suppose that the lemma is false and that there are elements $u_1, \ldots, u_m$ in h with $m > r$ and with ranks increasing with their subscripts. If $\delta \neq 0$, δu_i has the rank of u_i. It follows from Lemma 4.3 that any proper sum $\delta_i u_i$ has the rank of the non-null term of highest index i, and hence in particular is not o since $\rho(0)$ does not exist. The elements u_i are accordingly independent, contrary to the hypothesis that the dimension of h is r.

Let there be given an operator subgroup g of G and a set of subgroups $h(\alpha)$ of g, α being an enumerating index in a simply ordered set. The group g is said to be a *direct sum*

$$g = \sum_{\alpha} h(\alpha) \tag{4.2}$$

of the groups $h(\alpha)$ if each element u of g is a finite sum of elements from the groups $h(\alpha)$, and if there exists no relation of the form $u_{\alpha_1} + \ldots + u_{\alpha_m} = 0$ in which the α_i' s are distinct and u_{α_i} is a non-null element from the group $h(\alpha_i)$.

THEOREM 4.3. — *Let g be an operator subgroup of* G *whose dimension is at most alef-null. If each element of g save the null element has a rank, and if these ranks satisfy conditions* I, II, III, *then g is a direct sum of suitably chosen maximal subgroups $g(\rho)$ of elements of g with the respective ranks* ρ.

We shall prove the theorem in the case where the dimension r of g is alef-null. The case where r is finite admits a similar proof. The proof here given is due in essence to R. Baer, and was communicated by Baer to the writer.

Since r is alef-null there exists a maximal linearly independent set of elements $a_1, a_2, \ldots$ of g. These elements form a base for g. With Baer we form a new base $(b) = b_1, b_2, \ldots$ as follows. We set $b_1 = a_1$. For each $n > 1$ we set

$$b_n = a_n + \delta_i a_i \qquad (i = 1, \ldots, n-1), \tag{4.3}$$

choosing the elements δ_i so as to make the rank of the right member of (4.3) the least possible. Such a choice is possible, for the right member of (4.3) can take on at most n different ranks in accordance with Lemma 4.4. By virtue of this choice of the elements b_n we can affirm the following :

1° *The ranks of elements of g of the form*

$$b_n + e_i b_i \qquad (i = 1, \ldots, n-1;\ e_i \subset \Delta) \tag{4.4}$$

are at least the rank of b_n.

For the elements (4.4) are of the form $a_n + \delta'_i a_i$ where $i = 1, \ldots, n-1$, and so have ranks at least that of b_n. Let $\rho_1, \rho_2, \ldots$ be the set of distinct ranks of the elements of (b). Let g_k denote the subgroup of g generated by the subset of elements of (b) with the rank ρ_k. It is clear that g is the direct sum

$$g = \sum_k g_k. \tag{4.5}$$

We continue by establishing statements 2° and 3° as follows.

2° *Each non-null element of g_k has the rank ρ_k.*

3° *The group g_k is a maximal subgroup of g with the rank ρ_k.*

To establish 2° let x be an arbitrary non-null element of g_k. In the linear representation of x in terms of the generators b_j of g_k there is a term with greatest subscript j, say n, among terms with non-null coefficients. Let δ be the reciprocal of the coefficient of b_n. Then $\rho(x) = \rho(\delta x)$. In terms of the $b'_j s$, δx is of the form (4.4) and so has a rank at least ρ_k. Hence $\rho(x) \geq \rho_k$. But as an element in g_k the rank of x is at most ρ_k by virtue of rank condition II. Hence $\rho(x)=\rho_k$, and the proof of 2° is complete.

To establish 3° let u be any element of g with rank ρ_k. The element u is a sum $u=u_1+\ldots+u_m$ of non-null elements from different groups g_k, Without loss of generality we can suppose that the ranks of the elements u_i increase with their subscripts. It follows from Lemma 4.3 that $\rho(u)=\rho(u_m)$ so that $\rho(u_m)=\rho_k$. It follows from 2° that u_m is in g_k. We set $u_1+\ldots+u_{m-1}=w$. If $m=1$, then $w=0$. Otherwise $\rho(w)=\rho(u_{m-1})\neq\rho_k$, so that w is not in g_k. Thus $u=u_m+w$, where u_m is in g_k and w is not in g_k or null. Statement 3° follows, and the proof of the theorem is complete.

It is by virtue of the theorems of this section that we shall be able to give precise conditions under which « there are at least as many critical points of type k as there are independent non-bounding k-cycles of dimensions k ». Understanding that the *kth connectivity* of M is the dimension of the *kth* homology group, the following corollary of Theorem 4.3 is a statement of this type.

Corollary 4.3. — *The sum of the dimensions of maximal groups $g(s)$ of non-bounding k-cycles with the respective cycle limits s is at least the smaller of the two numbers alef-null and the kth connectivity* R_k *of* M.

If R_k is at most alef-null the corollary follows from the theorem. If R_k exceeds alef-null, there exists a subgroup g of non-bounding k-cycles of dimension alef-null. Applying the theorem to g with $\rho=s$, the corollary results again as stated (¹).

As a consequence of Theorem 4.1 the dimension of $g(s)$ in the above corollary depends only on s and not on the particular maximal group chosen. A much more general application of rank theory to cycles and caps on M is to be found in M [7]. In particular the latter paper contains a group theoretic formulation of the relations between maximal groups of caps of the different dimensions. When the dimensions of the groups are finite, these general group relations

(¹) We point out that ranks of ordinary singular cycles also satisfy the rank conditions, and that Corollary 4.3 is accordingly true for such cycles. Vietoris cycles enter essentially when questions of F-accessibility enter.

imply the author's first relations between the type numbers and the connectivities of M. See M [1].

5. **Sufficient conditions for accessibility.** — We shall show that the hypothesis of F-accessibility is satisfied provided the subsets $F \leq c$ are compact for each $c < 1$. We shall need various lemmas and theorems in topology.

LEMMA 5.1. — *If the vertices of an algebraic k-cycle u of norm* $\frac{e}{3}$ *are displaced a distance at most* $\frac{e}{3}$ *to define an algebraic k-cycle fu, then* $u \sim_e fu$.

The lemma is an immediate consequence of the existence of a deformation chain Du belonging to u and to the given mapping (displacement) of the vertices of u. Such a chain satisfies (1.4), and the lemma follows directly.

LEMMA 5.2. — *Let e be a positive constant and V a compact metric space. The dimension of a maximal group h of algebraic k-cycles of norm* $\frac{e}{3}$ *on V independent with respect to e-homologies on V is finite.*

Since V is compact there exists a finite set B of points of V such that each point of V has a distance from B less than $\frac{e}{3}$. Let g denote the group of algebraic k-cycles on B. It is clear that the dimension of g is finite. Let z be an algebraic k-cycle of norm $\frac{e}{3}$ on V. Let the vertices of z be mapped onto a subset of the points of B, each vertex of z corresponding to a vertex of B at a distance not exceeding $\frac{e}{3}$. Suppose z is thereby replaced by a cycle fz. It follows from the preceding lemma that $z \sim_e fz$ on V. But fz is in the group g and accordingly admits a representation $fz = \delta_i z_i$, where the elements z_i form a base for g. Hence $z \sim_e \delta_i z_i$. It follows that the dimension of h is at most the dimension of g, and hence finite.

Reduction sets W. — Let V be a compact subset of M. Corresponding to each positive δ let W(δ) be a group of algebraic k-cycles on V of norm δ, such that W(η) is a subgroup of W(δ) whenever $\eta < \delta$. Such a set of groups $W(\delta)$ will be called a *reduction set* W. Concerning these reduction sets we have the following lemma.

LEMMA 5.3. — *Corresponding to an arbitrary positive constant e, there exists a positive constant $\delta < e$ such that to each cycle $w(\delta)$ in $W(\delta)$ and each positive constant $\eta < \delta$ there corresponds at least one cycle $w(\eta)$ in $W(\eta)$, e-homologous to $w(\delta)$ on V.*

The basic idea in the proof of this lemma was communicated to the writer by Professor E. Čech.

For each integer n let $e_n = e3^{-n}$. Let ω denote the subgroup of cycles of $W(e_1)$ e-homologous to zero on V. Let h_1 denote the group $W(e_1)$ mod ω. It follows from Lemma 5.2 that the dimension of h_1 is finite. Recall that h_1 is a group of classes of k-cycles. For each integer $n > 1$ let h_n be the subgroup of those classes of h_1 which contain at least one cycle of $W(e_n)$. We see that

$$h_1 \supset h_2 \supset h_3 \supset \ldots. \tag{5.1}$$

There must accordingly exist a finite integer r such that

$$\dim h_r = \dim h_{r+1} = \ldots.$$

But two abelian operator groups with coefficients in a field and with equal finite dimensions will be identical if one group is a subgroup of the other. Hence $h_r = h_{r+1} = \ldots.$

The constant e was arbitrary and $e_r = e3^{-r}$. Let δ be any positive constant less than e_r. The cycle $w(\delta)$ of the lemma is in $W(e_r)$, and hence in some class of h_r. Corresponding to the constant $\eta < \delta$ of the lemma let p be an integer so large that $e_p < \eta$. Then $e_p < \delta < e_r$ so that $p > r$ and $h_p = h_r$. The cycle $w(\delta)$ is in a class of $h_r = h_p$, and by virtue of its definition this class of h_p contains at least one k-cycle $w(e_p)$ of $W(e_p)$. Since $e_p < \eta$, the cycle $w(e_p)$ is a cycle $w(\eta)$. The cycles $w(\delta)$ and $w(\eta)$ are in the same class of h_p and hence of h_1. That $w(\delta) \sim_e w(\eta)$ follows from the definition of h_1, and the proof of the lemma is complete.

Let $u = (u_n)$ be a k-cycle with carrier C. Let ζ be an algebraic k-cycle on C of norm e, such that $u_n \sim_e \zeta$ for all integers n exceeding some integer N. We then write.

$$u \sim_e \zeta \quad \text{(on C)}. \tag{5.2}$$

If v is any k-cycle such that $u \sim v$ on C, (5.2) implies that

(5.3) $$v \sim_e \zeta \quad \text{(on C)}.$$

Suppose in particular that

(5.4) $$u \sim 0 \quad \text{(mod V on C)},$$

where V is a compact subset of C. I say that (5.4) implies the existence of an algebraic k-cycle $z(\delta)$ on V of arbitrary norm δ such that

(5.5) $$u \sim_\delta z(\delta) \quad \text{(on C)}.$$

Without loss of generality we can suppose that the norm e_n of the homology connecting u_n with u_{n+1} on C tends monotonically to zero as n becomes infinite and that corresponding to e_n, there exists in accordance with (5.4), a relation of the form $\beta w_n = u_n - z_n$, where w_n is an algebraic $(k+1)$-chain of norm e_n on C and z_n is an algebraic k-cycle of norm e_n on V. If $e_n < \delta$ and $m \geqq n$, it follows that $u_m \sim_\delta z_n$ (on C), and setting $z_n = z(\delta)$, (5.5) holds.

We have obtained (5.5) as a consequence of (5.4). The following theorem gives a deeper consequence of (5.4).

THEOREM 5.1. — *If u is a k-cycle with carrier* C, *homologous to zero* mod V *on* C *where* V *is a compact subset of* C, *then* $u \sim v$ on C, *where v is a k-cycle on* V.

We introduce a reduction set W as follows. Let $W(\delta)$ be the group of algebraic k-cycles $w(\delta)$ of norm δ on V such that

$$w(\delta) \sim_\delta 0 \quad \text{(on C)}.$$

To prove the theorem we shall first give an inductive definition of a sequence of positive numbers e_n tending to zero as n becomes infinite. We take e_0 as an arbitrary positive number. Lemma 5.3 applies to the reduction set W. In particular we can set $e = e_{n-1}$ in Lemma 5.3, supposing e_{n-1} already defined. Lemma 5.3 then affirms the existence of a constant $\delta < e$. We take $e_n < \delta$. We also suppose that e_n tends to zero as n becomes infinite.

By virtue of (5.5) there exists an algebraic k-cycle z_n on V of norm e_n such that

(5.6) $$u \sim_{e_n} z_n \quad \text{(on C)}.$$

We shall give an inductive definition of a Vietoris k-cycle $v = (v_n)$ on V. We begin by setting $v_1 = z_1$. Suppose that the components v_i, where $i = 1, \ldots, m$, have been defined in such a way that

(5.7) $$u \sim_{e_i} v_i \quad (\text{on C}; \ i = 1, \ldots, m),$$

(5.8) $$v_{i+1} \sim_{e_{i-1}} v_i \quad (\text{on V}; \ i = 1, \ldots, m-1).$$

We shall define v_{m+1} and show that the relations (5.7) and (5.8) hold for all integers i.

It follows from (5.6) and (5.7) that $z_{m+1} - v_m$ is in $W(e_m)$. We shall apply Lemma 5.3 with $e = e_{m-1}$. By virtue of our choice of $e_m < \delta$ where δ is the δ in the lemma we can apply the lemma with

(5.9) $$z_{m+1} - v_m = w(\delta).$$

Setting $\eta = e_{m+1}$ Lemma 5.3 affirms the existence of a cycle $w(\eta)$ in $W(e_{m+1})$ such that

(5.10) $$w(\delta) \sim_{e_{m-1}} w(\eta) \quad (\text{on V}).$$

Proceeding inductively we define v_{m+1} by the relation

(5.11) $$v_{m+1} = z_{m+1} - w(\eta).$$

Since $w(\eta) \sim_\eta 0$ on C with $\eta = e_{m+1}$,

(5.12) $$v_{m+1} \sim_{e_{m+1}} u \quad (\text{on C})$$

by virtue of (5.6) and (5.11), thus establishing (5.7) for $i = m + 1$. Upon adding (5.9), (5.10) and (5.11) we find that

(5.13) $$v_{m+1} \sim_{e_{m-1}} v_m \quad (\text{on V}),$$

establishing (5.8) for $i = m$. The algebraic k-cycles v_i thus satisfy (5.7) and (5.8) for all positive integers m. By virtue of (5.8) the components v_i admit connecting homologies on V, and by virtue of (5.7) $u \sim v$ on C as stated.

The principal theorem of this section is as follows.

THEOREM 5.2. — *If the subsets* $F \leqq c$ *are compact for* $c < 1$, *the hypothesis of* F-*accessibility is satisfied.*

We begin by proving the following statements.

(α). *Under the hypothesis of the theorem the distances of points* *on* $F \leqq c + e$, $e > 0$, *from* $F \leqq c$ *tend to zero uniformly as e tends to zero.*

If (α) were false there would exist an infinite sequence of positive constants e_n decreasing to zero and corresponding to e_n a point p_n on $F \leqq c + e_n$ such that the distance of p_n from $F \leqq c$ is bounded from zero for all n. This is impossible if $c = 1$. Suppose then that $c < 1$. For n sufficiently large, $c + e_n < 1$ and the points p_n lie on the compact subset $F \leqq c + e_n$. Let q be a limit point of the points p_n. We have $F(q) \leqq c + e_n$. But e_n is arbitrarily small so that $F(q) \leqq c$. On the other hand p must be at a positive distance from the set $F \leqq c$, since the distance of p_n from the set $F \leqq c$ is bounded from zero. From this contradiction we infer the truth of (α).

(β). *If c is the cycle limit of a non-bounding k-cycle u, then* $u \sim 0 \bmod F \leqq c$.

