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Mémorial des sciences mathématiques, fascicule 73 (1935)
<http://www.numdam.org/item?id=MSM_1935__73__1_0>

MÉMORIAL
DES
SCIENCES MATHÉMATIQUES

PUBLIÉ SOUS LE PATRONAGE DE

L'ACADÉMIE DES SCIENCES DE PARIS,
DES ACADÉMIES DE BELGRADE, BRUXELLES, BUCAREST, COÏMBRE, CRACOVIE, KIEW, MADRID, PRAGUE, ROME, STOCKHOLM (FONDATION MITTAG-LEFFLER),
DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, AVEC LA COLLABORATION DE NOMBREUX SAVANTS.

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FASCICULE LXXIII

Approximation by Polynomials in the complex Domain

By J.-L. WALSH
Associate Professor of Mathematics in Harvard University

PARIS
GAUTHIER-VILLARS, ÉDITEUR
LIBRAIRE DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE
Quai des Grands-Augustins, 55.

1935

AVERTISSEMENT

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APPROXIMATION BY POLYNOMIALS

IN THE COMPLEX DOMAIN

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INTRODUCTION.

In the past quarter-century the theory of approximation in the complex domain has developed from a few scattered results (due notably to Runge, Hilbert, and Faber) into an extensive, coherent structure. Several outstanding factors have contributed to this evolution : 1° Montel's book (1910) made the previous results available, added important new results, and has had a profound effect on later work; 2° the study of approximation in the real domain — notably by S. Bernstein but also by Borel, Jackson, de la Vallée Poussin, Lebesgue, Montel, and others — has supplied methods and a structural model of significance in the complex domain; 3° the great extent of the theory of expansions in real orthogonal functions (Schmidt, Riesz-Fischer, etc.) has been the inspiration for a corresponding development in the complex domain; 4° progress in other parts of the theory of functions of a complex variable, notably in conformal mapping and in Montel's theory of normal families, has made possible a corresponding advance in the study of approximation.

It is the object of the present essay to set forth at least in broad outlines the present status of the theory of approximation, particularly *in the sense of uniform approximation to a given function in a given limited region*. Due to lack of space, we are not able even to state all interesting results in detail. The choice of material for detailed proof, summarizing, or bare reference is not necessarily intended to indicate relative importance; indeed, such choice is to some extent arbitrary or accidental, depending on simplicity of exposition. Broadly speaking, we have tried to indicate *typical methods* in the discussion of each kind of problem, and also have been at especial pains to point out numerous problems which await solution. Many results here set forth have not been previously published.

Various topics have been deliberately left out of consideration. Approximation of real functions, even though the behavior of such functions for complex values has some significance, is set forth in books by Borel, de la Vallée Poussin [2], S. Bernstein [2], and Jackson [5], so that we have not considered it advisable to include that subject. We emphasize the fact that our primary topic is *approximation* of functions, not series of polynomials or expansion of functions. Thus representation of functions in series of polynomials which do not involve uniform approximation has been omitted; this omission is not serious, for everything known in the field is taken into account in important recent papers by Hartogs and Rosenthal [1] and Lavrentieff [1]. Likewise the detailed discussion of the modern special theory of Taylor's series : summability, behavior on the circle of convergence, gap theorems, overconvergence, zeros of approximating polynomials, limits on coefficients, exceptional values, etc., has been omitted. Finally, of the two major problems of the theory of interpolation : *a.* the existence of functions with certain properties taking on prescribed values in given points; *b.* the approximate representation of a given function $f(z)$ by means of other functions required to coincide with $f(z)$ in certain points — it is only the second of these problems with which we shall be concerned. In particular, we do not treat factorial series and their generalizations involving interpolation in an unlimited (unbounded) set of points; the interested reader may refer to Nörlund [1, 2, 3].

Emphasis on *degree of approximation*, especially on the *greatest geometric degree of convergence*, exerts a unifying influence on all the subject matter that we present. The entire theory of approximation in the complex domain is still growing rapidly, and this essay will achieve its purpose if it portrays that growth to the reader (1).

(1) For help with the manuscript of this essay, the writer wishes to express his thanks to Miss H. G. Russell, and to the Milton Fund of Harvard University.

CHAPTER 1.

POSSIBILITY OF APPROXIMATION.

1. Point sets : preliminary definitions. — We shall be concerned in the present essay entirely with the plane of the complex variable. A *point set* C is an arbitrary aggregate of points. Its *complement* consists of all points of the plane not belonging to C. A *neighborhood* of a point P is the interior of a circle whose center is P. A *limit point* of C is a point P (whether belonging to C or not) in whose every neighborhood lie points of C other than P. A *boundary point* of C is a point P in whose every neighborhood lie points of C and points of its complement. A point P is *exterior* to C if there exists some neighborhood of P containing no point of C. A point P is *interior* to C if there exists some neighborhood of P containing only points of C. A set C which contains its limit points is *closed*. A set C whose elements are all interior points of C is *open*. A set is *limited* if it lies within some circle.

A *Jordan arc* is a one-to-one continuous transform of a line segment, that is, a point set which can be represented

$$(1) \qquad x=f_1(t), \qquad y=f_2(t) \qquad (0 \leqq t \leqq 1),$$

where $f_1(t)$ and $f_2(t)$ are continuous functions of t and where the system (1) has at most one solution t for given x and y.

A point set C is *connected* if any two points of C can be joined by a Jordan arc consisting only of points of C. A *region* is an open connected set. A *closed region* need not be a region, but is a region closed by the adjunction of its boundary points.

A *Jordan curve* is a one-to-one continuous transform of a circumference, that is, a point set which can be represented

$$(2) \qquad x=f_1(\theta), \qquad y=f_2(\theta),$$

where $f_1(\theta)$ and $f_2(\theta)$ are continuous functions of θ with period 2π and where any two solutions θ of the system (2) for given x and y differ by an integral multiple of 2π. We shall use the term *contour* in the sense of rectifiable Jordan curve.

A Jordan curve C is known (theorem of Jordan) to separate the plane into precisely two regions, one limited (finite) and the other not limited (infinite), respectively the *interior* and *exterior* of C. A *Jordan region* is a limited (finite) region bounded by a Jordan curve. A Jordan arc cannot separate the plane into two or more distinct regions.

A function is *analytic at a point* if it can be expanded in Taylor's series valid throughout a neighborhood of that point. A function is *analytic on a point set* if it is analytic at each point of the set.

An expression of the form $a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ is called a *polynomial in z* and in particular a *polynomial of degree n*.

The term *integrable* refers to integration in the sense of Lebesgue.

2. **Runge : approximation to analytic functions.** — In the study of the possibility of approximation to a given function the fundamental theorems were given by Runge in his classical paper [1] of 1885. These theorems are of the greatest importance in the present essay; we omit the proofs, however, because they are to be found in many standard works [such as Picard, 1; Montel, 1]. Moreover, the method of Hilbert, which we shall consider later in some detail, also includes a proof of Runge's theorems. We give the name *Runge's first theorem* to the following, although Runge's own statement was somewhat different in content :

If the function $f(z)$ is analytic in a closed Jordan region C, *then in that closed Jordan region $f(z)$ can be uniformly approximated as closely as desired by a polynomial in z.*

Runge's theorem is more readily proved for the case of a convex region than for the general case [Painlevé, 1].

The two concepts, possibility of uniform approximation by a polynomial with an arbitrary small error, and uniform expansion in a series or sequence of polynomials, are of course equivalent (without reference to the present situation), in the sense that each implies the other directly.

Runge's theorem specifies that the region under consideration shall be a *Jordan* region; it is essential that the region not be an arbitrary simply-connected region, as we shall now illustrate by an example [Walsh, 2]. Let C be a strip, closed at one end, which winds around the outside of the unit circle $\gamma : |z| = 1$ infinitely often and approaches γ

The circle γ belongs to the *closed* region C. Choose $f(z) = \frac{1}{z}$; there exists no sequence of polynomials approaching $f(z)$ uniformly in the closed region. For if we assume the existence of such a sequence $p_n(z)$, it converges uniformly on γ and we may integrate term by term over γ,

$$\lim_{n \to \infty} \int_\gamma p_n(z)\, dz = \int_\gamma \frac{dz}{z}, \qquad 0 = 2\pi i,$$

which is absurd.

Runge's theorem was extended by him to include simultaneous approximation in several distinct regions :

Let functions $f_1(z), f_2(z), \ldots, f_\mu(z)$ be analytic in mutually exclusive closed Jordan regions $C_1, C_2, \ldots, C_\mu$. Then in these closed regions the functions $f_k(z)$ can be simultaneously uniformly approximated with an arbitrarily small error by a polynomial in z.

We state for reference the two classical theorems of Weierstrass on approximation, generalizations of which are to be mentioned later.

I. *If the function $f(z)$ is continuous on the interval $a \leqq z \leqq b$, then on this interval the function $f(z)$ can be approximated as closely as desired by a polynomial in z.*

II. *If the function $f(\theta)$ is continuous for all real values of θ and periodic with period 2π, then for all values of θ the function $f(\theta)$ can be approximated as closely as desired by a trigonometric polynomial of the form $\sum_{n=0}^{N}(a_n \cos n\theta + b_n \sin n\theta)$.*

By virtue of the equations which hold on the unit circle

$$\sin n\theta = \frac{(z^n - z^{-n})}{2i}, \qquad \cos n\theta = \frac{(z^n + z^{-n})}{2},$$

$$z^n = \cos n\theta + i \sin n\theta, \qquad z^{-n} = \cos n\theta - i \sin n\theta,$$

this second theorem can be expressed :

If the function $f(z)$ is continuous on the unit circle $C : |z| = 1$, then on C the function $f(z)$ can be approximated as closely as desired by a polynomial in z and $\frac{1}{z}$.

3. Approximation to analytic functions; more general point sets. — The theorems already mentioned are of much importance in answering such questions as the following, which we now consider :

When can a function defined and continuous on a given point set be uniformly approximated by a polynomial on that point set? Of what point sets C is it true that EVERY *function analytic on C can be uniformly approximated by a polynomial on that point set? Of what point sets C is it true that every fuction* CONTINUOUS *on C can be so approximated? Of what closed* REGIONS *C is it true that an arbitrary function analytic in the corresponding closed region can be uniformly approximated in the closed region? Of what closed regions C is this true for an arbitrary function analytic interior to C, continuous in the corresponding closed region?*

Approximation on unlimited point sets is easily disposed of :

A necessary and sufficient condition that a function $f(z)$ can be uniformly approximated on an unlimited point set C as closely as desired by a polynomial in z is that $f(z)$ itself should be on C a polynomial in z.

The proof is not difficult and is left to the reader.

Uniform approximation by a polynomial of a function $f(z)$ on a limited point set C which is not closed is equivalent to uniform approximation to an extension of the function $f(z)$ on the set C′ composed of C and the limit points of C. If $\varepsilon > 0$ is preassigned, a sequence of polynomials $p_n(z)$ converging to $f(z)$ uniformly on C satisfies the inequality $|p_n(z) - p_m(z)| \leqq \varepsilon$, z on C, for $n > N$, $m > N$, where N depends only on ε. This inequality, valid on C, is also valid on C′, so the sequence converges uniformly on C′. The function $f(z)$ is naturally continuous on C, and if that function is defined on C′ by means of the sequence of polynomials (or, what is equivalent, by the requirement of continuity on C′) then $f(z)$ can be uniformly approximated by a polynomial on C′. Thus, in our future consideration of approximation, it is sufficient to study approximation of continuous functions on closed limited point sets.

The study of approximation of functions merely continuous on a given closed set is much more delicate than the study of functions assumed analytic on the given set, so we turn first to the latter problem.

Let C be an arbitrary closed limited point set and let the function $f(z)$ be analytic in every point of C. Then a necessary and sufficient condition that $f(z)$ can be uniformly approximated on C as closely as desired by a polynomial in z is that C should separate no singularity of $f(z)$ (considered as one or more monogenic analytic functions) from the point at infinity.

Let C′ denote the complement of the infinite region whose boundary consists of points of C but which contains in its interior no points of C. Then the condition (otherwise expressed) is *that $f(z)$ can be extended analytically from C along paths of C′ so as to be analytic on the entire set C′.* Naturally, if C falls into several distinct components, the monogenic analytic function defined on one component may have a singularity on or interior to another component.

The example given in paragraph 2 is of interest in this connection.

The condition of the theorem is sufficient. If the condition is satisfied, the function $f(z)$ can be extended so as to be analytic and single-valued not merely on C but everywhere within a positive distance δ of C. Hence there exist one or more mutually exterior Jordan curves $C_1, C_2, \ldots, C_\mu$ such that every point of C lies interior to some curve, and such that $f(z)$ is analytic on and within each curve. Then by Runge's theorem (§ 2) the function $f(z)$ can be uniformly approximated on and within the curves C_k, hence on C.

The condition of the theorem is necessary. Let P be a singularity of $f(z)$ separated from the point at infinity by C and assume that $f(z)$ can be expressed on C as a uniformly convergent series of polynomials; we shall reach a contradiction. The point P is not a point of C, but lies interior to a limited simply connected region R bounded entirely by points of C. In fact, the points which can be joined to P by Jordan arcs not meeting C form a region which contains P in its interior and whose boundary consists wholly of points of C. That region need not be simply connected, but if it is multiply connected and if suitable points are adjoined to the region, there results a region R with the properties mentioned.

The series of polynomials representing $f(z)$ on C converges uniformly on the boundary of R, hence uniformly in the closed region R, and represents a function $\Phi(z)$ analytic interior to R, continuous in the corresponding closed region, equal to $f(z)$ on C. The function $f(z)$ is analytic on C, hence analytic in a suitable neighborhood of the boundary of R. The function $f(z) - \Phi(z)$ is analytic in a subregion of R adjoining the boundary of R and approaches zero when z interior to R approaches the boundary of R. We shall prove that $f(z) - \Phi(z)$ vanishes identically interior to R.

LEMMA. — *If the function $\Psi(z)$ is analytic in an annular region interior to but bounded in part by $\gamma : |z| = 1$ and if $\Psi(z)$ is continuous in the corresponding closed region and zero on γ, then $\Psi(z)$ vanishes identically in the original annular region.*

The function $\Psi(z)$ can be extended analytically across γ by Schwarz's principle of reflection; the function $\Psi_1(z)$:

$$\Psi_1(z) = \Psi(z), \qquad z \text{ on and within } \gamma,$$

$$\Psi_1(z) = \overline{\Psi\left(\frac{1}{\bar{z}}\right)}, \qquad z \text{ exterior to } \gamma,$$

is analytic in an annular region which contains γ in its interior and vanishes on γ, hence vanishes identically.

If R is mapped onto the interior of the unit circle, the function $f(z) - \Phi(z)$ corresponds to a function $\Psi(z)$ which satisfies the hypothesis of the lemma, when $\Psi(z)$ is suitably defined on γ, so $f(z) - \Phi(z)$ vanishes identically in an annular region interior to R, hence throughout R. Thus P, not a singularity of $\Phi(z)$, cannot be a singularity of $f(z)$, and the theorem is completely proved.

An immediate consequence of the theorem is : *Let* C *be an arbitrary closed limited point set. A necessary and sufficient condition that every function $f(z)$ analytic on* C *can be uniformly approximated on* C *as closely as desired by a polynomial in z is that* C *should be the complement of an infinite region.* Otherwise expressed, this condition is that C should not separate the plane.

4. Jordan configurations. — The results subsequent to Runge's theorems that we have considered [due primarily to Walsh, 2, 3, 6; but see also Hartogs and Rosenthal, 1] are entirely satisfactory so far as concerns approximation of functions analytic at every point of the closed considered. If one desires more refined results, methods correspondingly more refined must be used. Here the modern theory of conformal mapping is of service, particularly results of Carathéodory, Courant, and Lindelöf, and one can prove :

If the function $f(z)$ is analytic in a Jordan region, continuous in the corresponding closed region, then in that closed region $f(z)$ can be approximated as closely as desired by a polynomial in z (1).

This is a generalization of Runge's theorem.

If $f(z)$ is continuous on the Jordan curve C, in whose interior the origin lies, then on C the function $f(z)$ can be approximated as closely as desired by a polynomial in z and $\frac{1}{z}$.

This is a generalization of Weierstrass's theorem on approximation by trigonometric polynomials (§ 2).

If $f(z)$ is continuous on a Jordan arc C, then on C the function $f(z)$ can be approximated as closely as desired by a polynomial in z.

The special case where C is an interval of the axis of reals gives us Weierstrass's theorem on approximation by polynomials.

The three theorems just stated are due to Walsh [2, 3]; they are all contained in the following more general theorem [Walsh, 6] :

Let C be a closed point set composed of a finite number of Jordan arcs and regions, which separates no pair of points not belonging to C. Then an arbitrary function analytic in the interior points of C and continuous on C can be uniformly approximated on C as closely as desired by a polynomial in z.

5. **Further results.** — From the discussion of paragraph 3 we see that *if a closed limited point set C has the property that every*

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(1) We remark in passing that this yields an immediate proof of a general form of the Cauchy-Goursat theorem : *If $f(z)$ is analytic interior to a rectifiable Jordan curve C, continuous in the corresponding closed region, then we have* $\int_C f(z)\,dz = 0$.

function continuous on C *can be uniformly approximated on* C, *then* C *must have no interior points and cannot separate the plane*. Lavrentieff [Hartogs and Rosenthal, 1] has stated the converse, that *an arbitrary function continuous on the closed limited set* C, *which has no interior points and does not separate the plane*, *can be uniformly approximated on* C. This result is highly important; it is to be hoped that publication of the proof will not be long delayed. The special case that C is of superficial measure zero has been treated by Hartogs and Rosenthal [1].

