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MORE ON THE TOURNAMENT EQUILIBRIUM SET[1]

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RÉSUMÉ — Un peu plus à propos de l'ensemble d'équilibre d'un tournoi.
Schwartz (1990) a proposé un nouveau concept de solution pour le choix dans un tournoi, appelé Tournament Equilibriium Set (Ensemble d'équilibre d'un tournoi). Il a posé quatre problèmes à propos de cette solution. L'article introduit des questions supplémentaires et établit quelques relations logiques entre ces questions.

SUMMARY — *Schwartz (1990) proposed a new solution concept for choosing from a tournament ; called the Tournament Equilibrium Set. He stated four problems concerning this solution. In this paper we introduce further questions and demonstrate some logical relationship between these questions.*

1. INTRODUCTION

Given a finite set X of outcomes (candidates, decisions) a tournament T on X expresses decisive preference judgements for all pairs of outcomes. It is defined formally by a binary relation U over X which is assumed to be complete and asymmetric : $x\ U\ y$ means x dominates y.

When there is no outcome that dominates every other outcome there is no straightforward notion of winner(s) for T. A large litterature is devoted to the question of designing some principles for selecting the winner(s). This research is discussed at some length in Moulin (1986) and more recently in Laffond, Laslier and Le Breton (1991).

The purpose of this paper is to investigate further two recent solutions : the tournament equilibrium set proposed by Schwartz (1990) and the minimal covering set proposed by Dutta (1988). The recursive definition of the tournament equilibrium set makes apparently difficult to verify which properties are satisfied or not by this solution. Schwartz proves that the tournament equilibrium set is a subset of the Banks set investigated by Banks (1985) and states four problems. The three first ones deal with properties of the tournament equilibrium set. We add two further questions and we establish some logical relationships between four of these

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questions. The fourth problem concerns the relationship between the tournament equilibrium set and the minimal covering set. We also prove that this question is logically related to the previous ones.

Except from proving that the questions raised by Schwartz (1990) are logically related, this paper shows that either the tournament equilibrium set induces a controversial correspondence or it induces a correspondence satisfying nice properties and refining most of the existing solution correspondences.

The paper is organized as follows. In section 2 we present the definitions and notations used in the paper. Then in section 3 we state and prove the results we have obtained.

2. NOTATIONS AND DEFINITIONS

In this section we gather the main notations and definitions used in the paper.

A digraph (directed graph) D is a pair (X, U) where X is a finite set of elements called vertices and U is a set of ordered pairs (x, y) of vertices called arcs. If the arc (x, y) is present we say that x dominates y. The number of vertices of D is the order of D.

A subdigraph of D is a digraph $D' = (X', U')$ where $X' \subseteq X$ and $U' \subseteq U$. Given $X' \subseteq X$ we denote by D/X' the subdigraph $(X', U/X')$ where U/X' is the restriction of U to X'. A path of D is a subdigraph $D' = (X', U')$ where $X' = \{x_1, ..., x_n\}$ and $U' = \{(x_i, x_{i+1}), 1 \leq i \leq n - 1\}$. If we add the arc (x_n, x_1) we obtain a cycle of D.

A digraph D is strong if for any two vertices x and y, D contains a path from x to y and a path from y to x. A component of a digraph is a maximal (with respect to inclusion) strong subdigraph. The components of D can be labelled $D_1, D_2, ..., D_k$ such that no vertex of D_i dominates a vertex of D_j if $j < i$. The components D_i such that no vertex of D_j $j \neq i$ dominates a vertex of D_i will be called maximal. The number of maximal components of D is called the index of D an is denoted by $i(D)$; their union is called the Top-Set of D and is denoted by $TS(D)$.

In his social choice paper, Schwartz (1990) uses a different vocabulary for the same notions :

Definition 2.1. : Let $D = (X, U)$ be a digraph. A subset A of X is said retentive if $\nexists$ $(x, y) \in U$ with $x \notin A$ and $y \in A$.