Statement (β) is trivial if $c = 1$. We suppose then that $c < 1$. Let e be an arbitrary positive constant. We seek to prove that there exists a positive integer N such that the components u_n of u for which $n > N$ satisfy the homology

$$u_n \sim_e 0 \quad (\bmod F \leqq c). \tag{5.14}$$

Let δ be so small a positive constant that each point on $F \leqq c + \delta$ is at a distance less than $\frac{e}{3}$ from $F = c$. It follows from the definition of a cycle limit that there exists a k-cycle v in the homology class of u on $F \leqq c + \delta$, and hence with vertices at distances at most $\frac{e}{3}$ from $F \leqq c$. Let the integer N be chosen so large that

$$v_n \sim_{e/3} u_n \quad (n > N). \tag{5.15}$$

Let v_n be mapped onto a chain fv_n on $F \leqq c$ by replacing each vertex of v_n by a nearest point on $F \leqq c$. It follows from Lemma 5.1 that $v_n \sim_e fv_n$. Combining this homology with (5.15), (5.14) is obtained as required.

Te prove the theorem we must show that there is a k-cycle v on $F \leqq c$ in the homology class of u. If $c = 1$, we can take v as u. If $c < 1$, we infer from (β) that $u \sim 0 \bmod F \leqq c$, and conclude from

Theorem 5.1 that there exists a k-cycle v on $F \leq c$ homologous to u. The proof of the theorem is complete.

6. **Homology groups of dimension at most alef-null.** — Theorem 4.3 concerns the decomposition of a group of dimension at most alef-null. The most important application of Theorem 4.3 is to a maximal group H_k of non-bounding k-cycles on $F < 1$. In this section we shall give conditions under which the dimension of H_k is at most alef-null. These conditions are that the sets $F \leq c < 1$ be compact and locally F-connected in a sense which we now define. *Cf.* Lefschetz [2]. These conditions will be shown to be fulfilled in the « locally convex » variational problem.

Let E_n, $n > 0$, be an n-simplex, $\bar{E}_n$ its closure. Let $\bar{E}_n$ be continuously mapped onto M. The resulting image of the boundary of E_n will be termed a *singular* $(n-1)$-sphere and the image of E_n a *singular n-cell.* We shall say that the singular $(n-1)$-sphere bounds the singular n-cell. The images of the vertices of E_n form a vertex n-cell on M *spanned*, as we shall say, by the singular n-cell, and *superficially spanned* by the singular $(n-1)$-sphere.

Let p be a point of M at which $F(p) = c$. The set M will be said to be *locally* F-*connected* of order $m > 0$ at p if corresponding to each positive constant e there exists a positive constant δ such that each singular $(n-1)$-sphere on the δ-neighborhood of p and on $F \leq c + \delta$ bounds an n-cell of norm e on $F \leq c + e$. The constant δ depends upon c, m, e, and p. In this section we shall assume that M is locally F-connected of all orders $m > 0$ at each point of the subset $F < 1$. If the subset $F \leq c < 1$ is compact it is clear that for c, m, and e fixed the preceding constant δ may be choson so as to be independent of p on $F \leq c$.

We shall say that a set Z of algebraic cells can be e-spanned on a subset B of M if the vertex i-cells of cells of Z can be successively spanned in the order of their dimensions by singular i-cells of norm e on B with the following properties. The singular $(k-1)$-cells spanning the vertex $(k-1)$-cells of a vertex k-cell x combine (with proper closure) to form a singular $(k-1)$-sphere superficially spanning x while this $(k-1)$-sphere bounds the singular k-cell spanning x.

Let z be an algebraic μ-chain. A process by virtue of which each vertex k-cell ($k = 0, \ldots, \mu$) is e-spanned will be termed an *e-spanning* of z. Corresponding to an e-spanning of z there exists a new chain z^*, termed the *first subdivision* of z, and constructed as follows. We first replace each algebraic 0-cell of z by itself. Let u be an arbitrary algebraic k-cell of z with $k > 0$, and let $S(u)$ be the singular k-cell spanning u. Let P_u be an arbitrary point of $S(u)$. Proceeding inductively we suppose that the algebraic $(k-1)$-cells of βu have been replaced by a set of algebraic $(k-1)$-cells forming an algebraic $(k-1)$-cycle v. Let

$$(6.1) \qquad \delta A_1 \ldots A_k \qquad (\delta \subset \Delta)$$

represent an arbitrary term in the « reduced » form of v. We replace u by a sum

$$(6.2) \qquad \Sigma \delta P_u A_1 \ldots A_k$$

of terms obtained by adding P_u as in (6.2) to each term (6.1) in the reduced form of v. It is clear that the boundary of the chain (6.2) is v. Moreover the inductive hypothesis that βu is replaced by an algebraic $(k-1)$-cycle v is readily verified when k is replaced by $k+1$. *Cf.* proof of Lemma 6.1. The inductive definition of z^* is complete.

Upon referring to the k-simplices of which the above singular k-cells u are images, it appears that the point P_u can serve as a new vertex of a subdivision of the singular k-cell $S(u)$ into a set of new singular k-cells determined as the join (relative to straightness on the defining k-simplex) of P_u and the singular $(k-1)$-cells on the previously subdivided boundary of $S(u)$. These new singular k-cells will span the new algebraic μ-cycle z^*, and may serve to define a subdivision of z^*, or as we shall say, a « second subdivision » of z. An algebraic k-cycle z which can be e-spanned thus admits an infinite sequence of corresponding subdivisions $z_1, z_2, \ldots$. If the new vertices P_u of these subdivions are properly chosen the norm of z_n will tend to zero as n becomes infinite. We admit only such infinite sequences of subdivisions.

Lemma 6.1. — *The boundary of a subdivision of an algebraic k-chain z is the corresponding subdivision of the boundary of z.*

We shall designate the first subdivision of a chain by prefixing the letter φ. It is understood that we are concerned with a unique spanning of all algebraic cells involved and with the corresponding first subdivision. Let u_i be an arbitrary algebraic k-cell for which φu_i exists. As seen in connexion with (6.2)

$$\beta\varphi u_i = \varphi\beta u_i. \tag{6.3}$$

Moreover for any finite sum $\delta_i u_i$, $\varphi\delta_i u_i = \delta_i\varphi u_i$.
Upon using these relations we find that

$$\beta\varphi\delta_i u_i = \beta\delta_i\varphi u_i = \delta_i\beta\varphi u_i = \delta_i\varphi\beta u_i = \varphi\beta\delta_i u_i,$$

and the proof of the lemma is complete.

LEMMA 6.2. — *If z^* is a subdivision arising from an e-spanning on* B *of an algebraic μ-cycle z, then*

$$z^* \sim_e z \qquad (\text{on B}). \tag{6.4}$$

Lemma 6.2 follows from the theory of deformation chains and in particular from (1.4). For the vertices of z^* can be mapped onto the vertices of z as follows. Each vertex of z^* not a new vertex P_u shall be mapped onto itself, while P_u shall be mapped onto an arbitrary vertex of u. The algebraic k-cycle thereby replacing z^* « reduces » to z. The deformation chain Dz of (1.4) has the norm e so that $z^* \sim_e z$, as stated.

(α). Let $z_1, z_2, \ldots$ be an infinite sequence of subdivisions of an algebraic μ-cycle z e-spanned on B. Let K be the sum of the closures of the singular μ-cells spanning the respective vertex μ-cells of z. The set K is compact. Upon applying the preceding lemma with K replacing B we see that the successive subdivisions z_n admit connecting homologies on K with norms which tend to zero as n becomes infinite. Hence the sequence z_n defines a Vietoris μ-cycle with carrier $K \subset B$. We denote such a cycle by $V(z)$, and term $V(z)$ a Vietoris μ-cycle *derived* from z.

LEMMA 6.3. — *Let u and v be algebraic m-cycles which bound an algebraic $(m+1)$-chain w admitting a spanning on* B *by virtue of which u is η-spanned on* B *and a Vietoris cycle (v_n) is*

« *derived* » *from* v. *Then*

$$u \sim_{\eta} v_r \quad \text{(on B)} \tag{6.5}$$

for all sufficiently large integers r.

Corresponding to the given spanning of w let (w_n) and (u_n) be infinite sequences of subdivisions of w and u respectively. If r is so large that the norm of w_r is less than η, then $u_r \sim_{\eta} v_r$ on B in accordance with Lemma 6.1. By virtue of Lemma 6.2, $u \sim_{\eta} u_r$ on B for all integers r. Combining these homologies, (6.5) follows as stated.

(β). Under the hypotheses that the sets $F \leq c < 1$ are compact and that M is locally F-connected of all orders at points of $F < 1$, the following statements are readily seen to be true. Corresponding to a positive constant e, a constant $c < 1$ and a positive integer m there exists a positive constant σ and a positive function $\theta(\eta)$ defined for $0 < \eta \leq \sigma$, tending to zero with η and possessing the following property. The set of all algebraic cells on $F \leq c$ of dimensions at most $m + 1$ and norms σ admits an e-spanning on $F \leq c + e$ in which a singular k-cell spanning a vertex k-cell of norm η has a norm $\theta(\eta)$. The function $\theta(\eta)$ depends on c, e and m.

The group $\Gamma_m(c)$. Under the hypotheses of (β) and corresponding to the constants c, m and σ of (β) let $(a_1, \ldots, a_q) = (a)$ be a maximal linear set of algebraic m-cycles of norm $\frac{\sigma}{3}$ on $F \leq c$, independent with respect to σ-homologies on $F \leq c$. That the number of cycles in such a linear set is finite follows from Lemma 5.2. Let the set of all vertex k-cells of norm σ on $F \leq c$ for which $0 < k \leq m + 1$ be e-spanned in accordance with the conditions of paragraph (β). So spanned let $V(a_i)$ be a Vietoris m-cycle derived from a_i in accordance with (α). Corresponding to each algebraic m-cycle $u = \delta_i a_i$ we set

$$V(u) = \delta_i V(a_i).$$

When $u \neq 0$, $V(u)$ is a Vietoris m-cycle « derived » from u. The Vietoris m-cycles $V(u)$ form a group $\Gamma_m(c)$ with the finite base $V(a_1), \ldots, V(a_q)$.

THEOREM 6.1. — *If the subsets* $F \leq c < 1$ *of* M *are compact and* M *is locally* F-*connected of all orders at points of* $F < 1$, *then for* *a fixed $m > 0$ and $c < 1$ any Vietoris m-cycle $y = (y_n)$ on $F \leqq c$ is homologous on $F \leqq c + e$ to a cycle $V(u)$ of the group $\Gamma_m(c)$.*

Without loss of generality we can assume that the norms of the components (y_n) of y are at most $\frac{\sigma}{3}$ where σ is the constant described in (β). The algebraic m-cycle y_n together with an algebraic m-cycle (¹) of the form $u = \delta_i a_i$ bounds an algebraic $(m+1)$-chain w_n on $F \leqq c$, of norm σ by virtue of the choice of the base (a). If w_n is spanned with the aid of the singular cells described in (β), u will be similarly spanned (¹) and determine the Vietoris m-cycle $V(u)$. If e_n is the norm of y_n, y_n will thereby be spanned by singular cells of norm $\eta = \theta(e_n)$. It follows from Lemma 6.3 that y_n is η-homologous on $F \leqq c + e$ to each component V_r of $V(u)$ for which r is sufficiently large. Recalling that $\theta(e_n)$ tends to zero with e_n we see that the Vietoris m-cycles y and $V(u)$ are homologous on $F \leqq c + e$.

The principal theorem of this section is as follows.

THEOREM 6.2. — *If each subset $F \leqq c < 1$ of M is compact and M is locally F-connected of all orders at points of $F < 1$, the dimension of the mth homology group of $F < 1$ is at most alef-null.*

Let S be a maximal linear set of Vietoris m-cycles on $F < 1$, non-bounding on $F < 1$. The number of cycles of S on the respective sets $F \leqq 1 - \frac{1}{n}$, $n = 1, 2, \ldots$ is finite by virtue of the preceding theorem. Hence the number of cycles in H is at most alef-null.

It is clear that the proofs of the two preceding theorems make use of local F-connectedness merely of the orders 1 to $m + 1$.

PART II.

CRITICAL POINTS.

7. F-deformations. — The fundamental theorem of Part II is that each cap limit is assumed by F at some homotopic critical point provided F is upper-reducible. We shall presently define a homotopic

(¹) Independent of n and w_n.

critical point. In the following section upper-reducibility will be defined and the theorem proved. We do not assume that the sets $F \leqq c < 1$ are compact. We shall begin by defining F-deformations. They are abstract generalizations of deformations along the orthogonal trajectories of the manifolds $F =$ constants when such manifolds and trajectories exist.

Let E be a subset of M. We shall admit deformations D of points initially on E which replace a point p on E at the time $t = 0$ by a point $q = q(p, t)$ $(p \subset E;\ 0 \leqq t \leqq \tau)$ on M at the time t, where t varies on the closed interval $(0, \tau)$. We shall suppose that τ is a positive constant and that $q(p, t)$ is a continuous point function of its arguments. Such deformations will be termed *admissible*. The curve $q = q(p, t)$ obtained by holding p fast and varying t will be termed the *trajectory* T defined by p. If a point q precedes a point r on the trajectory T, q will be termed an *antecedent* of r.

We shall say that the deformation D admits a *displacement function* $\delta(e)$ on E, if whenever q is an antecedent of r such that $qr > e > 0$, then

$$F(q) - F(r) > \delta(e), \tag{7.1}$$

where $\delta(e)$ is a positive single-valued function of e. An admissible deformation of E which possesses a displacement function on each compact subset of E will be termed an F-*deformation* of E. A deformation in which $q(p, t) \equiv p$ is an F-deformation and will be termed a *null* deformation.

If F is continuous and E compact any admissible deformation such that $F(q) > F(r)$ whenever q is an antecedent of r distinct from r, is an F-deformation. This follows from the uniform continuity of F on a compact set. If however F is merely lower semi-continuous the situation is not so simple as examples will show. See Example 9.1, M [7].

A point p will be said to be *homotopically ordinary* if some neighborhood of p relative to $F \leqq F(p)$ admits an F-deformation which displaces p. A point which is not homotopically ordinary will be termed *homotopically critical*. Simple examples of homotopic critical points are maximum and minimum points and saddle points of surfaces.

To illustrate these ideas we shall consider the case of a function

$F(x_1, \ldots, x_n)$ of class C^2 in an open region R of the space (x). We term $(x) = (a)$ *differentially critical* if the first partial derivatives of F vanish at (a). Otherwise we term (a) *differentially ordinary*. We shall show that a point which is differentially ordinary is homotopically ordinary.

To that end let $x_i = x_i(a_1, \ldots, a_n, t) = x_i(a, t)$ be a trajectory defined by the differential equations

$$\frac{dx_i}{dt} = -F_{x_i} \quad (i = 1, \ldots, n) \tag{7.2}$$

with the initial conditions $x_i(a, 0) = a_i$. A deformation which replaces (a) by the point $[x(a, t)]$ is an F-deformation neighboring any ordinary point (a^0) of R, provided t be restricted to a sufficiently small interval $(0, \tau)$. For under such conditions it follows from (7.2) that

$$\frac{dF}{dt} = -F_{x_i}F_{x_i} < \text{const.} < 0.$$

It is thereby seen that a point which is differentially ordinary is homotopically ordinary. Hence a homotopic critical point of F is a differential critical point. The converse is not true. For example $x = 0$ is a differential critical point of $F = x^3$, but not a homotopic critical point.

We now develop certain properties of F-deformations. Let D be a deformation of a set A. The set of final images of points of A under D will here be denoted by DA. Let $B_1, \ldots, B_n$ be a set of F-deformations, such that B_1 is applicable to A, B_2 is applicable to B_1A, or more generally B_{i+1} is applicable to $B_iB_{i-1} \ldots B_1A$. In such a case the deformations $B_1 \ldots B_n$ will be said to define the product deformation $\Delta = B_n \ldots B_1$ of A. Under Δ a point p is deformed under B_1 into B_1q, the point B_1q is then deformed under B_2 into B_2B_1q, and so on until $B_{n-1} \ldots B_1q$ is deformed under B_n into the final image of q under Δ.

