MATHÉMATIQUES ET SCIENCES HUMAINES

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Mathématiques et sciences humaines, tome 127 (1994), p. 49-58
<http://www.numdam.org/item?id=MSH_1994__127__49_0>

Math. Inf. Sci.hum. (32e année, n°127, *1994*, p.49-58)

PATHS THROUGH FIXED VERTICES IN EDGE-COLORED GRAPHS

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RÉSUMÉ — Chaînes alternées passant par des sommets donnés dans des graphes arêtes-colorés.
Nous étudions le problème de trouver dans un graphe arêtes-coloré une chaîne alternée joignant deux sommets donnés et passant par des sommets donnés (une chaîne est alternée si deux arêtes adjacentes arbitraires ont des couleurs différentes). Plus précisément nous démontrons que ce problème est NP-complet dans le cas de graphes 2-arêtes-colorés.

Ensuite nous montrons que le problème de l'existence d'une telle chaîne est polynomial dans le cas où l'on se restreint aux graphes complets 2-arêtes-colorés.

Nous étudions également le problème de trouver une (s,t)-chaîne (c'est-à-dire une chaîne de longueur s+t qui se partage en deux sous-chaînes monochromatiques de couleurs différentes) joignant deux sommets donnés et passant par des sommets donnés, dans un graphe complet arêtes-coloré.

ABSTRACT — *We study the problem of finding an alternating path having given endpoints and passing through a given set of vertices in edge-colored graphs (a path is alternating if any two consecutive edges are in different colors). In particular, we show that this problem in NP-complete for 2-edge-colored graphs.*

Then we give a polynomial characterization when we restrict ourselves to 2-edge-colored complete graphs.

We also investigate on (s,t)-paths through fixed vertices, i.e. paths of length s+t such that s consecutive edges are in one color and t consecutive edges are in another color.

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[1] This work was done while the authors were visiting the University of Orsay.

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1. INTRODUCTION AND TERMINOLOGY

Research in social sciences often deals with relations of opposite content, e.g., "love" - "hatred", "likes" - "dislikes", "tells the truth to" - "lies to" etc. A good model for representing such relations is a so-called signed graph (a graph in which we associate to each edge one of the signs "+" or "-"), i.e., a 2-edge-colored graph. In this work we deal with problems directly linked to the existence of two relational patterns in a 2-edge-colored graph: the alternating paths (cycles) and the $(s,\ t)$-paths (cycles). Pictorially an alternating path (cycle) is a path (cycle) any two adjacent edges of which are in distinct colors. An $(s,\ t)$-path (cycle) is a path (cycle) of length $s+t$ such that s consecutive edges are in one color and t consecutive edges are in another color. The notion of an $(s,\ t)$-path (cycle) is directly related to the balance of a graph introduced by Cartwright and Harary [7] (and originated in psychology [7, 9, 13, 17, 19]). We recall that a 2-edge-colored graph is balanced if, and only if, in each cycle the number of edges with color "-" is even. In a recent work [16], it has been shown that a 2-edge-colored complete graph contains an $(s,\ t)$-hamiltonian cycle (with s and t non-fixed) if, and only if, the graph is unbalanced. In a further result the same authors showed that, for s and t fixed, a sufficiently large signed complete graph contains an $(s,\ t)$-cycle if, and only if, the graph is unbalanced.

In this work we study the problem of finding alternating as well as $(s,\ t)$-paths having given endpoints and passing through a given set of vertices in 2-edge-colored graphs. For further results on the subject the reader is encouraged to consult [1-6, 8, 11, 12, 14-16, 18, 20].

Formally, in what follows, unless otherwise specified, we denote the vertex-set, the edge-set and the order of a graph G by $V(G)$, $E(G)$ and $n(G)$, respectively. When just one graph is under discussion, we usually write V, E and n instead of $V(G)$, $E(G)$ and $n(G)$, respectively.

Let A, B denote non-empty subsets of V. The graph induced in G by A is denoted by $G[A]$. The set of all edges that have one endpoint in A and the other one in B is denoted by AB. If $A=\{x\}$, then for simplicity we may write xB instead of $\{x\}B$.