Remark 2.2. : Schwartz considers the retentive subsets which are minimal with respect to inclusion ; it is easy to verify that the two notions of "minimal retentive subset of X" and of "maximal component of D" are equivalent.

Definition 2.3. : A tournament is a digraph $D = (X, U)$ where U is asymmetric and complete (in particular, for no $x \in X$ we have $(x, x) \in U$).

We denote by $\mathcal{T}$ the set of all tournaments and by $\mathcal{T}_n$ the set of all tournaments of order n.

Definition 2.4. :A solution correspondence (over $\mathcal{T}$) is a multivalued mapping S which associates to any tournament $T = (X, U)$ a subset $S(T)$ of X : $S(T)$ is the choice set or set of winners of T (according to S).

It is not easy to figure out what could be a "good" solution correspondence. An approach now quite developed among the researchers consists in identifying the properties satisfied by the solution correspondence[2]. We shall not discuss here this important question ; we just recall below some properties discussed in this paper (See Moulin, 1986).

Definition 2.5. : A solution correspondence S is said to be monotonic if : $\forall T = (X, U) \in \mathcal{T}$, if $x \in S(T)$ and $T' = (X, U') \in \mathcal{T}$ is such that $(x, y) \in U$ implies $(x, y) \in U'$ for all $y \neq x$ and $T \,/\, X - \{x\} = T' \,/\, X - \{x\}$, then $x \in S(T')$.

Definition 2.6. : A solution correspondence S is said to satisfy the strong superset property if : $\forall T = (X, U) \in \mathcal{T}$, $\forall A \subseteq X$ such that $A \cap S(T) = \emptyset$, $S(T) = S(T \,/ X - A)$.

Definition 2.7. : A solution correspondence S is said to be independent with respect to non winners if for every tournament $T = (X, U)$ and $T' = (X, U')$ on X such that for all $x \in S(T)$, for all $y \in X$, $(x\, U\, y) \Leftrightarrow (x\, U'\, y)$, then $S(T) = S(T')$.

The following remark will be useful in the sequel.

Remark 2.8. : If a solution correspondence S is monotonic and satisfies the strong superset property then S is independent with respect to non-winners.

Proof : Let $T = (X, U) \in \mathcal{T}, x, y \in X - S(T), (y,x) \in U$ and let $T' = (X, U')$ be defined by :

$$\begin{cases} T' \mid X - \{x\} = T \mid X - \{x\} \\ T' \mid X - \{y\} = T \mid X - \{y\} \\ (x,y) \in U' \end{cases}$$

By monotonicity, $y \in S(T') \Rightarrow y \in S(T)$, thus $y \notin S(T')$. So by the strong superset property $S(T') = S(T' \mid X - \{y\})$. But $T' \mid X - \{y\} = T \mid X - \{y\}$, thus $S(T') = S(T \mid X - \{y\})$. By the strong superset property, $y \notin S(T)$ implies $S(T \mid X - \{y\}) = S(T)$, hence $S(T) = S(T')$. ■

Given a solution correspondence S we define inductively $S^k(T)$ for all $k \geq 1$ by $S^1(T) = S(T)$ and $S^{k+1}(T) = S(T / S^k(T))$. Clearly, $S^{k+1}(T) \subset S^k(T)$ and this sequence becomes stationnary for k greater than the order of T. We further define $S^\infty(T)$ by $\cap_k S^k(T)$.

Definition 2.9. : A solution correspondence S is said to be idempotent if $S^2(T) = S(T)$" $T \in \mathcal{T}$

Remark 2.10. : Of course if S satisfies the strong superset property then S is idempotent.

3. THE RESULTS

In this section we state some results concerning a solution correspondence introduced recently by Schwartz (1990) and its relationship with another one introduced by Dutta (1988).