LEMMA 7.1. — *The product $\Delta = B_n \ldots B_1$ of F-deformations B_i with various domains of applicability is an F-deformation of any set A to which Δ is applicable.*

That Δ is a continuous deformation of A is clear at once. Let C be a compact subset of A. We continue by showing that Δ admits a displacement function $\delta(e)$ belonging to C.

Let $\delta_1(e)$ be a displacement function for B_1 applied to C. For $i > 1$ let $\delta_i(e)$ be a displacement function for B_i applied to $B_{i-1} \ldots B_1 C$. Let q be an antecedent of r under Δ on a trajectory λ whose initial point is $p \subset C$. If $qr > e$, at least one of the deformations B_i must have displaced a successor of p on λ between q and r a distance greater than $\frac{e}{n}$. Since the change of F as q moves along λ from q to r is the sum of the changes in F under the different deformations B_i we see that

$$F(q) - F(r) > \min \delta_i\left(\frac{e}{n}\right) \qquad (i = 1, \ldots, n). \tag{7.3}$$

The right member of (7.3) thus serves as a displacement function $\delta(e)$ for C, and the proof of the lemma is complete.

The following lemma concerns the extension of an F-deformation beyond its original domain of definition.

LEMMA 7.2. — *Let* A *and* B *be subsets of* M *such that* $A \subset B$, *and let* A_e *be an e-neighborhood of* A *relative to* B *which admits an* F*-deformation* D *on* M. *There exists an* F*-deformation* θ *of* B *which deforms points initially on* $A_{e/3}$ *as does* D, *and subjects the points of* B *not on* $A_{2e/3}$ *to the null deformation.*

Suppose the time t in D varies on the interval $(0, \tau)$. Under θ the time t shall likewise vary on $(0, \tau)$. Points of B initially on $A_{e/3}$ shall be deformed under θ as under D while points of B not on $A_{2e/3}$ shall be held fast. For points q of B whose distance $d(q)$ from A is such that

$$\frac{e}{3} < d(q) \leqq \frac{2e}{3} \tag{7.4}$$

we define θ as follows. Let t_q divide the interval $(0, \tau)$ in the ratio inverse to the ratio in wich $d(q)$ divides the interval (7.4). Under θ points q of B which satisfy (7.4) initially shall be deformed as under D until t reaches t_q, and shall be held fast thereafter. It follows that θ deforms points initially on B continuously. Further if C is any compact subset of B the closure K relative to C of $C.A_{2e/3}$ is a compact set on A_e. One sees that θ admits the same displacement functions on C as does D on K. Thus θ is an F-deformation of B, as stated.

Let D be an F-deformation of a set A. An F-deformation θ of the set A will be said to be *related* to D if the trajectory of each point p of A under θ is a subarc of the trajectory of p under D and if there exist positive constants σ and τ with the following property. Every point of A which is displaced under D a distance at most σ has the same trajectory under θ as under D, while points of A which are at any time displaced a distance exceeding σ under D are at some time displaced a distance exceeding τ under θ. We shall need the following lemma.

LEMMA 7.3. — *Corresponding to an* F*-deformation of a set* A *and a positive constant e, there exists a « related »* F*-deformation θ of* A *in which no point of* A *is displaced a distance greater than e.*

It might seem that one could obtain θ from D by merely shortening the time interval for D, but this is hardly the case since there may exist points of A which are displaced early in D while other points are not displaced at all during the first part of the time interval, but are displaced during the latter part of the time interval. This difficulty will be met by making a change of parameter on the trajectories of D, passing from t to an intrinsic parameter μ, with the property that the point $q(\mu)$ on a given curve is displaced whenever μ is varied. This parametization in terms of μ has other important properties of great use in our variational theory. We shall describe this parameterization and its properties.

μ-parameterizations. — For each t on an interval (o, a), with $a \geqq o$, let $q(t)$ be a point on M which varies continuously with t. The set of points $q(t)$ taken in the order of the corresponding values of t will be termed a *parameterized* curve (written p-curve). In general p-curves will be denoted by Greek letters α, β, γ, etc. while points on M will be denoted by letters p, q, r, etc.

The Fréchet distance $\eta\zeta$ between two p-curves η and ζ will now be defined. Suppose η and ζ are represented in the forms

$$p = p(t) \qquad (o \leqq t \leqq a), \tag{7.5'}$$

$$q = q(u) \qquad (o \leqq u \leqq b), \tag{7.5''}$$

respectively. Suppose first that a and b are positive and let ω represent any sense-preserving homeomorphism between the intervals $(0, a)$ and $(0, b)$. Let $d(\omega)$ be the maximum distance between points of η and ζ which correspond under ω. The *Fréchet distance* $\eta\zeta$ shall be the greatest lower bound of the numbers $d(\omega)$ for all admissible ω. If $a=0$, η reduces to a point and the distance $\eta\zeta$ shall be the distance of this point from ζ. When $b=0$, $\eta\zeta$ is similarly defined.

We understand that the p-curves η and ζ are *identical* if and only if $a=b$ and $p(t)\equiv q(t)$. It is readily seen that $\eta\zeta$ may be zero without η and ζ being identical. However one notes that $\eta\zeta=\zeta\eta$ and if λ is a third p-curve that

$$\eta\zeta \leqq \eta\lambda + \lambda\zeta. \tag{7.6}$$

The set of p-curves at a null Fréchet distance from a given p-curve will be called a *curve class*, or more briefly a *curve*.

We are seeking a parameterization $q(\mu)$ in which q is displaced when μ is varied. Parameterizations in terms of arc length have this property when they exist. Such parameterizations however fail to exist for certain curves, and fail to have the important property that points on η and ζ bearing the same parameter s will be arbitrarily near for η fixed and $\eta\zeta$ sufficiently small. It is however possible to single out from each curve class α a unique p-curve φ with the desired properties. The parameter of φ will be denoted by μ and termed *μ-length*, and φ will be called a *μ-curve*. The characteristic properties of these μ-curves are as follows.

a. If $\eta: p=p(t)$ is an arbitrary p-curve of the curve class α, the corresponding μ-curve φ takes the form

$$q=q(\eta,\mu)=p[t(\mu)] \qquad [0\leqq\mu\leqq\mu(\eta)],$$

where $t(\mu)$ is a continous non-decreasing function of μ on the closed interval $[0, \mu(\eta)]$.

b. The value of μ at an arbitrary point q on φ satisfies the condition

$$\frac{d}{2}\leqq\mu\leqq d, \tag{7.7}$$

where d is the diameter of the set of points preceding q on φ.

c. The limit $\mu(\eta)$ is a continous function of η, for η on the Fréchet space of p-curves, and is independent of η in the curve class α.

d. For η fixed $q(\eta, \mu)$ is constant with respect to μ on no subinterval of $[o, \mu(\eta)]$.

e. The function $q(\eta, \mu)$ is continuous for η on the Fréchet space and μ on $[o, \mu(\eta)]$.

The proofs of these properties are to be found in M [6]. *Cf.* Whitney [1, 2] and Fréchet [2]. Whitney deals with curves without multiple points. The intrinsic parameterizations of Fréchet do not have the property *e*. We add the following definition. The *distance* between two curve classes λ and ζ shall be the Fréchet distance between any two p-curves in the classes λ and ζ. It is thereby uniquely defined.

Proof of Lemma 7.3. — Under D a point p of A determines a trajectory which we denote by $\eta(p)$. Regarding $\eta(p)$ as a p-curve in the Fréchet space, recall that $\eta(p)$ varies continously with p. Let $q = q(\eta, \mu)$ be the « μ-curve » in the curve class of $\eta(p)$. Recall that $q(\eta, \mu)$ is continuous in its arguments provided μ varies on the interval $[o, \mu(\eta)]$. Set

$$\mu[\eta(p)] = \bar{\mu}(p) \tag{7.8}$$

whenever the left member of (7.8) is less than $\frac{e}{2}$, where e is the constant e of the lemma. Otherwise let $\mu(p) = \frac{e}{2}$. Set

$$q[\eta(p), t\bar{\mu}(p)] = r(p, t) \qquad (p \subset A\,;\ o \leqq t \leqq 1).$$

The point function $r(p, t)$ defines a deformation θ of A satisfying the lemma as we shall see.

It is clear that the point $r(p, t)$ varies continuously on M for p on A and $o \leqq t \leqq 1$. Moreover θ is an F-deformation of A. For its trajectories are in the curve classes of subarcs of trajectories of D so that θ admits the same displacement functions as does D. To show that θ is « related » to D we shall first show that the constant σ in the definition of the term « related » can be taken as $\frac{e}{2}$. For a point p

which is displaced at most $\frac{e}{2}$ under D thereby defines a trajectory η on which $\mu(\eta) \leqq \frac{e}{2}$ by virtue of (7.7) so that p has the same trajectory under θ as under D. The remaining points p of A give rise to trajectories η under D for which $\mu(\eta) > \frac{e}{4}$ by virtue of (7.7). Such points will define trajectories T under θ on which μ will increase beyond $\frac{e}{4}$. The diameters of these trajectories T will then be at least $\frac{e}{4}$ again by virtue of (7.7). Thus θ is « related » to D.

Finally no point p of A is displaced a distance greater than e under θ. For p is deformed under θ along a trajectory on which μ never exceeds $\frac{e}{2}$ so that the diameter of this trajectory is at most e, and the proof of Lemma 7.3 is complete.

Lemma 7.4. — *If an* F-*deformation* Δ *of a compact set* A *on* $F \leqq c$ *carries* A *into a set d-below* c, *any* « *related* » F *deformation* θ *of* A *will carry* A *into a set d-below* c.

The set A is the sum $A' + A''$ of sets of the following nature. A' consists of points which are displaced at least a distance τ under θ, where τ is a positive constant, while A'' consists of points deformed under θ as under Δ. The closure $\overline{A}'$ of A' is compact. Its points are displaced under θ at least the distance τ and so are deformed under θ onto $F \leqq c - \delta(\tau)$, where $\delta(e)$ is the displacement function of θ on $\overline{A}'$.

8. **Upper-reducibility and the fundamental theorem.** — The function F will be said to be *upper-reducible* at p if corresponding to each constant $c > F(p)$ some neighborhood of p relative to $F \leqq c$ admits an F-deformation onto a set d-below c. If F is upper-reducible at each point p of a set B, F will be said to be upper-reducible on B.

A function F which is lower semi-continuous is not necessarily upper-reducible. For example let M be the semi-circle $x^2 + y^2 = 1$ with $y \geqq 0$. Let $F(x, y) = y$ on M for $x \neq 0$, and let $F(0, 1) = 0$. The function $F(x, y)$ is lower semi-continuous. It is not upper-reducible at $(0, 1)$. A function which is upper-reducible is not necessarily lower semi-continuous. For example let M be the semicircular disc $x^2+y^2\leqq 1$ with $y\geqq 0$. Let $F(x, y)=0$ on M for $x\neq 0$, and let $F(0, y)\equiv y$. The function F is upper-reducible without exception. It is not lower semi-continuous on the positive y axis.

A function which is continuous at p is upper-reducible at p. For if $c>F(p)$, any sufficiently small neighborhood of p will be d-below c, and will remain there under the null F-deformation. We shall see that the functionals of ordinary variational theory are upper-reducible, together with the more general functionals of our abstract locally convex variational theory.

Lemma 8.1. — *Let* C *be a compact subset of* $F\leqq c$ *which contains no homotopic critical points at which* $F=c$. *If* F *is upper-reducible at points of* C, *there exists an* F-*deformation* Δ *of* $F\leqq c$ *in which* C *is carried into a set d-below c.*

Corresponding to each point p of C some spherical neighborhood $V(p)$ of p relative to $F\leqq c$ admits an F-deformation $D(p)$ onto a set d-below c. This is true if $F(p)=c$ since p is then homotopically ordinary, and it is true if $F(p)<c$ since F is upper-reducible at p. Let $V'(p)$ and $V''(p)$ be respectively spherical neighborhoods of p with radii one third and one sixth that of $V(p)$.

Since C is compact there exists a finite set of the neighborhoods $V''(p)$, say $V''(p_1), \ldots, V''(p_n)$ which covers C. Upon setting $A=p_i$ in Lemma 7.2 we infer the existence of an F-deformation B_i of $F\leqq c$ which deforms $V'(p_i)$ as under $D(p_i)$. Let e be the minimum of the radii of the neighborhoods $V''(p)$. By virtue of Lemma 7.3 there exists an F-deformation θ_i « related » to B_i under which no point of $F\leqq c$ is displaced a distance exceeding $\frac{e}{n}$. The product deformation $\Delta=\theta_n\ldots\theta_1$ is an F-deformation of $F\leqq c$ by virtue of Lemma 7.1. I say that Δ carries C into a set d-below c.

Under Δ, $V''(p_i)$ is deformed on $V'(p_i)$ since Δ displaces no point a distance greater than e. The deformation θ_i being « related » to B_i deforms any compact subset of $V'(p_i)$ into a set d-below c in accordance with Lemma 7.4. Hence Δ deforms any compact subset of $V''(p_i)$ into a set d-below c. But C is covered by the neighborhoods $V''(p_i)$, and so is deformed by Δ into a set d-below c. The proof of the lemma is complete.

The main theorem of Part II is as follows :

THEOREM 8.1. — *If* F *is upper-reducible, each cap limit c is assumed by* F *in at least one homotopic critical point p.*

Let u be a k-cap with cap limit c. By definition u admits a compact carrier $\varkappa$ on $F \leqq c$ while βu lies on $F \leqq b'$, where b' is some constant less than c. Suppose the theorem false. According to the preceding lemma there will then exist an F-deformation Δ of $F \leqq c$ in which k is carried into a set on $F \leqq b''$, where $b'' < c$. If u_n is the n th component of u, there exists a deformation chain $\Delta_n u_n$ of norm e_n belonging to Δ and u_n such that

$$\beta \Delta_n u_n = u_n - f u_n - \Delta \beta u_n, \tag{8.1}$$

where $f u_n$ is the final image of u_n under Δ and where e_n tends to zero with n. Let b be the larger of the constants b' and b''. It follows from (8.1) that

$$u_n \sim_{e_n} 0 \qquad (\text{on } F \leqq c \bmod F \leqq b). \tag{8.2}$$

According to (8.2), u is c-homologous to zero, contrary to the hypothesis that the cap limit of u is c. We infer the truth of the theorem.

9. Critical sets and their type groups. — In this section we are concerned with the counting and classification of critical points. Consider for example a harmonic function $u(x, y)$ which has a critical point at the origin at which all of the partial derivatives up to but not including those of the n th order vanish. The function $u(x, y)$ is the real part of an analytic function of the complex variable $z = x + iy$ which has a zero of the $(n - 1)$st order at $s = 0$. If one wishes to use the critical point theory to derive and extend the classical theorems on the number of zeros of an analytic function in a given region, it will be necessary to count the origin as if it were a non-degenerate critical point of index one taken $n - 1$ times,

A non-degenerate differential critical point of index k naturally counts as just one such point. In the general case where a critical set is highly degenerate or is more than 0-dimensional, it is possible to give a group theoretic local topological mode of counting and classifying critical sets. Corresponding to each dimension k one associates with the given critical set σ a class of isomorphic groups of k-caps termed the k th type groups of σ. The dimension of these groups is the k th type number of σ. For the case of the harmonic function in the preceding paragraph the 1st type number will be $n-1$, the other type numbers zero. The k th type number of a non-degenerate differential critical point of index 1 is 1, the other type numbers 0. A critical set with k th type number μ_k can be regarded as equivalent (with respect to certain properties) to μ_k non-degenerate critical points of index k. Our mode of counting critical sets will lead to a proof of the following important theorem.