A k-edge-coloring (or, for simplicity, a k-coloring) of G is a mapping c from E onto the set of *"colors"* $\{1,\ 2,\ \cdots,\ k\}$. If $e \in E(G)$, then $c(e)$ is the color of the edge e. For any $v \in V$ and any color i, let the color-i neighborhood of v be defined as $N_i(v)=\{a \in V\backslash\{v\} \mid c(va)=i\}$. For any non-empty subset A of V, we define $N_i(A)=\bigcup_{a\in A} N_i(a)$. We let G^c denote a graph G colored by a k-edge-coloring c. A complete graph K_n colored by a k-edge-coloring c is denoted by K_n^c.

Let x and y be two distinct vertices of G^c and let S be a subset of $V(G^c)\setminus\{x,\ y\}$. An arbitrary simple path between x and y in G^c passing through all vertices of S is denoted by $P_{x,\ S,\ y}$. Whenever S contains only a few vertices, say $S=\{z_1,\ z_2,\ z_3\}$, then for simplicity, we write $P_{x,\ z_1,\ z_2,\ z_3,\ y}$ instead of $P_{x,\ S,\ y}$, replacing S in the notation by a sequence of its elements in any order.
Let us note that all paths and cycles considered in this paper are supposed to be elementary.

Definition of an $(s_1, s_2, \cdots, s_{a+1})$-path.
Let $P = x_0 x_1 \cdots x_\ell$ be a path in G^c. Suppose that P is non-monochromatic and let $\{x_{i-j} \mid j = 1, 2, \cdots, a\}$ denote the set of its alternating vertices, where $i_j < i_{j+1}$, $j = 1, 2, \cdots, a-1$. The alternation sequence of P is defined to be the sequence $< i_1, i_2 - i_1, i_3 - i_2, \cdots, i_a - i_{a-1}, \ell - i_a >$ of $a+1$ terms, and P is called a $(i_1, i_2 - i_1, i_3 - i_2, \cdots, i_a - i_{a-1}, \ell - i_a)$-path. If P is monochromatic its alternation sequence is defined to be the one-term sequence $< \ell >$.
Observe that an $(s_1, s_2, \cdots, s_{a+1})$-path P in G^c is:
(1) Monochromatic iff $a = 0$ and
(2) alternating iff $a \geq 1$ and $s_i = 1$, for all $i = 1, 2, \cdots, a+1$.

Here we investigate the following problem:

PROBLEM 1.1. *Let $S = \{x_1, x_2, \cdots, x_\ell\}$ be a set of ℓ specified vertices in a k-edge-colored graph G^c. Let x and y be two distinct fixed vertices in $V(G^c) \setminus S$. Under which conditions does there exist a sequence $(s_1, s_2, \cdots, s_{a+1})$ such that there exists an $(s_1, s_2, \cdots, s_{a+1})$-path between x and y containing the vertices of S?*

In the sections that follow, we study the complexity of the above problem for small values of ℓ, and for the two special cases:
(1) $s_i = 1, i = 1, 2, \cdots, a+1$, and
(2) $a = 1$.

2. NP-COMPLETENESS RESULTS

The following theorem of [15] is used in this section.

THEOREM 2.1. *The following problem Π is $NP-$complete.*
Instance. *A complete graph K_n, a set $C = \{1, 2, \cdots, k\}$ of $k \geq 4$ colors, a k-edge-coloring $c : E(K_n) \to C$ of K_n, four distinct vertices x_1, x_2, y_1, y_2 in K_n^c, a fixed permutation $q = (c_1, c_2, \cdots, c_k)$ of the colors of C.*
Question. *Does K_n^c contain two vertex-disjoint alternating paths from x_1 to y_1 and from x_2 to y_2 respectively, such that the sequence of colors of each is a concatenation of a number of copies of q?*

We start with an $NP-$completeness result in edge-colored graphs.

THEOREM 2.2. *Let G^c be a 2-edge-colored graph. Let x, y and z be three distinct vertices of G^c. Deciding whether there exists an alternating path from x to y through z in G^c is NP-complete.*

PROOF. Our problem obviously belongs to NP. To prove that it is NP-complete, we transform the following so-called local path problem (LPP) [10], into an instance of our problem: Given three distinct vertices x, y and z in a directed graph D, deciding if there is a directed path from x to y through z in D is NP-complete.

