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RELATIVISTIC FUZZY SETS. TOWARD A NEW APPROACH TO SUBJECTIVITY IN HUMAN SYSTEMS

GUY JUMARIE*

1. INTRODUCTION

Many systems, and maintly those which explicitely deal with human factors, are subjectivistic in their nature in the sense that their definitions and their characteristics, say their observed features, are essentially varying from an observer to another one. As a matter of fact, until now, the main trend in mathematical modelling has been the systematic use of absolute equations which does not involve, either implicitely or explicitely, the relativity of the systems under consideration, and one exception (may be the only one) is the fuzzy set theory which tries to provide a new approach to precision and significance regarding the structure of complex systems. Unfortunately, as presently stated, the theory appears rather as a generalization of available results by everywhere substituting the range $[0, 1]$ for the set $\{0, 1\}$ via the concept of characteristic functions of sets; moreover the advantages of this approach with regard to existing theories like catastrophe theory, stability theory, contingent differential equations, etc... is not clear all, and in any case no comparison has been done at date by the advocates of fuzzy sets.

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In fact, there is a basic fundamental need to explain how the above substitution $\{0, 1\} \leftarrow [0, 1]$ should be carried out depending upon the observer who considers the set; problem which, in our opinion, is the genuine motivation of fuzziness, and whose a solution is herein proposed.

After a critical review of fuzzy set theory, we give a brief background on the special relativistic physics along with the new concept of relativistic fuzzy sets which we derived from the Einsteinian framework, and then we carefully identify a fuzzy set as subjectivistic fuzzy set, therefore we obtain various ways to associate a set with a given membership function. It is shown that the basic concept should be that of fuzziness function the practical meaning of which is exhibited.

The essential feature of this approach lies in the following. A relativistic fuzzy set has a fuzziness which depends explicitely upon the observer who considers it, composition laws for observers are given, the concepts of dependent and independent observers are exhibited, therefore a new possible way to modelling systems involving subjective factors as semantics of languages in linguistics for instance.

List of symbols and notations

The numbers in parentheses refer to the equation where the symbol first appears.

A	class or set A,
(A/R')	relativistic fuzzy set A given the observer R',
$\Phi_A(x)$	characteristic function of the set A, (4.2)
$f_A(x)$	membership function of the set A, (5.5)
$\bar{A}$	complement of A
$\Phi'_A(x/R')$	relativistic characteristic function (RCF) of the class A given the observer R', (4.2)
$u_A(x/R')$	relativistic fuzziness function (RFF) of the class A given the observer R', (4.4)

$\rho(u_A)$	relativistic term $[1 - u_A(x/R')]^{-1/2}$, (4.4)
(AB/R',R")	intersection of the sets (A/R') and (B/R"), (4.9)
$u_B(x/(A,R')/R")$	relativistic fuzziness function of B given R", conditional to the pair (A/R'), (4.11)
$u_{AB}(x/R',R")$	relativistic fuzziness function of the intersection of the set (A/R') and (B/R"), (4.11)
(A+B/R',R")	union of the set (A/R') and (B/R"), (4.16)

2. A CRITICAL REVIEW OF FUZZY SETS

The concept of fuzzy sets has been introduced by Zadeh [14] a few years ago to explicitely take into account the fact that in many practical instances, the boundary of the classes or sets under consideration are not clearly defined, therefore a kind of unaccurateness, say a fuzziness, which is essentially different from probabilistic uncertainty and thus requires a specific representation. Broadly speaking the characteristic function Φ_A : $E \to \{0, 1\}$, $x \mapsto \Phi_A(x)$, of a class A E is replaced by a more general function say a *membership function* f_A : $E \to [0, 1]$, $x \mapsto f_A(x)$; the set is straightforwardly identified with its membership function to define a *fuzzy set* (FS) and the algebra for FS's is constructed vis composition rules on the membership functions.

The theory is under extensive investigations by a certain school of scientists (see for instance the textbooks by Kaufman [10] and by Negoita and Ralescu, [12] to who "such methods could open many new frontiers in psychology, sociology, political science, philosophy, physiology, economics, linguistics, operation research, management science and other fields, and provide a basis for the design of systems for superior in artificial intelligence to those we can conceive today" (foreword by Zadeh in 10).

We are not convinced at all by such a statement and we are rather inclined to think that Fuzzy Set Theory, in its present form, cannot be fruitfully applied just because of its flaw in experimental foundations; probability can be defined either theoretically as a mesure or empirically as a frequency, fuzziness is defined theoretically only by using functionals which does not refer explicitely to the sets under consideration. In fact, all the problem stems from the very nature of the FS concept : does it apply at the stage of describing an object which itself is not accurately well defined, or else should it be rather considered as the unaccurate result of the observation of an object which is on the contrary well defined ? More specifically, the present framework of the FS theory appeals the following comments.

(i) As defined, the membership function does not explicitely refer to the set from which it is derived in such a manner that the theory appears rather as a subjective approach in the sense that the class which is associated with a given membership function is defined subjectively only to a given observer.

(ii) The theory seems to be an absolute theory which by no way refers to the observer who considers the class under observation. This feature is intrinsically very questionable because when, for instance, I consider the "class of beautiful women", a given woman who appears beautiful to me, may be not so to my brother; in other words, the membership function of this class should explicitely depend upon the observer. This absoluteness, of course, is not consistent with the apparent subjectivity of the theory.

(iii) The FS theory aims to be a theory of fuzziness which should apply when the probabilistic framework does not hold. But while probability distributions are defined independently of characteristic functions, membership function are merely a somewhat arbitrary extension which explicitely refers to these characteristic functions. We think we should have a new concept for fuzziness, concept which would be the counterpart of the probability concept and which would allow for simultaneously expending fuzziness and probability in the same framework.

(iv) Starting from FS as the basic concept, the advocates of the theory tend to convert every mathematical question to a problem related to classes therefore the possibility of introducing the corresponding fuzzy version and in this way appeared such tentative theories as decision-making in fuzzy environment (Bellman and Zadeh, [1]), approximate reasoning (Zadeh [15]), possibility theory (Zadeh, [16])etc... . This point of view also is subject to contention. First, it is not sure at all that classes or sets should be the basic mathematical concept; second, in numerous cases, the model under observation is well defined and it is its observation process by an external observer which involves fuzziness, so that we should have a theory for fuzzy observation which would apply to general systems.

(v) While the catastrophe theory initiated by Thom [13] is basically a theory of discontinuities which aims to provides an approach to modelling general systems, FS's on the contrary smooth the discontinuities which define an object therefore an apparent contradiction between the two approaches. This seems to strenghten the idea that FS's should be rather considered as the result of observation processes which involve the observer as parameter.

(vi) The idea of using FS in linguistics stems from expressions like *young, very young, may be young, more or less young, quite young,* etc... in which the predicate *very, may be, more or less* and *quite* can be thought of as different fuzziness degrees for the label *young*. Here again, the fuzziness of the predicate *very*, for instance, will be varying from a speaker to another one, therefore the necessity to have a membership function which explicitely depends upon the observer. It seems that this need has been perceived by Zadeh (1975) which tries to fulfill it by introducing the concept of "fuzzy sets with fuzzy membership functions" but this model itself fails in the way that the fuzziness of the membership functions should again explicitely depend upon the observer, and it is not the case.