Let S be an arbitrary solution correspondence over $\mathcal{T}$ and $T = (X, U) \in \mathcal{T}$. We define a binary relation $D(S, T)$ as follows : given $x, y \in X$, $x\, D(S, T)\, y$ iff $x \in S(T \,/ V(y))$, where

[2]In the case where the list of properties fully describes the given solution correspondence we have a so called axiomatic characterization.

$V(y)$ is the set of predecessors of y : $V(y) = \{x \in X : (x,y) \in U\}$. This defines a digraph $D_S(T) \equiv (X, D(S, T))$ and a new solution correspondence S^* by $S^*(T) = TS(D_S(T))$.

Schwartz (1990) proposed to consider the unique solution correspondence S satisfying S = S*. He called it the Tournament Equilibrium set and denoted it by $TEQ(T)$. This fixed point presentation must not hide that TEQ is in fact defined inductively : if we know how to calculate the Tournament Equilibrium set for all tournaments of order n, let us take $T = (X, U)$ a tournament of order $n+1$, we can calculate $TEQ\ (T\ /\ V(y))$ for all $y \in X$. This gives the digraph $D_{TEQ}(T)$ and thus its top-set, $TEQ(T)$. It is not clear from this abstract definition which properties are satisfied (or not) by TEQ and indeed Schwartz (1990) provides few information on his concept, except some insights concerning its location with respect to previous solutions. He formulates the three following questions.

Question 1 : Does $i(D_{TEQ}(T))$ equals 1 for all $T \in \mathcal{T}$?

Question 2 : Is it true that $\forall T = (X,U) \in \mathcal{T}, \forall x \in X - TEQ(T), \exists y \in TEQ(T) : y\ D(TEQ, T)\ x$?

Question 3 : Is TEQ idempotent ?

We add to these questions two new questions dealing with two properties that are satisfied in all the examples we have examined.

Question 4 : Is TEQ monotonic ?

Question 5 : Does TEQ satisfy the strong superset property ?

The following propositions show that four of the above five questions at least are logically related. Two lemmas will be useful.

Lemma 3.1. Let $T = (X,U)$ and $T' = (X,U')$ be two tournaments, and A a subset of X ; if for all x in A, $TEQ\ (T\ / V(x)) = TEQ\ (T'\ / V'(x))$ then A is a minimal retentive set of $D_{TEQ}(T)$ if and only if A is a minimal retentive set of $D_{TEQ}(T')$.

Proof : Suppose that A is a minimal retentive set of $D_{TEQ}(T)$. (i) Then, $\forall x \in A$, $\forall y \in X-A$, $y \notin TEQ(T / V(x)) = TEQ\ (T' / V'(x))$; this shows that A is a retentive set of $D_{TEQ}\ (T')$ (ii) If A is not minimal retentive in $D_{TEQ}\ (T')$, then there exists $A \subset A$, $A' \neq A$, A' retentive for $D_{TEQ}(T')$. But then, from (i), it appears that A' is retentive also for $D_{TEQ}(T)$, this is impossible since A is minimal retentive for $D_{TEQ}(T)$. ■

Lemma 3.2. Let $T = (X,U)$ be a tournament, A a minimal retentive set of $D_{TEQ}\ (T)$, $z \notin A$. Note $T\text{-}z = T \mid X\text{-}\{z\}$. If for all $x \in A$, $TEQ\ (T \mid V(x)) = TEQ\ (T\ / V(x) - \{z\})$, then A is a minimal retentive set of $D_{TEQ}(T\text{-}z)$.

Proof : Similar to the proof of lemma 3.1.

Proposition 3.3. If TEQ satisfies the strong superset property, then for all $T \in T$, $i\ (D_{TEQ}(T)) = 1$.