Consider now an arbitrary instance of LPP in a directed graph D. Let G^c denote the 2-edge-colored graph obtained from D as follows : Split each arc of D into two parts, the first part (containing the tail of the arc) being colored 1 and the other part being colored 2; furthermore add a new vertex x' and join this vertex with x by an edge in color 2. Clearly, this construction can be done in polynomial time.

Now, if there is a directed path $P_{x,\ z,\ y}$ in D, then clearly there is an alternating path $P_{x',\ z,\ y}$ in G^c. Conversely, if there is an alternating path $P_{x',\ z,\ y}$ in G^c, then we can easily deduce the existence of a directed path $P_{x,\ z,\ y}$ in D. This completes the proof. □

THEOREM 2.3. *The following problem is NP−complete.*
Instance. *A complete graph K_n , a set $C = \{1,\ 2,\ \cdots,\ k\}$ of $k \geq 4$ colors , a k-edge-coloring $c : E(K_n) \rightarrow C$ of K_n, three distinct vertices x, z and y in K_n^c, a fixed permutation $q = (c_1,\ c_2,\ \cdots,\ c_k)$ of the colors of C.*
Question. *Does K_n^c contain an alternating path $P_{x,\ z,\ y}$ whose sequence of colors is the concatenation of a number of copies of q?*

PROOF. The transformation is established from the problem Π of Theorem 2.1 above. Consider an arbitrary instance of Π, by fixing four vertices $x_1,\ x_2,\ y_1,\ y_2$ in K_n^c and by assuming, without loss of generality, that the required sequence of colors is $q = (1,\ 2,\ \cdots,\ k)$. Let now $K_{n+k-1}^{c^*}$ denote a k-edge-colored complete graph obtained from K_n^c by adding $k-1$ new vertices $w_1,\ w_2,\ ...,\ w_{k-1}$ and the corresponding edges and then coloring the edges by $c^* : E(K_{n+k-1}) \rightarrow C$ which is an extension of c defined as follows:
(1) $c^*(w_iw_j) = max(\{i,\ j\})$ if $\mid i-j \mid = 1$ and $c^*(w_iw_j) = 1$ if $\mid i-j \mid > 1,\ i,\ j = 1,\ 2,\ \cdots,\ k-1$;
(2) $c^*(y_1w_1) = 1$;
(3) $c^*(x_2w_{k-1}) = k$;
(4) The edges between w_1 and $V(K_n^c)-\{y_1\}$, and between each w_i and $V(K_n^c)$, $2 \leq i \leq k-2$, are colored k.
(5) The edges between w_{k-1} and $V(K_n^c) \setminus \{x_2\}$ are colored 1.
Clearly, the above transformations can be done in polynomial time.

Fix now three vertices x', y', z' in $K_{n+k-1}^{c^*}$ by setting $x' = x_1$, $y' = y_2$, $z' = w_2$. It is easy to see that $K_{n+k-1}^{c^*}$ contains an alternating path $P_{x',\ z',\ y'}$ whose sequence of colors is the concatenation of a number of copies of q if, and only if, K_n^c contains two vertex-disjoint alternating paths from x_1 to y_1 and from x_2 to y_2 respectively, such that the sequence of colors of each is a concatenation of a number of copies of q. □

PROBLEM 2.4. *Is the following problem NP-complete ?*
Instance. *A complete graph K_n , a set $C = \{1,\ 2,\ \cdots,\ k\}$ of $k \geq 4$ colors , a k-edge-coloring $c : E(K_n) \rightarrow C$ of K_n, three distinct vertices x, z and y in K_n^c, a positive integer t.*
Question. *Does K_n^c contain an alternating path $P_{x,\ z,\ y}$ such that each color appears at least t times on $P_{x,\ z,\ y}$?*

3. POLYNOMIAL CHARACTERIZATIONS

In the last section we have shown that the alternating $P_{x,\ z,\ y}$-path problem with a particular sequence of colors is NP-complete for k-edge-colored complete graphs, $k \geq 4$. In Theorem 3.2 and Corollary 3.3 of this section, we show that the alternating $P_{x,\ z,\ y}$-path problem is no longer NP-complete, if we restrict ourselves to the case of 2-edge-colored complete graphs. However, we note that in the case of three colors, the problem is still open.

In view of the proof of Theorem 3.2 below, we prove the following lemma.