Relatively to another topic, these predicates above are considered as operators which modify the meaning of their operands in a context way which is given by the algebra of fuzzy sets. But this algebra itself in an arbitrary generalization of the usual set algebra, it is based on an intuition which is not truly supported by experience, so that the quantitative linguistics which we can derive from it is not quite sure. In this way, it seems that multiplicative values for the predicates would be very interesting since it would allow for a direct use of arithmetic, but this possibility is subject to introducing a new concept of fuzziness function with composition laws different from the present ones for membership functions.

The following framework is an attempt to avoid some of these difficulties.

3. THE LORENTZ OBSERVATION PROCESS

Consider two co-ordinate reference frames C : (0;x,y,z) and C' : (0';x', y',z'); and assume that C moves with respect to C' in such a manner that the following conditions are satisfied : 0 moves on the axis 0'x' with the constant velocity u while C and C' remain parallel, namely 0x//0'x', 0y//0'y', 0z//0'z'. Let M denotes a point mass which moves in the frame C with the constant velocity v//0x given an observer R who is at rest in 0. In the framework of the special relativity, the point mass is depicted as a point event (x,y,z,t), where t denotes time, which lies on a Riemannian manifold endowed with the geodesic

$$d\sigma^2 = c^2\, dt^2 - (dx^2 + dy^2 + dz^2) \qquad (3.1)$$

in which c represents the light velocity. $d\sigma$ is the differential of the geodesic observed by R, and for an observer R' who is at rest in 0', one will have

$$d\sigma'^2 = c^2 dt'^2 - (dx'^2 + dy'^2 + dz'^2). \qquad (3.2)$$

It is easy to show that one has necessarily

$$d\sigma^2 = d\sigma'^2, \qquad (3.3)$$

condition which is satisfied when and only when the following equations are themselves satisfied, which are

$$x' = \rho(u)\ (x + ut) \qquad (3.4)$$

$$y' = y \qquad (3.5)$$

$$z' = z \qquad (3.6)$$

$$t' = \rho(u)\ [\ t + (u/c^2)x] \qquad (3.7)$$

with

$$\rho(u) \triangleq [1 - (u^2/c^2)]^{-1/2}. \qquad (3.8)$$

Consequently, the velocity v' of the particle as observed by R' is

$$v' = (v + u)\ /\ (1 + \frac{uv}{c^2})\ . \qquad (3.9)$$

Equations (3.4) - (3.7) are refered to as the Lorentz-equations, and they express the variables (x',y',z',t') observed by R' in terms of those observed by R, say (x,y,z,t).

Our basic idea is to use this relativistic model to describe any general system S which involves relativistic features due to human factors, and our interpretation of the Lorentz equations is pictured in Fig. 1.

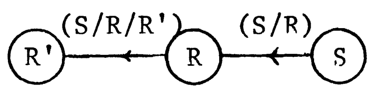

Fig. 1. Lorentz observation process

R observes S to define (S/R), *i.e.* S given R : this corresponds to the quartet (x,y,z,t) of the particle above. At a second level, R' uses the observation (S/R) to define (S/R/R') *i.e.* S given R' via R : this corresponds to the quartet (x', y', z', t') of the point mass M.

This observation process will be refered to as a Lorentz observation process. For more details see Jumarie [2], [3] , [6] .

4. RELATIVISTIC FUZZY SETS

In this section, we shall carefully derived the new concept of relativistic fuzzy sets that we further outlined elsewhere [4] .

4.1 Derivation of relativistic fuzzy sets

Consider the following observation process.

Let A denote a mathematical class (we mean a properly well defined class) in a space of objects E whose the generic element is denoted by x. An observer R' examines an element $x \in E$ and tries to determine whether $x \in A$ or $x \notin A$. To this end, he will measure the value of the *characteristic function* $\{\Phi_A : E \to [0, 1], x \mapsto \Phi_A(x)\}$ of the set A in x, and he will so obtain an observed value $\Phi_A(x/R')$ involving explicitely the subjective factors due to R'. The process is thus the Lorentz observation process (A/x/R'), (not (x/A/R') since R' observes the characteristics of A) and the Lorentz equations will express $\Phi_A(x/R')$ in terms of $\Phi_A(x)$ subject to the condition that we preliminarily identify the other variables of the system A.

In fact, let $\bar{A}$ and $\Phi_{\bar{A}}(x)$ denote respectively the mathematical complement of A and its characteristic function; a careful physical analysis suggests the following geodesic for A, that is

$$d\sigma^2(A/x) = c^2 \; d\Phi_A^2(x) - d\Phi_{\bar{A}}^2(x) \; , \tag{4.1}$$

so that equations (3.4) and (3.7) then are

$$\Phi'_A(x/R') = \rho(u) \left[\Phi_A(x) + u(x/R') \; \Phi_{\bar{A}}(x) \right] \tag{4.2}$$

$$\Phi'_{\bar{A}}(x/R') = \rho(u) \left[\Phi_{\bar{A}}(x) + \frac{1}{c^2} u(x/R') \; \Phi_A(x) \right] \tag{4.3}$$

Moreover, by applying equation (4.2) to $\bar{A}$, one can show (Jumarie [4]) that the constant c is necessarily the unit, therefore we have the following definition.

Definition 1. Relativistic Fuzzy Sets RFS. Let $A \subset E = \{x\}$ denote a given set in E, and let $\Phi_A(x)$ and $\Phi_{\bar{A}}(x)$ denote the characteristic function of A and of its complement $\bar{A}$ respectively. An observer R', who observes A, will measure $\Phi_A(x)$ and $\Phi_{\bar{A}}(x)$ in the form of the *relativistic characteristic functions*, RCF in the following, $\Phi'_A(x/R')$ and $\Phi'_{\bar{A}}(x/R')$ defined as

$$\Phi'_A(x/R') = \rho(u_A)\,[\Phi_A(x) + u_A(x/R')\,\Phi_{\bar{A}}(x)] \qquad (4.4)$$

$$\Phi'_{\bar{A}}(x/R') = \rho(u_A)\,[\Phi_{\bar{A}}(x) + u_A(x/R')\,\Phi_A(x)] \qquad (4.5)$$

where $u_A : E \to [0, M] \subset [0, 1]$, $x \mapsto u(x/R')$ is a function, the fuzziness function, which is indexed by R' and which characterizes the *fuzziness* of the observation of A by R', and where $\rho(u_A)$ is given by equation (2.8). ■

Definition 2. We shall refer to the function $u_A(./R')$: $E \to [0, M] \subset [0, 1]$, $x \mapsto u_A(x/R')$ as the *relativistic fuzziness function* RFF, *of the set A given the observer R'.* ■

In the following, R' and R'' will denote two differents observers.

4.2 Operations on relativistic fuzzy sets

Definition 3. The RFS (A/R') is *empty* if and only if

$$\Phi'_A(x/R') = 0, \forall\, x \in X \qquad (4.6)$$

The RFS (A/R') is *contained* in the RFS (B/R'') written as (A/R') (B/R'') if and only if

$$\Phi''_B(x/R'') = 0 \qquad (4.7)$$

for every x such that

$$\Phi'_A(x/R') = 0 \qquad (4.8)$$

The RFS's (A/R') and (B/R'') are *equal*, written (A/R') = (B/R'') if and only if (A/R') (B/R'') and (B/R'') (A/R'). ■

As it is evident, this definition is far from the definition for usual fuzzy sets (in Zadeh sense) in the way that it explicitely refers to the absolute set which it is stemmed from. Indeed, recall that, in this theory, the equality of two fuzzy sets A and B is given by the equality of their membership functions $f_A(x)$ and $f_B(x)$ while A is contained in B when one has $f_A(x) \leq f_B(x)$, $x \in E$.