Proof : The proof is by induction on the order n of T. It is trivial if $n=1$. Assume that for all $T \in \mathcal{T}_n$, $i\ (D_{TEQ}(T)) = 1$ and let $T = (X,U) \in \mathcal{T}_{n+1}$. Suppose that there are two distinct minimal retentive components of $D_{TEQ}(T)$, A and B. From lemma 3.2. and the strong superset property we can assume $X = A \cup B$. Now let $\gamma \notin X$ and $Y = X \cup \{\gamma\}$, and consider the tournament $T' = Y, U')$ defined by

$$\begin{cases} T' \mid X = T \\ \forall a \in A, (a,\gamma) \in U' \\ \forall b \in B, (\gamma,b) \in U' \end{cases}$$

Then for all $a \in A$, $V(a) = V'(a)$ and $T \mid V(a) = T \mid V'(a)$ thus A is a minimal retentive set of $D_{TEQ}(T')$. Let $b \in B$, if $\gamma \notin TEQ(T' \mid V'(b))$, then by the strong superset property, $TEQ(T' \mid V'(b)) = TEQ(T \mid V(b)) \subset B$.

Hence, γ is a vertex in $V'(b)$ which dominates every vertex of $TEQ(T \mid V'(b))$, the reader will easily verify that this is impossible. Hence : $\forall b \in B$, $\gamma \in TEQ(V'(b))$.

Clearly, $TEQ(T' \mid V'(\gamma) \subset A$. We obtain that $TEQ(T') = A$. By the strong superset property applied to the tournament T' of order $n+2$, $A = TEQ(T') = TEQ(T'-\gamma) = TEQ(T)$. ■

Remark 3.4. Let us note $SSP(n)$ the assertion "The strong superset property is true for all tournaments of order $\leq n$" and similary *Indice (n)* "For all tournament T of order $\leq n$, $i(D_{TEQ}(T)) = 1$". We just proved the implication :

$$\left.\begin{matrix} Indice\ (n) \\ SSP(n+2) \end{matrix}\right\} \Rightarrow Indice\ (n+1).$$

Proposition 3.5. If TEQ satisfies : $\forall T \in \mathcal{T}$, $i(D_{TEQ}(T)) = 1$, then TEQ satisfies the strong superset property.

Proof : By induction on the order of the tournament ; we prove the implication :

$$\left.\begin{matrix} Indice\ (n) \\ SSP(n) \end{matrix}\right\} \Rightarrow SSP\ (n+1)$$

Let $T = (X,U) \in \mathcal{T}_{n+1}$, $z \in X$, $z \notin TEQ(T)$. From the definition of TEQ, it follows that for all $x \in TEQ(T)$, $z \notin TEQ(T \mid V(x))$. From $SSP(n)$, we deduce that $TEQ(T \mid V(x)) = TEQ(T \mid V(x)-z\})$. If $A \subset X$ is a minimal retentive set of $D_{TEQ}(T)$, lemma 3.2. ensures that A is also a minimal retentive set of $D_{TEQ}(T-z)$. From *Indice (n)* there is only one such A, thus $A = TEQ(T) = TEQ(T-z)$. ■

Proposition 3.6.. If $\forall T \in \mathcal{T}$, $i(D_{TEQ}(T)) = 1$ then TEQ is monotonic.

Proof : By induction on the order, n, of T. We shall prove the implication :

$$\left.\begin{matrix} Indice\ (n+1) \\ SSP(n) \\ Monotone\ (n) \end{matrix}\right\} \Rightarrow Monotone\ (n+1)$$

From the proofs of the previous propositions, this is sufficient. Let $T = (X,U) \in \mathcal{T}_{n+1}$ and $x \in TEQ(T)$, $y \in X$ such that $(y,x) \in U$. Consider the tournament $T' = (X,U')$ defined by :

$$\begin{cases} T \mid X-\{x\} = T \mid X-\{x\} \\ \forall z \neq y, (x,z) \in U' \Leftrightarrow (x,z) \in U \\ (x,y) \in U'' \end{cases}$$

We have to prove that $x \in TEQ(T')$.