LEMMA 3.1. *Let x, z, y be three specified distinct vertices in a k-edge-colored complete graph K_n^c, $k \geq 2$. There exists an alternating path $P_{x,\ z,\ y}$ if, and only if, there exists an alternating path $P'_{x,\ z,\ y}$ containing at least one of the edges xz, zy in K_n^c such that $V(P'_{x,\ z,\ y}) \subseteq V(P_{x,\ z,\ y})$.*

PROOF. The existence of $P'_{x,\ z,\ y}$ clearly implies the existence of $P_{x,\ z,\ y}$ in K_n^c. Conversely, suppose that K_n^c contains an alternating path $P_{x,\ z,\ y}$ and suppose it to be of shortest length. Set $P_{x,\ z,\ y} = xx_1x_2\cdots x_i\cdots x_py$, where for the sake of homogeneity we identify z with x_i on this path. Assume by contradiction that $1 < i < p$. Let q, r denote the colors of the edges $x_{i-1}x_i$ and x_ix_{i+1}, respectively. Since $P_{x,\ z,\ y}$ is alternating, $q \neq r$. If the edge xx_i is colored otherwise than r, then $xx_i\cdots x_py$ is shorter than $P_{x,\ z,\ y}$, a contradiction to the minimality property of $P_{x,\ z,\ y}$. Similarly we obtain a contradiction if we assume that the color of the edge yx_i is other than q. It follows that the path xx_iy is alternating and shorter than $P_{x,\ z,\ y}$, a final contradiction. This completes the proof of the lemma.

THEOREM 3.2. *Let x, y, z be three distinct vertices in a 2-edge-colored complete graph K_n^c. There is an alternating path $P_{x,\ z,\ y}$ in K_n^c if, and only if, one of the following three conditions holds:*
(1) $c(xz) \neq c(yz)$.
(2) $c(xz) = c(yz) = 1$, say, and either $N_1(x) \cap N_2(z) \neq \emptyset$ or $N_1(y) \cap N_2(z) \neq \emptyset$.
(3) $c(xz) = c(yz) = 1$, say, $N_1(x) \cap N_2(z) = \emptyset = N_1(y) \cap N_2(z)$, and in the following sequence of subsets of K_n^c, there is a positive integer $t = 2s - \delta$, $\delta = 0$ or 1, so that $C_t \neq \emptyset$ and either C_t is not contained in $N_{1+\delta}(x) \cap N_{1+\delta}(y)$ or the complete subgraph of K_n^c generated by $C_{2-\delta} \cup \cdots \cup C_{2s-\delta}$ is not monochromatic:

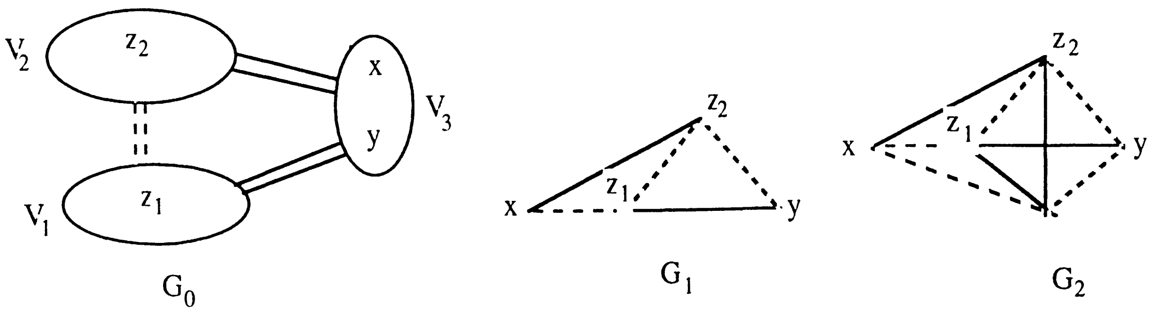
$$
\begin{aligned}
C_1 &= N_2(z),\\
C_2 &= N_1(C_1),\\
C_3 &= N_2(C_2)\backslash C_1,\\
C_4 &= N_1(C_3)\backslash C_2,\\
&\vdots\\
C_{2i+1} &= N_2(C_{2i})\backslash(C_1 \cup C_3 \cup \ldots \cup C_{2i-1}),\\
C_{2i+2} &= N_1(C_{2i+1})\backslash(C_2 \cup C_4 \cup \ldots \cup C_{2i}),\\
&\vdots
\end{aligned}
$$