By virtue of condition (4.6), $\Phi'_A(x/R')$ will define a *relativistic fuzzy singleton* if the equation

$$\Phi'_A(x/R') = 1$$

is satisfied for only one $x_o \in X$.

Similary, $\Phi_A(x/R')$ will define a *relativistic fuzzy countable set*, if the above equation is satisfied on a countable set in X. ■

Definition 4. Let $u_B(x/(A,R')/R'')$ denote the *fuzziness function of* B *given* R'', *conditional to the pair* (A/R'), which is involved by the observation process in which R'' determines the B-membership of an element x, given that R', has already determined the A-membership of this element. Then the intersection, denoted (AB/R',R''), of the two RFS's (A/R') and (B/R''), is defined by the relativistic characteristic functions

$$\Phi'_{AB}(x/R',R'') = \rho(u_{AB})\left[\Phi_{AB}(x) + u_{AB}(x/R',R'')\Phi_{\overline{AB}}(x)\right] \qquad (4.9)$$

$$\Phi'_{\overline{AB}}(x/R',R'') = \rho(u_{AB})\left[\Phi_{\overline{AB}}(x) + u_{AB}(x/R',R'')\Phi_{AB}(x)\right] \qquad (4.10)$$

where $u_{AB}(x/R',/R'')$, which is the fuzziness function of AB given the pair (R',R''), is defined as

$$u_{AB}(x/R',R'') = \frac{u_A(x/R') + u_B(x/(A,R')/R'')}{1 + u_A(x/R')\, u_B(x/(A,R')/R'')} \quad . \blacksquare \qquad (4.11)$$

Remark that this intersection is not commutative; generally one has

$$(AB/R',R'') \neq (BA/R'',R') \;,$$

and the equality holds when and only when the following equations are satisfied;

$$u_A(x/(B,R'')/R') \equiv u_A(x/R') \qquad (4.12)$$

$$u_B(x/(A,R')/R'') \equiv u_B(x/R'') \;, \qquad (4.13)$$

in other words, when the pair (B,R'') does not interact with the observation process (A/x/R') and similary when (A,R') does not interact with (B/x/R''). Roughly, it appears a new concept of independance which involves both the set and the observer.

Definition 5. Two RFS's (A/R') and (B/R'') are said to be *independent relativistic fuzzy sets* whenever equations (4.12) and (4.13) are satisfied. ■

Definition 6. Two observers R' and R" are said to be *independent observers* whenever the following equations are satisfied for every $A \subset E$:

$$u_A(x/(A,R'')/R') \equiv u_A(x/R') \tag{4.14}$$

$$\equiv u_A(x/(A,R')/R''). \blacksquare \tag{4.15}$$

These concepts are quite meaningful by considering the "class of kind women" and the "class of beautiful women": it may happens that a preliminary appreciation by my brother about the kindness of a given woman can effect my posterior judgment concerning the beautifulness of this same woman.

Definition 7. The union (A + B/R',R") of the two RFS's (A/R') and (B/R") is defined by the RCF's

$$\Phi_{A+B}(x/R',R'') = \rho(u_{AB})\left[\Phi_{A+B}(x) + u_{AB}(x/R'/R'')\Phi_{\overline{A+B}}(x)\right] \tag{4.16}$$

$$\Phi_{\overline{A+B}}(x/R',R'') = \rho\,(u_{AB})\left[\Phi_{\overline{A\pm B}}(x) + u_{AB}(x/R',R'')\Phi_{A+B}(x)\right] \tag{4.17}$$

where $u_{AB}(x/R',R'')$ again is defined by (3.11). ∎

This definition is a direct consequence of equations (4.9) and (4.10) via the equality $\overline{A+B} = \bar{A}\bar{B}$.

4.3 <u>Composition laws for observers</u>

The symmetry of the formulation with respect to the set and the observer, more explicitely, the discrimination between (A/R') and (A/R") provides a composition law for composite observers, which is specific to the present approach.

Indeed, consider the observation process in which an observer R" determines the A-membership of an element x, given that another observer R' has alrealy determined the A-membership of this element. Then, according to definition 4, the resulting fuzziness function $u_A(x/R',R'')$so involved is

$$u_A(x/R',R'') = \frac{u_A(x/R') + u_A(x/(A,R')/R'')}{1 + u_A(x/R')\ u_A(x/(A,R')/R'')} \tag{4.18}$$

4.4 A few specific features for RFS

(i) Relativistic fuzziness function RFF and relativistic characteristic functions are illustrated on Fig. 2.

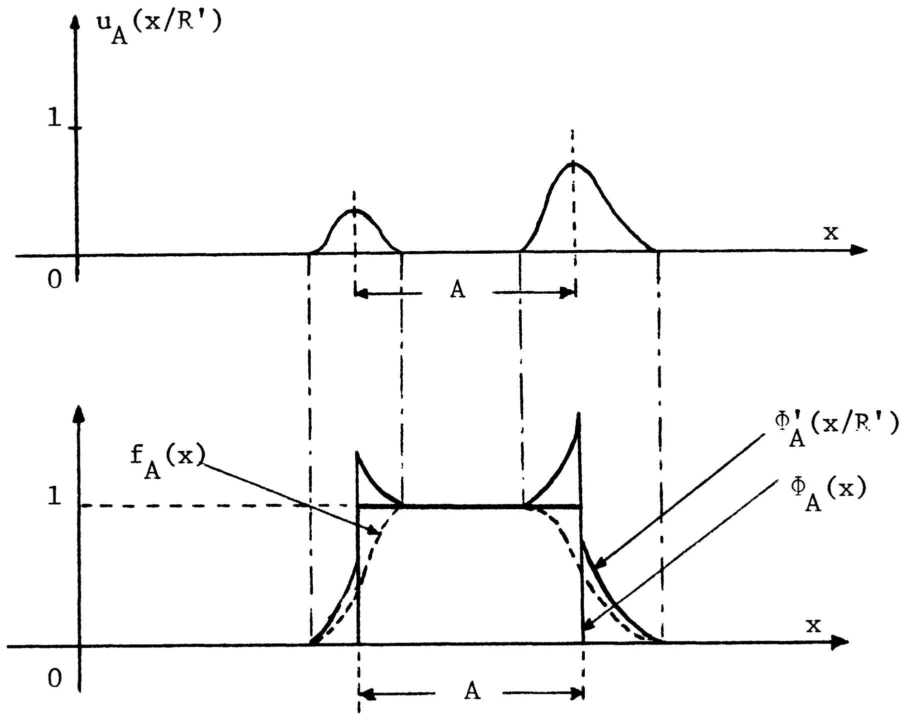

Fig. 2. Fuzziness function, relativistic characteristic function and membership function.