THEOREM 9.1. — *If* M *is* F*-accessible and* F *is upper-reducible on* $F < 1$, *the sum of the kth type numbers of the respective critical sets on* $F < 1$ *is at least the smaller of the two cardinal numbers, alef-null and the kth connectivity of* $F < 1$.

We begin with several definitions. By the *complete critical* set ω at the level c is meant the set of all homotopic critical points at which $F = c$. It follows from the definition of a critical point that ω is closed relative to the set $F = c$. By a *critical set* σ at the level c will be meant any subset of ω which is closed in ω and at a positive distance from $\omega - \sigma$. A neighborhood of σ which is at a positive distance from $\omega - \sigma$ will be termed *separate*.

Let R be a subset of M. If we regard R as a space M, a k-cap *relative* to R has a new meaning dependent on R. The type groups which we shall associate presently with each critical set σ should be definable in terms of the values of F on arbitrarily small neighborhoods of σ. To this end the following theorem is fundamental.

THEOREM 9.2. — *Let* U *be a separate neighborhood of a critical set* σ *at the level* c *and suppose that* F *is upper-reducible on* $F \leqq c$.

a. If u *is a* k*-cap relative to* U *with cap limit* c, u *is a* k*-cap relative to* M.

b. If u *is a* k*-cap relative to* U *with cap limit* c, u *is* c*-homologous on* U *to a* k*-cap on an arbitrarily small neighborhood of* σ.

In proving this theorem the closure of the e-neighborhood of a subset g of M will be denoted by g^e.

Proof of (a). — If (a) were false there would exist a formal $(k+1)$-chain w on $F \leq c$ together with a formal k-chain y d-below c such that

$$\beta w = u - y. \tag{9.1}$$

We shall show that there then exists a formal $(k+1)$-chain w'' on U and a formal k-chain y'' on U and d-below c such that

$$\beta w'' = u - y'', \tag{9.1$'$}$$

thereby contradicting the hypothesis that u is a k-cap relative to U with cap limit c.

Let ω be the complete critical set at the level c and let $\tau = \omega - \sigma$. Let $\varkappa$ be a carrier of u on U and on $F \leq c$. Let e be a positive constant so small that σ^{3e} and $\varkappa^{3e}$ are on U while τ^{3e} and U are disjunct. Without loss of generality we can suppose that w and u admit the norm $e/3$.

Let B be a carrier of w on $F \leq c$ with $B \supset \varkappa$. The sets B and $\varkappa$ are compact. Let C be the closure of $B - B\omega^e$. The set C is compact and contains no points of ω. According to Lemmas 8.1, 7.3 and 7.4 there exists an F-deformation θ of $F \leq c$ which carries C into a set d-below c but displaces points of $F \leq c$ distances less than $e/3$. The images of u and w under θ then admit the norm e. We see that $B - C$ is on ω^e and is deformed on ω^{2e} under θ. Indicating final images under θ by prefixing f we see that

$$fB = f(B - C) + fC \subset \omega^{2e} + fC \tag{9.2}$$

where fC is a compact set d-below c.

Let the nth components of the formal chains w, u, etc., be denoted by w_n, u_n, etc. Corresponding to θ let D_n be a deformation operator belonging to u_n with norm e_n tending to zero as n becomes infinite. We have the relation

$$\beta D_n u_n = u_n - fu_n - D_n \beta u_n \qquad (n \text{ not summed}). \tag{9.3}$$

From the commutativity of β and f, and from (9.1) we see that

$$\beta f w_n = f\beta w_n = fu_n - fy_n. \tag{9.4}$$

Upon adding the extreme members of (9.3) and (9.4) we find that

$$\beta w'_n = u_n - y'_n \tag{9.5}$$

where

$$y'_n = fy_n + D_n \beta u_n, \qquad w'_n = D_n u_n + f w_n.$$

We set $y' = (y'_n)$ and $w' = (w'_n)$. The formal chain y' is d-below c since y and βu are d-below c. Moreover $D_n u_n$ is on $\varkappa^e$ since u is on $\varkappa$, while fw is on fB since w is on B. Hence w' admits a carrier $\varkappa'$ such that

$$(9.6) \qquad \varkappa' \subset \varkappa^e + f\mathrm{B} \subset \varkappa^e + \omega^{2e} + f\mathrm{C}.$$

Thus cells of w' intersect $\varkappa^e + \omega^{2e}$ or are on fC.

Let w'' be the formal k-chain obtained by dropping all cells from the components of w' save those which meet $\varkappa^e + \sigma^{2e}$. The cells of w' admit the norm e so that w'' has the intersection of $\varkappa'$ with $\varkappa^{2e} + \sigma^{3e}$ for a carrier. This carrier is on U. With w'' so defined y'' in $(9.1)'$ is d-below c. For

$$(9.7) \qquad \beta w'' = \beta w' + \beta(w'' - w'),$$

or upon using (9.5),

$$(9.8) \qquad \beta w'' = u - y' + \beta(w'' - w').$$

To show that $\beta(w'' - w')$ in (9.8) is d-below c, note that cells of $w'' - w'$ are on $\varkappa'$ [see (9.6)] but do not meet $\varkappa^e + \sigma^{2e}$, and accordingly intersect τ^{2e} or are on fC. Referring to (9.7) recall that cells of $\beta w''$ do not intersect τ^{2e} nor do cells of $\beta w'$ [see (9.5)] excepting at most cells of y'. Relation (9.7) then implies that $\beta(w'' - w')$ has a carrier d-below c. Returning to (9.8) we conclude that y'' in $(9.1)'$ is d-below c. Statement (a) follows as indicated.

Proof of (b). — With $\mathrm{B} = \varkappa$ (9.3) holds without the assumption that (a) is false. Hence u and fu are c-homologous on U. But fu is on fB. Upon referring to (9.2) we see that cells of fu either intersect σ^{2e} or are on fC. Upon dropping all cells of fu which do not intersect σ^{2e} one obtains a k-cap v on σ^{3e}, c-homologous to fu on U. Since σ^{3e} is on an arbitrarily small neighborhood of σ for e sufficiently small, we conclude that u is c-homologous on U to a k-cap v on an arbitrarily small neighborhood of σ. The proof of (b) is complete.

Let σ be a critical set at the level c. A k-cap u with cap limit c will be said to be *associated* with σ if u is c-homologous to a k-cap on an arbitrarily small neighborhood of σ. Two isomorphic groups of

k-caps with cap limit c will be said to be *cap-isomorphic* if corresponding elements are c-homologous. A maximal group of k-caps associated with a critical set σ will be called a kth *type group* of σ. It follows at once from the definitions that any two kth type groups of σ are cap-isomorphic. The dimension of a kth type group of σ will be called the kth *type number* of σ. That these type numbers and type groups (with cap-isomorphic groups regarded as equivalent) depend only upon F in arbitrarily small neighborhoods of σ is shown by the following theorem.

THEOREM 9.3. — *A maximal group g of k-caps with cap limit c relative to any separate neighborhood of a critical set σ at the level c is a kth type group of σ.*

It follows from Theorem 9.2 that each k-cap of g is associated with σ. That g is a maximal group of k-caps associated with σ on M is seen as follows. Any k-cap associated with σ is c-homologous on M to a k-cap v on U. Since v is a k-cap relative to M it is a k-cap relative to U. Since g is maximal among k-caps relative to U with cap limit c, for some k-cap w in g, $v - w$ is c-homologous to zero, and hence not associated with σ on M. Hence g is a kth type group of σ.

THEOREM 9.4. — *Let the complete critical set ω at the level c be represented as a sum of disjunct critical sets σ^i, $i = 1, \ldots, n$, and let g^i be a kth type group of σ^i. Then the groups g^i admit a direct sum g, which is a kth type group of ω.*

Without loss of generality we can suppose that the k-caps of g^i lie on neighborhoods U^i of the respective sets σ^i at positive distances from each other. For Theorem 9.3 affirms that there exists a kth type group on U^i, cap-isomorphic with g^i. We suppose then that the k-caps of g^i are on U^i. Let $u^1, \ldots, u^r$ be k-caps belonging to distinct groups g^i. Let U be the sum of the neighborhoods U^i. That $u^1 + \ldots + u^r = u$ is a k-cap with cap limit c relative to U may be seen as follows. The neighborhoods U^i are at a positive distance from each other so that if u were c-homologous to zero ou U, then u^j would be c-homologous to zero on U^j for each u^j, contrary to the nature of u^j. Hence u is a k-cap with cap limit c relative to U. It follows from Theorem 9.2 (a) that u is a k-cap with cap limit c relative to M. Thus the groups g^i sum to a group g of k-caps with cap limit c. By virtue of Theorem 9.2 (b) each k-cap of g is associated with ω.

That g is a maximal group of k-caps associated with σ may be seen as follows. Each k-cap v with cap limit c is c-homologous to a k-cap on the above neighborhood U of ω by virtue of Theorem 9.2 (b). Since the groups g^i are maximal on their respective neighborhoods U^i, v is then c-homologous to a sum of k-caps of the respective groups g^i. Thus g is maximal as stated. The proof of the theorem is complete.

Corollary 9.4. — *The kth type number of a complete critical set ω is the sum of the kth type numbers of any finite set of disjunct critical sets summing to ω.*

Proof of theorem 9.1. — We base the proof of this theorem on Corollary 4.3 and the following statement.

(α). *Under the hypotheses of the theorem the kth type number of a complete critical set at the level $c < 1$ is at least the dimension of a maximal group g of non-bounding k-cycles with cycle limit c.*

Each k-cycle of g is homologous to a k-cycle on $F \leq c$. Without loss of generality we can suppose that g consists of cycles on $F \leq c$. The cycles of g are then canonical (§ 2). They are also k-caps with cap limit c by virtue of Theorem 3.1. There accordingly exists a kth type group of ω with g as a subgroup. Statement (α) is accordingly true. Theorem 9.1 now follows from Corollary 4.3.

10. Non-degenerate critical points. — We shall apply the preceding theory to the case where M is a *regular n-manifold* of class C^3, that is to the case where M is a compact Hausdorff topological space with the following properties. Some neighborhood of each point p of M can be mapped homeomorphically onto a region U of a euclidean n-space of rectangular coordinates (x) such that whenever points (x) and (z) belong to two such neighborhoods and define the same points on M, the relation between the coordinates (x) and (z) is given by a non-singular transformation $z^i = z^i(x)$ of class C^3. Neighboring any point q of U we admit any system of coordinates (z) obtainable from the coordinates (x) by a non-singular transformation $z^i = z^i(x)$ of class C^3. We suppose moreover that the function F on M reduces to a function $\varphi(x)$ of class C^3 in terms of each admissible set of coordinates (x).

A differential critical point (a) of $\varphi(x)$ will be termed a *differential critical point* of F. The point (a) will be termed *degenerate* if the hessian of φ vanishes at (a). We assume that the differential critical points of F are non-degenerate. In such a case F is termed non-degenerate. As in the introduction the *index* of a differential critical point (a) shall be defined as the index of the quadratic form whose coefficients are the elements of the hessian of φ at (a). From the non-degeneracy of F it follows that the critical points of F are isolated.

Since M is compact the critical points of F are finite in number. We shall show that a non-degenerate differential critical point σ is a homotopic critical point, and shall evaluate the jth type number of σ. We begin with the following lemma.

LEMMA 10.1. — *If $\varphi(x)$ has a non-degenerate critical point of index k at the point $(x) = (0)$, there exists a non singular transformation $y_i = y_i(x)$ of class* C^1 *under which*

$$(10.1) \qquad \varphi(x) - \varphi(0) = -y_1^2 - \ldots - y_k^2 + y_{k+1}^2 + \ldots + y_n^2$$

neighboring $(x) = (0)$.

Employing Taylor's formula with the integral form of the remainder (JORDAN, *Cours d'Analyse*, vol. I, p. 249) we find that

$$(10.2) \qquad \varphi(x) - \varphi(0) = a_{ij}(x) x_i x_j \qquad (i, j = 1, \ldots, n),$$

$$(10.3) \qquad a_{ij}(x) = \int_0^1 (1-u)\, \varphi_{x_i x_j}(ux_1, \ldots, ux_n)\, du.$$

It follows from (10.3) that $a_{ij}(x)$ is of class C^1 neighboring $(x) = (0)$ and that $a_{ij}(0) = \frac{1}{2} \varphi_{x_i x_j}(0)$.

In particular the determinant $|a_{ij}(0)| \neq 0$.

If the coefficients $a_{ij}(x)$ were constants, the Lagrange mode of reduction would carry φ into the form (10.1). Proceeding formally as if the coefficients $a_{ij}(x)$ were constants we can still effect this reduction. In particular if $a_{11}(0) \neq 0$, the substitution

$$(10.4) \qquad z_1 = \frac{a_{1j}x_j}{a_{11}}, \qquad z_2 = x_2, \qquad \ldots, \qquad z_n = x_n$$

reduces φ to the form

$$(10.5) \qquad \varphi(x) - \varphi(0) = a_{11}z_1^2 + b_{ij}(x)z_i z_j \qquad (i, j = 2, \ldots, n).$$

If at least one of the coefficients $a_{rr}(0) \neq 0$, a substitution of the form (10.4) is applicable after interchanging the variables x_1 and x_r. If each of the coefficients $a_{rr}(0) = 0$, at least one of the coefficients $a_{1r}(0) \neq 0$. After a change of variables of the form $x_1 = x'_1 - x'_r$, $x_r = x'_1 + x'_r$, a substitution of the form (10.4) will again be possible. Thus in any case one is led to a quadratic remainder of the form $b_{ij}(x)z_i z_j$ $(i, j = 2, \ldots, n)$ to which the same method of reduction is applicable. Transformations such as these clearly are non-singular and of class C^1 neighboring the origin in the respective spaces (x), (z), etc., and lead to a representation of $\varphi(x)$ of the form

$$\varphi(x) - \varphi(0) = c_i(z)z_i^2 \qquad [c_i(0) \neq 0],$$

where the coefficients $c_i(z)$ are of class C^1. A further reduction to the form (10.1) is immediate, and the proof of the lemma is complete.

THEOREM 10.1. — *A non-degenerate differential critical point σ of index k is a homotopic critical point whose jth type number equals the Kronecker δ^j_k.*

We suppose that $F(\sigma) = 0$ and that σ is represented by the point $(y) = (0)$ in a coordinate system (y) in which F takes the form

$$(10.6) \qquad F = -y_1^2 - \ldots - y_k^2 + y_{k+1}^2 + \ldots + y_n^2,$$

as described in Lemma 10.1. Let r be so small a positive constant that the transformation of the lemma holds whenever $y_i y_i \leq r^2$. Let A denote the set of points (y) for which

$$(10.7) \qquad F \leq 0, \qquad y_i y_i < r^2$$

and let A′ denote the subspace of A on which $F < 0$. The set A forms a neighborhood of σ relative to $F \leq 0$ free from differential critical points other than σ, and hence free from homotopic critical points with the possible exception of σ. We shall determine the dimension μ_j of a maximal group of j-caps relative to A.

If the index k of the critical point σ is zero, A consists of the point σ alone, and the theorem is immediate. We suppose therefore that $k > 0$. We note that A can be radially deformed on itself onto the origin. We continue with a proof of the following statements.

(*a*). *The homology groups of* A′ *are isomorphic with those of the* $(k-1)$ *sphere;* (*b*). *If* u *is a* j*-cap relative to* A, $\beta u \not\sim 0$ *on* A′. (*c*). *If* u *is a* j*-cap relative to* A, $j = k$. (*d*). *A maximal group of* k*-caps relative to* A *has the dimension* $\mu = 1$.