PROOF.
Sufficiency: If (1) holds, then clearly the path xzy is alternating. If (2) holds, then take $u \in N_1(x) \cap N_2(z) \neq \emptyset$ (respectively $u \in N_1(y) \cap N_2(z) \neq \emptyset$) and the path $xuzy$ (respectively the path $xzuy$) is alternating. Suppose that (3) holds. Let $t = 2s - \delta$ (with $\delta = 0$ or 1) be the smallest positive integer satisfying (3). Then $C_i = \emptyset$ for all $1 \leq i \leq t$. Since $C_i \subset N_1(x) \cap N_1(y)$ if $1 \leq i \leq t$ is even and $C_i \subset N_2(x) \cap N_2(y)$ if $1 \leq i \leq t$ is odd, it follows that the sets $C_1, C_2, \cdots, C_t$ are pairwise disjoint. If C_t is not contained in $N_{1+\delta}(x) \cap N_{1+\delta}(y)$, take z_t in $C_t \setminus (N_{1+\delta}(x) \cap N_{1+\delta}(y))$ (we take $z_t = x$ if $x \in C_t$, or $z_t = y$ if $y \in C_t$). Take $z_{t-1} \in C_{t-1}$ so that $c(z_{t-1}z_t) = 1 + \delta$, $z_{t-2} \in C_{t-2}$ so that $c(z_{t-2}z_{t-1}) = 2 - \delta$ and so on, and $z_1 \in C_1$ so that $c(z_1 z_2) = 1$. If $z_t \in N_{2-\delta}(x)$, then $xz_t z_{t-1} \cdots z_1 zy$ is the desired path. If $z_t \in N_{2-\delta}(y)$, then $xzz_1z_2 \cdots z_t y$ is the desired path. Finally, suppose that $C_t \subset N_{1+\delta}(x) \cap N_{1+\delta}(y)$ but the complete subgraph of K_n^c generated by $C_{2-\delta} \cup \cdots \cup C_{2s-\delta}$ is not monochromatic. Take z_t, z_{t+1} in $C_{2-\delta} \cup \cdots \cup C_{2s-\delta}$ so that $c(z_t z_{t+1}) = 2 - \delta$. Since the complete graph $C_{2-\delta} \cup \cdots \cup C_{2s-2-\delta}$ is colored by $1 + \delta$ and $C_t \subset N_{1+\delta}(x) \cap N_{1+\delta}(y)$, we have z_t, $z_{t+1} \in C_t$. Now, take $z_{t-1}, \cdots, z_1$ as above. Then $xzz_1z_2 \cdots z_{t+1}y$ is the desired path.
Necessity: Let $P_{x,\, z,\, y}$ denote an alternating path of shortest length from x to y through z in K_n^c. From Lemma 3.1, $P_{x,\, z,\, y}$ is either of the form $xzz_1...z_k y$ or of the form $xz_k...z_1 zy$. Without loss of generality, we may assume that $P_{x,z,y} = xzz_1...z_k y$. If $P_{x,\, z,\, y}$ is of the form xzy, then (1) holds. If $P_{x,z,y}$ is of the form $xzz_1 y$, then (2) holds. Now, suppose $k \geq 2$ and let us assume, without loss of generality, that $c(xz) = 1$. In this case, $N_1(y) \cap N_2(z) = \emptyset = N_1(x) \cap N_2(z)$, since otherwise there would be an alternating path of length 3 from x to y through z. So, both $N_1(x)$ and $N_1(y)$ are contained in $N_1(z)$. Construct the finite sequence $C_1, \cdots, C_{k-1}$ of subsets of K_n^c as in the statement of this theorem. Set $k - 1 = 2s + \delta$, $\delta = 0$ or 1. From the sufficiency part and the fact that k is the smallest among all possible lengths of alternating paths from x to y through z, $C_{1+\delta} \cup \cdots \cup C_{2i-1+\delta}$ is contained in $N_{2-\delta}(x) \cap N_{2-\delta}(y)$ and is monochromatic for all $1 \leq i \leq s$, and $C_{2-\delta} \cup \cdots \cup C_{2i-\delta}$ is contained in $N_{1+\delta}(x) \cap N_{1+\delta}(y)$ and is monochromatic for all $1 \leq i \leq s$. Moreover, $C_{2-\delta} \cup \cdots \cup C_{2s-\delta} \cup C_{2s+\delta}$ is also contained in $N_{1+\delta}(x) \cap N_{1+\delta}(y)$. If $C_{2-\delta} \cup \cdots \cup C_{2s-\delta} \cup C_{2s+\delta}$ is not monochromatic we are done from the sufficiency part. So, we assume that $C_{2-\delta} \cup \cdots \cup C_{2s-\delta} \cup C_{2s+\delta}$ is monochromatic. Construct C_k as in the statement of the theorem. Then, $z_k \in C_k$. Note that $z_i \in C_i$ for all $1 \leq i \leq k-1$. Also note that $c(z_{k-1}z_k) = 2 - \delta$. If C_k were contained in $N_{2-\delta}(x) \cap N_{2-\delta}(y)$ the path $xzz_1 \cdots z_k y$ would not be alternating, contradicting our assumption. Therefore, C_k is not contained in $N_{2-\delta}(x) \cap N_{2-\delta}(y)$ and the proof is complete. □