Clearly, one has

$$\Phi'_A(x/R') = \Phi_A(x) + \varepsilon_A(x/R') \tag{4.19}$$

where $\varepsilon_A(x/R')$ denotes a positive error term. Furthermore, $u_A(x/R')$ represents the grade of the fuzziness of x given R'; in this way it is likely that the closer x will be to the boundary of A, the larger $u_A(x/R')$ will be, in such a manner that, generally, the RFF will have the convexity property, say

$$u_A(\lambda_1 x_1+\lambda_2 x_2/R') \geq \lambda_1 u_A(x_1/R') + \lambda_2 u_A(x_2/R')$$

with

$$\lambda_1 + \lambda_2 = 1 \ , \quad 0 < \lambda_1, \lambda_2 < 1 \ ;$$

and its maximum value will be reached on the frontier of A.

(ii) By setting c = 1 in equation (2.8), one has

$$0 \leq u_A \ (x/R') \leq 1 \tag{4.20}$$

and equation (4.18), for instance, shows that the upper bound for $u_A(x/R')$ can be approached when the observer is a composite one $(R', R'', \ldots R^{(n)})$. In the same way, consider an element x_o (if it exists) which belongs to the intersection of an infinite set $\{A_n, n \in N\}$ of classes A_n with their respective RFF $u_{A_n}(x/R_n)$; then, according to equation (4.11), the resulting value for the RFF $u_{A_1 A_2 A_3 ..}(x_o/R_1 R_2 R_3 \ldots)$ in x_o approaches the unit.

(iii) Remark that, due to its physical meaning itself, the RFF cannot be consider neither as a probability density nor as a probability distribution. In fact, it would be rather close to the entropy concept as pointed out in the following comment.

(iv) De Luca and Termini [11] "tried to introduce" a measure $d(f_A)$ of the fuzziness of a fuzzy finite set (in Zadeh sense) $A = \{x_i, i = [1, N]\}$ in the form

$$d(f_A) = H(f_A) + H(1-f_A)$$

where $H(f_A)$ is formally defined as

$$H(f_A) \triangleq -k \sum_{i=1}^{N} f_A(x_i) \ln f_A(x_i)$$

with k denoting a positive constant. According to these authors such a definition could be motivated by the following properties which are desirable for this measurement concept. (Here we drop the subscript A to simplify)

P_1: d(f) is zero when and only when f(x) takes on the value 0 or 1.

P_2: d(f) assumes its maximum value when and only when f takes on the value 1/2.

P_3: d(f) must be greater or equal to d(f*) where f* is any sharpened version of f, that is any fuzzy set such that $f^*(x) \geq f(x)$ if $f(x) \geq 1/2$ and $f^*(x) \leq f(x)$ if $f(x) \leq 1/2$.

As the authors themselves mentionned it, this definition holds only for finite sets, and its generalization to continuous sets is not obvious at all. Our concern is that assumptions P_1 and P_2 are very questionable because they contradict the intuitive fact that in numerous cases the fuzziness of the observer has its higher value on the boundary of the class, boundary which is not necessarily always defined by the condition $f_A(x)=1/2$. Here the membership function is the basic concept from where fuzziness is derived formally, while, in the relativistic approach, it is the fuzziness which is fundamental and does not necessarily appeal to a concept of characteristic function.

(v) The reader may be surprized by the fact that the RFF is the same for the intersection (AB/R',R") and for the union (A+B/R',R") of two given RFS's (A/R') and (B/R"). Fig. 3 illustrates what happens, in the case where (A/R') and (B/R") are independent relativistic fuzzy sets with independent observers (see definitions 5 and 6)for which one has

$$u_{AB}(x/R',R'') = \frac{u_A(x/R') + u_B(x/R'')}{1 + u_A(x/R')\ u_B(x/R'')} .$$

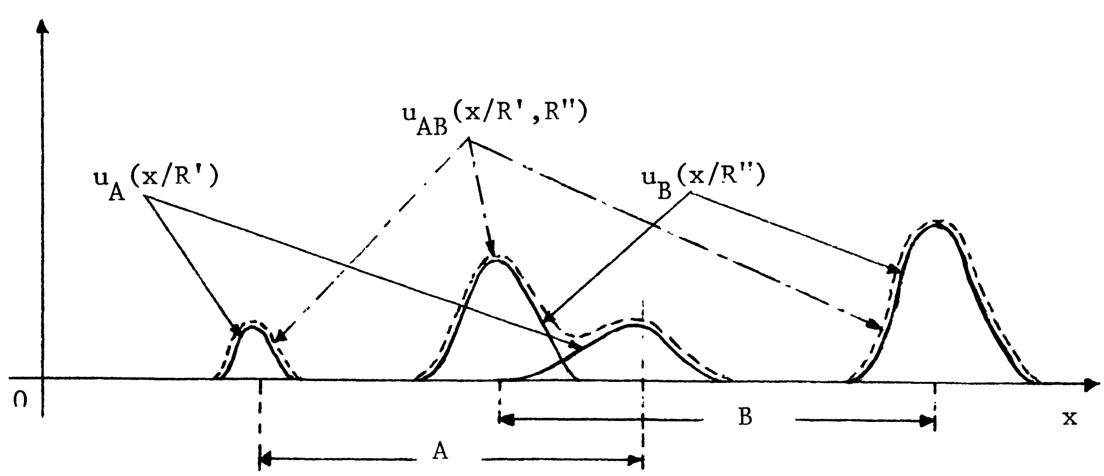

Fig. 3 . Pictural representation for $u_{AB}(x/R',R'')$.

The resulting RFF $u_{AB}(x/R',R'')$ is plotted in dashed line; it illustrates the respective contributions of the differents fuzziness components on A and B respectively to the resulting fuzziness with regard to AB and A+B. Another way to grasp that is to remember the identity

$$\Phi_{A+B}(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{AB}(x)$$

which exhibits the connection between A, B, A+B and AB.

From a philosophical point of view, the dualistic feature in equations (4.4) and (4.5) represents the facts that, when an observer R' determines whether an element x belongs to A; more or less consciously, he simultaneously determines whether x does not belong to $\bar{A}$.

5. RELATIVISTIC FUZZY SETS AND FUZZY SETS

The purpose of this section is two-fold: first, we shall exhibit the differences between relativistic fuzzy sets and fuzzy sets; and second, we shall show that, in a certain sense, fuzzy sets can be derived from relativistic fuzzy sets, therefore a possible way to explicitely introduce relativistic features in fuzzy sets.

5.1 Relativistic fuzziness function and membership function

In his pioneering work, Zadeh (1965) pointed out that the "range of the membership function $f_A(x)$ of a set A can be taken to be suitable partially ordered set", but for convenience he restricted the range to the unit interval. As a matter a fact, the RCF, as defined by equation (3.4) may be greater than the unit therefore an apparent discrepancy between the two approaches, which requires some explanations.

The important fact is that $f_A(x)$ is defined as an arbitrary generalization of $\Phi_A(x)$, while $\Phi'_A(x/R')$ results from the relativistic observation of the set A, following the Lorentz observation process. In Zadeh's approach the fuzziness of the set A is measured by $f_A(x)$ itself while, in the relativistic approach, it is $u_A(x/R')$ instead of $\Phi'_A(x/R')$ which measures this fuzziness. Remark that both f_A and u_A are to be determined by experiments, but u_A is subject to composition laws for the observers, while f_A is not.