The first and last questions mentioned in paragraph 3 are still unanswered, although the following contribution is due to Farrell [1]:

Let C *be a limited simply connected region and let* $w = \Phi(z)$ *map* C *conformally onto* $|w| < 1$. *In order that* $\Phi(z)$ *when suitably defined in the corresponding closed region* $\overline{C}$ *can be uniformly approximated in* $\overline{C}$ *by a polynomial in* z, *it is necessary and sufficient that every point of the boundary of* C *be contained in just one boundary element* (*Primende*) *and that the boundary of* C *be also the boundary of an infinite region*. *An arbitrary function analytic interior to such a region* C, *continuous in* $\overline{C}$ *and constant on each boundary element*, *can also be uniformly approximated in* $\overline{C}$.

A few other problems deserve mention. Approximation not by an arbitrary polynomial (that is, a linear combination of the functions $1, z, z^2, \ldots$) but by a linear combination of functions $1, z^{\lambda_1}, z^{\lambda_2}, \ldots$ has been studied by Carleman [1]. Approximation by polynomials whose roots lie in an infinite sector or are subject to broader conditions has been studied by Lindwart and Pólya [1]. Approximation in a region by polynomials not vanishing in that region has been studied by Walsh [13]. Some open questions of interest are: Determine the regions C with the property that an *arbitrary* function $f(z)$ analytic and limited interior to C can be represented in C by a sequence of polynomials $p_n(z)$ such that we have

$$(3) \qquad \overline{\lim_{n\to\infty}}\,[\max|p_n(z)|,\ z \text{ in } C] = \overline{\text{Bound}}\,[|f(z)|,\ z \text{ in } C];$$

compare Carleman [1]. Given an arbitrary region C and a function $f(z)$ analytic and limited interior to C; when can the *given* function

$f(z)$ be represented in the given region C by a sequence of polynomials $p_n(z)$ such that (3) is valid? If $f(z)$ and its derivative $f'(z)$ are analytic interior to a Jordan region and continuous in the corresponding closed region C, can $f(z)$ and $f'(z)$ be expressed in C as the uniform limit of the respective sequences $p_n(z)$, $p'_n(z)$? The answer is affirmative if the boundary of C is rectifiable.

What can be said of the possibility of approximation to a function $f(z)$ analytic in a region by a polynomial $p(z)$, as measured not by max $|f(z)-p(z)|$ but by the line or surface integral of $|f(z)-p(z)|^p$, $p>0$, taken over the boundary (assumed rectifiable) of the region or over the region itself provided the corresponding integral of $|f(z)|^p$ exists? Results for a Jordan region for the case $p=2$ have been given by Smirnoff [1] and Carleman [1] respectively. If approximation is measured by a line integral and if $f(z)$ is not analytic on the boundary, then boundary values in some sense are to be used in the line integral.

CHAPTER II.

DEGREE OF CONVERGENCE. OVERCONVERGENCE.

We have hitherto considered primarily the question of *possibility* of approximation. We turn now to the study of *degree* of approximation — study of the asymptotic behavior of the error of approximation : max$[|f(z)-p_n(z)|$, z on C$]$, as n becomes infinite.

6. **Lagrange-Hermite Interpolation Formula.** — *Let the distinct points $z_1, z_2, \ldots, z_p$, indices $\nu_1, \nu_2, \ldots, \nu_p$, and values $w_k, w_k^{(1)}, \ldots, w_k^{(\nu_k)}$, for $k=1, 2, \ldots, p$ be given, where*

$$(\nu_1+1)+(\nu_2+1)+\ldots+(\nu_p+1)=n+1.$$

Then there exists a unique polynomial $p(z)$ of degree n which satisfies the equations

$$p^{(m)}(z_k)=w_k^{(m)} \qquad (m=0, 1, 2, \ldots, \nu_k;\ k=1, 2, \ldots, p),$$

where the notation $p^{(m)}(k)$ indicates the m-th derivative of $p(z)$ at the point $z=z_k$, and $p^{(0)}(z_k)=p(z_k)$.

The determination of the polynomial $p(z)$ depends on the solution of a system of $n+1$ linear equations for the $n+1$ coefficients a_μ of $p(z)$. The vanishing of the determinant Δ of the coefficients of the a_μ in this system is a necessary and sufficient condition for the existence of a polynomial of degree n not identically zero which vanishes, together with its first ν_k derivatives, in the point z_k; such a polynomial is known not to exist. Hence Δ is different from zero, and the required polynomial $p(z)$ exists and is unique.

As a convention, we frequently express these requirements on $p(z)$ by saying that $p(z)$ shall take on prescribed values in the points z_k, counted *of respective multiplicities* ν_k+1. In particular the values $w_k^{(m)}$ may be the derivatives $f^{(m)}(z_k)$ of a given function $f(z)$. In this case the polynomial $p(z)$ is said to *coincide with* or to *interpolate to* the function $f(z)$ in the points z_k, considered of respective multiplicities ν_k+1.

For our present purposes, we require a formula for the polynomial $p(z)$ of degree n which coincides with the function $f(z)$ (analytic on and within a contour C) in the points (not necessarily distinct) $z_1, z_2, \ldots, z_{n+1}$ interior to C. We have

$$(1) \qquad f(z)-p(z)=\frac{1}{2\pi i}\int_C \frac{(z-z_1)\ldots(z-z_{n+1})f(t)\,dt}{(t-z_1)\ldots(t-z_{n+1})(t-z)}$$

z interior to C, as we shall proceed to verify. It is seen by inspection that $f(z_k)=p(z_k)$, $k=1, 2, \ldots, n+1$. Moreover, if $f(z)$ is expressed by Cauchy's integral taken over C, the resulting expression for $p(z)$ is clearly a polynomial of degree n. The conditions stated determine $p(z)$ uniquely if the z_k are distinct. The reader will verify the fact that the formula (1) is valid even if the points z_k are not all distinct (1) [Hermite, **1**].

Formula (1) is also correct if C is composed of several distinct contours bounding one or more distinct finite regions, provided $f(z)$ is analytic (and single valued) in each closed region, and the points z_k all lie in these regions.

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(1) Thanks to the convention made relative to multiple points z_k, it is easily proved from (1) that the polynomial $p(z)$ is a continuous function of the z_k. This continuity is uniform for z in any limited region, and for the z_k on any closed point set interior to C.

7. Expansion in powers of a polynomial. — We shall use the term *lemniscate* to indicate a locus $|z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_\nu)|=\mu>0$; a lemniscate thus consists of one or more contours which are mutually exterior except that each of a finite number of points may belong to several contours. By the *interior* of this lemniscate we mean the interiors of the contours composing it :

$$|(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_\nu)|<\mu.$$

If the function $f(z)$ is analytic on and within the lemniscate Γ : $|p(z)|=\mu$, where $p(z)=(z-\alpha_1)\dots(z-\alpha_\nu)$, then interior to Γ the function $f(z)$ can be expanded in a series of polynomials of which the nth term is a polynomial of degree $\nu-1$ multiplied by the $(n-1)$ st power of $p(z)$. The sum $S_n(z)$ of the first n terms of the series coincides with $f(z)$ in the points $\alpha_1, \alpha_2, \dots, \alpha_\nu$, each counted of multiplicity n. Moreover, for z on the set C :

$$|p(z)|\leqq\mu_1<\mu$$

we have

$$|f(z)-S_n(z)|\leqq M\left(\frac{\mu_1^n}{\mu^n}\right), \tag{2}$$

where M, independent of n and z, is suitably chosen.

The Cauchy-Taylor development is a special case of the series considered here, and for that case these properties are well known.

Define $S_n(z)$ as a polynomial of degree $\nu n-1$ which coincides in the points α_k (each counted of multiplicity n) with $f(z)$. Then the polynomial $S_n(z)-S_{n-1}(z)$ is of degree $\nu n-1$ and has $n-1$ roots in each point α_k. Thus we can write the formal expansion in the form

$$f(z)=q_1(z)+q_2(z)[p(z)]^1+q_3(z)[p(z)]^2+\dots,$$

where $q_k(z)$ is a polynomial of degree $\nu-1$.

We have by (1),

$$f(z)-S_n(z)=\frac{1}{2\pi i}\int_\Gamma\frac{(z-\alpha_1)^n(z-\alpha_2)^n\dots(z-\alpha_\nu)^n f(t)}{(t-\alpha_1)^n(t-\alpha_2)^n\dots(t-\alpha_\nu)^n(t-z)}\,dt.$$

For z on C and t on Γ we clearly have $|p(z)|\leqq\mu_1<\mu=|p(t)|$, which implies convergence as stated.

We shall need to apply the theorem just proved in a case where the number ν is of no significance, so we express our result in a slightly less precise form, in which ν no longer occurs. The function $\sqrt[\nu]{p(z)}$ is analytic (although not necessarily single-valued) and without branch points exterior to C. The function

$$w = \sqrt[\nu]{\frac{p(z)}{\mu_1}} = \Phi(z)$$

maps conformally (not necessarily uniformly if C falls into several distinct pieces) the exterior of C onto the exterior of the circle $|w| = 1$ so that the points at infinity correspond to each other. The lemniscate Γ can be expressed

$$|\Phi(z)| = \sqrt[\nu]{\frac{\mu}{\mu_1}},$$

and the right hand member of (2) can be written as $M\sqrt[\nu]{\frac{\mu_1}{\mu}}\left(\sqrt[\nu]{\frac{\mu_1}{\mu}}\right)^{\nu n-1}$; the exponent is precisely the degree of the polynomial $S_n(z)$.

Let us now introduce the notation

$$\begin{aligned}
p_{\nu-1}(z) &= p_\nu(z) = \ldots = p_{2\nu-2}(z) = S_1(z),\\
p_{2\nu-1}(z) &= p_{2\nu}(z) = \ldots = p_{3\nu-2}(z) = S_2(z),\\
p_{3\nu-1}(z) &= p_{3\nu}(z) = \ldots = p_{4\nu-2}(z) = S_3(z),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;
\end{aligned}$$

here $p_n(z)$ is a polynomial of degree n, and we can write for z on C

$$(3) \qquad |f(z) - p_n(z)| \leqq M_1\left(\sqrt[\nu]{\frac{\mu_1}{\mu}}\right)^n, \qquad M = M_1\left(\frac{\mu_1}{\mu}\right)^{\frac{\nu-2}{\nu}}.$$

To be sure, we have defined no polynomials of degrees $n = 0, 1, \ldots, \nu - 2$, but any polynomials whatever of these degrees can be used, and (3) will be valid for z on C for all n if M_1 is suitably modified.

Let C_R denote the locus $|\Phi(z)| = R > 1$. We have shown that if $f(z)$ is analytic on and within C_R, then polynomials $p_n(z)$ of respective degrees $n = 0, 1, 2, \ldots$ exist such that

$$|f(z) - p_n(z)| \leqq \frac{M_1}{R^n}$$

is valid for z on C.

The method we have used is due to Jacobi [1], Hilbert [1], and Montel [1], although these other writers did not emphasize the precise inequalities involved.

We can now generalize, to the case that C is no longer the interior of a lemniscate but is the interior of an arbitrary Jordan curve.

8. **Expansion in arbitrary Jordan regions.** — It is Hilbert's important achievement [1] to have shown that an arbitrary Jordan curve can be uniformly approximated by a lemniscate. More explicitly, *let* C *be a Jordan curve and* $\delta > 0$ *arbitrary. Then there exists a lemniscate* C′ *consisting of a single contour such that every point of* C *is interior to* C′ *but within a distance* δ *of* C′, *and every point of* C′ *is within a distance* δ *of* C. Hilbert's proof is based on the logarithmic potential and will be omitted here. As Faber suggests, it would be worth while to give an elementary proof of this result.

It is to be noted that a proof of Runge's first theorem (§ 2) is now easy to supply. Let $f(z)$ be analytic in the closed Jordan region C; choose δ in Hilbert's result less than half the shortest distance from C to a singularity of $f(z)$. Hilbert's lemniscate $C' : |p(z)| = \mu$ contains C in its interior, and $f(z)$ is analytic on and within C′. A suitably chosen lemniscate $|p(z)| = \mu_1 < \mu$ also contains C in its interior. The approximation studied in paragraph 7 now yields Runge's theorem. We shall obtain, however, a more precise result.

Let us say that a variable Jordan curve $C^{(n)}$ *approaches* the fixed Jordan curve C provided the *écart* of C and $C^{(n)}$ approaches zero, where the écart of C and $C^{(n)}$ is the distance to C of the most distant point of $C^{(n)}$ plus the distance to $C^{(n)}$ of the most distant point of C. Denote by C_R as before the curve $|\Phi(z)| = R > 1$, where the function $w = \Phi(z)$ maps conformally and uniformly the exterior of C onto the exterior of $|w| = 1$ so that the points at infinity correspond to each other, and similarly for $C_R^{(n)}$. Then *if* $C^{(n)}$ *approaches* C, *the curve* $C_R^{(n)}$ *approaches* C_R, *and monotonic approach of* $C^{(n)}$ *to* C *implies monotonic approach of* $C_R^{(n)}$ *to* C_R. This fact is by no means elementary; the proposition follows from the fundamental work of Lebesgue [1] on harmonic functions, or from the study of conformal mapping of variable regions as made by Carathéodory [1].

Not merely is it true that a Jordan curve C can be approximated by a lemniscate $C^{(n)}$ and that the curve C_R is approximated by the curve $C_R^{(n)}$; the reasoning can be applied to more general point sets. Let us prove :

Let C *be an arbitrary closed limited point set* (*not a single point*) *whose complement is simply connected. Let $w = \Phi(z)$ map the complement of* C *onto the region $|w| > 1$ so that the points at infinity correspond to each other. Let C_R denote the curve $|\Phi(z)| = R > 1$. If the function $f(z)$ is analytic on and within C_R, then there exist polynomials $p_n(z)$ of respective degrees $n = 0, 1, 2, \ldots$ such that the inequality*

$$|f(z) - p_n(z)| \leqq \frac{M}{R^n}, \qquad z \text{ on } C,$$

is valid, where M *is independent of n and z.*

The function $w = \frac{\Phi(z)}{\rho}$ maps the exterior of C_ρ onto the exterior of $|w| = 1$ so that the points at infinity correspond to each other, so $[C_\rho]_R$ is the curve $\left|\frac{\Phi(z)}{\rho}\right| = R$, or the curve $C_{\rho R}$. Thus for a suitable choice of $\rho > 1$, the function $f(z)$ is analytic on and within $[C_\rho]_R$. There exists a lemniscate C' consisting of a single contour interior to this C_ρ containing C in its interior; indeed there exists such a lemniscate between C_ρ and $C_{(\rho+1)/\rho}$. The curve C'_R is interior to $C_{\rho R}$. Our theorem now follows from the last result of paragraph 7.

This theorem is due to Faber [1] (implicitly) and Szegö [1] (explicitly) in the case that C is a Jordan region, and to Walsh [4] in the general case.

9. **Overconvergence.** — We now prove, for application in studying the converse of the results just proved, the following

Lemma. — *If* C *is a closed limited point set (not a single point) whose complement is simply connected and if the polynomial* $P(z)$ *of degree n satisfies the inequality* $|P(z)| \leqq L$, *for z on* C, *then we have*

$$(4) \qquad |P(z)| \leqq LR^n, \qquad z \text{ on or within } C_R.$$

This lemma was proved by S. Bernstein [1] in the case that C is a line segment. The present method of proof is due to M. Riesz [1] and to Montel [2]. The lemma itself was proved by Faber [4] and Szegö [4] in cases slightly less restrictive than here, and by Walsh [4] in the general case. Let us proceed to the proof.

The function $\frac{P(z)}{[\Phi(z)]^n}$ is analytic exterior to C even at infinity (if suitably defined there), where $\Phi(z)$ has the same meaning as before. No maximum of the modulus of this function can occur exterior to C, unless the modulus is everywhere constant. When z approaches C, the modulus can approach no limit greater than L. Thus the inequality $\left|\frac{P(z)}{[\Phi(z)]^n}\right| \leqq L$ is valid for z exterior to C and (4) is valid for z on C_R. The modulus of $P(z)$ can have no maximum interior to C_R, and hence (4) is also valid for z interior to C_R; the proof is complete.

The following theorem is a consequence of the Lemma :

If the polynomials $p_n(z)$ of respective degrees $n = 0, 1, 2, \ldots$ satisfy the inequalities

$$|f(z) - p_n(z)| \leqq \frac{M}{R^n} \tag{5}$$

for z on an arbitrary closed limited set C whose complement is simply connected, then the sequence $p_n(z)$ converges for z interior to C_R, uniformly for z on any closed point set interior to C_R. The function $f(z)$ (or its analytic extension) is analytic interior to C_R.

From the inequalities for z on C

$$|f(z) - p_n(z)| \leqq \frac{M}{R^n}, \qquad |f(z) - p_{n+1}(z)| \leqq \frac{M}{R^{n+1}},$$

we have for z on C

$$|p_{n+1}(z) - p_n(z)| \leqq \frac{1}{R^n}\left(M + \frac{M}{R}\right).$$

For z on C_{R_1}, $R_1 < R$, we have by the Lemma

$$|p_{n+1}(z) - p_n(z)| \leqq \frac{R_1^n}{R^n}\left(M + \frac{M}{R}\right);$$

the sequence converges to $f(z)$ on C; the theorem follows at once.

The theorem was proved in the special case that C is a line segment by S. Bernstein [1], and in the general case by Walsh [4].

We shall use the term *overconvergence* for this phenomenon, namely, that certain sequences known to converge on a given point set C with a certain degree of convergence necessarily converge on a point set containing C in its interior. The term overconvergence is used by Ostrowski (*see* paragraph 17) in a different connection.