Step 1 : Observe that for all $z \in X - \{x\}$, we have :

$$x \notin TEQ(T' \mid V'(z)) \Rightarrow TEQ(T \mid V(z)) = TEQ(T' \mid V'(z)) \qquad (1)$$

Indeed, if $z \neq y$ and $x \in TEQ(T \mid V(z))$, $V(z) = V'(z)$ and by the monotonicity property applied to $T \mid V(z)$, $x \in TEQ(T \mid V'(z))$. Thus if $x \notin TEQ(T \mid V'(z))$, $x \notin TEQ(T \mid V(z))$ and by the strong superset property, $TEQ(T' \mid V'(z)) = TEQ(T' \mid V'(z) - \{x\})$ and $TEQ(T \mid V(z)) = TEQ(T \mid V(z) - \{x\})$.

But of course $T' \mid V'(z) - \{x\} = T \mid V(z) - \{x\}$, so $TEQ(T \mid V'(z)) = TEQ(T \mid V(z))$.

For the case $z = y$, if $x \notin TEQ(T \mid V'(y))$, $TEQ(T \mid V'(y)) = TEQ(T \mid V'(y) - \{x\}) = TEQ(T \mid V(y))$.

Step 2 : If $x \notin TEQ(T')$, (1) and lemma 3.1. prove that $TEQ(T')$ contains a minimal retentive subset of $TEQ(T)$. From the property *Indice (n+1)*, we get that $TEQ(T') = TEQ(T)$, a contradiction. ■

Remark 3.7. Let us note $INW(n)$ the assertion that TEQ satisfies the independence with respect to non-winners, for all tournaments of order $\leq n$. In the proof of remark 2.8., it has been established that :

$$\left.\begin{array}{l} Monotone\ (n) \\ SSP(n) \end{array}\right\} \Rightarrow INW(n+1)$$

This remark will be used in the proof of the following proposition.

Proposition 3.8. If TEQ is monotone, then $\forall T \in \mathcal{T}$ $i(D_{TEQ}(T)) = 1$.

Proof : We shall prove the implication :

$$\left.\begin{array}{l} INW(n) \\ Monotone\ (n+1) \end{array}\right\} \Rightarrow Indice(n+1)$$

Let $T = (X,U) \in \mathcal{T}_{n+1}$, assume that $i(D_{TEQ}(T)) > 1$ and let A be a minimal retentive subset of $D_{TEQ}(T)$ and let $y \in TEQ(T)$, $y \notin A$. Consider the tournament T' defined by :

$$\begin{cases} T' \mid X\text{-}\{y\} = T \mid X - \{y\} \\ \forall z \in X\text{-}A,\ z \neq y \Rightarrow (y,z) \in U' \\ \forall z \in A,\ (y,z) \in U' \Leftrightarrow (y,z) \in U. \end{cases}$$

By monotonicity applied to T, $y \in TEQ(T')$. For every $x \in A$, $V(x) = V'(x)$ and $TEQ(T \mid V(x)) \subset A$ t hus by the independence with respect to non-winners applied to $T \mid V(x)$, $TEQ(T \mid V'(x)) = TEQ(T \mid V(x))$. From lemma 3.1., A is a minimal retentive set of $D_{TEQ}(T')$. By construction of T', either y is a Condorcet winner of T', in which case A cannot be a retentive set of $D_{TEQ}(T')$, or $\varnothing \neq V'(y) \subset A$, in which case $TEQ(T' \mid V'(y)) \subset A$ and thus $y \notin TEQ(T')$, a contradiction.

Now we have the following implications :

$$\left.\begin{matrix} Indice\ (n) \\ SSP\ (n) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} SSP\ (n+1) \\ INW\ (n) \\ Monotone\ (n+1) \end{matrix}\right\} \Rightarrow Indice\ (n+1)$$

The result follows by induction. ■

Thus we can summary our results by :

Theorem 3.8. Let *TEQ* be the solution correspondence over the tournaments defined above ; the three first following properties are equivalent and imply the two others.