The graphical representation for $f_A(x)$ and $\Phi_A'(x/R')$, as given in Fig. 2, suggests to associate with a given $\Phi_A'(x/R')$ a membership function $\hat{f}_A(x/R')$ defined as follows:

$$\hat{f}_A(x/R') \triangleq \Phi_A'(x/R') \quad , x \varepsilon \bar{A}$$

$$= u_A(x/R') \cdot \left[1 - u_A^2(x/R')\right]^{-1/2} \tag{5.1}$$

$$\hat{f}_A(x/R') \triangleq 1 - \left[\Phi_A'(x/R') - \Phi_A(x)\right] \quad , x \varepsilon A$$

$$= 2 - \left[1 - u_A^2(x/R')\right]^{-1/2} \tag{5.2}$$

Conversely, given a Zadeh membership function $f_A(x)$, we can obtain a corresponding RFF $\hat{u}_A(x)$ by using the converses of the equations (5.1) and (5.2) to obtain

$$\hat{u}_A(x) = f_A(x) \,/\, \left[1 + f_A^2(x)\right]^{1/2} \quad , x \varepsilon \bar{A} \tag{5.3}$$

$$= \left[3 - 4f_A(x) + f_A^2(x)\right]^{1/2} \,/\, \left[2 - f_A(x)\right], \; x \varepsilon A \tag{5.4}$$

The correspondence between $\hat{f}_A(x/R')$ and $\Phi_A'(x/R')$ is pictured on Fig.4

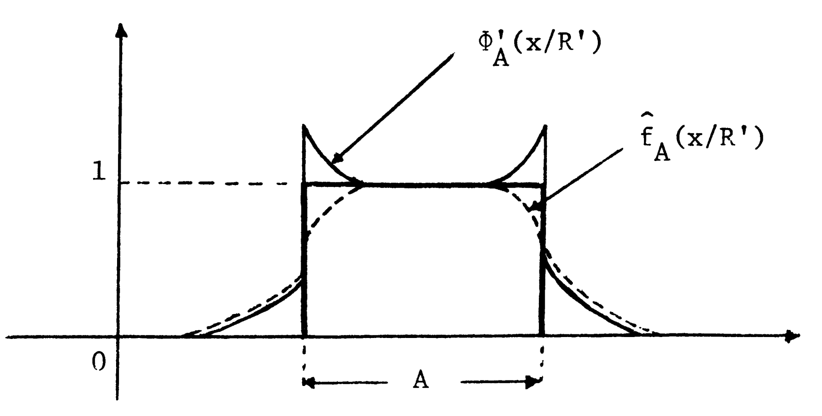

Fig. 4. Membership functions derived from relativistic characteristic function.

Such as defined, $\hat{f}_A(x/R')$ is generally discontinuous on the boundary of A. There is continuity when and only when the respective right-hand side terms of equations (5.1) and (5.2) have the same value on the boundary of A, that is

$$u/(1-u^2)^{1/2} = 2 - [1/(1-u^2)^{1/2}]$$

or else

$$u_A(x/R') = 3/5, \ x \in \text{boundary of } A.$$

5.2 Identification of a set defined by a membership function

Strictly speaking, identification in fuzzy set framework is not meaningful since membership function is an abstract generalization which does not explicitely refer to the set which it is associated with; nevertheless the comparison in subsection 4.1 provides some approaches which can be helpful in some applications.

The question is how to associate a set A with a given membership function $f_A(x)$. Of course, there is not one answer only to this question, various solutions are proposed below which, if they are close each other, can enforce the possibility for only one admissible class.

(i) In the relativistic framework, the boundary of A is defined by the discontinuity set of $\Phi'_A(x/R')$. Analogously, since $f_A(x)$ is generally continuous, assuming further that it is differentiable, we can define the boundary of an hypothetical corresponding class A as being the set B_1,

$$B_1 \triangleq \{x \in E \mid \dot{f}_A(x) \text{ maximum } \} \qquad (5.5)$$

where the dot in $\dot{f}_A(x)$ denotes the derivative of $f_A(x)$.

(ii) Another way to define A consists of using the equivalent RFF $\hat{u}_A$ (x) defined in sub-section 5.1 as follows. Indeed, according to a remark in subsection 3.4, *generally*, $u_A(x/R')$ will achieve its optimum value on the boundary of A; and in such a case, for every x_o on this frontier, we shall have

$$u_A(x_o^- /R) = u_A(x_o/R)$$
$$= u_A(x_o^+/R) \qquad (5.6)$$

where $x_o^- \varepsilon \bar{A}$ and $x_o^+ \varepsilon A$.

Applying equation (5.6) to $\hat{u}_A(x)$ yields

$$\frac{f_A(x_o)}{1+f_A^2(x_o)} = \frac{3 - 4f_A(x_o) + f_A^2(x_o)}{[2 - f_A(x_o)]^2}$$

that is

$$f(x_o) = 3/4$$

With this criterion, the boundary B_2 is the set

$$B_2 \triangleq \{x \ \varepsilon \ X \mid f_A(x) = 3/4\} \qquad (5.7)$$

(iii) In the particular case where $f_A(x)$ has a continuous set of discontinuity points, then this latter should be identified as the boundary of A.

Fig. 5 gives an illustration for the results i) and ii).

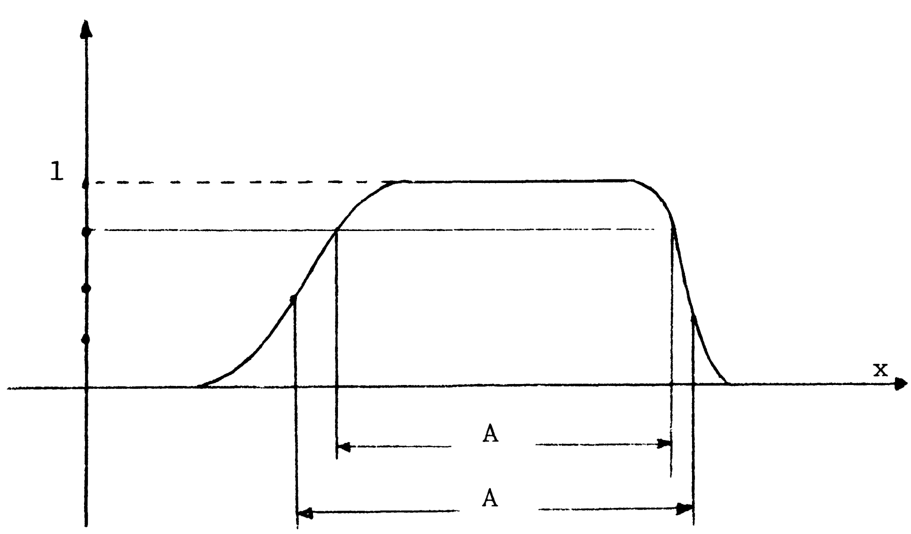

Fig. 5. Derivation of a set from a fuzzy set

6. FUZZINESS FUNCTION, MEMBERSHIP AND INCERTITUDE

In this section, we shall discuss the physical meaning of the fuzziness function $u_A(x/R')$, seeking practical ways to determine it.