10. Greatest geometric degree of convergence. — *Let* C *be an arbitrary closed limited point set (not a single point) whose complement is simply connected and let* $f(z)$ *be analytic within* C_ρ *but (considered as a monogenic analytic function) have a singularity on* C_ρ, *where* ρ *is finite or infinite. Then* (§ 8) *if* $R < \rho$ *is arbitrary, there exist polynomials* $p_n(z)$ *of respective degrees* $n = 0, 1, 2, \ldots$ *such that* (5) *is valid for* z *on* C, *where* M *depends on* R *but not on* n *or* z. *Moreover, there exist* (§ 9) *no polynomials* $p_n(z)$ *such that* (5) *is valid with* $R > \rho$.

A less specific theorem but one of some interest is :

Let C *be an arbitrary closed limited point set (not a single point) whose complement is simply connected. A necessary and sufficient condition that* $f(z)$ *be analytic on* C *is that there exist polynomials* $p_n(z)$ *of respective degrees* $n = 0, 1, 2, \ldots,$ *such that* (5) *is valid for* z *on* C *for some* R *greater than unity.*

These two theorems were given by S. Bernstein [1] in the case that C is a line segment (here C_ρ is an ellipse whose foci are the ends of the segment) and in the more general case by Walsh [4]. The first part of the second theorem was later proved by Szegö [5] in an elementary way, without the use of conformal mapping.

Thus far we have made the polynomials $p_n(z)$ to depend on $R < \rho$, a restriction which we shall remove in Chapter III. A sequence of polynomials $p_n(z)$ of respective degrees $n = 0, 1, 2, \ldots$ which converges to the given function $f(z)$ so that (5) is valid on C for *every* $R < \rho$ where M depends on R but not on n or z is said to converge to $f(z)$ on C with the *greatest geometric degree of convergence;* this concept appears throughout the remainder of the present essay. Such a sequence converges everywhere interior to C_ρ, uniformly on any closed set interior to C_ρ, and converges also on every C_σ ($\sigma < \rho$) with the greatest geometric degree of convergence. Condition (5) may here be replaced by

$$\overline{\lim_{n \to \infty}} \sqrt[n]{\mu_n} = \frac{1}{\rho}, \qquad \mu_n = \max[|f(z) - p_n(z)|, \ z \text{ on } C].$$

A sequence of polynomials which converges with the greatest geometric degree of convergence may or may not converge at individual points exterior to C_ρ; compare paragraph 34. It is not known whether a sequence of polynomials can converge with the greatest geometric degree of convergence and can still converge in a *region* exterior to C_ρ. although it can be shown that convergence like a convergent geometric series in a region or on a Jordan arc exterior to C_ρ is impossible. Moreover. the following theorem can also be proved [Walsh, 9] :

A sequence of polynomials which converges on C *with the greatest geometric degree of convergence can converge uniformly in no region which contains in its interior a point of* C_ρ.

It would be of interest to study the implication of (5) not for all n (or for all $n > N$) but for an arbitrary infinite sequence of numbers n.

11. Approximation on more general point sets. — In paragraphs 8, 9, 10 we have studied for the sake of simplicity approximation on a point set C whose complement is simply connected. Not merely the results but also all the methods can be directly extended to a more general closed limited point set C, provided C is bounded by a finite number of mutually exterior Jordan curves, or more generally, provided the complement K of C is connected and *regular* in the sense that Green's function with pole at infinity exists for K. There exists a function $w = \Phi(z)$ mapping K onto the exterior of $|w| = 1$ so that the points at infinity correspond to each other, and $|\Phi(z)|$ is single valued in K even if $\Phi(z)$ is not; the reasoning is nevertheless valid. We leave to the reader the care of verifying the possibility of this extension; the details are given by Walsh and Russell [1].

Let C *be an arbitrary closed limited point set whose complement is connected and regular and let* $f(z)$ *(considered as one or more monogenic functions) be analytic and single-valued within* C_ρ *but not within any* $C_{\rho'}$, $\rho' > \rho$. *Then if* $R < \rho$ *is arbitrary, there exist polynomials* $p_n(z)$ *of respective degrees* $n = 0, 1, 2, \ldots$ *such that* (5) *is satisfied for* z *on* C, *where* M *depends on* R *but not on* n *or* z. *Moreover, there exist no polynomials* $p_n(z)$ *such that* (5) *is valid on* C, *where* R *is greater than* ρ.

The curve (or curves) C_ρ is characterized by the fact that the function $f(z)$ (when suitably extended analytically from C along paths interior to C_ρ) is analytic and single valued interior to C_ρ, but is not analytic or is not single valued or fails in both particulars interior to every $C_{\rho'}$, $\rho' > \rho$, when extended from C along paths interior to $C_{\rho'}$. Thus, (*a*) at some point P of C_ρ the function $f(z)$ has a singularity for analytic extension along paths interior to C_ρ terminating in P; or (*b*) the curve C_ρ has at least one multiple point Q, and there is disagreement at Q among the various analytic extensions of $f(z)$ from the various parts of C to Q along paths belonging to the several regions separated and bounded by C_ρ; or (*c*) both (*a*) and (*b*) occur.

It is worth noticing that this theorem applies to the case where C consists of a finite number of segments of the axis of reals.

The discussion just outlined for the case that the complement K of C is connected and regular can be modified to yield certain results for the case that K is connected and not regular. If K is not regular, there exists a sequence of *regular* regions $K^{(1)}$, $K^{(2)}$, ... interior to K, $K^{(n)}$ interior to $K^{(n+1)}$, such that every point of K is interior to some $K^{(n)}$. Denote by $C^{(n)}$ the complement of $K^{(n)}$. Then the curves $C_R^{(n)}$, where $R > 1$ is fixed and n becomes infinite, approach monotonically a limit Γ_R which is independent of the particular choice of the $K^{(n)}$. The limit Γ_R may consist of the boundary of C, may consist of part of the boundary of C together with curves or parts of curves lying in K, and may consist entirely of curves in K. Each point of C is either a point of Γ_R or is separated from the point at infinity by Γ_R. In any case, it follows from the methods and results already given that if $f(z)$ is analytic on and within Γ_R, there exist polynomials $p_n(z)$ of respective degrees n such that (5) is valid for z on C; if (5) is valid for z on C, then $f(z)$ is analytic interior to Γ_R.

Study of this situation in more detail is an interesting open problem; compare paragraph 29.

CHAPTER III.

BEST APPROXIMATION.

12. Tchebichef approximation. — As we have already remarked, the polynomials $p_n(z)$ considered in paragraphs 8-11 are not uniquely determined and in fact largely arbitrary. Nevertheless, unique polynomials with minimizing properties exist which are included in the category already studied.

Minimizing polynomials (where the usual measure of approximation is considered) were first studied by Tchebichef; their existence and uniqueness were established by Kirchberger in the real case and by Tonelli [1] and later de la Vallée Poussin [1] in the complex case :

Let the function $f(z)$ be defined and continuous on the closed limited point set C containing at least $n+1$ points. Then there exists one and only one polynomial $\pi_n(z)$ of degree n such that

$$\mu_n = \max[|f(z) - \pi_n(z)|, \ z \text{ on } C] \tag{1}$$

is less than the corresponding expression for any other polynomial of degree n.

The function $|f(z) - \pi_n(z)|$ takes on the value μ_n in at least $n+2$ points of C if C contains $n+2$ or more points. The polynomial $\pi_n(z)$ is called the *Tchebichef polynomial of degree n for approximation to $f(z)$ on* C. Relatively few properties of the individual $\pi_n(z)$ are known, such as the distribution of the points of C where $|f(z) - \pi_n(z)| = \mu_n$, or when the Tchebichef polynomial of degree n for approximation to $f(z)$ on C is also the Tchebichef polynomial of degree n for approximation to $f(z)$ or $f_1(z)$ on C_1; but if C has infinitely many points and μ_n approaches zero as n becomes infinite, the sequence $\pi_n(z)$ converges to $f(z)$ on C more rapidly than any other sequence of polynomials of respective degrees n. *If $f(z)$ can be approximated on* C *as closely as desired by a polynomial in z, then the polynomial $\pi_n(z)$ approaches $f(z)$ uniformly on* C, as the reader will easily prove. This remark has obvious application to the configurations of Chapter I.

It is entirely possible to use not (1) but

$$\max[n(z)|f(z) - \pi_n(z)|, \ z \text{ on } C]$$

as a measure of the approximation of $\pi_n(z)$ to $f(z)$ on C, where $n(z)$ is continuous and positive on C. This may be called *approximation in the sense of Tchebichef with the norm function $n(z)$*. If C contains at least $n+1$ points, and if $f(z)$ is continuous on C, a unique polynomial $\pi_n(z)$ of best approximation exists [Walsh, 12].

It is clear that *if* C *is a closed limited point set (not a single* *point*) *whose complement is simply connected, and if* $f(z)$ *is analytic on* C, *then the sequence* $\pi_n(z)$ *of best approximation in the sense of Tchebichef with a norm function converges to* $f(z)$ *on* C *with the greatest geometric degree of convergence.* For, under the hypothesis $0 < N_1 \leqq n(z) \leqq N_2$, z on C, and $R < \rho$, we have from paragraph 10 :

$$n(z)|f(z) - p_n(z)| \leqq \frac{MN_2}{R^n}, \qquad z \text{ on } C,$$

whence for the polynomials $\pi_n(z)$ of *best* approximation,

$$n(z)|f(z) - \pi_n(z)| \leqq \frac{MN_2}{R^n}, \qquad z \text{ on } C.$$

We can now write

$$|f(z) - \pi_n(z)| \leqq \frac{MN_2}{N_1 R^n}, \qquad z \text{ on } C,$$

as we were to prove. Consequently the properties mentioned in paragraph 10 are all valid for the sequence $\pi_n(z)$.

Proof of the convergence interior to C_ρ is due to Bernstein [1] for the case $n(z) \equiv 1$ and C a line segment, and to Faber [3] (by a method different from the one just used) for the case that $n(z) \equiv 1$ and C is an analytic Jordan curve; the present more general results are due to Walsh [7, 8].

It is also clear that in the study of approximation in the sense of Tchebichef $[n(z) \equiv 1]$ in a closed limited region to a function $f(z)$ analytic interior to that region, it is immaterial whether approximation is measured in the closed region or on the boundary, for the expression $|f(z) - p_n(z)|$ has its maximum value on the boundary.

13. Other measures of approximation. — The Tchebichef measure of approximation is the one used directly in the study of uniform convergence, but there are other measures which have the advantage of facility of computation of the polynomials involved. For the sake of generality we shall introduce norm functions in these other measures of approximation. Each norm function $n(z)$ or $n(w)$ is assumed positive and continuous where defined.

a. If C is a rectifiable Jordan curve, the integral

$$\int_C n(z)|f(z) - p_n(z)|^p\, ds \qquad (p > 0),$$

is of significance as a measure of approximation of the polynomial $p_n(z)$ to the function $f(z)$ continuous on C. The most important special case is $n(z) \equiv 1$, $p = 2$.

b. Let C be an arbitrary limited closed region. A suitable measure of the approximation of the polynomial $p_n(z)$ to the function $f(z)$ continuous on C is $\int\int_C n(z) |f(z) - p_n(z)|^p dS \quad (p > 0)$.

c. Let C be an arbitrary limited closed point set (not a single point) whose complement is simply connected. Let the exterior of C be mapped onto the exterior of $\gamma : |w| = 1$ so that the points at infinity correspond to each other. We consider as measure of approximation of the polynomial $p_n(z)$ to the function $f(z)$ analytic on C the integral

$$(2) \qquad \int_\gamma n(z) |f(z) - p_n(z)|^p ds \qquad (p > 0).$$

To be sure, the function $f(z) - p_n(z)$ is not necessarily defined on γ but the limit exists along the radius almost everywhere on γ (Fatou); it is these values that are to be used in (2).

These three measures *a*, *b*, *c* are all distinct and each has advantages over the other two.

d. [generalization of (*a*)]. If C is a rectifiable Jordan curve, a rectifiable Jordan arc, or more generally if C consists of a finite number of rectifiable Jordan arcs or curves and is the boundary of a closed limited point set whose complement is simply connected, then the integral used in (*a*) is still a natural measure of the approximation of the polynomial $p_n(z)$ to $f(z)$ on C.

e. If C is an arbitrary limited simply connected region, map C onto the *interior* of the unit circle $\gamma : |w| = 1$, and choose

$$(3) \qquad \int_\gamma n(w) |f(z) - p_n(z)|^p ds \qquad (p > 0),$$

as the measure of approximation of $p_n(z)$ to $f(z)$, assumed analytic interior to C. Of course the functions which appear in (3) need not be defined on γ, but the limits may exist almost everywhere in the sense of radial approach and it is those boundary values that are to be used in (3). To be sure, this measure of approximation depends on the particular point z_0 interior to C which is made to correspond to $w = 0$, but best approximation for a particular choice of z_0 with a *particular* norm function $n(w)$ is equivalent to best approximation for an arbitrary choice of z_0 with a *suitable* norm function.

f. Under the transformation just used in (*e*), we may employ

$$\int\int_{|w|\leq 1} n(w)\,|f(z)-p_n(z)|^p\, dS \qquad (p>0),$$

as a measure of approximation.

g. Even if the complement K of the given set C is not simply connected, the method of Tchebichef applies directly if K is connected and regular; methods *a-f* can be extended so as to apply with the analogous restrictions if K is of finite connectivity.

In each of the cases *a-g* if $f(z)$ is continuous on C, there exists at least one polynomial of degree n of best approximation to $f(z)$ on C, although in case (*c*) we require the function $f(z)$ to be defined continuously in the closed neighborhood of C exterior to C. The existence of this polynomial is conveniently although not necessarily proved by the use of Montel's theory of normal families of functions [*see* for instance Walsh, 12]. The polynomial of best approximation when approximation is measured in the sense of least p-th powers, $0<p<1$, need not be unique; the uniqueness when $p>1$ results from the fact that if two polynomials give equal approximation, then half their sum gives better approximation; the general inequality

$$(4) \qquad \left|\frac{(\alpha+\beta)}{2}\right|^p < \frac{1}{2}[|\alpha|^p+|\beta|^p] \qquad (\alpha\neq\beta;\ p>1),$$

can be applied directly.

Best approximation in the sense of least p-th powers has been studied in some detail in the real case by Jackson [5]. It is interesting that in the case of approximation to a continuous function on a Jordan arc C, the polynomial of degree n of best approximation in the sense of least p-th powers approaches as p becomes infinite the polynomial of degree n of best approximation to that function on C in the sense of Tchebichef [Pólya, 1; Julia, 2]. This result extends to each of the cases *a-g*, even with a norm function. Here we have a theoretical method for the determination of the polynomial of best approximation in the sense of Tchebichef, for the determination of the polynomial of best approximation in the sense of least p-th powers is an algebraic problem.

The general problem of studying the dependence on $f(z)$, $n(z)$, n, p, C, of the polynomial of degree n of best approximation on C in the sense of least p-th powers would yield results of much interest. These polynomials of best approximation are intimately connected with interpolation, whether interpolation is the only requirement on the polynomial or an auxiliary condition. For instance, best approximation on C with a fixed norm function which becomes infinite at a point of C implies interpolation in that point. Best approximation corresponding to a variable norm function when that function becomes infinite at one point and not at other points is likewise related to interpolation in that point. The theorem proved at the end of paragraph 30 also deserves mention here.

14. Approximation measured by line integrals. — We proceed to study convergence of the sequence of polynomials of best approximation in the various cases mentioned. In case (*a*) the following lemma is useful :

Lemma. — *Let* C *be an arbitrary rectifiable Jordan curve and* C′ *an arbitrary closed point set interior to* C. *If the function* P(z) *is analytic on and within* C *and if we have*

$$\int_C |P(z)|^p\, ds \leqq L^p \qquad (p > 0), \tag{5}$$

then we have

$$|P(z)| \leqq L'L, \qquad z \text{ on } C',$$

where L′ *depends on* C, C′, *and* p, *but not on* P(z) *nor* L.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the zeros of P(z) (necessarily finite in number) interior to C and let $w = \Phi(z)$ denote a function which maps the interior of C onto the interior of $|w| = 1$. We set

$$F(z) = \frac{P(z)}{\prod \dfrac{\Phi(z) - \Phi(\alpha_i)}{1 - \overline{\Phi(\alpha_i)}\,\Phi(z)}}, \tag{6}$$

where F(z) is defined at the points α_i so as to be analytic there and where Π is extended over all the zeros of P(z) interior to C with proper allowance for their multiplicities. Then F(z) is analytic and has no zeros interior to C; the function $[F(z)]^p$ is analytic within C, continuous on and within C when suitably defined on C:

$$[F(z)]^p = \frac{1}{2\pi i}\int_C \frac{[F(t)]^p\,dt}{t-z}, \qquad z \text{ interior to } C.$$

The relation $|F(z)| = |P(z)|$ is valid for z on C, so by (5) we have $|F(z)| \leqq L'L$, z on C'. From (6) follows the inequality $|P(z)| \leqq |F(z)|$, z interior to C, so the Lemma follows at once.