The reader will find it helpful to make a diagram of the sets A and A′ in the case where $F = y_1^2 - y_1^2$ in the $y_1 y_2$-plane.

Proof of (*a*). — The space A′ can be radially deformed on itself onto the subspace

$$F < 0, \qquad 0 < y_i y_i \leq \frac{r^2}{2} \tag{10.8}$$

holding this subspace fast. The set (10.8) can then be deformed onto the subset

$$0 < y_1^2 + \ldots + y_k^2 \leq \frac{r^2}{2}, \qquad y_{k+1} = \ldots = y_n = 0, \tag{10.9}$$

holding $y_1, \ldots . y_k$ fast and letting each $|y_j|$ for which $j = k+1, \ldots, n$ decrease a unit of time at a rate equal to the initial value of $|y_j|$. The set (10.9) can be radially deformed on itself onto its spherical boundary

$$y_1^2 + \ldots + y_k^2 = \frac{r^2}{2}, \qquad y_{k+1} = \ldots = y_n = 0. \tag{10.10}$$

The preceding deformations leave the $(k-1)$-sphere (10.10) fixed.

Statement (*a*) follows from the following readily established principle : (*i*). When a space Σ can be continuously deformed on itself onto a subspace S holding S fast, the k th homology groups of Σ and S are isomorphic for each k.

Proof of (*b*). — We shall assume (*b*) false and seek a contradiction. Let $\varkappa$ then be a carrier of the homology $\beta u \sim 0$ on A′. The set $\varkappa$

is d-below 0. Let u_n be the nth component of u. There exists an algebraic j-chain w_n on χ bounded by βu_n with norm e_n tending to zero as n becomes infinite. We have in $u_n - w_n$ an algebraic j-cycle on A. Let θ be a radial deformation of A into the origin, and let f be prefixed to denote final images under θ. Let D_n be a deformation operator belonging to θ and to u_n and w_n. Upon referring to (1.4) we see that $\beta D_n(u_n - w_n) = u_n - w_n$ (n not summed) since $f(u_n - w_n)$ coincides with the origin and is null. Since w_n is on χ we conclude that $u \sim 0$ on A mod χ, contrary to the assumption that u is a j-cap relative to A. We infer the truth of (b).

Proof of (c). — If u is a j-cap relative to A, $\beta u \not\sim 0$ on A′ in accordance with (b). It follows from (a) that $j = 1$ or k. If (c) is false, $j = 1$ and $k > 1$. For $k > 1$, the space A′ admits the connected deform (10.10) so that $\beta u \sim 0$ on A′. From this contradiction we infer that (c) is true.

Proof of (d). — We begin by showing that $\mu \geqq 1$. The set A′ contains a non-bounding $(k-1)$-cycle v on the $(k-1)$-sphere χ defined by (10.10). Let v_n be a component of v of norm e_n, and let

$$(10.11) \qquad \delta A_1 \ldots A_k \qquad (\delta \subset \Delta)$$

represent an arbitrary term in the reduced form of v_n. Let P denote the origin in the space A. The sum

$$(10.12) \qquad \Sigma\, \delta P A_1 \ldots A_k$$

of terms derived from the terms (10.11) of v_n by adding the vertex P will be an algebraic k-chain z_n such that $\beta z_n = v_n$. Corresponding to each algebraic cell α of z_n there is a straight cell (possibly degenerate) on A whose vertices are the vertices of α. It is accordingly possible to subdivide z_n by introducing the barycenters of cells of z_n as new vertices, so that after a finite number of subdivisions z_n is replaced by an algebraic chain u_n of norm e_n. We shall perform this subdivision without introducing any new vertices corresponding to cells of v_n so that we shall still have $\beta u_n = v_n$.

I say that (u_n) is a Vietoris k-cycle mod χ. For the homologies connecting the components of v_n are defined by relations of the form $\beta w_n = v_{n+1} - v_n$, where w_n is an algebraic k-chain on χ. One can insert the vertex P in each term of w_n as in (10.12) obtaining thereby a $(k+1)$-chain w'_n such that $\beta w'_n = -z_{n+1} + z_n + w_n$. One can then subdivide w'_n as we have subdivided z_n including the new vertices of cells of z_{n+1} and z_n in the process, obtaining thereby a new $(k+1)$-chain w''_n with $\beta w''_n = -u_{n+1} + u_n + w_n$. Hence (u_n) is a Vietoris k-cycle mod $\varkappa$, as stated.

We conclude by showing that $\mu = 1$. Let u and v be two k-caps relative to A. The boundaries βu and βv are $(k-1)$-cycles on A′. It follows from (a) that there exists a proper homology

$$\delta_1 \beta u + \delta_2 \beta v \sim 0 \qquad (\text{on } A'). \tag{10.13}$$

Now $\delta_1 u + \delta_2 v$ is a k-cycle z mod A′. But $\beta z \sim 0$ on A′ in accordance with (10.13) so that it follows from (b) that z cannot be a k-cap relative to A. Hence $\mu = 1$, and the proof of (d) is complete.

We return to the proof of the theorem. We have seen that there is at least one k-cap relative to A with cap limit 0. It follows from Theorem 3.1 that there must be at least one homotopic critical point at the level 0. This point must be σ. That the jth type number of σ is δ^j_k now follows from (d), and the proof of Theorem 10.1 is complete.

There is at most a finite number of critical points of a non-degenerate function F, so that the sum of the kth type numbers of the critical points of F is finite. The preceding theorem and Theorem 9.1 accordingly have the following corollary.

Corollary 10.1. — *The number* M_k *of critical points of index* k *of a non-degenerate function* F *is at least the* k*th connectivity of* M.

Suppose that the preceding manifold M lies in an $(n+1)$-dimensional euclidean space E. Let Q be a point in E, not on M. We shall apply the preceding theory to determine the minimum number of normals to M which pass through Q. The function $F(p)$ shall be the distance from Q to an arbitrary point on M and will be of class C^3 in terms of the local coordinates of M. One sees that a necessary and sufficient condition that a point q on M be a critical point of F is that q be the foot of a normal from Q to q. Recall that there are n focal points (centers of principal normal curvature) of M and q on the normal to M at q. Some of these focal points may however be at the

« point at infinity » on the normal. One can easily prove the following (M[2], p. 403) : The index of a critical point q of F equals the number of focal points of M and q which lie on the normal Qq between Q and q exclusive. The point q is a degenerate critical point if and only if Q is a focal point of M and q. We can accordingly state the following theorem.

THEOREM 10.2. — *Suppose* Q *is not a focal point of* M. *Of the normals from* Q *to* M *cutting* M *orthogonally at points q on* M *the number on which there are k focal points of* M *and q between* Q *and q is at least the kth connectivity of* M.

Suppose that the preceding manifold M again lies on an $(n+1)$-dimensional euclidean space E, and that it is the homeomorph of an n-sphere. We shall consider chords which cut M orthogonally at both ends and term such chords critical chords. Let p and q be arbitrary points of M and let $F(p, q)$ be the distance between p and q. When $p \neq q$ a necessary and sufficient condition that F have a critical point in terms of the local coordinates of p and q is that the chord $\overline{pq}$ be critical. The space of the independent variables is the space of the pairs (p, q) with p and q on M, (p, q) identified with (q, p), and with cells on which $p = q$ regarded as null. Its connectivities in the field of integers mod 2 were first shown by the writer to be null (M[3]) except that $R_n = R_{n+1} = \ldots = R_{2n} = 1$. For Vietoris cycles these connectivities are the same, as one can readily show. Hence we have the following theorem.

THEOREM 10.3. — *In case the chord length* $F(p, q)$ *is non-degenerate* $(p \neq q)$, *and* M *is the homeomorph of an n-sphere, there exists a set of critical chords of* M *which correspond respectively to critical points of* $F(p, q)$ *whose indices run from n to $2n$ inclusive.*

Applications of the critical point theory to harmonic functions of two or three variables have been made by Kiang.

PART III.

VARIATIONAL THEORY.

11. **The space Ω.** — We shall apply the preceding theory to the problem of finding extremals joining two fixed points a and b. The underlying space shall be a *connected* metric space Σ with a symmetric distance function pq. The space of all sensed curves (curve classes, § 7) joining a to b on Σ will be denoted by $\Omega(a, b)$. The metric of $\Omega(a, b)$ will be that defined by the Fréchet distance between curves. The space $\Omega(a, b)$ will replace the space M of the preceding theory. The function F will be defined for each element λ of Ω and will be a generalized « length ». The case of the space Ω and corresponding function F is typical of a class of general boundary value problems defined and discussed in Chapter VII of M [5]. The space of closed curves leads to difficulties of a different character and will not be discussed here. See Chapter VIII of M [5].

Before introducing F it will be desirable to investigate the manner in which the kth homology group $H^k(a, b)$ of $\Omega(a, b)$ depends on the points (a, b). Understanding that the symbol $\ominus$ is read « is isomorphic with » we see that

$$H^k(a, b) \ominus H^k(b, a). \tag{11.1}$$

We shall extend this result by proving the following theorem.

THEOREM 11.1. — *If Σ is arcwise connected,*

$$H^k(a, b) \ominus H^k(c, d), \tag{11.2}$$

where (a, b) and (c, d) are arbitrary pairs of points on Σ.

We begin by showing that

$$H^k(a, b) \ominus H^k(c, b). \tag{11.3}$$

Let Θ be a curve joining c to a and λ a curve joining a to b. Let $\Theta\lambda$ denote the curve obtained by tracing Θ and λ successively. The curve $\Theta\lambda$ joins c to b and lies on $\Omega(c, b)$. We thus have a continuous map of $\Omega(a, b)$ onto $\Omega(c, b)$ in which λ on $\Omega(a, b)$ is replaced by $\Theta\lambda$ on $\Omega(c, b)$. A cycle z on $\Omega(a, b)$ will thereby be replaced by a cycle on $\Omega(c, b)$ which we denote by Θz. Let Θ^{-1} be the curve obtained by reversing the sense of Θ. Let u be a k-cycle on $\Omega(c, b)$. Then $\Theta^{-1} u$ is a k-cycle on $\Omega(a, b)$ and $\Theta\Theta^{-1} u$ is a k-cycle on $\Omega(c, b)$. We shall prove the following :

(*i*). *The cycle u can be continuously deformed on $\Omega(c, b)$ into $\Theta\Theta^{-1} u$.*

To that end let $p = p(\tau)$ with $0 \leqq \tau \leqq 1$ be a p-curve in the curve class Θ and let Θ_t denote the curve class defined by $p = p(\tau)$ when $0 \leqq \tau \leqq t$. Let D be the deformation of $\Omega(c, b)$ in which each curve λ on $\Omega(c, b)$ is replaced at the time t by the curve $\Theta_t\Theta_t^{-1}\lambda$, $(0 \leqq t \leqq 1)$. Under D, u is deformed into $\Theta\Theta^{-1} u$.

We return to the proof of (11.3) and set up an isomorphism between the groups $H^k(a, b)$ and $H^k(c, b)$. To a cycle z of $\Omega(a, b)$ shall correspond the cycle Θz of $\Omega(c, b)$. If $\beta z = u$ on $\Omega(a, b)$ it follows from the fact that β and Θ are commutative that $\beta\Theta z = \Theta\beta z = \Theta u$ on $\Omega(c, b)$. Thus bounding cycles go into bounding cycles, and each homology class of $H^k(a, b)$ determines a unique homology class of $H^k(c, b)$. That the above mapping is an operator homomorphism follows from the fact that $\Theta(u + v) = \Theta u + \Theta v$, $\Theta(\delta u) = \delta(\Theta u)$ for formal k-chains u and v on $\Omega(a, b)$. It remains to show that this homomorphism is an isomorphism.

The mapping leads to each homology class y of $H^k(c, b)$. In particular if u is a cycle of y, $\Theta^{-1} u$ is on $\Omega(a, b)$. Its image $\Theta\Theta^{-1} u$ on $\Omega(c. b)$ is homologous to u, and so is in the given homology class y. To show that the homomorphism is one-to-one we have merely to show that the null class of $H^k(c, b)$ is the map of the null class only in $H^k(a, b)$. Let u be a cycle on $\Omega(c, b)$ with $u \sim 0$. Suppose that u is the image of v on $\Omega(a, b)$. Then $\Theta v \sim u$ on $\Omega(c, b)$, and hence $\Theta v \sim 0$. It follows that $\Theta^{-1}\Theta v \sim 0$ on $\Omega(a, b)$. But $\Theta^{-1}\Theta v \sim v$ on $\Omega(a, b)$ so that $v \sim 0$. Thus the mapping Θ defines an isomorphism, and (11.3) is proved.

To establish (11.2) we note that the operation of replacing a by c in $H^k(a, b)$ or of interchanging a and b leads to a group isomorphic with $H^k(a, b)$. We can thus successively replace (a, b) by (c, b),

(b, c), (d, c) and (c, d), obtaining finally a group $H^k(c, d)$ isomorphic with $H^k(a, b)$. The proof of the theorem is complete.

12. The secondary metric $[pq]$. — The function F will be defined in term of the secondary metric $[pq]$. The new distance $[pq]$ shall be defined for each pair of points of Σ and have the following properties

$$[pq] = 0 \quad \text{if } p = q, \qquad [pq] > 0 \quad \text{if } p \neq q, \qquad [pr] \leqq [pq] + [qr].$$

The ordinary distance will again be denoted by pq.

We do not assume that $[pq] = [qp]$. We suppose that $[pq]$ is continuous in p and q for p and q on Σ. *Cf.* Menger [4].

The function $F(\lambda)$ will be defined for each curve λ of $\Omega(a, b)$. Let ζ be any p-curve in the curve class λ. Let $(p) = p_0, p_1, \ldots, p_n$ be a set of successive points on ζ. The set (p) will be termed a *partition* of λ of norm δ if the maximum of the distances $p_i p_{i+1}$ is less than δ. We term

$$S = \Sigma [p_i p_{i+1}] \qquad (i = 0, 1, \ldots, n-1)$$

a sum *approximating* $J(\lambda)$, and define $J(\lambda)$ as the least upper bound of such sums S for all partitions of λ. We term $J(\lambda)$ the J-*length* of λ. The J-length may be finite or infinite. We set

$$F(\lambda) = \frac{J(\lambda)}{1 + J(\lambda)}$$

when $J(\lambda)$ is finite, and set $F(\lambda) = 1$ when $J(\lambda)$ is infinite.

The proof of the following theorem can be readily supplied by any reader familiar with the ordinary theory of length.

THEOREM 12.1. — *The* J-*length* $J(\lambda)$ *is the limit of any sequence of sums* S_n *approximating* $J(\lambda)$, *provided the norm* δ_n *of the corresponding partitions tends to zero as n becomes infinite.*

The following theorem is of course well-known.

THEOREM 12.2. — *The* J-*length* $J(\lambda)$ *is a lower semi-continuous function of* λ *in the space of the curves* λ.

Corresponding to each curve η and constant $a < J(\eta)$ we shall show that there exists a positive constant δ' such that $J(\zeta) > a$ whenever $\eta\zeta < \delta'$. Let b be a constant between a and $J(\eta)$. By virtue of the definition of $J(\eta)$ there exists a partition $p_0, \ldots, p_n$ of η such that

$$\Sigma[p_i p_{i+1}] > b \qquad (i = 0, \ldots, n-1). \tag{12.1}$$

For n fixed we choose δ so that $0 < 2n\delta < b - a$. If $\eta\zeta < \delta'$ there exists a homeomorphism T between p-curves of the curve classes η and ζ such that the correspondent q_i of p_i has a distance from p_i at most δ'. For fixed points p_i we suppose δ' so small that the secondary distances $[p_i q_i]$ are less than δ. From the triangle axiom we infer that

$$\begin{aligned} &[q_i q_{i+1}] > [p_i p_{i+1}] - 2\delta \qquad (i \text{ not summed}), \\ &[q_i q_{i+1}] > [p_i p_{i+1}] - 2n\delta \qquad (i \text{ summed}). \end{aligned} \tag{12.2}$$

It follows from (12.1) and (12.2), together with the relation $2n\delta < b - a$, that

$$J(\zeta) \geqq b - (b - a) = a$$

and the proof of the theorem is complete.