6.1 General Comments

The fuzziness function appears related to the *incertitude*, the doubt that the observer R' has concerning the membership of the element x. Mainly it is positive, it equates zero when R' has no doubt on the membership of x, and it likely assumes its maximum value in the neighbourhood of the boundary of A. This type of doubt is basically different, in nature, from random uncertainty and to avoid confusing, we shall refer to it as "incertitude".

In FS theory, the basic concept is that of membership functions, while in the RFF set theory, it is the fuzziness function which is the starting concept.

Of course, either of these situations will hold depending upon the experimental framework as illustrated, for instance, by a photoelectric cell and a learning process with teacher. Indeed, the cell will takes the value 1 when it is subject to light (white color) and will remain at rest in 0 when immersed in darkness, (black color); but it may happen that the state of the cell continuously vary in the range $[0, 1]$, for instance when the cell is sensitive to grey colors; and in such a case the model straight - forwardly involves the membership function of the cell.

The case of a student who is learning under the direction of a supervisor is quite the opposite, in the way that he can himself give an estimate of his incertitude. To the question of the teacher who asks "are you certain of the membership of A", the student can answer in gradual forms as "very certain, certain, almost sure, vague, in to minds, etc..." and prior numerical values for these predicates will yield the function $u_A(x/R')$.

6.2 Analog computer

Another practical example which directly involves the fuzziness function is given by the analog computer, Fig. 6.

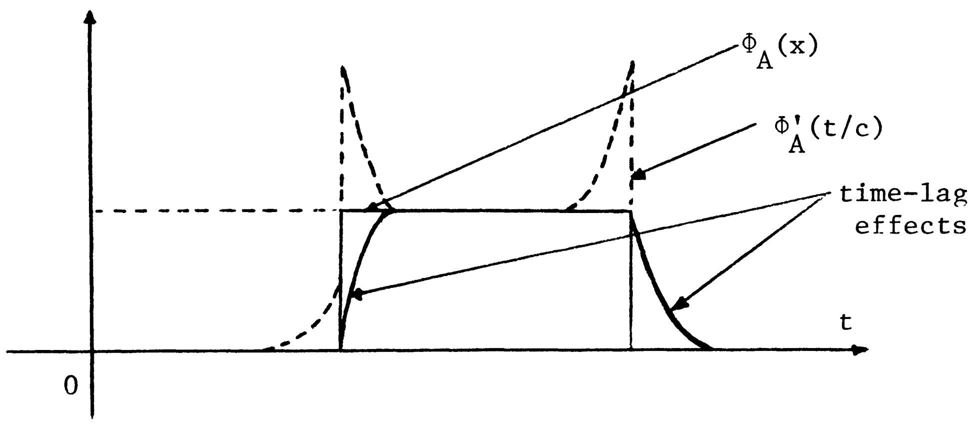

Fig. 6. Analog computer and fuzziness function

A typical response of an analog computer to a characteristic function in time, is given in Fig. 6. Because of the time-lags of the physical elements which compose the computer, the output is smoothed with an ascending slope and a descending slope, and these slopes can be interpresented in terms of fuzziness function as shown by the relativistic characteristic function $\Phi'_A(t/\text{computer})$ reconstructed in Fig. 6.

6.3 <u>Fuzziness function via membership function</u>

The difficulty then arises when one has to define the fuzziness function of a device which is described by its membership function. In the preceding section we gave a first approach which was suggested by the pictural representation of $f_A(x)$ and $\Phi'_A(x/R')$ but it may be not satisfactory in some instances as the functions so determined can involve discontinuities. So we have to define other ways to link $f_A(x)$ and $u_A(x/R')$, and a possible approach is the following.

Starting from $f_A(x/R')$ which is the membership function as observed by the observer R', the RFF u_A can be viewed as a functional of the derivative $\dot{f}_A(x/R') \triangleq df_A\ (x/R')\ /\ dx$, say $u_A(f_A)$, which would satisfy the following desirable conditions:

(i) $u_A(f_A) > 0 \ , \ \dot{f}_A > 0$

(ii) $u_A(0) = 0$

(iii) $\sup u_A(f_A) \leq 1 \ , \ f_A \geq 0$

(iv) $u_A(f_A)$ is decreasing with its argument $\dot{f}_A$.

An example, for instance, is

$$u_A(x/R') = a\,[1 - \exp(\dot{f}^2)] \tag{6.1}$$

where the constant a is such that

$$a \leq 1 \,/\, [1 - \exp(-\sup \dot{f}^2)] \tag{6.2}$$

6.4 Incertitude and subjectivity

A few time ago, the concept of *swiftness* has been proposed for educational purpose, to get a better understanding of the famous composition law of velocities in the special relativity. We would have two distinct concepts which are the "velocity" and the "swiftness", and they would be related by the equation

$$\text{velocity} = \text{th (swiftness)} \tag{6.3}$$

where th(·) denotes the hyperbolic tangent function. Swiftness is additive, therefore the composition law for velocity via the well known equation

$$\text{th}(a+b) = (\text{tha}+\text{thb}) \,/\, (1+\text{th}(a+b)). \tag{6.4}$$

Likewise, we shall infer that we have the concepts of fuzziness and of incertitude and that they are related by the equation

$$\text{fuzziness} = \text{th (incertitude)} \tag{6.5}$$

and, while the incertitude is additive, the fuzziness is relativistic. But the fuzziness itself, in our mind, is nothing else but the subjectivity, in other words we would have

$$\text{subjectivity} = \text{th (incertitude)} \tag{6.6}$$

6.5 Membership, subjectivity, incertitude

The reader may wonder why do we need to introduce the new concept of subjectivity since it is defined in term of incertitude so that, at first glance, it brings nothing else more.

Here still, the analogy with relativistic physic is direct. As the position of a particle can be determined by its velocity only, so the membership function of an element with respect to a class will be determined by the fuzziness function only. We have not direct composition laws for the membership function itself; and all we can do is to derive them from the composition law of the fuzziness function.

For instance, assume that equation (6.1) holds. For an interval, one has

$$\dot{f}_A(x/R') = \pm \left[\text{Log } (1-u_A(x/R')/a)\right]^{1/2}$$

where the sign + is on the lower side of the range, while - holds on its upper side, therefore $f_A(x/R')$ itself. This supposes that there is a cut-off point or a cut-off region where the + sign switches to - , and of course, this point or region is defined by the condition.

$$u_A(x/R') = 0.$$

When no such a point exists, all we can do is to select either + or - , in other word, the set in question can be thought of as a fuzziness spot inside a larger set.

7 ON THE SOUNDNESS OF THE MODEL

This section is devoted to a few questions related to the validity of the model.

7.1 On the use of the relativistic framework

The use of the mathematical formulation of the special relativity to describe subjectivistic feature in medelling systems is not a mere analogy, but rather is a consequence of a careful analysis that we performed in our approach to general systems [2] , [3] , [6] . Broadly speaking, a system is defined by its inside and its outside, and the structural variable of the outside has properties very similar to those of time in relativistic physics therefore the use of this latter.

Nevertheless, from on epistemological standpoint why not we use this material to investigate human sciences where the observer, as a reference, has a basic importance like in relativistic physics ? One can already does it, and then check the results so obtained. In this way, we invite the reader to consider the present paper as the first step above.