We can now easily show that *if $f(z)$ is analytic on and within* C, *and if approximation is measured as in* (a), *then the* (*for* $p \leqq 1$, *any*) *sequence* $\pi_n(z)$ *of polynomials of best approximation of respectives degrees n converges on* C *with the greatest geometric degree of convergence.* The notation of paragraphs 8-11 involves the mapping of the *complement* of a given set C onto the exterior of the unit circle. Under the present circumstances (and below) we map similarly the *exterior* of C. In the notation analogous to that of paragraph 10, let $R < \rho$ be arbitrary; choose R_1, $R < R_1 < \rho$. For polynomials $p_n(z)$ of paragraph 10 we have

$$\int_C n(z)\,|f(z) - p_n(z)|^p\,ds \leqq \frac{M_1}{R_1^{np}},$$

where M_1 is a suitably chosen constant; this inequality holds *a fortiori* if the $p_n(z)$ are replaced by the $\pi_n(z)$. The general inequalities

$$(7) \qquad \begin{cases} |\chi_1 + \chi_2|^p \leqq 2^{p-1}|\chi_1|^p + 2^{p-1}|\chi_2|^p & (p > 1), \\ |\chi_1 + \chi_2|^p \leqq |\chi_1|^p + |\chi_2|^p & (0 < p \leqq 1), \end{cases}$$

yield by virtue of the boundedness of $\dfrac{1}{[n(z)]}$

$$\int_C |\pi_{n+1}(z) - \pi_n(z)|^p\,ds \leqq \frac{M_2}{R_1^{np}}.$$

By the Lemma we have

$$(8) \qquad |\pi_{n+1}(z) - \pi_n(z)| \leqq \frac{M_3}{R_1^n}, \qquad z \text{ on } C'.$$

By inequality (8) we have (§ 9):

$$|\pi_{n+1}(z) - \pi_n(z)| \leqq \frac{M_3}{R^n}, \qquad z \text{ on } C'_{\frac{R_1}{R}}.$$

This inequality is therefore valid for z on C, for C' can be chosen

(§ 8) so that C is interior to $C'_{R_1/R}$; the limit of $\pi_n(z)$ on C is $f(z)$; the theorem follows at once.

The theorem we have just proved was established by other methods by Szegö [1] in the important case $n(z) \equiv 1$, $p = 2$, C an analytic Jordan curve, by Szegö [2] in the case that C is a circle, $p = 2$ [with only very light restrictions on $n(z)$], and by Smirnoff [1] in the case $n(z) \equiv 1$, $p = 2$, C an arbitrary rectifiable Jordan curve satisfying certain mild restrictions. We shall discuss details of these cases later (§ 20).

15. Approximation measured by surface integrals. — The proof in case (b) is somewhat different in detail from the proof just given; let us indicate the modifications.

LEMMA. — *If the function* $P(z)$ *is analytic interior to a region* C *and if we have*

$$\iint_C |P(z)|^p \, dS \leqq L^p \qquad (p > 0), \tag{9}$$

and if C′ *is an arbitrary closed point set interior to* C, *then we have*

$$|P(z)| \leqq LL', \qquad z \text{ on } C',$$

where L′ *depends on* C′ *but not on* $P(z)$.

The integral $\frac{1}{2\pi}\int_0^{2\pi} |P(z_0 + re^{i\theta})|^p \, d\theta$, $p > 0$, is well known to be a non-decreasing function of r, in an arbitrary circle K which together with its interior lies interior to C. Here (r, θ) are polar coordinates with pole at the point z_0. The limit of this integral as r approaches zero is obviously $|P(z_0)|^p$, so we have

$$|P(z_0)|^p \leqq \frac{1}{2\pi}\int_0^{2\pi} |P(z_0 + re^{i\theta})|^p \, d\theta.$$

Multiply both members by $r\,dr$ and integrate from zero to k, the radius of K; we find

$$\frac{k^2}{2}|P(z_0)|^p \leqq \frac{1}{2\pi}\iint_K |P(z)|^p \, dS,$$

or by (9),

$$|P(z_0)|^p \leqq \frac{1}{\pi k^2}\int\int_K |P(z)|^p\, dS \leqq \frac{1}{\pi k^2}\int\int_C |P(z)|^p\, dS \leqq \frac{L^p}{\pi k^2}.$$

This inequality is valid for every z_0 interior to C provided merely that the distance from z_0 to the boundary is not less than k, and hence holds for suitable choice of k for every z_0 of C′.

The application of this lemma is not greatly different from that already given in case (a) and is left to the reader.

The measure of approximation (b) was used by Carleman [1] for $n(z) \equiv 1$, $p = 2$, but without a proof of our result on degree of convergence and overconvergence. A less general form of the Lemma is given by him and another by Julia [3].

16. **Approximation measured after conformal mapping.** — Case (c) is of some interest (*see* paragraph 21), so the treatment will be outlined.

Lemma. — *Let* C *be an arbitrary closed limited point set (not a single point) whose complement is simply connected and let* $z = \Psi(w)$ *represent the inverse of the usual* $w = \Phi(z)$ *which maps the complement of* C *onto* $|w| > 1$. *If* $P(z)$ *is a polynomial of degree n and if we have*

$$\int_\gamma |P(z)|^p\, |dw| \leqq L^p, \qquad p > 0, \qquad \gamma : |w| = 1,$$

then we have also

$$|P(z)| \leqq LL'R^n, \qquad z \text{ on } C_R,$$

where L′ *depends on* R *but not on* $P(z)$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ denote the zeros of $P[\Psi(w)]$ exterior to γ. Assume purely for definiteness $P^{(n)}(z) \not\equiv 0$. The function

$$(10) \qquad Q(w) = P[\Psi(w)]\,\frac{(1-\bar\alpha_1 w)\ldots(1-\bar\alpha_k w)}{w^n(w-\alpha_1)\ldots(w-\alpha_k)}$$

is analytic and different from zero exterior to γ (if properly defined at the points α_i). For the values taken on by normal approach to γ we have $|Q(w)| = |P[\Psi(w)]|$. By hypothesis we have then

$$(11) \qquad \int_\gamma |Q(w)|^p\, |dw| \leqq L^p.$$

Let us use Cauchy's integral for the function $\frac{[Q(w)]^p}{w}$

$$\frac{[Q(w)]^p}{w} = \frac{1}{2\pi i}\int_\gamma \frac{[Q(v)]^p\,dv}{(v-w)v} \qquad (|w|>1).$$

The Lemma now follows easily by means of (10) and (11).

The application of this Lemma is entirely similar to the treatment in paragraph 14 and is left to the reader.

The discussion in case (*d*) (including the important case of approximation on a line segment) is only a slight modification of the discussion just given.

17. **Summary.** — We have discussed in detail Tchebichef approximation, and also methods (*a*), (*b*), (*c*). It is true, however, and can be similarly proved, that *in each of the cases* $a-g$, *if the function* $f(z)$ *is analytic on and within* C, *and if* C_ρ *is defined as in paragraphs* 10 *and* 11, *then the sequence of polynomials of best approximation converges to* $f(z)$ *on* C *with the greatest geometric degree of convergence.* The sequence thus converges interior to C_ρ, uniformly on any closed point set interior to C_ρ, but uniformly in no region containing in its interior a point of C_ρ.

The case $p=2$ will be discussed in more detail in Chapter IV.

The general results on methods $a-f$ are given by Walsh [9]; those on methods (*g*) [except the extensions of (*c*) and (*d*)] by Walsh and Russell [1]; those on (*g*), extensions of (*c*) and (*d*), are due to Walsh and still unpublished.

Study of degree of convergence and of actual convergence of the sequences of polynomials of best approximation must then be regarded as fairly satisfactory so far as concerns *regions* of uniform convergence. The question of convergence or divergence on and exterior to C_ρ is unanswered except under certain circumstances for $p=2$ (*see* paragraphs 20 and 21).

If the given function $f(z)$ is not known to be analytic on and within C, but is merely analytic interior to C and continuous on the corresponding closed set, the discussion already given is essentially valid in cases (*a*), (*b*) (if C is a Jordan region), (*d*) (if C consists of Jordan curves), (*e*), (*f*) (if C is a Jordan region), (*g*) [in the cases corresponding to (*a*), (*b*), (*d*), (*e*), (*f*), with the restrictions just mentioned], to prove convergence to $f(z)$ interior to C, uniform convergence on any closed point set interior to C, of the polynomials of best approximation; for Tchebichef approximation with a norm function, convergence is naturally uniform on the closed set C. Let us treat case (a) for instance. We set

$$\mu_n = \int_C n(z)\,|f(z)-\pi_n(z)|^p\,|dz| \qquad (p > 0),$$

where $\pi_n(z)$ is the polynomial of degree n of best approximation to $f(z)$ on C. It is known (§ 4) that polynomials $p_n(z)$ exist such that $\lim_{n\to\infty}[\max|f(z)-p_n(z)|,\ z \text{ on } C] = 0$. Hence μ_n approaches zero with $\frac{1}{n}$; $\lim_{n\to\infty}\int_C |f(z)-\pi_n(z)|^p\,|dz| = 0$. The stated convergence of the sequence $\pi_n(z)$ now follows from a slight extension of the Lemma of paragraph 14.

If the given function is analytic interior to C and is integrable on C together with its p-th power, then in the case $p=2$ further results have been obtained, to be discussed in paragraphs 20, 21. It would be of much interest to extend that study to include the general value of $p > 0$.

For the method (a), when C is the unit circle, $n(z)\equiv 1$, $p=2$, certain subsequences of the polynomials of best approximation (partial sums of the Maclaurin series) may converge uniformly even exterior to C_ρ, as was pointed out by Porter and later studied in more detail by Ostrowski and others. Is this behavior typical of general sequences of polynomials of best approximation?

In the study of approximation on an interval C of the axis of reals, S. Bernstein [2, 3] has developed highly interesting facts on the relation between the behavior of the function on C_ρ (particularly with reference to poles) and the degree of approximation (not necessarily geometric) on C; see also Mandelbrojt [1], Geronimus [1], Achieser [1, 2]. Almost nothing is known of the corresponding facts in the more general situation, especially concerning continuity properties of $f(z)$ on C_ρ (existence of generalized derivatives, Lipschitz conditions, etc.) although reference should be made to two notes by Jackson [2, 4]. Moreover, Faber [3] has indicated a method of some generality.

Can Bernstein's results be extended to the other measures of approximation that we have considered, and to more general point sets C? When $f(z)$ is analytic merely *interior* to the Jordan region C, what is the relation between behavior of $f(z)$ on the boundary and continuity properties of the boundary on the one hand, to the degree of convergence in C and convergence on the boundary on the other hand?

From our present viewpoint, the Hadamard theory of the Cauchy-Taylor series is of much interest. Let C be the unit circle $|z| = 1$ and let $f(z)$ be analytic interior to C, defined on C by normal approach, and integrable together with its square on C. The polynomial $p_n(z)$ of degree n of best approximation to $f(z)$ on C in the sense of least squares is (§ 19) the sum of the first $n+1$ terms of the Maclaurin development $f(z) = \Sigma a_n z^n$. The measure of approximation is

$$\int_C |f(z) - p_n(z)|^2 \, ds = 2\pi[|a_{n+1}|^2 + |a_{n+2}|^2 + \ldots],$$

so the Hadamard theory gives not only a relation between singularities of $f(z)$ and asymptotic properties of the a_n, but indirectly *a relation between singularities of $f(z)$ and degree of approximation on* C, and this is true whether the singularities of $f(z)$ lie on or exterior to C. The writer is not aware that this connection has been worked out in detail, either for approximation in the sense of least squares or for other measures of approximation.

Our results on various methods of approximation are clearly not exhaustive. Another measure of approximation — say for the case that C is the unit circle — is

$$\begin{aligned}&\max[n_1(z)|f(z)-p_n(z)|^p,\ z \text{ on } C] + \int_C n_2(z)|f(z)-p_n(z)|^p \, ds \\ &+ \iint_{|z|\leqq 1} n_3(z)|f(z)-p_n(z)|^p \, dS \\ &+ \max[n_4(z)|f(z)-p_n(z)|^p,\ -1 \leqq z \leqq +1] \\ &+ n_5\left|f\left(\frac{1}{3}\right) - p_n\left(\frac{1}{3}\right)\right|^p.\end{aligned}$$

The fundamental facts we have developed concerning convergence are valid also in this case. What is the most general measure of approximation and what is the most general norm function for which these characteristic results apply?

CHAPTER IV.

POLYNOMIALS BELONGING TO A REGION.

Let C be a region or more generally an arbitrary point set. If the polynomials $p_0(z)$, $p_1(z)$, $p_2(z)$, ... are such that an arbitrary function $f(z)$ analytic on C can be expanded on C in a series of the form

$$f(z) = a_0 p_0(z) + a_1 p_1(z) + a_2 p_2(z) + \ldots \qquad (a_k \text{ constant}),$$

then the polynomials $p_k(z)$ are said to *belong to the region or point set* C. An obvious illustration is that C is the set $|z| \leqq 1$ and $p_k(z) = z^k$; the series is the Maclaurin development of $f(z)$.

Polynomials belonging to a region are particularly useful in *effectively* determining an expansion of a given function valid in that region, and hence deserve to be studied in some detail. Indeed, the polynomials most easily determined are frequently those of Faber.

18. Faber's Polynomials. — Let C be an arbitrary analytic Jordan curve of the z-plane, let the function

$$(1) \qquad z = \theta(t) = \frac{1}{t} + a_0 + a_1 t + a_2 t^2 + \ldots = \frac{1}{t} + E(t), \qquad |t| \leqq r,$$

map the exterior of C onto the *interior* of the circle $|t| = r$; then r is uniquely determined. Let us suppose, moreover, a suitable translation to be given to C so that the coefficient a_0 vanishes. The polynomial $P_n(z)$ of degree n is now uniquely determined by the fact that the coefficient of z^n in $P_n(z)$ is unity, and that by (1) we have

$$P_n(z) = \frac{1}{t^n} + t E_n(t), \qquad |t| \leqq r,$$

where $E_n(t)$ is a series of positive powers of t. It follows that $P_0(z) = 1$, $P_1(z) = z$, and that the recursion formula is valid

$$P_{n+1}(z) = z P_n(z) - a_1 P_{n-1}(z) - a_2 P_{n-2}(z) - \ldots - a_n.$$

Cauchy's integral formula in the z-plane, where the integral is taken over C, involves when transformed to the t-plane the function

$$\frac{\theta'(t)}{z-\theta(t)} = \frac{1}{t} + \frac{z - E(t) - tE'(t)}{1 + tE(t) - tz},$$

and the polynomials $P_n(z)$ are also the respective coefficients of the terms t^{n-1} in the development here.

The polynomials $P_n(z)$ satisfy the important relations

$$\lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} = \frac{1}{t}, \qquad \lim_{n \to \infty} \sqrt[n]{|P_n(z)|} = \frac{1}{|t|},$$

where z is any point exterior to C, and where t and z are connected by (1). If $f(z)$ is analytic interior to C_ρ but has a singularity on C_ρ, the development $f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$, $a_n = \frac{1}{2\pi i}\int_{|t|=r} f[\theta(t)]\, t^{n-1}\, dt$, is unique and valid for z interior to C_ρ, uniformly interior to $C_{\rho'}$, $\rho' < \rho$. The series converges on C with the greatest geometric degree of convergence, diverges exterior to C_ρ, and the coefficients satisfy the relation $\overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|} = \frac{r}{\rho}$.

The fundamental properties of these polynomials were developed by Faber [1, 2, 3]; interesting generalizations also exist [Faber, 2; Montel, 1; König and Krafft, 1].

Faber's polynomials are of great interest, and have exerted a large influence in the history of the theory of functions. We might well have used them in paragraph 7 instead of Hilbert's method of interpolation, were it not for the fact that Hilbert's method applies equally well to simultaneous approximation in several distinct regions, whereas Faber's method has not yet been extended to apply to this more general case. Faber's polynomials have the advantage over those of Hilbert of applying directly to a region bounded by an *arbitrary analytic Jordan curve* instead of by a lemniscate.

Faber [1] formulates the problem of determining polynomials belonging to *the most general* Jordan region. That problem seems never to have been completely solved by Faber's method. It is quite conceivable that modern results on conformal mapping would yield a solution, and it would be of much interest to investigate this question.

Be that as it may, there are now other solutions of Faber's problem. The first of these is due to Fejér, and will be discussed in paragraph 28. Other solutions will be discussed in paragraphs 21 and 25.

19. Orthogonality. — A set of functions $p_0(z), p_1(z), p_2(z), \ldots$ all integrable together with their squares on a curve C is said to be *orthogonal on* C provided we have $\int_C p_k(z)\overline{p_n(z)}\,ds = 0$, $k \neq n$, and is said to be *normal on* C if we have $\int_C p_k(z)\overline{p_k(z)}\,ds = 1$.

A finite number of normal orthogonal functions $p_n(z)$ on C are *linearly independent* on C in the sense that

$$(2) \qquad \int_C |a_0 p_0(z) + a_1 p_1(z) + \ldots + a_n p_n(z)|^2\,ds = 0$$

implies $a_k = 0$; indeed this integral has the value

$$|a_0|^2 + |a_1|^2 + \ldots + |a_n|^2.$$

If the functions $p_0(z), p_1(z), \ldots$ are normal and orthogonal on C, and if $f(z)$ is integrable with its square on C, then

$$(3) \qquad \begin{cases} f(z) \sim a_0 p_0(z) + a_1 p_1(z) + a_2 p_2(z) + \ldots, \\ a_k = \int_C f(z)\overline{p_k(z)}\,ds, \end{cases}$$

is called the *formal expansion* of $f(z)$ on C. If the sign $\sim$ (here used simply to denote formal correspondence) can be replaced by the equality sign and if the series converges uniformly on C, then the coefficients a_k are unique and given by the formulas indicated. For we can multiply the *equation* (3) through by $\overline{p_k(z)}\,ds$ and integrate over C term by term.