LEMMA 12.1. *On any compact subset* A *of* Σ, pq *is less than a prescribed positive constant e whenever* $[pq]$ *is less than a suitably chosen positive constant* δ *where* δ *depends upon e but not upon the choice of* p *and* q *on* A.

If the lemma were false there would exist an infinite sequence of pairs of points p_n, q_n of A such that $[p_n q_n]$ tends to zero as n becomes infinite, while $p_n q_n$ is bounded from zero for all n. The pairs p_n, q_n would then have at least one cluster pair p^0, q^0 since A is compact. We see that $[p^0 q^0] = 0$ since $[pq]$ is a continuous function of p and q. Hence $p^0 = q^0$ so that $p^0 q^0 = 0$. The distance $p_n q_n$ cannot then be bounded from zero. From this contradiction we infer the truth of the lemma.

13. Finite J-compactness of Σ. — If for each fixed point p of Σ and finite constant c the subset $[pq] \leqq c$ of Σ is compact, Σ will be said to be *finitely* J-*compact*. If Σ is compact it is clear that it is finitely J-compact. If Σ is a euclidean n-space and $[pq]$ is the ordinary distance, Σ is finitely J-compact. We assume that Σ is finitely J-compact.

In proving the next theorem we shall need several new terms. An ordered set of $n+1$ successive points on a curve λ, including the end points of λ and dividing λ into n successive arcs of equal J-length, will be termed an *n-set* of λ. We continue with the following lemma.

LEMMA 13.1. — *The curves λ of $\Omega(a, b)$ on which* J *is at most a finite constant c are divided by n-sets into arcs whose diameters tend to zero uniformly as n becomes infinite.*

By the J-*diameter* of a point set A is meant the least upper bound of distances $[pq]$ between pairs of points of A. Observe that the J-diameter of a curve is at most its J-length. If $J(\lambda) \leqq c$, the J-diameters of the arcs h_i into which λ is divided by an n-set are at most c/n and so tend to zero uniformly with n. But the points of Σ on curves issuing from a with $J \leqq c$ are points p of the set $[ap] \leqq c$, and this set is compact since Σ is finitely J-compact. Each of the above arcs h_i is on this compact set. It follows from Lemma 12.1 that the diameter of h_i is less than a prescribed positive constant provided n is greater than some integer N dependent only on c. The proof of the lemma is complete.

The following theorem is well-known when the primary and secondary metrics are identical. Its proof here depends upon the finite J-compactness of Σ.

THEOREM 13.1. — *The set of curves of $\Omega(a, b)$ whose* J-*lengths are at most a finite constant c is compact relative to the metric of $\Omega(a, b)$.*

Let λ be an infinite sequence of curves of $\Omega(a, b)$ with J at most c. Because of the finite J-compactness of Σ there will exist a subsequence $(\lambda)_1$ of λ such that the 2-sets on curves of (λ) converge to a set of three points on Σ. Proceeding inductively we see that there will exist a set $(\lambda)_1, (\lambda)_2, \ldots,$ of subsequences of λ such that $(\lambda)_m$ is a subsequence of $(\lambda)_{m-1}$ and the 2^m-sets on the curves of $(\lambda)_m$ converge to a set of points $p_m^0, \ldots, p_m^{2^m}$ on Σ. We shall define a p-curve $p = p(t)$ on Σ with $0 \leqq t \leqq 1$. For each $m > 0$ we set

$$(13.1) \qquad p\left(\frac{r}{2^m}\right) = p_m^r \qquad (r = 0, 1, \ldots, 2^m),$$

observing that the definitions (13.1) are consistent for successive values of m.

If t' and t'' are any two values of t for which $p(t)$ is defined we see that

$$[p(t')p(t'')] \leqq c \mid t'' - t' \mid. \tag{13.2}$$

Let t^* be an arbitrary value of t on the interval (0, 1) and let (t_n) be an infinite sequence of values of t for which $p(t)$ is defined and which tend to t^* as n becomes infinite. It follows from (13.2) that the points $p(t_n)$ form a Cauchy sequence relative to the secondary metric. But these points lie on the compact subspace of Σ consisting of points p such that $[ap] \leqq 0$. It follows from Lemma 12.1 that the points $p(t_n)$ form a Cauchy sequence relative to the metric pq and converge to a point q independent of the sequence (t_n) converging to t^*. We set $p(t^*) = q$ and observe that (13.3) then holds for all values of t' and t'' on the interval (0,1).

Let ζ be the curve defined by $p = p(t)$. Let (e_k) be a sequence of positive constants tending to zero as k becomes infinite. Corresponding to e_k Lemma 13.1 implies the existence of an integer $m = m_k$ so large that each of the arcs into which a curve η of Ω for which $J \leqq c$ is divided by its 2^m-set has a diameter at most e_k. We suppose m also so large that the arcs of $p(t)$ for which

$$\frac{r-1}{2^m} \leqq t \leqq \frac{r}{2^m} \qquad (r = 1, \ldots, 2^m) \tag{13.3}$$

have diameters at most e_k. With m so chosen let η_k be a curve of $(\lambda)_m$ such that the points of the 2^m-set of η_k are at distances at most e_k from the corresponding points p_m^r on ζ. If η_k^r is the r th of the arcs into which a 2^m-set divides η_k and ζ' is the arc of ζ for which (13.3) holds we see that for $m = m_k$ the distance of η_k^r from ζ_r is at most $3e_k$. It follows that the distance $\eta_k\zeta$ is at most $3e_k$. The sequence η_k thus converges to ζ. By virtue of the lower semi-continuity of $J(\lambda)$, $J(\zeta) \leqq c$. The set of curves of $\Omega(a, b)$ for which $J \leqq c$ is accordingly compact, and the proof of the theorem is complete.

Finite J-compactness thus implies the compactness of the subsets $F \leqq c < 1$. But we have seen in Theorem 5.2 that the compactness of the subsets $F \leqq c$ implies F-accessibility, in this case F-accessibility of $\Omega(a, b)$. Hence we have the following corollary of the theorem.

COROLLARY 13.1. — *The finite* J-*compactness of* Σ *implies the* F-*accessibility of* $\Omega(a, b)$.

14. Spaces Σ locally J-convex. — The principal hypotheses of the general theory were the F-accessibility of the space and the upper-reducibility of F. In our variational theory we have seen that the finite J-compactness of Σ implies the F-accessibility of $\Omega(a, b)$. Upon adding the assumption that Σ is locally J-convex the upper-reducibility of F will be implied as we shall see.

A simple sensed curve λ joining two points p and q will be termed a *right* arc if a point r lies on λ when and only when

$$[pq] = [pr] + [rq]. \tag{14.1}$$

We assume that Σ is *locally* J-*convex* in the following sense. With each point p of Σ there shall be associated a positive number $\rho(p)$ continuous in p and such that when $q \neq p$ and $[pq] \leqq \rho(p)$, p can be joined to q on Σ by a right arc $E(p, q)$, every subarc of which is a right arc. We term $E(p, q)$ an *elementary* arc joining p to q, applying this term only in the case where $[pq] \leqq \rho(p)$.

The condition on an elementary arc that every subarc be a right arc is a consequence of the other conditions on an elementary arc in the case where $\rho(p)$ is a constant. That subarcs of a right arc are not necessarily right arcs is shown by an example due to Dr. Busemann. Let the space M be a unit segment $0 \leqq x \leqq 1$ of a straight line. Let the secondary distance $[xy]$ between two points x and y, $x < y$, on this segment be defined by the formula

$$[xy] = |x - y| \left\{ 1 + \frac{1}{3} \min(x, 1 - y) \right\}.$$

When either point is an end point of the unit segment this distance reduces to $|x - y|$ so that the whole arc is a right arc. One sees however that subarcs for which $x + y \neq 1$ are not right arcs. If Σ is a regular manifold of class C^3 geodesic arcs of suitably restricted lengths are elementary arcs, as we shall see in § 16. The following theorem is an immediate consequence of the definition of an elementary arc.

THEOREM 14.1. — *There is at most one elementary arc joining a point p to a point q.*

THEOREM 14.2. — *The* J-*length of an elementary arc* $E(p, q)$

equals $[pq]$ and is a proper minimum relative to the J-lengths of all curves which join p to q on Σ.

Let $p_1, \ldots, p_n$ be a set of points which appear in the order written on $E(p, q)$. Any segment of an elementary arc is a right arc. It then follows inductively for $n = 1, 2, \ldots,$ that

$$[pq] = [pp_1] + [p_1p_2] + \ldots + [p_nq]. \tag{14.2}$$

Hence the J-length of $E(p, q)$ equals $[pq]$.

Let λ be any « curve » joining p to q. It follows from the definition of J-length that $J(\lambda) \leqq [pq]$. It remains to show that $J(\lambda) > [pq]$ when α is not the elementary arc $E(p, q)$. The proof falls into two cases.

Case I. The curve λ contains a point s not on $E(p, q)$. Case II. Each point of λ is on $E(p, q)$ but $\lambda \not\equiv E(p, q)$. In Case I, $J(\lambda) \geqq [ps] + [sq] > [pq]$, and the proof is complete. In Case II there must be distinct points r and s on λ and on $E(p, q)$ which appear in the order rs on $E(p, q)$ but in the order sr on λ. Then

$$J(\lambda) \geqq [ps] + [sr] + [rq] > [ps] + [rq] > [ps] + [sq] = [pq],$$

and the proof is complete.

THEOREM 14.3. — *Corresponding to any one-to-one continuous representation $r = r(t)$ of an elementary are joining p to q, the distance $[pr(t)]$ is a continuous increasing function of t.*

The continuity of $[pr(t)]$ is a consequence of the continuity of $r(t)$, and of $[pr]$ as a function of p and r. To show that $[pr(t)]$ is an increasing function of t we note that

$$[pr(t')] = [pr(t)] + [r(t)\, r(t')] \tag{14.3}$$

when $0 \leqq t < t'$, so that

$$[pr(t')] > [pr(t)],$$

and the proof is complete.

We are able to prove a theorem which is much stronger than Theorem 14.3. To formulate this theorem let $E(pq)$ be an elementary arc with variable end points p and q. Let t be a number between 0 and $[pq]$ inclusive. Let the point r on $E(p, q)$ for which $[pr] = t$ be denoted by $f(p, q, t)$.. Our theorem is as follows.

THEOREM 14.4. — *The point function* $f(p, q, t)$ *is continuous in its arguments for* $[pq] \leqq \rho(p)$ *and* $0 \leqq t \leqq [pq]$ *and for p and q on any compact subset* A of Σ.

We shall begin by proving the following statement.

(α) *The points* $f(p, q, t)$ *of the theorem lie on a compact subset of* Σ.

Since A is compact its J-diameter is a finite number d. If b is the maximum of $\rho(p)$ for p on A, the points $f(p, q, t)$ have secondary distances at most b from A, and accordingly at most $d+b$ from any fixed point of A. Since Σ is finitely J-compact the points $f(p, q, t)$ of the theorem lie on a compact subset of Σ, as stated in (α)

To establish the theorem let p_n, q_n, t_n be a sequence of sets p. q, t admitted in the theorem and possessing a limit set p^0, q^0, t^0. Set $r_n = f(p_n, q_n, t_n)$. By virtue of (α) there is a subsequence of the points r_n with a limit point r^0. For simplicity we assume that the sequence r_n converges to r^0. We shall show that

$$r_0 = f(p^0, q^0, t^0), \tag{14.4}$$

thereby establishing the theorem. Upon letting n become infinite in the relation $[p_n q_n] = [p_n r_n] + [r_n q_n]$, we infer that

$$[p^0 q^0] = [p^0 r^0] + [r^0 q^0]. \tag{14.5}$$

We also have the relation

$$[p^0 q^0] = \lim [p_n r_n] = \lim t_n = t^0. \tag{14.6}$$

From (14.5) and (14.6) we see that r^0 is the point on the elementary arc $E(p^0, q^0)$ at which $[p^0 q^0] = t^0$. Hence (14.4) holds, and the theorem is true.

Under the hypotheses that Σ is connected, finitely J-compact, and locally J-convex we could prove that the space $\Omega(a, b)$ is separable. We shall not use this fact and accordingly omit the proof. The following lemma will suggest the well-known Theorem of Osgood in the classical variation theory. It will be used in studying certain basic deformations.

LEMMA 14.1. — *Let* H *be a compact subset of* Σ. *Corresponding* *to* H *and a positive constant e there exists a positive constant* δ *such that when* ζ *and* η *are respectively an arbitrary curve and an elementary arc with common and points, with first end point on* H, *and with* $\zeta\eta \geqq e$, *then*

$$J(\zeta) - J(\eta) \geqq \delta. \tag{14.7}$$

The J-lengths of elementary arcs with end points on H are at most a positive constant ρ. To prove the lemma we accordingly need consider only those curves ζ whose J-lengths are at most 2ρ. The set B of such curves with end points on H is readily seen to be compact by an obvious extension of the proof of Theorem 13.1. Let A be the subset of curves of B whose end points can be joined by an elementary arc η such that $\zeta\eta \geqq 0$. The set A is closed in B, and accordingly compact. For ζ, on A and η the corresponding elementary arc, $J(\eta)$ is a continuous function $\varphi(\zeta)$ of ζ. The difference $\Delta(\zeta) = J(\zeta) - \varphi(\zeta)$ is lower semi-continuous. Hence $\Delta(\zeta)$ assumes its minimum m on some curve of A. But we have seen that each elementary arc η affords a proper minimum to J so that $m > 0$. Thus (14.7) holds with $\delta = m$.

15. **The upper-reducibility of F on $\Omega(a, b)$.** — In this section we shall derive a number of consequences of the finite J-compactness and local J-convexity of Σ including the upper-reducibility of F on $\Omega(a, b)$. We first define a basic deformation.

The deformation $\theta(r)$. — Let A be any compact set of curves of $\Omega(a, b)$. The curves λ of A can be represented in the form

$$p = \varphi(\mu, \lambda) \qquad [0 \leqq \mu \leqq \mu(\lambda)],$$

where μ is the intrinsic parameter defined in § 7. For λ on A and $0 \leqq \mu \leqq \mu(\lambda)$ the function $\varphi(\mu, \lambda)$ is uniformly continuous. The set of all points on the curves of A is readily seen to be compact. There accordingly exists a constant $\delta > 0$, such that on arcs of λ for which $\Delta\mu \leqq \delta$ successive points p and q satisfy the condition $[pq] < \rho(p)$. Let M be an upper bound of $\mu(\lambda)$ for λ on A. Let $(r_1, \ldots, r_n) = (r)$ be a set of positive numbers such that

$$r_1 + \ldots + r_n = 1, \qquad M r_i \leqq \delta \qquad (i = 1, \ldots, n). \tag{15.1}$$

Corresponding to the numbers (r) we shall define a deformation $\theta(r)$ of A. We shall refer to the numbers (r) as the *ratio set* defining $\theta(r)$.