7.2 Toward prior and posterior subjectivity

Despite its rigorous mathematical foundations by Kolmogorov, the concept of probability is basically subjectivistic, in the sense that its prior asymptotic value is what we believe it should be rather them what it is actually. Of course, one can consider, and it is customary to do it, that this asymptotic probability is an objective probability, but this is quite useless since we never repeat a same experiment on infinite number of times, from a practical viewpoint !

One may expect to have similar difficulties in the practical estimation of the relativistic fuzziness function $u_A(x/R')$. It is likely that we shall have new concepts such as "prior subjectivity" and "posterior subjectivity" which further remain to define fully.

7.3 On an apparent contradiction of the approach

A somewhat inconsistency which seems to stem from the definition of RFS is the following. The RCF such as defined has discontinuities on the boundary of the set, that is to say where one may expect that the incertitude achieves its maximum value. But if there is discontinuity, there is discrimination and this is exactly the classical concept of class !!

In fact, this inconsistency is only apparent. We have indentified the characteristic functions of a set and of its complement as the state variables of this set, only to be in a position to apply the Lorentz transformation. The use of this transformation has exhibited the importance of the relativistic fuzziness function of a set, and it is this RFF which is basic in our approach. As a matter of fact, a RFS is defined by its RFF only, and it is exactly this way that we have followed to define our subjectivistic calculus ! [4]

It is therefore clear that the boundary of a RFS is not exactly well defined, and all we can say is that it lies on the domain of the RFF, that is to say on the range where the fuzziness is different from zero. There remains the task of estimating the set itself given its RFF, and it is this problem which has been addressed in the sub-section 5.2.

We shall conclude this remark by emphasizing that the basic concept of our approach is the RFF rather than the RCF. The relativistic characteristic function has been utilized to construct the relativistic fuzziness function, and once the latter has been obtained, we deleted the former.

7.4 Research program

A problem which contributes as an essential part in the flaw of the present fuzzy set theory is that of defined the aggregation of membership functions by human beings. This cannot be done without supplementary hypotheses and it is exactly the advantage of our approach which states theses assumptions in a natural way via the special relativity, whereby all our results stem from.

The question now is to verify that this hypothesis coincides with real human aggregation behaviour; and this can be made only if we develop a method to determine experimentally the relativistic fuzziness function.

As a matter of fact, if we work with equation (6.5), defined the fuzziness is reduced to defining the incertitude, so that the problem would be considerably simplified.

For instance, in a frequency framework similar to the frequency approach to probability, if we take the incertitude in the form $-q(x/R')$ $\mathrm{Log}\ q(x/R')$ where $q(x/R')$ denotes the frequency of membership, then one has

$$u(x/R') = -\ \mathrm{th}\ \ q(x/R')\ \mathrm{Log}\ q(x/R') \quad ;$$

therefore an approach to the empirical determination of relativistic fuzziness function would be as follows:

(i) Definition of basic interesting fuzzy terms of fuzzy sets;

(ii) Selection of appropriate samples of elements (or keywords) to define with regard to the fuzzy classes above;

(iii) Empirical definition of the frequency of memberships for each sample by a representative class of human beings.

This approach leads to a first subjectivistic characterization of a human being given a fuzzy matter; and by making a systematic clustering, we can characterize everybody in every situation by a *prior relativistic fuzziness function*. Of course, such a procedure assume implicitely that the behaviour, or yet the assessment of somebody is repeatedly reproductive in the presence of the same causes. This appears to be true in some cases, for instance it is customary to talk about the "type of female beauty" for a given man. But it may also be wrong in some scarce cases, and sometimes the definition of woman beautifulness may change for a given observer. In other words, we should have too the concept of *posterior relativistic fuzziness function*, and these would remain to deepen the relation between *prior* and *posterior subjectivity*.

8 APPLICATIONS

This new model for fuzzy sets has been derived as a by-product of a new approach to general systems and to information theory so that its first direct consequences are rather related to these topics instead of to human sciences, properly speaking, as psychology for instance. Furthermore, it appears that its application to psychology cannot be made independently of the general system model above which basically involves the coupling effects between the observer and the observable. It follows that our first results deal with that part of science which is viewed as objective, but it nevertheless gives an idea of which kind of applications may be expected.

8.1 Application to Renyi entropy

In order to define the informational uncertainty on a random experiment α which can provide the issues A_1, A_2, ... , A_n with the probabilities p_1, p_2, ... , p_n; Renyi proposed to use the entropy $H_r(A)$ defined as

$$H_r(\alpha) = \frac{1}{1-r} \operatorname{Log} \sum_i p_i^r$$

where r denotes a constant different from the unit. This constant appears as the result of an integration so that the theory cannot ascribe it any physical significance.

By using our relativistic model, we have shown [7] that this coefficient is nothing else but the efficiency coefficient of the observation of the process α by the observer R which considers α: and in effect, it depends explicitely upon both α and R, say $r(\alpha/R)$. When $r < 1$, there is loss of information and R receives the amount $H_r(\alpha)$ of information only; while when $r = 1$, there is no less, and R receives exactly $H_r(\alpha)$. The case $r > 1$ corresponds to a subjective process in which R gets more information from α than this latter contains.

Until now, this entropy of Renyi was remainded a purely mathematical curiosity; so one can expect that this practical significance can open new ways for its future applications in communications.

8.2 Application to societal systems

Societal systems basically involve subjectivistic factors. Consider four individuals (A, B, C, D) which are first isolated each other, and then can spontaneously combine by themselves to get various hierachical structures. The problem is to determine which structural combination will be reached.

The model that we have proposed [6] to solve this problem involves explicitely relativistic fuzzy coefficients such that K(A/B) , K(D/C), ... which have practical meanings similar to r in the Renyi entropy. Given different prior values for the coefficients K(./.), we obtain final structures which are quite meaningful from a practical standpoint. For instance, we have statements like "if A and B have the same organizing potential level and if they are isolated, they will remain so".

8.3 Application to subjectivity in information

The following problem is well known. People who live in a city A speak the true while people who live in another city B tell lie. A man knows that he is presently in A or in B, but he does not know exactly in which one. He meets somebody, and the problem is to ask this latter only one question to determine the city.

An example of such a question is "Do you live in the city where we are presently" ?

The answer to this question contains a certain amount of information I(Q), that one which is sufficient to solve the problem, but it is well known that, given the question and its answer, a questionner will be able to determine the city while another one will be not. Thus the amount of information contained in Q depends explicitely upon the observer R, say I(Q/R) and all the problem is to define the analytical expression for this latter.

We have shown [3] that the information which is effectively observed by R is in the form

$$I(Q/R) = K(Q/R)\ I(Q)$$

where K(Q/R) denotes a coefficient which involves explicitely the fuzziness coefficient u(Q/R). Moreover, we have given the explicite expression for K(Q/R) in terms of various probabilities and we have shown quantitatively how the subjectivity effects the information. The coincidence with the shannon theory is pointed out.

8.4 Appraisals in probability

As a result of the application of our relativistic model to the Renyi entropy, the probability p^r which is involved by this latter can be viewed as an observed probability. More specificly, if we repeatedly make the experiment A defined by the prior probability set $\{p_1, p_2, \ldots, p_n\}$, and if we observe the i-th issue A_i; then at the m-th trial, we shall observe the probability $p_i^{r(m)}$, where r(m) depends explicitely upon the order m of the trial in question.