COROLLARY 3.3. *There is an algorithm of complexity $O(n^3)$ for finding an alternating path $P_{x,\, z,\, y}$ (if any) in a 2-edge-colored complete graph.*

PROOF. Clearly, Condition (2) of Theorem 3.2 requires $O(n^2)$ time. Condition (3) needs $O(n^3)$ time, since the length of the sequence of C_i's is $O(n)$, while specifying each C_i requires at most $O(n^2)$ operations; therefore the decision algorithm costs $O(n^3)$ operations in the worst case. The algorithm for finding the desired path (if any) follows directly from the proof of Theorem 3.2. □

We shall finish this section with a result on $(s,\ t)$-paths through fixed vertices. In order to state the next theorem we define a 2-edge-colored complete graph G_0 as follows (see Figure 1):



The edges xy of both G_1 and G_2, omitted here, are colored arbitrarily either 1 or 2. Dashed lines represent color 1, while solid lines represent color 2.
Figure 1

Partition the vertex set of a complete (non-colored) graph of order $n \geq 5$ into three subsets V_1, V_2 and V_3 such that $| V_3 |= 3$, say $V_3 = \{x,\ y,\ w\}$. Then color the edges between V_1 and V_2 by 1, the edges between V_3 and $V_1 \cup V_2$ by 2 and color one of the edges wx, wy by color 1. All other edges are colored arbitrarily. Clearly the obtained 2-edge-colored graph G_0 has no $(s,\ t)$-path between x and y containing a vertex z_1 of V_1 and a vertex z_2 of V_2.

THEOREM 3.4. *Let K_n^c be a 2-edge-colored complete graph, $n \geq 4$. Let $x,\ y,\ z_1,\ z_2$ be distinct fixed vertices of K_n^c.*
(i)For some s and t, there exists an $(s,\ t)$-path $P_{x,\ z_1,\ y}$ in K_n^c if, and only if, K_n^c contains two distinct edges e_1 and e_2, other than xy, such that e_1 is adjacent to x, e_2 is adjacent to y and $c(e_1) \neq c(e_2)$.
(ii) For some s and t, there exists an $(s,\ t)$-path $P_{x,\ \{z_1,\ z_2\},\ y}$ in K_n^c if, and only if, there exist the edges e_1 and e_2 of (i) and in addition K_n^c is not isomorphic to any of G_0, G_1, G_2 of Figure 3 (isomorphism here is considered in the usual sence and by taking into account the colors of edges.

PROOF. Necessity being obvious, let us prove the "if" case.
Proof of (i). Assume without loss of generality that the edge xz_1 is in color 1. If the edge yz_1 is in color 2, we have finished. Otherwise by the hypothesis there is a vertex w in $K_n^c - \{x,\ z_1\}$ such that at least one of the edges wy, wx is in color 2. Now, independently of the color of the edge wz_1, if $c(wy) = 2$ then $P_{x,\ z_1,\ y} = xz_1wy$, otherwise $P_{x,\ z_1,\ y} = xwz_1y$.
Proof of (ii). Let us assume without loss of generality that $c(z_1z_2) = 1$. If $c(xz_1) \neq c(yz_2)$, or $c(xz_2) \neq c(yz_1)$then clearly the path xz_1z_2y or xz_2z_1y is the desired one. Consequently, in what follows assume $c(xz_1) = c(yz_2)$ and $c(xz_2) = c(yz_1)$.
Assume first $c(yz_1) = c(xz_2) = c(xz_1) = c(yz_2) = 1$. By the hypothesis, there exists a vertex w in $K_n^c - \{x,\ y,\ z_1,\ z_2\}$ such that either wy or wx is in color 2. Now, independently of the color of the edge wz_2, either the path xz_1z_2wy or the path xwz_1z_2y is the desired one.