Under the hypothesis that the $p_k(z)$ are normal and orthogonal on C and that $f(z)$ together with its square is integrable on C, *the best approximation to $f(z)$ on* C *in the sense of least squares by a linear combination of the functions $p_0(z), p_1(z), \ldots, p_n(z)$ is given by*

$$s_n(z) = a_0 p_0(z) + a_1 p_1(z) + \ldots + a_n p_n(z), \qquad a_k = \int_C f(z)\overline{p_k(z)}\,ds.$$

Indeed, we have clearly

$$\begin{aligned} &\int_C |f(z) - \Sigma \lambda_k p_k(z)|^2\,ds \\ &= \int_C [f - \Sigma \lambda_k p_k][\bar{f} - \Sigma \overline{\lambda_k}\overline{p_k}]\,ds \\ &= \int_C f\bar{f}\,ds - \Sigma \lambda_k \int_C \bar{f} p_k\,ds - \Sigma \overline{\lambda_k} \int_C f \overline{p_k}\,ds + \Sigma \lambda_k \overline{\lambda_k} \\ &= \int_C f\bar{f}\,ds - \Sigma a_k \overline{a_k} + \Sigma (a_k - \lambda_k)(\overline{a_k} - \overline{\lambda_k}), \end{aligned}$$

and this last expression is a minimum when and only when we have $\lambda_k = a_k$, $k = 0, 1, 2, \ldots, n$.

The remarkable fact just proved is noteworthy not merely in giving a simple formula for the linear combination of best approximation in the sense of least squares, but also in that the *coefficients* a_k *for best approximation do not depend on* n, for $n \geqq k$.

It is a familiar and easily proved fact that the functions $1, z, z^2, \ldots$ are orthogonal on an arbitrary circle $C : |z| = R$ whose center is the origin. Consequently, if $f(z)$ is an arbitrary function analytic on and within C, then the polynomial

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

$$a_k = \frac{1}{2\pi R^{2k}} \int_C f(z) \bar{z}^k \, ds = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}} \, dz,$$

is both the polynomial of degree n of best approximation to $f(z)$ on C in the sense of least squares, and the sum of the first $n+1$ terms of the Maclaurin development of $f(z)$.

If functions $q_0(z), q_1(z), q_2(z), \ldots$ integrable and with an integrable square be given on C all linearly independent in the sense mentioned in connection with (2), then by the well known Gram-Schmidt method of orthogonalization may be found functions (unique except for arbitrary multiplicative constants of unit modulus) $p_0(z), p_1(z), \ldots$, normal and orthogonal on C, such that $p_n(z)$ is a linear combination of the functions $q_0(z), q_1(z), \ldots, q_n(z)$; reciprocally, $q_n(z)$ is a linear combination of the functions $p_0(z)$, $p_1(z), \ldots, p_n(z)$. Thus, if $f(z)$ is integrable and with an integrable square on C, it possesses a formal expansion (3) in terms of the functions $p_k(z)$, and the sum of the first $n+1$ terms of this expansion is the linear combination of the first $n+1$ functions $q_k(z)$ of best approximation to $f(z)$ on C in the sense of least squares.

If $a_0 q_0(z) + a_1 q_1(z) + \ldots + a_n q_n(z)$ is the linear combination of the functions $q_0(z), \ldots, q_n(z)$ of best approximation to $f(z)$ on C in the sense of least squares, the following equation is then valid :

$$\int_C [f(z) - a_0 q_0(z) - \ldots - a_n q_n(z)] \overline{q_k(z)} \, |dz| = 0$$
$$(k = 0, 1, 2, \ldots, n).$$

It is likewise true, and was proved by Jackson [1] for the case of real functions, that if $a_0 q_0(z) + \ldots + a_n q_n(z)$ is the linear combination of the functions $q_0(z), \ldots, q_n(z)$ of best approximation to $f(z)$ on C in the sense of least *p-th* powers ($p > 0$), then we have

$$\int_C |f - a_0 q_0 - \ldots - a_n q_n|^{p-2} [f - a_0 q_0 - \ldots - a_n q_n] \overline{q_k} |dz| = 0$$
$$(k = 0, 1, \ldots, n).$$

20. **Szegö's polynomials.** — It is obvious that the results described in paragraph 19 can be applied to expansions in polynomials; the $q_k(z)$ are the functions z^k, $k = 0, 1, 2, \ldots$, and the functions $p_k(z)$ are then normal and orthogonal polynomials of respective degrees k. It is the great merit of Szegö [1] to have applied the method of orthogonalization, long familiar in the real domain, to these functions $q_k(z) \equiv z^k$ considered on an arbitrary rectifiable Jordan curve in the z-plane, and to have studied the corresponding expansions.

Szegö's fundamental theorem regarding approximation (all but the last part is proved in paragraph 14) is that *if* C *is an analytic curve and if* $f(z)$ *is analytic within* C_ρ *but has a singularity on* C_ρ, *then its formal expansion in terms of the orthogonal polynomials* $p_n(z)$ *belonging to* C *converges to* $f(z)$ *interior to* C_ρ, *uniformly on any closed set interior to* C_ρ, *and diverges exterior to* C_ρ.

Szegö obtains other results. If the polynomials $p_n(z)$ are orthogonal on C but instead of being normal are subject to the restriction $\int_C |p_n(z)|^2 ds = l$, where l is the length of C, then under the hypothesis given on $f(z)$ we have

$$f(z) = \sum a_n p_n(z), \qquad a_n = \frac{1}{l} \int_C f(z) \overline{p_n(z)}\, ds,$$

$$\overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|} = \frac{1}{\rho}, \qquad \sum |a_n|^2 = \frac{1}{l} \int_C |f(z)|^2 ds.$$

If C is an analytic curve, then a function which maps the interior of C onto the interior of $|w| = 1$ so that $z = a$ corresponds to $w = 0$ is, for z interior to C,

$$\varepsilon \frac{2\pi}{l} \frac{1}{K(a, a)} \int_a^z [K(a, z)]^2 dz, \qquad |\varepsilon| = 1,$$

$$K(a, z) = \sum_{n=0}^{\infty} \overline{p_n(a)} p_n(z).$$

The following asymptotic representation is valid uniformly exterior to C, if in $p_n(z)$ the coefficient of z^n is chosen positive :

$$p_n(z) = e^{i\left(n+\frac{1}{2}\right)\alpha}\sqrt{\frac{l}{2\pi}}\sqrt{\Phi'(z)}[\Phi(z)]^n[1+\varepsilon_n(z)],$$

where $w = \Phi(z)$ maps the exterior of C onto the exterior of $|w| = 1$ so that the points at infinity correspond to each other, and where $\varepsilon_n(z)$ approaches zero. As a corollary we have

$$\Phi(z) = e^{-i\alpha}\lim_{n\to\infty}\frac{p_{n+1}(z)}{p_n(z)}, \qquad z \text{ exterior to C.}$$

Smirnoff later [1, 2] treated Szegö's problem by a new method, and obtained sharper — perhaps the sharpest possible — results relative to expansion in a region bounded by an arbitrary rectifiable Jordan curve, of a function not necessarily analytic in the closed region. This treatment requires delicate study of conformal mapping and of the representation of functions by Cauchy's integral. Smirnoff's primary results are : (*i*) the extension of Szegö's theorem on approximation to the case of an arbitrary rectifiable Jordan curve (proved in paragraph 14; Smirnoff makes an additional assumption regarding the curve), (*ii*) proof that if $f(z)$ is analytic interior to the analytic Jordan curve C and if $f(z)$ has boundary values almost everywhere on C and is represented by Cauchy's integral over C, then the formal development of $f(z)$ is valid for z interior to C, uniformly on any closed set interior to C. Under the hypothesis of (*ii*) and if the integral $\int_C |f(z)|^2\,|dz|$ of the boundary values exists, then the formal development converges in the mean to $f(z)$ on C, and this implies convergence to $f(z)$ of the formal development interior to C, uniformly on any closed point set interior to C; this is also true (although not mentioned by Smirnoff) even if a positive continuous norm function is introduced. The corresponding result for the most general rectifiable Jordan curve C is doubtful.

21. Analogous results. — In paragraph 19 we considered line integrals as the measure of approximation and hence also in defining the corresponding condition of orthogonality. One may likewise use surface integrals with no modification in the formal work. This measure of approximation was used for analytic functions by Bergman, Bochner, and Carleman [1]; the last-named carried through the analogue of Szegö's study for the new measure of orthogonality. Carleman's chief results in this connection are that if $f(z)$ is analytic interior to a Jordan region C and if the integral $\int\int_C |f(z)|^2 dS$ exists, then the formal expansion of $f(z)$ in the new orthogonal polynomials $p_n(z)$ converges to $f(z)$ interior to C, uniformly on any closed set interior to C. He also develops an asymptotic formula for these polynomials analogous to that of paragraph 20. Proof of overconvergence in the case that $f(z)$ is analytic on and within C [Walsh, 9] is given in paragraph 15.

It is worth noticing that the polynomials $1, z, z^2, \ldots$ are orthogonal not merely on the *circumference* $|z| = r$ but over the *area* $|z| \leqq r$, as the reader can easily verify, so that Taylor's series is a special case not only of Szegö's but also of Carleman's series.

The discussion of approximation on C in the sense of least squares is readily modified so as to apply to approximation on C in the sense of least squares *with a norm function* $n(z)$. For simplicity take $n(z)$ positive and continuous on C. The minimizing of the integral

$$\int_C n(z)\,|f(z) - \Sigma\, a_k p_k(z)|^2\, ds = \int_C |\sqrt{n(z)}f(z) - \Sigma\, a_k p_k(z)\sqrt{n(z)}\,|^2\, ds$$

is most conveniently studied if the functions $p_k(z)\sqrt{n(z)}$ are orthogonal on C, that is to say, if we have

$$\int_C n(z)\,p_k(z)\,\overline{p_n(z)}\,ds = 0 \qquad (k \neq n). \tag{4}$$

This relation is called orthogonality of the set $p_k(z)$ on C with *respect to the norm function* $n(z)$. A given set of linearly independent functions $q_k(z)$, integrable and with an integrable square on C, can obviously be orthogonalized on C with respect to $n(z)$ by orthogonalizing the set $q_k(z)\sqrt{n(z)}$. Thus there exists a theory of approximation in the sense of least squares with a norm function, a theory which is analogous to and can be derived from the corresponding theory without a norm function (with norm function unity).

It is clear that every case of approximation studied in Chapter III where an integral measure of approximation with the exponent $p = 2$

is involved, with or without a norm function, leads to a set of polynomials belonging to a region or other point set C, and thus to a solution of Faber's problem (§ 18) for the point set C. We have, then, a solution of Faber's problem if C is bounded by a rectifiable Jordan curve (Szego, Smirnoff, Walsh), if C is a Jordan region (Carleman), if C is an arbitrary point set whose complement is simply connected (Walsh; herein lies the chief interest of paragraph 16), and if C consists of several Jordan regions (Walsh and Russell). Faber's problem can similarly be solved for an arbitrary point set whose complement is regular (§ 11) and of finite connectivity.

Analogues of well known results on Taylor's series, such as gap theorems, study of functions with natural boundaries, zeros of approximating polynomials, summability, etc., have never been worked out for these more general series.

Of particular interest in connection with approximation in the sense of least squares with respect to a norm function is the case that C is the interval $-1 \leqq z \leqq +1$. Various norm functions lead to expansions in terms of well known polynomials; for instance, $n(z) \equiv 1$ corresponds to Legendre polynomials. The curves C_ρ turn out to be ellipses with foci at the points $+1$ and -1. The polynomials are the numerators in the expansion of $\int_{-1}^{+1} \frac{n(t)}{(t-z)} dt$ as a continued fraction. Expansions in terms of these polynomials have been widely studied, by C. Neumann [1], Heine [1], Frobenius, Laurent, Tchebichef, Pincherle, Darboux [1], Stieltjes, Picard, Markoff, Faber [2, 4], Blumenthal [1], Van Vleck, Nielsen, Fejér, Bernstein [4], Szegö [2], Angelesco [1], Abramesco [2], Jackson [3], Shohat [1], and many others. The most general results are those of Szegö, Faber, Jackson, Bernstein, and Shohat. We omit the detailed discussion of these polynomials; the behavior of the corresponding expansions of *analytic* functions is not essentially different (so far as is known) from the behavior discussed in paragraph 10, provided the function $n(z)$ is suitably restricted; for further information the reader is referred to two articles on orthogonal polynomials for this *Mémorial* to be written by Mr. Shohat.

If orthogonality is studied not on a single interval but on several intervals, the polynomials $[n(z) \equiv 1]$ are generalizations of those of Lamé; here too the expansions have been studied in detail, by Lindemann [1], Faber [4], Shohat [1], and others; compare paragraph 17.

The analogous situation (including the continued fraction) deserves further study, where the integral is extended not over a line segment but over one or more rectifiable arcs or curves.

22. T-polynomials. — Let C be a closed limited point set containing an infinite number of points. Then there exists precisely one polynomial of the form $T_n(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n$ whose maximum modulus g_n on C is least; indeed, the problem of determining $T_n(z)$ is that of approximating to the function z^n by a polynomial of degree $n - 1$, so by the results of Tonelli (§ 12) the existence and uniqueness of $T_n(z)$ are assured. The designation T-polynomial is fitting because Tchebichef determined these polynomials when C is a line segment; we use the shorter form to avoid confusion with the polynomials of paragraph 12. T-polynomials have a surprisingly large number of applications in various fields in the theory of functions.

Faber studied the T-polynomials in the complex domain in a noteworthy paper [3] and derived their fundamental properties for the case that C is a region bounded by an analytic Jordan curve. The relations

$$\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \frac{1}{r}, \qquad \lim_{n\to\infty} \sqrt[n]{g_n} = \frac{1}{r},$$

(in the notation of paragraph 18) are fundamental; generalizations are discussed in paragraph 29. The polynomials $T_n(z)$ differ relatively little from Faber's polynomials $P_n(z)$:

$$|T_n(z) - P_n(z)| < M\left(\frac{\rho}{r}\right)^n \theta^n, \qquad z \text{ on } C_\rho,$$

where M and $\theta < 1$ are suitably chosen. The relations

$$\lim_{n\to\infty} \frac{T_{n+1}(z)}{T_n(z)} = \frac{1}{t}, \qquad \lim_{n\to\infty} \sqrt[n]{|T_n(z)|} = \frac{1}{|t|}$$

are valid for z exterior to C, where t and z are related as in paragraph 18.

An arbitrary function analytic interior to C_ρ can be expanded in a unique series $\sum c_n T_n(z)$ entirely comparable in properties to the other series of Faber.

It would be of interest to determine the pairs of curves C and C′

for which all the T-polynomials are the same, or for which the T-polynomials are the same for an infinite sequence of degrees. In the latter case it is necessary that C be C'_R or that C′ be C_R for suitable R and it is sufficient that C and C′ be lemniscates of the form

$$C : |z^m + k_1 z^{m-1} + \ldots + k_m| = \mu,$$
$$C' : |z^m + k_1 z^{m-1} + \ldots + k_m| = \mu'.$$

23. Polynomials in z and $\frac{1}{z}$. — We have hitherto discussed primarily the expansion of given functions known to be analytic in some region. We consider now the further problem of expanding functions defined merely on a Jordan curve C. This work is particularly related to Weierstrass's theorem II of paragraph 2 and its generalization of paragraph 4.

If $f(z)$ is integrable together with its square on $C : |z| = 1$, then [F. and M. Riesz, 1] $f(z)$ can be expanded formally (§ 19) on C in each of the two orthogonal systems $1, z, z^2, \ldots; z^{-1}, z^{-2}, \ldots$ (orthogonal also to each other on C):

$$(5) \quad \begin{cases} f_1(z) = a_0 + a_1 z + a_2 z^2 + \ldots & (|z| < 1), \\ f_2(z) = a_{-1} z^{-1} + a_{-2} z^{-2} + \ldots & (|z| > 1), \end{cases} \qquad a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}} dz,$$

The two series (5) converge as indicated; the functions represented have as boundary values on C (found by normal approach almost everywhere) functions $f_1(z)$ and $f_2(z)$ also integrable together with their squares on C. On C we have almost everywhere

$$f(z) = f_1(z) + f_2(z),$$

so the sum of the two series (5) is a formal expansion of $f(z)$ on C. The functions $f_1(z)$ and $f_2(z)$ can also be represented

$$(6) \qquad \frac{1}{2\pi i} \int_C \frac{f(t)}{t - z} dt = \begin{cases} f_1(z) & (|z| < 1), \\ f_2(z) & (|z| > 1), \end{cases}$$

where the integral representing $f_2(z)$ is to be taken in the *clockwise* sense on C. If $f(z)$ is analytic or satisfies a Lipschitz condition on C, the same is true of $f_1(z)$ and $f_2(z)$, continuous respectively for $|z| \leq 1$ and $|z| \geq 1$, and (5) is valid also on C.

This entire situation is difficult to generalize to an arbitrary rectifiable Jordan curve C. Assume the origin to lie interior to C. The two sets of functions $1, z, z^2, \ldots; z^{-1}, z^{-2}, \ldots$ are not orthogonal to each other on such an arbitrary curve — indeed it seems not to be known on what curves these two sets are orthogonal to each other. If $f(z)$ satisfies a Lipschitz condition on an analytic Jordan curve C, then $f(z)$ can be split up as in (6) for the circle. Two developments can then be given, for $f_1(z)$ and $f_2(z)$ respectively, in terms of polynomials in z belonging to the interior of C and polynomials in $\frac{1}{z}$ belonging to the exterior of C. The determination of the coefficients directly in terms of $f(z)$ on C is not immediate, and here lies another open problem of interest. Perhaps for an arbitrary C there exists a weight function $n(z)$ such that all the functions $1, z, z^2, \ldots$ are orthogonal on C with respect to $n(z)$ to all the functions $z^{-1}, z^{-2}, \ldots$.