Let each curve λ of A be divided with respect to its μ-length into n curves for which the differences $\Delta\mu$ (measuring μ from the initial point of λ) are proportional to the respective numbers $r_1, \ldots, r_n$. Let λ_k be the k th of these curves and let p_k be the initial point of λ_k. For $0 \leqq t \leqq 1$ let p_t be a point on λ_k which divides λ_k with respect to μ-length in the ratio in which t divides the interval $(0, 1)$. At the time t let the arc $p_k p_t$ of λ_k be replaced by the elementary arc $E(p_k, p_t)$. The curve λ will thereby be deformed into the sequence of elementary arcs determined by the points p_k. We denote this deformation by $\theta(r)$.

Let η be an arbitrary curve of $\Omega(a, b)$. The deformation $\theta(r)$ can be defined for any sufficiently small neighborhood U of η. For the constant δ can be chosen so that on arcs of η for which $\Delta\mu \leqq \delta$ successive points p and q satisfy the condition $[pq] < \rho(p)$. Let U be a neighborhood of η so small that for λ on U, arcs of λ for which $\Delta\mu \leqq \delta$ at successive points p, q, again satisfy the condition $[pq] < \rho(p)$. Suppose U so small that for λ on U, $\mu(\lambda)$ has a finite upper bound M. The ratio set (r) will next be chosen so as to satisfy (15.1) taking n sufficiently large. The deformation $\theta(r)$ can then be defined as previously for curves λ initially on U.

Lemma 15.1. — *A deformation $\theta(r)$ is an* F-*deformation of any subset A of $\Omega(a, b)$ on which it is defined and on which* J *is at most a finite constant c.*

We must show that $\theta(r)$ admits a displacement function $\delta(e)$ corresponding to each compact subset B of A. Let η and ζ be images of a curve of B with ζ an antecedent of η under $\Theta(r)$ and $\zeta\eta > e > 0$. The curve η is obtained from ζ by replacing certain subarcs ζ_k of ζ by elementary arcs η_k where η_k joins the end points of ζ_k. If $\zeta\eta > e$, then for at least one of these subarcs $\eta_k \zeta_k > e$. The points of B are on a compact subset of Σ. It follows from Lemma 14.1 that there exists a constant $\delta > 0$ such that

$$J(\zeta_k) - J(\eta_k) \geqq \delta.$$

Hence

$$J(\zeta) - J(\eta) \geqq \delta, \qquad F(\zeta) - F(\eta) > \delta_1,$$

where δ_1 depends only on e and c, and the proof of the lemma is complete.

The function F can be shown to be upper-reducible on $\Omega(a, b)$. The proof of this fact however would require the use of deformations more complicated than the deformations $\theta(r)$. *Cf* [7], p. 443. Fortunately our present purposes will be served adequately by showing that F on the subspace $F < 1$ of $\Omega(a, b)$ is upper-reducible. The following theorem justifies the omission of the general proof of the upper-reducibility of F.

THEOREM 15.1. — *There is no cap limit with* $c = 1$ *relative to the function* F *on* $\Omega(a, b)$.

Suppose that the theorem is false and that u is a k-cap with cap limit 1. Let $\varkappa$ be a carrier of u. Since $\varkappa$ is compact the « ratio set » (r) can be so chosen that $\theta(r)$ is defined over $\varkappa$. On the trajectories of θ F never increases and $\varkappa$ is deformed into a compact set $\varkappa'$ of curves each of which is a sequence of n elementary arcs. The points of $\varkappa'$ form a compact set on Σ, so that the J-lengths of the above elementary arcs are less than some finite constant. Thus the final image of u lies on a subset of $\Omega(a, b)$ on which J is at most a finite constant. Hence u is c-homologous to zero with $c = 1$. We infer that $c = 1$ is not a cap limit.

We state the following principal theorem.

THEOREM 15.2. — *The function* F *on the subspace* $F < 1$ of $\Omega(a, b)$ *is upper-reducible.*

Let η be a curve of $\Omega(a, b)$ on $F < 1$ and let d and c be constants such that $F(\eta) < d < c < 1$. To establish the theorem we shall show that a deformation $\theta(r)$ defined on a sufficiently small neighborhood U of η relative to $F \leq c$ F-deforms U onto $F \leq d$. Under $\theta(r)$, η is F-deformed into a curve ζ composed of elementary arcs. Observe that $F(\zeta) \leq F(\eta) < d$. Let U be so small a neighborhood of η relative to $F \leq c$ that for λ on U, $\theta(r)$ F-deforms λ into a curve λ_1 for which $F(\lambda_1) < d$. This is possible since the vertices of the elementary arcs of λ_1 will lie arbitrarily near the corresponding vertices of ζ if U is sufficiently small. The theorem follows from the definition of upper-reducibility.

A homotopic critical point (*i.e* curve) of F on $\Omega(a, b)$ will be called a *homotopic extremal* of $\Omega(a, b)$. On the other hand a curve η will be called a *metric extremal* provided every closed subarc of η whose J-length is sufficiently small is an elementary arc. An elementary arc is a metric extremal in accordance with Theorem 14.2 More generally we have the following theorem.

THEOREM 15.3. — *Each homotopic extremal of $\Omega(a, b)$ is a metric extremal.*

This theorem will be proved merely for extremals of finite J-length. It is vacuously true for extremals of infinite J-length because it can be shown that (under our hypotheses) there are no homotopic extremals of infinite length. Cf M [7], Theorem 14.3.

Let λ be a curve of $\Omega(a, b)$ of finite J-length c, not a metric extremal. We shall show that λ is homotopically ordinary. Since λ is not a metric extremal there exists a subarc pq of λ which is not an elementary arc and whose J-diameter is less than $\rho(p)$. A suitably chosen F-deformation $\theta(r)$ of a neighborhood of λ on $J \leq c$ will replace the arc pq by the elementary arc $E(p, q)$. This deformation $\theta(r)$ displaces λ so that λ is homotopically ordinary. The theorem follows for extremals of finite J-length.

By virtue of Theorem 15.3 a homotopic extremal will have the same degree of regularity and differentiability as have elementary arcs. Since elementary arcs are minimizing arcs this means that a homotopic extremal in a classical problem will satisfy the Euler equations and have the differentiability of ordinary extremals.

Recalling the definition of local F-connectedness of § 6 we continue with a proof of the following theorem.

THEOREM 15.4. — *The space $\Omega(a, b)$ is locally* F*-connected of all orders.*

Let e be a positive constant. Let the e-neighborhood relative to $\Omega(a, b)$ of a curve η on $\Omega(a, b)$ be denoted by η_e. Let $A(c)$ be the subset of $\Omega(a, b)$ on $J \leq c$. We shall prove Theorem 15.4 by proving the following statement.

(a). *Corresponding to the constants c and e and any curve η*

of $A(c)$ *there exists a constant* $\delta > 0$ *such that* $\eta_\delta A(c+\delta)$ *can be deformed on* $\eta_e \cdot A(c+e)$ *into a single curve of* η_e.

Let $\theta(r)$ be a deformation of a neighborhood of η of the type previously defined. In the set (r) we suppose that $r_1 = \ldots = r_n$. Under $\theta(r)$ a curve λ is deformed into a succession of n elementary arcs with vertices on λ. If the number n is sufficiently large and if δ is at most a sufficiently small positive constant δ_1, $\theta(r)$ will deform η_δ on η_e. We suppose n and δ so chosen.

Let λ be an arbitrary curve of $A(c+\delta)$ on η_δ. Let η_1 and λ_1 be final images of η and λ respectively under $\theta(r)$. Let p_i and q_i be corresponding vertices of η_1 and λ_1 respectively. The points p_i being fixed by η and the choice of n the distances $[p_i q_i]$ will be arbitrarily small if δ is sufficiently small. If these distances $[p_i q_i]$ are sufficiently small the curves λ_1 can be deformed into η_1 as follows. As t varies from 0 to 1 inclusive, q_i shall be replaced by a point $q_i(t)$ which divides $E(p_i, q_i)$ in the same ratio with respect to J-length as that in which t divides the interval $(0,1)$. We replace the ith elementary arc of λ_1 by the elementary arc

$$E[q_i(t), q_{i+1}(t)] \tag{15.2}$$

at the time t. and denote the resulting deformation by Λ.

If the distances $[p_i q_i]$ are sufficiently small the elementary arcs (15.2) will exist and vary continuously with their end points. For their end points will be arbitrarily near the corresponding end points p_i of η and these end points satisfy the conditions $[p_i p_{i+1}] < \rho(p_i)$.

Il the constant δ is then sufficiently small and in particular $< \delta_1$, the deformation Λ will be possible, and will deform the curves λ_1 on $\eta_e \cdot A(c+e)$. The deformation $\theta(r)$ followed by Λ will satisfy (a), and the proof of the theorem is complete.

THEOREM 15.3. — *A maximal group of non-bounding k-cycles on the subset* $F < 1$ *of* $\Omega(a, b)$ *is a maximal group of non-bounding k-cycles on* $\Omega(a, b)$.

We have seen in the proof of Theorem 15.1 that any compact subset of $\Omega(a, b)$ admits a deformation of the type $\theta(r)$ into a set of points d-below 1. The deformations $\theta(r)$ never increase F along a trajectory so that sets on $F < 1$ are deformed on $F < 1$. Theorem 15.1 follows readily.

It will be useful at this point to give a resume of the implications of the finite J-compactness and local J-convexity of Σ. (*a*). Finite J-compactness implies the following : (1). The subsets $F \leqq c < 1$ of $\Omega(a, b)$ are compact. (2). The space $\Omega(a, b)$ is F-accessible. (*b*). Finite J-compactness and local J-convexity imply the following : (3) The function F is upper-reducible. (4) The space $\Omega(a, b)$ is locally F-connected. (5) The cycle limits are less than 1.

The following theorem combines the preceding conditions with the general theory of critical points of Part II.

THEOREM 15.6. — *If* Σ *is finitely* J-*compact and locally* J-*convex, then corresponding to each cycle limit* s *of a non-bounding* k-*cycle on* $\Omega(a, b)$ *there is at least one homotopic extremal on which* $F = s$. *Moreover the sum of the kth type numbers of the critical sets of* F *on* $F < 1$ *is at least the kth connectivity of* $\Omega(a, b)$.

Under the hypotheses of the theorem properties (1) to (5) preceding the theorem hold. It follows from the upper-reducibility of F (*cf.* Theorem 8.1) that there is at least one homotopic extremal on which F equals the cycle limit s. Properties (1) and (4) imply (*cf.* Theorem 6.2) that the connectivities of the subset $F < 1$ of $\Omega(a, b)$ are at most alef-null. According to Theorem 15.5 the connectivities of $\Omega(a, b)$ are then at most alef null. The concluding statement of the theorem follows from Theorem 9.1.

16. The theory under classical hypotheses. — We concern ourselves here with a regular manifold Σ of class C^5 defined as in § 10 in terms of overlapping coordinate systems except that we do not assume Σ compact. In every local coordinate system (x) we suppose that there is defined a function $F(x^1, \ldots, x^m, r^1, \ldots, r^m) = F(x, r)$ which is of class C^4 in (x, r) for (x) in the local coordinate system and for any set of numbers $(r) \neq (0)$. We require that F be an invariant. More precisely if

$$z^i = z^i(x) \qquad (i = 1, \ldots, m) \tag{16.1}$$

is an admissible transformation to a coordinate system (z) and $Q(z, \sigma)$ is the function replacing $F(x, r)$ we suppose that the relation

$$F(x, r) = Q(z, \sigma) \tag{16.2}$$

shall be an identity in (x) and (r) when (z) is given by (16.1) and (σ) is the contravariant tensor image of (r). We assume that F is positive and positive homogeneous of degree 1 in the variables (r). As is well-known,

$$| F_{r^i r^j}(x, r) | = 0 \qquad (i, j = 1, \ldots, m). \tag{16.3}$$

We assume however that the rank of the determinant (16.3) is $m - 1$ and that all of its characteristic roots save the null one are positive. As a consequence (M [5], Chapter V, § 7) of these hypotheses the classical Legendre and Weierstrass sufficient conditions are satisfied along any extremal of the integral

$$J = \int F(x, \dot{x})\, dt. \tag{16.4}$$

A curve λ whose closed subarcs are rectifiable in each coordinate system in which they lie will be termed *rectifiable*. If λ is rectifiable there will exist a p-curve η in the curve class of λ such that the coordinates of a point of any closed subarc η^* of η which lies in a coordinate system (x) are absolutely continuous functions of the parameter t. On η^* the integral J will have a determinate value $J(\eta^*)$ as a Lebesgue integral. To obtain $J(\lambda)$ one breaks η up into a finite set of arcs such as η^* and sums the corresponding values $J(\eta^*)$.

We assume that Σ is *arcwise connected*, and for any two points p and q of Σ we let the *distance* $[pq]$ be the greatest lower bound of J along all rectifiable curves which join p to q on Σ. The distance $[pq]$ is in general not symmetric in p and q. One shows readily that $[pq] = 0$ if and only if $p = q$ and that $[pq] \leqq [pr] + [rq]$. If one sets

$$pq = \max \{ [pq], [qp] \}$$

one obtains a new symmetric distance function. As before $pq = 0$ if and only if $p = q$. The distance pq also satisfies the triangle axion. For if $pq = [pq]$, $pq \leqq [pr] + [rq] \leqq pr + rq$, and if $pq = [qp]$ a similar result holds. We shall regard pq as defining the *metric* of Σ

and $[pq]$ as defining a *secondary metric*. From the fact that $[pq] \leqq pq$ and that $[pq]$ satisfies the triangle axiom it follows readily that $[pq]$ is a continuous function of p and q (with respect to the primary metric). The distances pq and $[pq]$ can accordingly be identified with the corresponding distances of the general theory.

We assume that Σ is finitely J-*compact in the sense of* § 13.

It follows that an extremal λ on which the parameter t is the value of the integral J can be continued for unrestricted positive values of t. For if $c > 0$ were a finite greatest lower bound of the values of t on λ and t_n were an increasing sequence of values of t tending to c, the corresponding points p_n on λ would have at least one cluster point q. The classical existence theorems for extremals applied to a neighborhood of q would then show that λ could be continued so that t exceeds c on λ, and our statement follows.

THEOREM 16.1. — *The space Σ is locally* J-*convex.*

To establish the theorem one must establish the existence of a function $\rho(p)$, positive and continuous in p such that whenever

$$0 < [pq] \leqq \rho(p) \tag{16.5}$$

p can be joined to q by a « right arc » (§ 14), every subarc of which is a right arc.

We begin with a statement of facts well-known in the classical theory. Cf. Cairns [1]. Let z be an arbitrary point of Σ. Corresponding to any sufficiently small neighborhood U of z there exists a positive constant σ with the following property. Any point p on U can be joined to any point q such that $0 < [pq] \leqq \sigma$ by a unique extremal λ with the following minimizing property. If η is any subarc of λ (including λ), if r is any point not on η, and ζ is a curve joining the end points of λ and passing through r, then $J(\zeta) > J(\eta) + e$, where e is a positive constant depending only on r and η.

This is the uniform minimizing property implied in the classical theory by Osgood's theorem and the usual field constructions in the small. It follows at once that λ is a right arc. We have yet to define the function $\rho(p)$ appearing in the definition of an elementary arc.

To define $\rho(p)$ let z be a fixed point of Σ and let S_n be the set of points of Σ whose secondary distances from z are between n and $n+1$ inclusive, $n \geqq 0$. Since S_n is compact it follows from the results of the preceding paragraph that there exists a positive constant σ_n such that any point p of S_n can be joined to any point of Σ whose secondary distance from p is at most σ_n by a unique right arc, every subarc of which is a right arc. We admit the possibility of the sets S_n being vacuous for sufficiently large integers n. In any case we can suppose that the constants σ_n do not increase with n. We now define a function $\varphi(s)$. We set $\varphi(n) = \sigma_n$ and for other positive values of s define $\varphi(s)$ by interpolating linearly between the successive values σ_n. At a point p whose distance from the fixed point z is s we set $\rho(p) = \varphi(s)$. The function $\rho(p)$ is readily seen to be continuous for p on Σ. The extremal arc issuing from p and consisting of points q such that (16.5) holds is thus an elementary arc in the sense of § 14, and the space Σ is locally J-convex.