(1) *TEQ* is a monotonic correspondence
(2) $"T \in \mathcal{T}\ i(D_{TEQ}(T)) = 1$
(3) *TEQ* satisfies the strong superset property
(4) *TEQ* is independent with respect to non-winners
(5) *TEQ* is idempotent.

To adress the last question raised by Schwartz (1990) some further definitions are needed.

Given $T = (X, U) \in \mathcal{T}$ and x, y in X we say that x covers y if $(x, y) \in U$ and $V(x) \subseteq V(y)$. The uncovered set of T, denoted by $UC(T)$ is the subset of vertices which are not covered.

Definition 3.9. : Given T in $\mathcal{T}$ a subset A of X is called a covering set of T if :

(i) $UC(T \mid A) = A$
(ii) $\forall x \in X - A, x \notin UC(T \mid A \cup \{x\})$.

We denote by $\mathrm{C}(T)$ the collection of covering sets of T[3]. Dutta (1988) proved the following result.

Proposition 3. 10. : $\forall T \in \mathcal{T}$, the family $\mathrm{C}(T)$ has a minimal element with respect to inclusion.

He called this element the minimal covering set of T and denoted it $MC(T)$.

Schwartz (1990) raised the following question.

Question 6 Is it true that $TEQ(T) = MC(T)\ \forall T \in \mathcal{T}$?

Dutta (1990) answered by the negative by providing a tournament T for which $TEQ(T)$ is a proper subset of $MC(T)$. This suggests the following revised question.

Question 6' Is it true that $TEQ(T) \subseteq MC(T)\ \forall T \in \mathrm{T}$?

We prove below that this question is also logically related to the previous ones. To this end the following simple lemma will be useful.

Lemma 3.11. : $\forall T \in \mathcal{T}$ if $x \in MC(T)$ then :

$$MC(T \mid V(x)) \subseteq MC(T) \cap V(x)$$

Proof : It suffices to show that $MC(T) \cap V(x)$ is a covering set of $T \mid V(x)$. Let $y \in V(x)$, $y \notin MC(T)$.

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[3]This family is non empty (Dutta (1988)).

There exists $z \in MC(T)$ such that z covers y in $T \mid MC(T) \cup \{y\}$. Thus in particular $z \in MC(T) \cap V(x)$ and z covers y n $T \mid (MC(T) \cap V(x)) \cup \{y\}$. This proves that $MC(T) \cap V(x)$ satisfies the part (ii) of definition 3.9 with respect to the tournament $T \mid V(x)$. If it satisfies the part (i) of this definition, it is done. Otherwise we consider $UC(T \mid MC(T) \cap V(x))$. It is easy to see that this new subset again satisfies the part (ii) of definition 3.9. If it does not satisfy (i) we continue. Of course the process stops after a finite number of steps. ■

Proposition 3.12. : If TEQ is monotonic, then for all $T \in \mathcal{T}$, $TEQ(T) \subseteq MC(T)$.

Proof: The proof is by induction on the order n of T. Let $T \in \mathcal{T}_{n+1}$.

Claim $\forall x \in MC(T)$ if $y\, D_{TEQ}(T)\, x$ then $y \in MC(T)$.

Since $y \in TEQ(T \mid V(x))$ we deduce from the induction assumption that $y \in MC(T \mid V(x))$ and thus from lemma 3. 11, $y \in MC(T) \cap V(x)$.

From this claim we deduce that there exists a minimal retentive subset of $D_{TEQ}(T)$ included in $MC(T)$. The conclusion then follows from proposition 3.8. ■

The monotonicity property is considered by many authors as a minimal property to be satisfied by a solution correspondence for being considered as acceptable. The main message conveyed by the results above can be summarized as follows. Either TEQ is not monotonic, in which case it is not a good correspondence. Or it is, in which case it has further nice properties and is a refinement of the minimal covering set. We hope that this paper will encourage further research in order to get a full solution to this problem.

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