Let C be an arbitrary analytic Jordan curve. Walsh [1, 5] has exhibited a set of polynomials in z and z^{-1} respectively $p_0(z)$, $p_1(z), \ldots; p_{-1}(z), p_{-2}(z), \ldots$ belonging respectively to the interior and exterior of C, such that an arbitrary function $f(z)$ satisfying a Lipschitz condition on C can be expanded on C in the sum of two corresponding series. The coefficients in the two series are found directly from $f(z)$ by integration on C, and the two resulting series converge respectively in the *closed* interior and exterior of C. This result is of interest; nevertheless the method is relatively artificial and the polynomials exhibit no natural extremal property. It is desirable to solve the same problem by use of other polynomials.

Ghika [1] has made a general study of expansions of functions defined on a curve C in terms of functions analytic interior to C or analytic exterior to C and zero at infinity, but seems not to have made application to expansions by polynomials.

We add a further remark relative to orthogonality on C :

Let $f(z)$ be integrable on an arbitrary rectifiable Jordan curve C *in whose interior the origin lies. A necessary and sufficient condition that $f(z)$ be orthogonal on C to each of the functions* $1, z, z^2, \ldots$ *is* $\int_C \frac{\overline{f(t)}}{t-z} |dt| \equiv 0$, *z exterior to* C. *A necessary and sufficient condition that $f(z)$ be orthogonal on* C *to each of the functions* $z^{-1}, z^{-2}, \ldots$ *is* $\int_C \frac{\overline{f(t)}}{t-z} |dt| \equiv 0$, *z interior to* C.

24. Open problems. — The polynomials $1, z, z^2, \ldots$ are orthogonal not merely on a single circle $|z| = 1$ but are orthogonal on every circle $|z| = R$. Other sets of polynomials are orthogonal on more than one curve with respect to suitable norm functions; for instance, the set $1, z-c, z(z-c), z^2(z-c), \ldots, c > 0$, is orthogonal on every circle $|z| = R > c$, with respect to the positive norm function $\frac{z}{(R^2 - cz)(z-c)}$. The following fundamental question is still unanswered : When does the same set of polynomials $p_k(z)$ result from orthogonalizing the set $1, z, z^2, \ldots$ on a curve C_1 with respect to a norm function $n_1(z)$ and also from orthogonalizing that set on another curve C_2 with respect to a norm function $n_2(z)$? Given C_1, can $n_1(z)$ always be found so that C_2 and $n_2(z)$ exist? Given C_1 and C_2, can $n_1(z)$ and $n_2(z)$ always be found?

The present writer has some partial results on this problem, hitherto unpublished, such as the following :

If a set of polynomials $p_k(z)$ results from orthogonalizing the set $1, z, z^2, \ldots$ on a curve C_1 with respect to the norm function $n_1(z)$ and also results from orthogonalizing that set on the curve C_2 with respect to the norm function $n_2(z)$, then either C_1 is a curve $(C_2)_R$ (notation of paragraph 8) or C_2 is a curve $(C_1)_R$.

Other problems worth studying are these : When are the polynomials of Faber orthogonal (with respect to a norm function) on the curve C or on a curve C_R? When is this true of the generalized Faber polynomials? When is a set of polynomials orthogonal in a region C in the sense of Carleman and also orthogonal on the boundary or on C_R in the sense of Szegö? When is a sequence of polynomials of the form

$$1, \quad (z-\alpha_1), \quad (z-\alpha_1)(z-\alpha_2), \quad (z-\alpha_1)(z-\alpha_2)(z-\alpha_3), \quad \ldots$$

— here the formal development is a series of interpolation; compare paragraph 25 — orthogonal on a curve C or on several curves with respect to a norm function? In any case, what can be said of the convergence *on the curve* C of the various developments of a function analytic interior to C, with certain continuity properties on C [studied for particular polynomials by Walsh, 1, 5]; what can be said of convergence on C_ρ (notation of § 10) if the function is analytic on and within C?

Many classical sets of orthogonal and other polynomials satisfy recurrence relations. Polynomials which satisfy such relations arise naturally also in connection with difference equations. The convergence of series of such polynomials was investigated by Poincaré [1] and Abramesco [1], and the analogous expansion of arbitrary functions by Pincherle [1]. These polynomials $p_n(z)$ share with orthogonal polynomials, Faber's polynomials, T-polynomials, and many sets of polynomials used in interpolation, the property that

$$\lim \sqrt[n]{|p_n(z)|} = |w(z)|$$

exists. It is clear that whenever this limit exists, an arbitrary series $\sum a_n p_n(z)$, where $\overline{\lim} \sqrt[n]{a_n} = \frac{1}{\mu}$, converges for $|w(z)| < \mu$ and diverges for $|w(z)| > \mu$. This fact can be used both to determine regions of convergence and divergence of a given series, and frequently to determine the asymptotic properties of the coefficients in the known expansion of a given function. Ordinarily the function $w(z)$ can be expressed as $r\Phi(z)$, where $\Phi(z)$ maps the exterior of some set C onto the exterior of the unit circle so that the points at infinity correspond to each other. When that occurs, when

$$\lim \sqrt[n]{|p_n(z)|} = |w(z)|$$

uniformly on every $C_\sigma (\sigma > 1)$, when $p_n(z)$ is of degree n, and when (as is usual) the sum $f(z)$ of the series has a singularity on $|w(z)| = \mu > |r|$, then the series converges to $f(z)$ on C with the greatest geometric degree of convergence. It would be decidedly worth while to bring not merely the convergence of such series but the corresponding expansions *of an arbitrary function* into a single theory.

Many, if not all, of the developments we mention in Chapters IV and V can be derived from the expansion

$$\frac{1}{t-z} = \Sigma p_n(z) q_n(t), \qquad z \text{ interior to C, } t \text{ on or exterior to C,}$$

where $p_n(z)$ is a polynomial, by applying Cauchy's integral over C or a contour containing C in its interior. The detailed study of the functions $q_n(t)$ and of expansions in terms of them has been made in but few cases, notably that in which the $p_n(z)$ are the Legendre polynomials, and in the general case should be quite fruitful. Here it is of interest to study orthogonality not merely in the sense of (4) but also in the sense $\int_C n(z) p_k(z) p_m(z)\, dz = 0$; compare the references given in paragraph 21, and Geronimus [2].

CHAPTER V.

INTERPOLATION.

The two principal methods for the *effective* determination of polynomials of approximation are the use of polynomials belonging to a region and the use of interpolation; some important polynomials, such as those used in the Taylor expansion, belong in both classes. We now turn to the determination of polynomials by interpolation.

25. Interpolation in arbitrary points. — The following problem is fundamental : Given a function $f(z)$. Let the polynomial $p_n(z)$ of degree n be defined by the requirement of coinciding with $f(z)$ in points $z_1^{(n)}, z_2^{(n)}, \ldots, z_{n+1}^{(n)}$, with the usual convention (§ 6) relative to multiple points $z_k^{(n)}$. To study the convergence of the sequence $p_n(z)$.

The sequence $p_n(z)$ clearly need not converge to $f(z)$ in an arbitrary region C provided merely $f(z)$ is analytic in C and the points $z_k^{(n)}$ are chosen in C. Indeed, let us choose $z_k^{(n)} = 0$, $f(z) = \frac{1}{(z-1)}$, so that $p_n(z)$ is the sum of the first $n+1$ terms of the Maclaurin development of $f(z)$. The function $f(z)$ is analytic over the entire plane except at $z = 1$, yet convergence of $p_n(z)$ to $f(z)$ takes place only in the circle $|z| < 1$. Another example, due to Méray, where the sequence $p_n(z)$ fails to converge to $f(z)$ except at a single point, is given in paragraph 31.

We have already (§ 7) seen an illustration of sequences found by interpolation, although the sequence $p_n(z)$ there considered was not originally defined for all values of n. That sequence can be readily replaced, however, by one whose convergence properties are essentially the same and which is defined directly by interpolation for all values of n. We need merely require [compare Martinotti, 1] that $p_n(z)$

should coincide with $f(z)$ in the first $n+1$ points of the sequence

$$(1) \qquad \alpha_1, \; \alpha_2, \; \ldots, \; \alpha_\nu, \; \alpha_1, \; \alpha_2, \; \ldots, \; \alpha_\nu, \; \alpha_1, \; \alpha_2, \; \ldots.$$

In a general way, the points $z_k^{(n)}$ of interpolation may happen to be so chosen that $z_k^{(n)} = z_k$ does not depend (for $n \geqq k-1$) on n. Whenever this occurs, the polynomial $p_n(z)$ is the sum of the first $n+1$ terms of a series of the form

$$(2) \qquad a_0 + a_1(z-z_1) + a_2(z-z_1)(z-z_2) \\ + a_3(z-z_1)(z-z_2)(z-z_3) + \ldots.$$

Indeed, the polynomials $p_{n-1}(z)$ and $p_n(z)$ are equal in the points z_1, z_2, $\ldots$, z_n, and their difference is a polynomial of degree n, hence a constant multiple of $(z-z_1)(z-z_2)\ldots(z-z_n)$. Series (2) is called a *series of interpolation*, in distinction to a *sequence of polynomials of interpolation* such as those defined by the general condition $p_n(z_k^{(n)}) = f(z_k^{(n)})$, $k = 1, 2, \ldots, n+1$.

Thus the polynomials of paragraph 7 can be replaced by polynomials corresponding to the sequence (1) which yield a series of interpolation of form (2). The polynomials $(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)$ then belong to the interior of the lemniscate. The series converges (proved as in paragraph 7) to $f(z)$ interior to the largest lemniscate $|p(z)| = \mu$ which contains in its interior no singularity of $f(z)$, converges on every point set $|p(z)| \leqq \mu' < \mu$ with the greatest geometric degree of convergence, and diverges (for the proof see paragraph 24) exterior to $|p(z)| = \mu$.

The series just discussed is typical of many series and sequences of interpolation corresponding to points $z_k^{(n)}$ whose asymptotic properties are comparable with those of the sequence (1). For detailed references the reader may consult Nörlund [1, 2]. Moreover, Poritsky [1] has studied the convergence of certain sequences, such as polynomials $p_{2n-1}(z)$ of respective degrees $2n-1$ determined by

$$p_{2n-1}^{(2i)}(a) = f^{(2i)}(a), \quad p_{2n-1}^{(2i)}(b) = f^{(2i)}(b), \qquad (i = 0, 1, \ldots, n-1).$$

The following result [Méray, 1] is one of the most interesting relative to interpolation in points not completely prescribed :

If all the points $z_k^{(n)}$ satisfy the inequality $|z_k^{(n)}| \leqq B$, if the function $f(z)$ is analytic for $|z| < T > 2B$, and if the polynomial

$p_n(z)$ of degree n is found by interpolation to $f(z)$ in the points $z_k^{(n)}$, $k=1, 2, \ldots, n+1$, then the sequence $p_n(z)$ converges to $f(z)$ for $|z|<T-2B$, uniformly for $|z|\leqq Z<T-2B$.

A slightly more general theorem than that of Méray has recently been proved by the present writer [10] :

Let R_1 and R_2 be closed limited point sets and let L be the set consisting of the closed interiors of all circles whose centers are points of R_1 and which pass through points of R_2. If $f(z)$ is analytic on the closed set L, and if the points $z_k^{(n)}$ have no limit point exterior to R_1, then the sequence of polynomials $p_n(z)$ defined by interpolation to $f(z)$ in the points $z_k^{(n)}$ converges to $f(z)$ uniformly for z on R_2.

26. **Interpolation in roots of unity.** — Given a function $f(z)$ analytic in a region C, it is quite conceivable that by proper choice of the points of interpolation, a sequence of polynomials might be constructed which would converge to $f(z)$ throughout C. A beautiful illustration was given by Runge [2], who considers a function analytic in the unit circle and chooses the $(n+1)$st roots of unity as points of interpolation. Let us prove :

Let the function $f(z)$ be analytic for $|z|<R>1$ but have a singularity on the circle $|z|=R$. Let $p_n(z)$ be the polynomial which of degree n coincides with $f(z)$ in the $(n+1)$st roots of unity. Then the sequence $p_n(z)$ converges to $f(z)$ on $C: |z|\leqq 1$ with the greatest geometric degree of convergence; consequently the sequence $p_n(z)$ approaches $f(z)$ everywhere interior to $|z|=R$, uniformly on any closed point set interior to the circle $|z|=R$.

Moreover, if $P_n(z)$ is the sum of the first $n+1$ terms of the Maclaurin development of $f(z)$, then we have

$$(3)\quad \lim_{n\to\infty}[p_n(z)-P_n(z)]=0 \quad \text{for } |z|<R^2, \quad \text{uniformly for} \quad |z|\leqq Z<R^2.$$

The convergence for $|z|\leqq 1$ is proved by Runge; the remainder of the theorem is due to Walsh [10]. We have, if Γ denotes a circle $|z|=R_1$, $1<R_1<R$,

$$f(z)-p_n(z)=\frac{1}{2\pi i}\int_\Gamma \frac{(z^{n+1}-1)f(t)\,dt}{(t^{n+1}-1)(t-z)}, \qquad z \text{ interior to } \Gamma,$$

$$f(z)-P_n(z)=\frac{1}{2\pi i}\int_\Gamma \frac{z^{n+1}f(t)\,dt}{t^{n+1}(t-z)}, \qquad z \text{ interior to } \Gamma.$$

Consequently we find by subtraction

$$p_n(z) - P_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(t^{n+1} - z^{n+1})f(t)\,dt}{t^{n+1}(t^{n+1}-1)(t-z)}, \qquad z \text{ interior to } \Gamma.$$

The integrand suitably interpreted has no singularity for $t = z$ and hence this equation is valid for all values of z. In particular for $|z| = Z > R_1$, the right-hand member approaches zero uniformly provided the expression $\frac{R_1^{n+1} + Z^{n+1}}{R_1^{n+1}(R_1^{n+1} - 1)}$ approaches zero, that is to say, provided we have $Z < R_1^2$. For $Z = 1$, the right-hand member is in modulus less than $\frac{M}{R_1^n}$, where M is suitably chosen. The sequence $P_n(z)$ converges to $f(z)$ on C with the greatest geometric degree of convergence. The proof is complete [1].

It is worth noticing that equation (3) enables one to conclude the divergence of the sequence $p_n(z)$ for $R < |z| < R^2$, Abel's theorem and its modified converse for the sequence $p_n(z)$, etc.

27. **Fejér's extension.** — Runge raised the question whether this example could not be generalized to apply to a more general region. In a brilliant paper, Fejér [1] considers an arbitrary Jordan curve C of the z-plane and the set of $n+1$ points in the z-plane (called *equally spaced* on C) which correspond to the points $w^{n+1} = 1$ when the exterior of C is mapped onto the exterior of $|w| = 1$ so that the points at infinity correspond to each other. The following theorem is proved with the aid of Hilbert's method (§ 8); only the first part of the theorem is stated by Fejér, but the second part [compare Szegö, 1] is an immediate consequence of his formulas :

Let C *be an arbitrary Jordan curve in the z-plane and $f(z)$ an arbitrary function analytic on and within* C. *Then if $p_n(z)$ is a polynomial of degree n found by interpolation to $f(z)$ in $n+1$ equally spaced points on* C, *the sequence $p_n(z)$ converges to $f(z)$ uniformly in the closed interior of* C. *More explicitly, the sequence $p_n(z)$ converges to $f(z)$ on* C *with the greatest geometric degree of convergence.*

[1] We remark that (3) is also valid if the $p_n(z)$ are determined by interpolation to $f(z)$ in the points $z^{n+1} = \alpha_n$, where $|\alpha_n| \leq 1$.

Fejér also shows, as a complement to Runge's result, that **if $f(z)$** is analytic interior to $C : |z| = 1$ and continuous in the closed region, then this sequence of polynomials $p_n(z)$ converges to $f(z)$ for $|z| < 1$, uniformly for $|z| \leq r < 1$. Convergence need not take place on C itself, as Fejér shows by an example. The corresponding result would be well worth studying for a more general Jordan or other region.

28. Interpolation and Jordan regions. — At the end of his paper, Fejér considers the points of interpolation $z_1, z_2, z_3, \ldots$ on an arbitrary Jordan curve C which correspond, under the conformal map previously described, to points $w = \varepsilon, \varepsilon^2, \varepsilon^3, \ldots$, where ε is no root of unity. He states without proof that if $f(z)$ is analytic on and within C, and if the polynomial $p_n(z)$ of degree n is determined by interpolation to $f(z)$ in the points $z_1, z_2, \ldots, z_{n+1}$, then the sequence $p_n(z)$ approaches $f(z)$ uniformly on and within C. The corresponding series is of form (2), so we have here not merely a series of interpolation but also a series of polynomials *belonging to the region* C, the first solution of Faber's problem (*see* paragraph 18) for an arbitrary Jordan region.

The following is a more general problem. Given a Jordan curve C. To characterize the sets of points

$$z_1^{(n)}, \quad z_2^{(n)}, \quad \ldots, \quad z_{n+1}^{(n)}, \tag{4}$$

on or within C such that *for an arbitrary function* $f(z)$ analytic on and within C, the polynomial $p_n(z)$ of degree n which coincides with $f(z)$ in the points $z_k^{(n)}$ has the property

$$\lim_{n \to \infty} p_n(z) = f(z), \qquad \text{uniformly for } z \text{ on or within C.} \tag{5}$$

Fejér considers further the choosing of points *on* the curve C. There results the following theorem; the sufficiency of the condition is due to Fejér, the necessity was proved later by Kalmár [1].