Assume next $c(xz_1) = 1 = c(yz_2)$ and $c(xz_2) = 2 = c(yz_1)$. If $n = 4$, then K_n^c is isomorphic to G_1. In the sequel suppose $n \geq 5$. If there is a vertex w in $R = K_n^c - \{x, y, z_1, z_2\}$ such that $c(wy) = 2$, then clearly the path xz_1z_2wy is the desired one. Assume therefore that all edges between y and $V(R)$ are in color 1. Similarly we may also assume that all edges between x and $V(R)$ are in color 1. Now if there is a color-1 edge between z_2 (or z_1) and a vertex say w of R, then the path xwz_2z_1y (or the path xz_2z_1wy) is the desired one. Otherwise, i.e., if all edges between $\{z_1, z_2\}$ and $V(R)$ are in color 2 then, in case $n = 5$, K_n^c is isomorphic to G_2 and in case $n \geq 6$ the path $xz_2wz_1w'y$ is the desired one, where w and w' are two arbitrarily chosen vertices of R.
Assume finally $c(xz_1) = c(yz_2) = c(xz_2) = c(yz_1) = 2$. Let S denote the subgraph induced by the color -1 edges of K_n^c and let V_1 denote the component of S which contains both z_1 and z_2 (z_1 and z_2 belong to a same component of S, since $c(z_1z_2) = 1$, by assumption). Set $V_2 = V \setminus V_1$. Clearly, if V_2 is not the empty set, all edges between V_1 and V_2 are in color 2 in K_n^c. If at least one of x, y belongs to V_1, say $x \in V_1$, then there exists a color-1 path having x as one endpoint and z_1 or z_2 as the other endpoint and containing both z_1 and z_2. Indeed, since V_1 is a connected component of S, there exists a color-1 path, say P, between x and z_1 in V_1 (and therefore in K_n^c). If z_2 is an internal vertex of P, then P is the desired path. Otherwise, we extend P to a new color-1 path P' between x and z_2 containg z_1, by adding the edge z_1z_2. The path P (or P') can be easily transformed to an (s, t)-path between x, y passing through both z_1 and z_2 by adding (the color-2) edge z_1y (or z_2y). Suppose therefore that $\{x, y\} \subseteq V_2$. By the hypothesis of case (ii) there exists $w \in V \setminus \{x, y\}$ such that one of the edges wx, wy is in color 1. Now if w were a vertex in V_1 then one of the vertices x, y would belong to V_1, a contradiction to our assumption. Thus $w \in V_2$ and therefore V_2 contains at least 3 vertices. Now if V_2 has at least 4 vertices, then the path ywz_1zz_2x (or the path xwz_1zz_2y) is the desired one, where z denotes any vertex in $V_2 \setminus \{x, y, w\}$. Consequently, in the sequel assume that V_2 has precisely 3 vertices, namely x, y and w. Now, if there is a monochromatic color-2 path between z_1 and z_2, say $z_1t_1 \cdots t_\ell z_2$, in V_1, then $ywz_1t_1 \cdots t_\ell z_2x$ satisfies the conclusion of the theorem. Otherwise, the vertices of V_1 can be partitioned into 2 subsets T_1 and T_2 such that $z_1 \in T_1$, $z_2 \in T_2$ and all edges between T_1 and T_2 are in color 1. In this final case K_n^c is isomorphic to G_0. □

Let us notice that the proof of the above theorem can be easily transformed to an algorithm for finding, the desired paths (if any). More precisely, the proof of (i) has complexity $O(n)$, while the proof of (ii) has complexity $O(n^2)$. The factor $O(n^2)$ in (ii) comes from the fact that we have to check if the graph induced by the color-1 (color-2) edges is connected. It also comes from the determination of a color-2 path between z_1 and z_2 in $V_1 \setminus \{x, w, y\}$.

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