For the purposes of the next theorem the value of the integral (16.4) taken along a curve η will be called the *integral* J-*length* $J(\eta)$, understanding that this J-length is infinite if η is not rectifiable. On the other hand the J-length of η as defined in § 12 will be called the *abstract* J-length $J^*(\eta)$. We shall prove the following theorem.

THEOREM 16.2. — *The integral* J-*length and the abstract* J-*length of a curve* η *are equal.*

We shall rely on the classical theory only to the extent of using the lower semi-continuity of the integral J-length. We first observe that if λ is an elementary arc its integral and its abstract J-lengths are equal. This is also true of a finite succession of elementary arcs. But corresponding to an arbitrary curve λ there existe a sequence $\lambda_1, \lambda_2, \ldots$, of curves each of which is a finite succession of elementary arcs whose vertices define a partition of λ, and which are such that the Fréchet distance $\lambda_n\lambda$ tends to zero as n becomes infinite, while $J^*(\lambda_n)$ tends to $J^*(\lambda)$ as n becomes infinite. From the minimizing properties of elementary arcs we infer that $J^*(\lambda_n) \leqq J(\lambda)$, and hence that

$$J^*(\lambda) \leqq J(\lambda). \tag{16.6}$$

But since the abstract and integral J-lengths are equal on elementary arcs and $J(\eta)$ is lower semi-continuous we conclude that the equality only can hold in (16.6), and the proof is complete.

Conjugate points and the index theorem. — A closed extremal segment will ordinarily be given in several overlapping coordinate systems. The coordinates of such an extremal will be of clase C^4 in terms of the J-length as a parameter. Let us term a coordinate system which is obtained from admissible coordinate systems by non-singular transformations of class C^r, $0 < r \leq 5$, a coordinate system of class C^r. Such coordinate systems are not admissible in the earlier sense unless $r = 5$, but nevertheless are useful. As shown in M [5], p. 108 a simple regular curve λ of class C^r is wholly contained in at least one coordinate system of class C^r. If λ is not simple it is possible to map a suitably chosen region R of a euclidean m-space (x) onto Σ by a transformation locally non-singular and of class C^r in such a fashion that λ is the image of a simple regular arc of class C^r in R. We term R a *special* coordinate system of class C^r containing λ.

In case λ is an extremal with initial end point A, the conjugate points of A on λ are defined as follows. Let (ρ) be the unit contravariant vector which gives the direction of λ at A in a special coordinate system (x) with $r = 4$. Suppose the J-length of λ is s. In the system (x) let the components r^i of the unit vectors neighboring (ρ) be regularly represented as functions $r^i(u)$ of class C^3 of $n = m - 1$ parameters (u). Suppose that (ρ) corresponds to $(u) = (0)$. In the system R the extremal issuing from A with the direction $r^i(u)$ can be represented in terms of (u) and its J-length t in the form $x_i = \varphi^i(t, u)$ where the functions φ^i are of class C^3 in terms of their arguments for t on the closed interval $(0, s)$ and (u) neighboring (0). The zeros of the jacobian

$$\Delta(t) = \frac{D(\varphi^1, \ldots, \varphi^m)}{D(t, u_1, \ldots, u_n)} \qquad [(u) = (0)]$$

on the interval $0 < t \leq a$ are isolated and define the *conjugate* points of A on λ. The order of any one of these zeros is at most $m - 1$. and will be termed the *order* of the corresponding conjugate point of λ. (M [5], p. 117).

An extremal on which the final end point b is not conjugate to the initial end point a will be termed *non-degenerate*. We shall show that a non-degenerate extremal λ of $\Omega(a, b)$ is a homotopic extremal of $\Omega(a, b)$ and determine the type numbers of λ. To that end the following construction and theorem are fundamental.

Let η be a finite extremal arc. We refer η to a special coordinate system (x) of class C^4 as previously. Let $a, a_1, \ldots, a_p, b$ be a set of successive points on η such that the segments into which η is thereby divided have J-lengths less than the minimum of $\rho(p)$ on η. We cut across η at the respective points a_q, $q = 1, \ldots, p$, by regular manifolds M_q of class C^3, not tangent to η at the points a_q. We suppose that M_q is regularly represented neighboring a_q by a set (u_q) of $n = m - 1$ parameters u_q^i in such a manner that the set $(u_q) = (0)$ determines a_q on η. The ensemble of the pn parameters u_q^i will be denoted by (z). The set (z) determines a set of points $Q_1, \ldots, Q_p$ on the respective manifolds M_q. If (z) is sufficiently near (0) successive points of the set $a, Q_1, \ldots, Q_p, b$ can be joined by extremals to form a broken extremal $E(z)$ whose J-length will be a function $I(z)$ of class C^3. Let t be the J-length along η measured from the initial point of η. Our theorem is as follows. (M [8]).

THEOREM 16.3. – *The point $(z) = (0)$ is a critical point of $I(z)$. It is degenerate if and only if the initial point $t = 0$ of η is conjugate to the final end point $t = t_0$ of η. The index of $(z) = (0)$ equals the number k of conjugate points of $t = 0$ on the interval $0 < t < t_0$ counting these conjugate points with their orders.*

The set $S(\eta)$ of broken extremals $E(z)$ used to define $J(z)$ is a subset of the curves of $\Omega(a, b)$ neighboring η. We term $S(\eta)$ a *canonical section* of $\Omega(a, b)$ neighboring η. The set $S(\eta)$ can be taken arbitrarily near η. « Relative » to $S(\eta)$ the terms homotopic extremal and the j th type number of η are well defined. With this understood we state the following lemma.

LEMMA 16.1. — *Relative to any canonical section of $\Omega(a, b)$ neighboring a non-degenerate extremal η, η is an isolated homotopic extremal. If there are k conjugate points of a on η, the j th type number of η relative to $S(\eta)$ equals δ_k^j.*

The broken extremal $E(z)$ is determined by (z), and for (z) sufficiently near (0) the relation between (z) and $E(z)$ is a one-to-one continuous mapping of a neighborhood N of $(z) = (0)$ in the space

(z) upon a canonical section $S(\eta)$ of $\Omega(a, b)$ neighboring η. Under this mapping $I(z) = J(E)$. The lemma follows from Theorems 16.3 and 10.1.

Lemma 16.2. — *Corresponding to any canonical section* $S(\eta)$ *of* $\Omega(a, b)$ *neighboring the extremal* η *any sufficiently small neighborhood* V *of* η *relative to the subset* $J \leqq J(\eta)$ *of* $\Omega(a, b)$ *can be* F-*deformed onto* $S(\eta)$ *without displacing* η.

We shall obtain the desired deformation as the product of two F-deformations of which the first shall be a deformation $\Theta(r)$ (cf. § 15) of a neighborhood of η. The final images under $\Theta(r)$ are broken extremals with vertices $(p_1, \ldots, p_n)$ neighboring a particular set of n vertices on η.

The deformation Z. — For the purpose of defining Z we *admit* any set A of curves λ of $\Omega(a, b)$ neighboring η such that λ intersects the respective manifolds M_q in unique points $p_q(\lambda)$ which vary continuously with λ on A and which divide λ into successive arcs λ_i with J-diameters less than the greatest lower bound of $\rho(p)$ on points of A. If the neighborhood V of the lemma is sufficiently small the final images under $\Theta(r)$ of curves of V will be admissible in the preceding sense. To define Z we deform each arc λ_i into the elementary arc which joins its end points exactly as in defining $\Theta(r)$. For V sufficiently small the product deformation $Z\Theta$ is well defined, and satisfies the lemma.

Recall that a non-degenerate extremal η of $\Omega(a, b)$ is isolated among extremals of $\Omega(a, b)$. For the hypothesis that b is not conjugate to a on η implies that the extremals issuing from a with directions sufficiently near that of η form a field near b, with η the only curve in their field to pass through b. For the purposes of the following proof the reader should recall the definition of a k-cap « associated » with a critical set σ relative to a subset S of $\Omega(a, b)$ containing σ. Let u be a k-cap with cap limit equal to the value c of J on σ. If u is « associated » with σ relative to S, then corresponding to each of σ's neighborhoods U relative to S, there shall exist a k-cap v relative to U, c-homologous to u on S. Recall also that the j th type number of σ is the dimension of a maximal group of j-caps associated with σ. With this understood the fundamental theorem of this section is as follows.

THEOREM 16.4. — *Each non-degenerate extremal* η *of* $\Omega(a, b)$ *is an isolated homotopic extremal of* $\Omega(a, b)$. *If there are* k *conjugate points of* a *on* η, *the* j*th type number of* η *is* δ_k^j.

That η is isolated among extremals of $\Omega(a, b)$ follows from the fact that its end points are not conjugate on η as noted. To continue let $S(\eta)$ be a canonical section of $\Omega(a, b)$ neighboring η. We have seen in Lemma 16.1 that η is a homotopic extremal relative to $S(\eta)$. We base the remainder of the proof on statements (i) and (ii).

(i). *Any* j*-cap* u *associated with* η *relative to* $S(\eta)$ *is a* j*-cap associated with* η *relative to* $\Omega(a, b)$.

It will be convenient to write u c-hom v when « u is c-homologous to v ». Set $J(\eta) = c$. If u is associated with η relative to $S(\eta)$, then u c-hom v on $S(\eta)$ where v is a j-cap on an arbitrarily small neighborhood of η. In particular we suppose that v is on the neighborhood V affirmed to exist in Lemma 16.2.

Suppose (i) false. Then u c-hom o on $\Omega(a, b)$. Hence v c-hom o on $\Omega(a, b)$, and it would follow from Theorem 9.2 (a) that v c-hom o on V, at least if V is a separate neighborhood of η as we suppose the case. But V can be F-deformed under $Z\Theta$ onto $S(\eta)$ as in Lemma 16.2. In this deformation let w be the final image of v. Since v c-hom o on V, w c-hom o on $S(\eta)$. Let $\varkappa$ be a carrier of v on $S(\eta)$ and $\varkappa'$ the point set swept out by $\varkappa$ under $Z\Theta$. The curves of $\varkappa'$ are broken extremals to which Z is applicable at least if V is sufficiently small as we suppose the case. But Z deforms $\varkappa'$ onto $S(\eta)$ leaving v and w fixed. Hence v c-hom w on $S(\eta)$. In résumé we have u c-hom v c-hom w c-hom o on $S(\eta)$. Thus u c-hom o on $S(\eta)$, contrary to the nature of u. We infer the truth of (i).

(ii). *Any* j*-cap* u *associated with* η *relative to* $\Omega(a, b)$ *is c-homologous on* $\Omega(a, b)$ *to a* j*-cap* v *associated with* η *relative to an arbitrary canonical section* $S(\eta)$ *of* $\Omega(a, b)$.

The k-cap u is c-homologous on $\Omega(a, b)$ to a k-cap on the neighborhood V of Lemma 16.2. Hence u can be F-deformed onto

$S(\eta)$ so that u is c-homologous to a j-cap v on $S(\eta)$. Moreover v is a j-cap relative to $S(\eta)$, for otherwise it could not be a j-cap relative to $\Omega(a, b)$. Finally v is associated with η relative to $S(\eta)$ in accordance with Theorem 9.2 (b), since η is the only extremal among curves of $S(\eta)$.

Proof of the theorem. — I say that η is not homotopically ordinary on $\Omega(a, b)$. Otherwise it would follow from the definition involved that some neighborhood N of η relative to $J \leq c$ on $\Omega(a, b)$ would admit an F-deformation onto a set d-below c. This is impossible since there is a k-cap associated with η relative to $S(\eta)$, and hence associated with η relative to $\Omega(a, b)$, as stated in (i). It now follows from (i) and (ii) that a maximal group of j-caps associated with η relative to $S(\eta)$ is a maximal group of j-caps associated with η relative to $\Omega(a, b)$. We can use Lemma 16.1 to conclude that the dimension of such a group is δ'_k. The proof of the theorem is complete.

THEOREM 16.5. — *If the manifold Σ of this section is the homeomorph of an m-sphere and if a and b are points of Σ which are conjugate on no extremal through a, then for every integer $k \equiv 0 \bmod m - 1$ there is at least one extremal joining a to b on which there are k conjugate points of a.*

The connectivities R_k of the space $\Omega(a, b)$ are all null except those for which $k \equiv 0 \bmod m - 1$, and the latter equal unity. Cf. M[5], Chapter VII, Theorem 15.1. Since a and b are never conjugate the critical sets of F on $\Omega(a, b)$ consist of isolated homotopic extremals. If $R_k = 1$, there must be a homotopic extremal η whose kth type number is at least one by virtue of Theorem 15.6. In accordance with Theorem 16.4 the kth type number of η is exactly one, and k is the number of conjugate points of a on η. The proof of the theorem is complete

The condition that a be conjugate to b on no extremal is not very restrictive. In fact the writer has shown that the set of points on Σ which are conjugate to a given point A on extremals through A has an m-dimensional measure zero on Σ (M[5], Chapter VII, Theorem 12.1). If the point b is conjugate to a on some extremal, b is nevertheless the limit point of points never conjugate to a. Using this fact one can prove the following. If γ_k is a non-bounding k-cycle on $\Omega(a, b)$, the cycle limit $J(a, b)$ of γ_k is a continuous function of a and b on Σ, and there exists a homotopic extremal γ_l whose J-length is $J(a, b)$ and on which there are at least k and at most $k + m - 1$ conjugate points of a (M [5], Chapter VII, Theorem 13.3).

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GLOSSARY.

Arc : elementary, 56; right, 56.

Boundary operator, 5.

Canonical section, *see* Section.

Cap : k-cap, 12; k-cap associated with a critical set, 41; k-cap relative to space R, 39.

Cap limit, 12; associated with a critical set, 41.

Cap-isomorphic groups, 42.

Carrier : of a k-cycle, 6; of an homology, 6; of a formal k-chain, 7.

Cell : algebraic k-cell, 5; null k-cell, 5; oriented k-cell, 5, (of norm e) 5; oriented $(k-1)$-cell, 5; singular n-cell, 25; vertex k-cell, 5.

Chain : k-chain of norm e, 5; algebraic k-chain, 6; boundary chain, 6; equality of chains, 5; formal k chains, 7, (sum of) 7; reduced chains, 5; sum of chains, 5.

Conjugate point, 68.

Connectivities, 19.

Coordinate system : admissible, 43; of class C′, 68; special, 68.

Critical chords, 49.

Critical points, 30.

Curve, 34; μ-curve, 34; rectifiable curve, 65.

Curve class, 34; partition of, 52.

Cycle : algebraic cycle, 6; canonical k-cycle, 11; cycle mod B on C, 6; non-bounding k-cycle, 7; rank of a k-cycle, 11; singular cycle, 3; Vietoris k-cycles mod B on C, 6, (homologous) 6, (derived from an algebraic k-cycle) 27.

Cycle bound, 10.

Cycle limit, 10.

Definitely below (d-below) a, 12.

Definitely-modulo (d-mod), 12.

Deformation : admissible, 30; D, 30; Λ, 63; $\theta(r)$, 59; Z, 70.

Deformation chain, 8, 9.

Deformation operator, 8, 9, 10.

Derived cycles, 27.

Displacement function, *see* Function.

Distance between curve classes, 35.

Distance, Frechet, 34.

TABLE OF CONTENTS.