Let C *be an arbitrary Jordan curve and let points* (4) *be chosen on* C. *A necessary and sufficient condition that* (5) *be valid for every* $f(z)$ *analytic on and within* C *is that the transforms of the points* $z_k^{(n)}$ *be uniformly distributed* (*see below*) *on* γ :

$|w| = 1$ *when the exterior of* C *is mapped onto the exterior of* γ *so that the points at infinity correspond to each other.*

The transforms $w_k^{(n)}$ of the points $z_k^{(n)}$ are said to be *uniformly distributed on* γ [Weyl] if and only if we have $\lim\limits_{n\to\infty} \frac{\nu_n}{n} = \frac{\sigma}{2\pi}$, where σ is the length of an arbitrary arc of γ and ν_n is the number of points $w_1^{(n)}$, $w_2^{(n)}$, ..., $w_{n+1}^{(n)}$ which lie on this arc.

Fejér points out that his discussion (§ 27) is also valid if C is a line segment instead of a Jordan curve. Indeed, the discussion is valid if C is a Jordan arc. It would be of interest to adapt his methods (§§ 27, 28) to the study of a more general point set C, even if the complement of C is not simply connected. Compare Szegö [3].

The general problem of the nature of the points (4) was further considered by Faber [3], who shows that if C is a Jordan curve and if the lemniscates $|(z - z_1^{(n)})(z - z_2^{(n)}) \ldots (z - z_{n+1}^{(n)})| = \frac{1}{r^{n+1}}$ (notation of paragraph 18) approximate C uniformly, then (5) is satisfied for every $f(z)$ analytic on and within C. It is also true in this case (although this is not mentioned by Faber) that the sequence $p_n(z)$ converges to $f(z)$ on C with the greatest geometric degree of convergence. It is not true (contrary to Faber's statement) that this condition of approximation to C by lemniscates is necessary for (5), for every $f(z)$ analytic on and within C.

Kalmár [1] gives the following solution of the general problem mentioned. *A necessary and sufficient condition that* (5) *should hold for every* $f(z)$ *analytic on and within the Jordan curve* C *is* (notation of paragraphs 8 and 18) *that we have for* z *exterior to* C

$$\lim_{n\to\infty} |(z - z_1^{(n)})(z - z_2^{(n)}) \ldots (z - z_{n-1}^{(n)})|^{\frac{1}{n+1}} = \frac{\Phi(z)}{r}.$$

In a paper as yet unpublished, Fekete has proved : *Let* C *be an arbitrary Jordan curve, and let the points* (4) *lie on or within* C. *Then a necessary and sufficient condition that we have* $\lim\limits_{n\to\infty} p_n(z) = f(z)$ *interior to* C (*not necessarily uniformly*) *for every* $f(z)$ *analytic on and within* C, *is*

$$\lim_{n\to\infty} \sqrt[n]{M_n} = \frac{1}{r}, \qquad M_{n+1} = \max[\,|(z - z_1^{(n)}) \ldots (z - z_{n+1}^{(n)})|, \ z \text{ on C}].$$

If this condition is satisfied, we have $\lim_{n \to \infty} p_n(z) = f(z)$ *uniformly on any closed set interior to* C, *for every such* $f(z)$. The quantity $\frac{1}{r}$ (§ 18) is also the transfinite diameter (§ 29) of C.

29. **The transfinite diameter.** — The term *transfinite diameter* was introduced by Fekete, and the fundamental properties which here concern us are primarily due to him and to Szegö, Pólya, and Pólya and Szegö [1]; the last paper includes a bibliography.

Let C be a closed limited point set containing an infinity of points. The absolute value of the Vandermonde determinant of order n has a maximum value for each n, where the $z_i^{(n)}$, $i = 1, 2, \ldots, n$, belong to C but are otherwise arbitrary :

$$(6) \qquad |V_n| = \prod_{i<j=1}^{j=n} |z_i^{(n)} - z_j^{(n)}| = d_n^{\frac{n(n-1)}{2}}$$

Then the quantities d_n decrease monotonically with $\frac{1}{n}$ and approach the limit d, the transfinite diameter of C. The term transfinite diameter is equivalent to the term *capacity* of the logarithmic potential. For this same point set C, let z_0 be a point exterior to C, and let $\alpha_n(z_0)$ and $\omega_n(z_0)$ denote the least upper bounds of $|p_n(z_0)|$ and $|p'_n(z_0)|$ respectively for all polynomials $p_n(z)$ of degree n whose absolute value on C is not greater than unity. Then $\lim_{n\to\infty} \sqrt[n]{\alpha_n(z_0)}$ and $\lim_{n\to\infty} \sqrt[n]{\omega_n(z_0)} = \alpha(z_0)$ exist and are equal; the value is finite or infinite according as d is positive or zero. In particular, if C is a set whose complement is simply connected, then for z_0 exterior to C we have $\alpha(z_0) = |\Phi(z_0)|$ in the notation of paragraph 8; this result is to be compared with the Lemma of paragraph 9.

The transfinite diameter of C has interesting properties relative to the T-polynomial $T_n(z)$ for C. If we set $\rho_n^n = \max\,[|T_n(z)|, z \text{ on C}]$, then we have $\lim_{n\to\infty} \rho_n = d$; compare paragraph 22. A connection of the transfinite diameter with interpolation was likewise indicated by Fekete [1]; the following is a generalization of his result which may be proved by his methods.

Let C *be a closed limited point set whose complement* K *is* *connected and regular* (§ 11), *and let* $f(z)$ *be analytic on* C. *Let* $P_n(z)$ *be the polynomial of degree* $n-1$ *of interpolation to* $f(z)$ *in the* n *points* $z_i^{(n)}$, $i = 1, 2, \dots, n$, *which give to* $|V_n|$ *its maximum value. Then the sequence* $P_n(z)$ *converges to* $f(z)$ *on* C *with the greatest geometric degree of convergence.*

The set of points $z_i^{(n)}$ here used is, in case C is a circle, identical with the set used by Runge and Fejér (§§ 26, 27).

The modification of this theorem and those of paragraph 11 to include the case where K is not regular would seem still to be closely connected with the concept of transfinite diameter.

30. A synthesis of interpolation and Tchebichef approximation. — We have seen many illustrations of the close connection between individual polynomials of interpolation and approximation.

Let $f(z)$ be analytic in the region C and let a closed point set C_n containing at least $n+1$ distinct points belong to C. Let $p_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on C_n in the sense of Tchebichef. If C_n contains precisely $n+1$ points, then $p_n(z)$ is the polynomial which coincides with $f(z)$ in those $n+1$ points and is found by interpolation. If C_n does not depend on n, then $p_n(z)$ is the polynomial of degree n of best approximation to $f(z)$ on C_n in the sense of Tchebichef as we have always considered it. In any case $p_n(z)$ exists and is unique, and the sequence $p_n(z)$ may converge to $f(z)$ in C.

The writer hopes to elaborate this remark (for instance where C_n consists of certain roots of unity) on another occasion. Here we merely present a relatively simple illustration.

If $f(z)$ *is analytic for* $|z| < R > 1$ *and if* $p_n(z)$ *is the polynomial of degree* n *of best approximation in the sense of Tchebichef to the function* $f(z)$ *on the point set* $C_n : |z| = r_n \leqq 1$, *then the sequence* $p_n(z)$ *converges to* $f(z)$ *for* $|z| < R$, *uniformly for* $|z| \leqq Z < R$. *In fact, the sequence* $p_n(z)$ *converges to* $f(z)$ *on* C : $|z| = 1$ *with the greatest geometric degree of convergence.*

Let $R_1 < R$ be arbitrary, $R_1 > 1$. Then in the Cauchy-Taylor development $f(z) = \Sigma a_n z^n$, the coefficients satisfy an inequality of the form $|a_n| \leqq \frac{M}{R_1^n}$. If $P_n(z)$ denotes the sum of the first $n+1$ terms of the development, we have for $|z| = r_n \leqq 1$

$$|f(z) - P_n(z)| \leqq |a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots| \leqq \frac{\dfrac{M r_n^{n+1}}{R_1^{n+1}}}{1 - \dfrac{r_n}{R_1}} \leqq \frac{M}{R_1 - 1} \frac{r_n^{n+1}}{R_1^n}.$$

Then by the definition of the polynomial $p_n(z)$ we have *a fortiori*

$$|f(z) - p_n(z)| \leqq \frac{M}{R_1 - 1} \frac{r_n^{n+1}}{R_1^n}, \qquad \text{for } |z| = r_n.$$

Combination of these two inequalities yields

$$(7) \qquad |p_n(z) - P_n(z)| \leqq \frac{2M}{R_1 - 1} \frac{r_n^{n+1}}{R_1^n}, \qquad \text{for } |z| = r_n,$$

whence by paragraph 9 we have

$$|p_n(z) - P_n(z)| \leqq \frac{2M r_n}{(R_1 - 1) R_1^n}, \qquad \text{for } |z| = 1.$$

The sequence $P_n(z)$ is known to converge to $f(z)$ on C with the greatest geometric degree of convergence, so our theorem is completely proved.

Another remark (to be elaborated elsewhere) emphasizes the close relationship for *polynomials of a given degree* between interpolation and approximation in the sense of Tchebichef :

Let $f(z)$ be analytic at the origin, let $p_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on the set $|z| = r$, and let $P_n(z)$ be the sum of the first $n + 1$ terms of the Maclaurin development of $f(z)$. Then as r approaches zero we have

$$\lim p_n(z) = P_n(z), \qquad \text{for all values of } z, \text{ uniformly for all values of } z \text{ on any closed limited point set.}$$

For suitably chosen M_1 and for suitably small r the derivation of (7) is valid, with only minor and obvious changes :

$$|p_n(z) - P_n(z)| \leqq M_1 r^{n+1}, \qquad \text{for } |z| = r.$$

Then for $|z| \leqq R_2 > r$ we have (§ 9) the inequality

$$|p_n(z) - P_n(z)| \leqq M_1 \frac{r}{R_2^n},$$

and the right hand member approaches zero with r.

31. Least squares and interpolation in roots of unity. — Our result of paragraph 26 brings out in a striking manner the close connection between approximation on the unit circle in the sense of least squares and interpolation in the roots of unity. Another result [Walsh, 14] also reveals this connection :

Let $f(z)$ be defined and continuous merely on the unit circle $C : |z| = 1$ and let the polynomial $p_n(z)$ of degree n be found by interpolation to $f(z)$ in the $(n+1)$st roots of unity. Then the sequence $p_n(z)$ approaches the limit $f_1(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt$ for $|z| < 1$, uniformly for $|z| \leqq r < 1$.

This function $f_1(z)$ is also the limit for $|z| < 1$, uniformly for $|z| \leqq r < 1$ of the sequence of polynomials $P_n(z)$ of degree n of best approximation to $f(z)$ on C in the sense of least squares.

The special case $f(z) = \frac{1}{z}$ was given by Méray [1], as an illustration to show that a sequence of polynomials found from a given analytic function by interpolation need not converge to that function in any region. We verify at once $p_n(z) = z^n$, $\lim_{n\to\infty} p_n(z) = 0$, for $|z| < 1$.

It would be of interest to study the corresponding situation for curves C more general than the unit circle.

32. Further remarks. — Certain other sets of points have been used for interpolation. Faber [3] shows that if C is an analytic Jordan curve and if $f(z)$ is analytic on and within C, then the sequence of polynomials $L_n(z)$ of degree n found by interpolation to $f(z)$ in the roots of $T_{n+1}(z)$ (*see* paragraph 22) converges to $f(z)$ on C with the greatest geometric degree of convergence; the sequence $L_n(z)$ diverges exterior to C_ρ(§ 10). Szegö [1] proves the exact analogue for interpolation in the roots of his polynomials. These are generalizations of known results for the case that C is a line segment. Interpolation in the roots of Shohat's polynomials (§ 35) would also be worth studying.

The open problems in addition to those mentioned are particularly concerned with convergence on and exterior to the boundary of the regions considered, in connection with interpolation in the special points considered and also in connection with interpolation in general points, especially study of the relation between continuity properties of $f(z)$ and convergence on the boundary *of the region of convergence*. Compare Fejér's remark, paragraph 27.

CHAPTER VI.

APPROXIMATION WITH AUXILIARY CONDITIONS AND TO NON-ANALYTIC FUNCTIONS.

33. **Approximation with interpolation to given function.** — Perhaps the most obvious form of auxiliary condition in the study of approximation by polynomials is to require the approximating polynomial $p_n(z)$ to coincide with the given function in ν fixed points independent of n. We have the theorem [Walsh, 5] :

If the function $f(z)$, defined on the bounded point set S, *can be approximated on* S *as closely as desired by a polynomial in z, and if distinct points $z_1, z_2, \ldots, z_\nu$ be chosen on* S, *then on* S *the function $f(z)$ can be approximated as closely as desired by a polynomial which satisfies the auxiliary conditions :*

$$(1) \qquad p(z_i) = f(z_i) \qquad (i = 1, 2, \ldots, \nu).$$

The proof follows easily by the Lemma [for instance Walsh, 5], whose proof we omit :

Lemma. — *If* R *and the distinct points $z_1, z_2, \ldots, z_\nu$ are fixed, if we have $|G_k| \leqq \eta$, $k = 1, 2, \ldots, \nu$, and if $G(z)$ denotes the polynomial of degree $\nu - 1$ which takes on the values G_k in the points z_k, then there exists a constant* M *independent of η such that we have $|G(z)| \leqq M\eta$, for $|z| \leqq R$.*

Choose R so that S lies interior to the circle $|z| = R$. Let $\varepsilon > 0$ be arbitrary. Then a polynomial $q(z)$ exists such that we have $|f(z) - q(z)| \leqq \frac{\varepsilon}{(1+M)}$, z on S. Define $G(z)$, a polynomial of degree $\nu - 1$, by the equations

$$G(z_k) = q(z_k) - f(z_k) \qquad (k = 1, 2, \ldots, \nu).$$

Then for the polynomial $p(z) = q(z) - G(z)$ we have by the Lemma,

$$|f(z) - p(z)| \leqq |f(z) - q(z)| + |G(z)| \leqq \frac{\varepsilon}{1+M} + \frac{M\varepsilon}{1+M} = \varepsilon, \qquad z \text{ on S}.$$

The proof is complete. Theorem and proof (and the Lemma) extend under suitable circumstances to the case of points z_k not necessarily distinct. Compare also the method used below.

In all the cases in which we have studied polynomials of best approximation (Chap. III) on a set C, the introduction of such auxiliary conditions as (1); where the points z_i (not necessarily distinct) are interior to C, does not alter the existence and uniqueness of polynomials of best approximation, provided the degree of the polynomial is at least $\nu - 1$. The new sequences of polynomials of best approximation converge in every case with the same degree of convergence as the old (geometric or not), and hence may exhibit the same phenomenon of overconvergence. We shall prove by way of illustration :

Let C be a closed limited point set whose complement is regular, let $f(z)$ be analytic and single-valued interior to C_ρ but not interior to any $C_{\rho'}$, $\rho' > \rho$. Let the points $z_1, z_2, \ldots, z_\nu$ (not necessarily distinct) belong to C. Then the sequence $\Pi_n(z)$ of polynomials of best approximation to $f(z)$ on C in the sense of Tchebichef with the auxiliary conditions $\Pi_n(z_k) = f(z_k)$, $k = 1, 2, \ldots, \nu$, converges to $f(z)$ on C with the greatest geometric degree of convergence.

Let $R < \rho$ be arbitrary. Let $p(z)$ be the polynomial of degree $\nu - 1$ which takes on the values $f(z_k)$ in the points z_k, and let us set $\Pi(z) = (z - z_1)(z - z_2)\ldots(z - z_\nu)$. The function $\dfrac{f(z) - p(z)}{\Pi(z)}$ is single-valued and analytic interior to C_ρ if suitably defined in the points z_k, so polynomials $p_n(z)$ of respective degrees 0, 1, 2, ... exist such that we have for z on C

$$\left| \frac{f(z) - p(z)}{\Pi(z)} - p_n(z) \right| \leqq \frac{M}{R^n},$$

$$|f(z) - [p(z) + p_n(z)\Pi(z)]| \leqq \frac{M_1}{R^n}.$$

This last inequality holds on C for the polynomials $p(z) + p_n(z)\Pi(z)$ of respective degrees $n + \nu$ which satisfy the auxiliary conditions, and hence the corresponding inequality holds for the polynomials $\Pi_n(z)$, $n = \nu, \nu + 1, \ldots$. The proof is complete.

34. Interpolation exterior to C. — Auxiliary conditions in the form of interpolation may also be given to apply *exterior* to the point set C on which approximation takes place. Such auxiliary conditions [either alone or in conjunction with such conditions as (1) on C] do not alter the *possibility* of uniform approximation on C, and they alter the *degree* and *region* of uniform convergence when and only when they prescribe values for the sequence of polynomials of best approximation in disagreement with the normal limit (interior to the usual C_ρ) of that sequence of polynomials. The first of these remarks is a simple modification of our first application of the Lemma, for by Runge's theorem regarding several Jordan regions, the requirement of approximation (hence of interpolation) exterior to C does not alter the possibility of approximation on C. The second of these remarks is more accurately described in the following theorem [a special case is given by Walsh, 8], easily applied to our other measures of approximation. We omit the proof, which is similar to the last proof given in paragraph 33.

Let the function $f(z)$ be analytic on the closed limited point set C, *whose complement is regular. Let $p_n(z)$ be the Tchebichef polynomial of degree n for approximation to $f(z)$ on* C *with the auxiliary conditions*

$$p_n(\alpha_i) = f(\alpha_i), \quad \alpha_i \text{ on C} \qquad (i = 1, 2, \ldots, k),$$
$$p_n(\beta_i) = \gamma_i, \quad \beta_i \text{ not on C} \qquad (i = 1, 2, \ldots, k').$$

Let σ denote the largest number such that $f(z)$ is single-valued and analytic interior to C_σ when extended analytically from C *along paths interior to C_σ, and such that C_σ contains within it no point β_i at which γ_i is different from the value $f(\beta_i)$ of this analytic extension of $f(z)$. Then if $R < \sigma$ is arbitrary, the inequality*

$$|f(z) - p_n(z)| \leqq \frac{M}{R^n}, \qquad z \text{ on C},$$

is valid for suitably chosen M. *The sequence $p_n(z)$ converges to $f(z)$ interior to C_σ, uniformly on any closed point set interior to C_σ, and converges uniformly in no region whose interior contains a point of C_σ.*

35. **General approximation with auxiliary conditions.** — Many extremal problems of the theory of functions are related to approximation by polynomials. The typical problem of this sort is

Problem I. — *Let* C *be* (*a*) *the boundary of a limited region or* (*b*) *a rectifiable Jordan curve or* (*c*) *an arbitrary closed limited region, and let the function* $f(z)$ (*analytic or not*) *be defined on* C. *Given also points* $z_1, z_2, \ldots, z_\nu$ *interior to* C *and functional values* $\gamma_1, \gamma_2, \ldots, \gamma_\nu$ *corresponding to these points. To study the function* $F(z)$ *analytic interior to* C *which takes on the values* γ_k *in the points* z_k *and which is the function of best approximation to* $f(z)$ *in the sense of* (*a*) *Tchebichef on* C, (*b*), (*c*) *least p-th powers integrated over* C.

In the most general Problem I as thus formulated we subject $F(z)$ to no restrictions other than those mentioned, except that the suitable measure of approximation (*a*), (*b*), (*c*), in some sense should exist; thus in cases (*a*) and (*b*) the function $F(z)$ is naturally to be defined on C in terms of the boundary values taken on as z interior to C approaches C. It is clear that Problem I (and similarly Problem II below) can be modified so as to subject $F(z)$ to still other restrictions — such as being different from zero or univalent (schlicht) interior to C, or that $F(z)$ can be approximated on C by polynomials as closely as desired, etc. — in all the cases (*a*), (*b*), (*c*). Merely for the sake of simplicity the norm functions are taken positive and continuous on C.

A slight modification of (*a*), itself of interest, is that C be an arbitrary limited region and that Tchebichef approximation be considered with reference to the entire region rather than its boundary. These two methods of approximation are equivalent if $f(z)$ is analytic interior to C and if the norm function for the region is the modulus of a function analytic interior to C.

The usual convention relative to multiple points z_k is assumed. In particular we may have $\nu = 0$, so that there are no auxiliary conditions. Or we may have $\nu \neq 0$, $f(z) \equiv 0$, so the problem is that of determining the admissible function (i. e., analytic interior to C and satisfying the auxiliary conditions) of minimum *norm* :

$$(a) \qquad \overline{\text{Bound}}\,[\,n(z)\,|\,F(z)\,|,\ z \text{ on } C\,],$$

$$(b) \qquad \int_C n(z)\,|\,F(z)\,|^p\,|\,dz\,| \qquad (p > 0),$$

$$(c) \qquad \iint_C n(z)\,|\,F(z)\,|^p\;dS \qquad (p > 0).$$

The function $F(z)$ of Problem I may also be restricted to be a polynomial of given degree :

PROBLEM II. — *Under the circumstances of Problem I, to study the polynomial* $p_n(z)$ *of degree* n $(n \geqq \nu - 1)$ *which satisfies the auxiliary conditions and is of best approximation to* $f(z)$ *on* C. *More particularly, to study* $\lim\limits_{n \to \infty} p_n(z)$, z *interior to* C, *and the relation of this limit to* $F(z)$. Under suitable restrictions the polynomial $p_n(z)$ exists, and is unique if $p > 1$; see for instance Walsh [9, 12].

Problem II is of interest not merely as a specialization of Problem I, but may also be of interest in connection with the general Problem I itself : to prove the existence of the function $F(z)$, to determine $F(z)$ effectively, and to derive a polynomial expansion of $F(z)$.

If $\nu = 0$ and $f(z)$ is analytic on and within C, Problem II is precisely the problem studied in Chapter III. If $f(z)$ is analytic on and within C and if we have $\gamma_k = f(z_k)$, then Problem II is the problem of paragraph 33. If we set $f(z) \equiv z^m$, $n = m - 1$, $\nu = 0$, the polynomial $p_n(z)$ of Problem II is the T-polynomial belonging to C in case $(a)[n(z) \equiv 1]$, a multiple of the Szegö-Smirnoff polynomial belonging to C in case $(b)(p = 2)$, a multiple of the Carleman polynomial belonging to C in case $(c)(p = 2)$, and is a polynomial studied by Shohat [1] in case $(b)(p \geqq 1)$.

Problems I and II can both be reduced to the case $\nu = 0$. Let $p(z)$ be the polynomial of degree $\nu - 1$ which satisfies the auxiliary conditions and let us set $\Pi(z) = (z - z_1)\ldots(z - z_\nu)$. We study on C

$$(2) \quad n(z)\,|f(z) - F(z)|^p = n(z)\,|\Pi(z)|^p \left| \frac{f(z) - p(z)}{\Pi(z)} - \frac{F(z) - p(z)}{\Pi(z)} \right|^p.$$

If $F(z)$ (which here need not be the solution of Problem I) is admissible, the function $\Phi(z) = \frac{[F(z) - p(z)]}{\Pi(z)}$ is analytic interior to C (when suitably defined in the points z_k). Reciprocally, if $\Phi(z)$ is given analytic interior to C, then the function

$$F(z) = \Phi(z)\Pi(z) + p(z)$$

is admissible. Thus, approximation to $f(z)$ on C by the function $F(z)$ with auxiliary conditions is equivalent to approximation on C with the norm function $n(z)\,|\,\Pi(z)\,|^p$ to the function $\frac{[f(z)-p(z)]}{\Pi(z)}$ by functions $\Phi(z)$ analytic interior to C but without auxiliary conditions. Moreover, if $F(z)$ is an admissible polynomial of degree n, the function $\Phi(z)$ is also a polynomial and is of degree $n-\nu$. The transformation just made of Problems I and II is valid in cases (a) (we set $p=1$), (b), and (c), but in (c) the new norm function is no longer *positive* on C and the new approximated function $\frac{[f(z)-p(z)]}{\Pi(z)}$ may not be integrable on C if the points z_k are not distinct.

The transformation (2) can be somewhat improved in cases (a) and (b) if we relinquish the requirement that $\Phi(z)$ be a polynomial whenever $F(z)$ is a polynomial, and if the given region is simply connected. Let the function $w=\eta(z)$ map the interior of C onto the interior of $|w|=1$; we may use (2) with $\Pi(z)$ replaced by

$$\Pi_1(z)=\frac{[\eta(z)-\eta(z_1)][\eta(z)-\eta(z_2)]\ldots[\eta(z)-\eta(z_\nu)]}{\left[1-\overline{\eta(z_1)}\,\eta(z)\right]\left[1-\overline{\eta(z_2)}\,\eta(z)\right]\ldots\left[1-\overline{\eta(z_\nu)}\,\eta(z)\right]},$$

and this function is of modulus unity on C.

Without going into details, let us simply remark that the natural method for proof of the existence of the function $F(z)$ of Problem I is the use of normal families [compare Walsh, **12**], and the natural method for proof of uniqueness is that used in paragraph **13**.

Even if $F(z)$ exists and is unique, we do not necessarily have

$$\lim_{n\to\infty} p_n(z)=F(z), \tag{3}$$

uniformly for z in C, as is shown by the example of paragraph **2**. Nevertheless, it frequently occurs that this equation is valid and even that the sequence $p_n(z)$ converges to $F(z)$ with the greatest geometric degree of convergence, so that overconvergence takes place.

Problems I and II in their most general form are still unsolved; even if C is a Jordan region these problems have not been completely treated in the literature, although the present writer has some unpublished results in this field. Let us indicate some results in connection with these problems.

36. Case (c) : surface integrals. — The first study (other than cases of Chap. III) of Problems I and II seems to be that of Bieberbach [1] : C is a limited simply connected region, $f(z) \equiv 0$, $n(z) \equiv 1$, $p = 2$, $\nu = 1$, $\gamma_1 = 1$, approximation is measured by (*c*). The function $F(z)$ exists and is unique, and turns out to be the derivative of the function which maps the interior of C onto the interior of a circle so that z_1 corresponds to the center. If the boundary of C is also the boundary of an infinite region. equation (3) is valid for z on C, uniformly for z on any closed point set interior to C. It is also true (although not indicated by Bieberbach) that if $F(z)$ is analytic in the closed region C, then the polynomial $p_n(z)$ converges to $F(z)$ in C with the greatest geometric degree of convergence. Thus overconvergence also takes place.

Julia [3, 4] has considered Bieberbach's problem with the requirement $p = 2$ omitted, and also the case $f(z) \equiv 0$, $n(z) \equiv 1$, with the boundary conditions $F(0) = 0$, $F'(0) = 1$, for general p. Both of these cases are intimately connected with the conformal mapping of C onto the unit circle. Kubota [1] considers Problem I with $f(z) \equiv 0$, $n(z) \equiv 1$, $\nu = 2$, $p = 2$, which is also related to conformal mapping.

Problem I has recently been studied in the case $p = 2$, $n(z) \equiv 1$, $\nu = 0$, by Wirtinger [1], where $f(z)$ is suitably restricted, and an explicit formula for $F(z)$ derived. Problem II is not mentioned by Wirtinger, but it follows from the results of Carleman [1] that if C is a Jordan region, then (3) is valid for z in C, uniformly for z on any closed point set interior to C.

Let us devote some attention to the most general Problems I and II for a Jordan region, where the measures of approximation to $f(z)$ have a meaning and where $p = 2$; we shall further choose $n(z) \equiv 1$, although that choice is only for simplicity. Approximation to $f(z)$ by polynomials in the sense of Problem II is a linear problem. More explicitly (notation of paragraph 35), any admissible polynomial $p_n(z)$, $n > \nu - 1$, is of the form

$$p(z) + a_0 \Pi(z) + a_1 z \Pi(z) + \ldots + a_{n-\nu} z^{n-\nu} \Pi(z).$$

Then by the Riesz-Fischer Theory the sequence of polynomials $p_n(z)$ of best approximation converges in the mean on C to some function $F(z)$:

$$\lim_{n \to \infty} \iint_C |F(z) - p_n(z)|^2 \, dS = 0. \tag{4}$$

It follows from the Lemma of paragraph 15 that the sequence $p_n(z)$ converges to $F(z)$ interior to C, uniformly on any closed point set interior to C. The limit function $F(z)$ is admissible and is the unique (as in paragraph 13) solution of Problem I. Indeed (paragraph 33 and Carleman [1]), the formal expansion by admissible polynomials in the sense of least squares of any admissible function $\Phi(z)$ $\left(\text{hence} \int\int_C |\Phi(z)|^2 dS\right.$ exists$\left.\right)$, converges in the mean to $\Phi(z)$ on C. Thus, if $F(z)$ is the solution of Problem I, then equation (4) is equivalent to

$$\lim_{n \to \infty} \int\int_C |f(z) - p_n(z)|^2 dS = \int\int_C |f(z) - F(z)|^2 dS$$

for the polynomials $p_n(z)$ of best approximation to $f(z)$ on C or for any other admissible polynomials; compare paragraph 14, inequality (7).

The fact that the $p_n(z)$ are the admissible polynomials of best approximation to $f(z)$ on C implies also that the $p_n(z)$ are the admissible polynomials of best approximation to $F(z)$ on C. Thus *the sequence $p_n(z)$ always converges to $F(z)$ interior to C, uniformly on any closed set interior to C; and if $F(z)$ is analytic in the closed region C, the sequence $p_n(z)$ converges to $F(z)$ on C with the greatest geometric degree of convergence*, so overconvergence takes place in the usual manner (§§ 10 and 33).

37. **Case (a) : method of Tchebichef.** — The Tchebichef measure of approximation has the advantage of being invariant under conformal transformation, and therefore Problem I for an arbitrary simply connected region is equivalent to a similar problem (the auxiliary conditions must be suitably modified) for the unit circle. For the unit circle, the case $f(z) \equiv 0$, $z_k = 0$ corresponds to the prescription of the first ν coefficients of the Taylor development of $F(z)$. Problem I for this case ($n(z) \equiv 1$) was studied by Carathéodory and Fejér (1911). This and the more general problem (z_k not necessarily zero) were later studied by Gronwall, Pick, Schur. Kakeya, Nevanlinna, F. Riesz, Walsh; Kakeya [1] gives a method for the effective determination of $F(z)$; for further references see Walsh [8].

Of particular interest (the interior of C simply connected) is the case $n(z) \equiv 1$, $f(z) \equiv 0$, and auxiliary conditions of the form

$$(5) \qquad F(\alpha) = 0, \qquad F'(\alpha) = 1.$$

Here the solution of Problem I is (as we shall prove) a function which maps the interior of C onto the interior of a circle, and this fact is the basis of the well known Fejér-Riesz proof [Radó, 1] of the fundamental mapping theorem for an arbitrary simply connected region. Indeed, an equivalent Problem I found by mapping C onto $\gamma : |w| < 1$ so that α corresponds to $w = 0$, involves the auxiliary conditions $F_1(0) = 0$, $F'_1(0) = a \neq 0$, $F_1(w) \equiv F(z)$. An arbitrary admissible function is of the form

$$F_1(w) = aw + a_2 w^2 + a_3 w^3 + \dots \qquad (|w| < 1).$$

The least upper bound of $|F_1(w)|$ for $|w| < 1$ cannot be assumed interior to γ (principle of maximum) so this least upper bound is approached when $|w|$ approaches unity. The least upper bound is the same for $F_1(w)$ and for the function $a + a_2 w + a_3 w^2 + \dots$, and hence the smallest least upper bound is $|a|$, attained only by the function $F_1(w) = aw$, which maps the interior of γ onto a circle.

Problems I and II for $n(z) \equiv 1$, $f(z) \equiv 0$, and the auxiliary conditions (5) were studied in detail by Julia [1]; Problem II leads to a development of the mapping function for C (bounding a simply connected region) in a series of polynomials, frequently valid throughout the interior of C. Problems I and II for these same conditions were considered later by Walsh [8], who studied convergence of the sequence $p_n(z)$ on the boundary of the region considered and proved overconvergence in case the boundary is an analytic Jordan curve. Walsh also studied the more general Problem I, $f(z)$ meromorphic interior to C, ν arbitrary, and determined the function $F(z)$, and studied convergence of $p_n(z)$ on the boundary and overconvergence. In particular the case $\nu = 0$, $f(z) = \frac{1}{(z - \alpha)}$ leads to a function $F(z)$ which maps the interior of C onto the *exterior* of a circle.

Problem I has never been studied for the most general limited function $f(z)$. Even in the case (5) and where C is a Jordan curve, the convergence of the sequence $p_n(z)$ on C is doubtful.

38. Case (b) **: line integrals.** — In the case $n(z) \equiv 1$, $p = 2$,

$\nu = 0$, C the unit circle, the function $F(z)$ is represented for $|z| < 1$ by Cauchy's integral $F(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt$; compare paragraph 23.

Kakeya [2, 3, 4] studied Problem I for the case $n(z) \equiv 1, f(z) \equiv 0$, C the unit circle; he proved the existence and uniqueness of the function $F(z)$, and in the case that the z_k are all distinct gave the effective determination of the function. F. Riesz [1] studied the case $n(z) \equiv 1, f(z) \equiv 0, p = 1$, C the unit circle. Kubota [1] studied the case $n(z) \equiv 1$, $f(z) \equiv 0$, $\nu = 1$ or 2, $p = 1$ or 2, C the unit circle, which is connected with conformal mapping. Takenaka [1, 2, 3] considered particularly the case $n(z) \equiv 1, f(z) \equiv 0, \gamma_k = 0$ for $k > 1$, C the unit circle and also $(p = 2)$ the case $n(z) \equiv 1$, $f(z) \equiv 0$, all the z_i distinct. Walsh [11] took up the more general case $n(z) \equiv 1$; $f(z) \equiv 0$, $p = 2$, C the unit circle, in connection with both Problems I and II.

Julia [4] studied both Problem I and Problem II again for $n(z) \equiv 1, f(z) \equiv 0$, auxiliary conditions $F(z_1) = 0$, $F'(z_1) = 1$, and for the auxiliary conditions $F(z_1) = 1$; here too there is intimate connection with conformal mapping.

The application of the Riesz-Fischer theory given at the end of paragraph 36 requires only minor and obvious modifications to apply in the present case, $p = 2$, provided that a function $F(z)$ is admissible if it is analytic interior to C and satisfies the auxiliary conditions, is represented interior to C by an integral $\frac{1}{2\pi i}\int_C \frac{\alpha(t)}{t-z}\,dt$ where $\alpha(z)$ is integrable together with its square on C, and where there exists a sequence of polynomials in z which converges in the mean to $\alpha(z)$ on C. This class of functions $F(z)$ has been studied in detail by Smirnoff [1]. In this modification of the discussion of paragraph 36, the Lemma of paragraph 14 is applied instead of that of paragraph 15.

39. **Further remarks.** — It will be noticed from the account given that much remains to be done in solving Problems I and II in the most general cases, and especially for multiply connected regions. These problems can be still further modified by requiring that the function $F(z)$ shall not vanish interior to C. Interpolation by non-vanishing functions has been considered by Kakeya [4] and approxi-

mation by Walsh [13], but these papers represent only the beginnings of the theory.

Broader problems may also be formulated by assigning as auxiliary conditions *non-successive* derivatives at various points. Or one may place restrictions on a function $F(z)$ and study the measures corresponding to (a), (b), (c) for $F'(z)$ or some other derivative. Many other extremal problems similar to I and II are also of importance.

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TABLE OF CONTENTS.

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