In a non subjectivistic framework, the coefficient r(m) is equivalent to a random coefficient whose the asymptotic value is the unit. In the special case of rolling a fair die, plotting $p_i^{r(m)}$, yields a diagram similar to that of the Fig. 7; and it looks like we know.

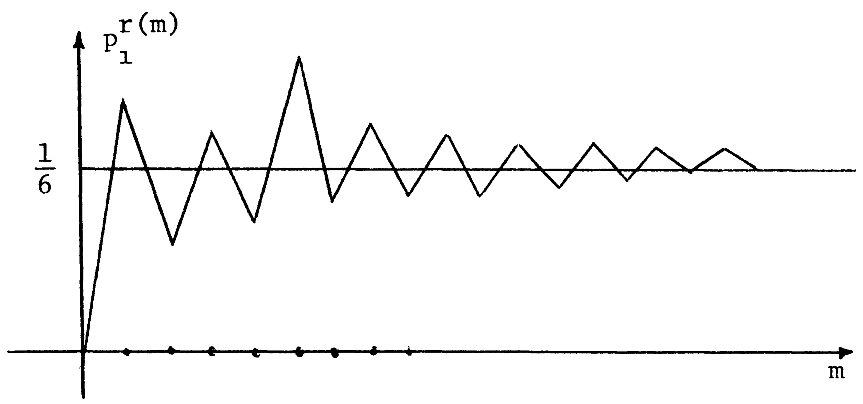

Fig. 7. Prior and observed probability

This being so, the comments which follow are on the edge of the objective science.

Assume that the source S of the random phenomenon reacts to the observer R, then the coefficient r is a relativistic coefficient r(S/R) and the probability which is issued by S is $p_1^{r(S/R)}$. Such a model would be involved , for instance, by psyphenomena, or yet, by the various paradoxes which occur in quantum mechanics like the so-called famous psi-collapse. In the physics of particles, for instance, S would be the particle itself and R would be the measurement device, so that r(S/R) would involve the interaction between the particle and the measurement device. In our framework, this interaction appears in a natural way, while presently, it is introduced in the form of additional assumptions in quantum mechanics .

9. AN INFORMATION THEORETIC APPROACH TO THE MODEL

In this section we shall derive our relativistic fuzzy set concept in an information theoretic framework, by using specific features for natural languages only.

We think we can assert, at no risk of making a grave mistake, that information is conveyed between humain beings mainly via *natural languages* (NL in the following). They are not the only ones; pictures and motion too convey information in human systems; but they are the main ones. In which way, how to describe NL is order that they provide a suitable approach to the dynamics of human systems?

9.1 Natural languages

Every NL can be considered as defined by a set $\Omega = \{\beta\}$ of symbols generically denoted by β, call them *words* to fix the thought, and a set $\Omega' = \{b\}$ of meanings whose the generic element is b. The structure of Ω, or else the *internal structure of the* NL is organized by a set of rules, the grammar rules, which are referred to as the *syntax* of the NL. In the same way Ω' is ruled by the so-called *semantics* which somewhat define all the admissible combinations of its elements. Each word in Ω is associated with one or several meanings in Ω', and it is customary in linguistics to refer to this correspondence as a *lexem*: shortly a lexem is a word considered in a given sense.

Formally, a NL is an application $L: \Omega \to \Omega'_p$ which maps Ω onto the set Ω'_p of the subset of Ω'.

Let us consider a second NL, $l : \Sigma \to \Sigma'_p$. If there exist two one-to-one applications $h : \Omega \to \Sigma$ and $h' : \Omega'_p \to \Sigma'_p$ such that the commutative diagram in Fig. 8.

$$\begin{array}{ccc} \Omega & \xrightarrow{\ L\ } & \Omega'_p \\ {\scriptstyle h}\downarrow & l & \downarrow{\scriptstyle h'} \\ \Sigma & \longrightarrow & \Sigma'_p \end{array}$$

Fig 8. Equivalent languages

holds, we shall say that L and l are *equivalent*. Remark that this diagram is exactly the basic diagram which defines the equivalence of applications in differential geometry (catastrophe theory)!

9.2 Natural languages and communication

A modelling of communication in human systems can be related to equivalent languages as above defined, in the following way. The speaker S says something in a language (L,Ω,Ω') to a listener R who himself has his own language (l,Σ,Σ'). We shall say that there is communication between S and R provided that R can define two one-to-one applications h and h' such that the diagram of Fig. 8 holds.∎

It is clear that with such a definition, there may be communication without mutual understandability, but in the present framework, communication with understandability appears as being a subclass of communication processes in the general sense of this term, so that we can restrict ourselves to this broad acceptation in a first step.

9.3 Subjectivity in communication

Subjectivity, what is it? We shall refer to subjectivity as to this ability that an observer, because of a certain prior internal model which is built in his own, has to interpret and recover the content of a message which is receiving, in such a manner that the understood meaning is not the actual meaning of the message as received, but is rather what the observer think it should be. In this way, a human receiver can rectify mistakes in an erroneous transmission, but it may also give a wrong significance to a message which is nevertheless correctly transmitted.

One of the striking feature of subjectivity is the following. When a message is sent by means of a certain natural language NL, to the observer who is receiving this message, this latter is defined locally in syntax and semantics, and it is precisely the role of subjectivity to define which part of the syntax and of the semantics in the NL is involved by the message in question. The message contains a certain amount of information which can be resolved into the information related to its syntax and the information related to its semantics. In the same way, there is a certain uncertainty on the meaning of the message: even when the message is suitably technically transmitted, the human receiver may have a certain doubt on its content. *Subjectivity works in such a way that when information increases then uncertainty decreases and conversely.*

This is a modelling of this feature which is given is the following subsection.

9.4 The framework

Despite it is by no means a strict necessity, we shall herein use the concept of *entropy* as defined by Shannon, and so mainly to fix the thought. We bear in mind: given a random experiment β which can yield the results $\{B_1, B_2, \ldots, B_n\}$ with the respective probabilities $p(B_1)$, $p(B_2), \ldots, p(B_n)$; the entropy $H(\beta)$ of β is defined as

$$H(\beta) := -\sum_{i=1}^{n} p(B_i) \log p(B_i)$$

(the symbol := means that the quantity on the left-hand side is defined by the expression on the right, while =: means that the right-hand side is defined by the expression on the left) where the logarithm function may have any basis.

H(β) measures the mount of uncertainty hat an observer has about the result of the experiment β; and also, it defined the amount of information which is contained in β.

9.5 Information and uncertainty in natural languages

A NL is pictured on the diagram of Fig 9

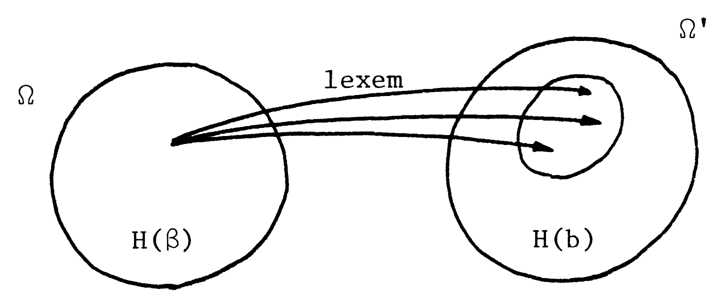

Fig 9. Syntax, semantics, lexem

In the standard way of the "mathematical theory of communication" of Shannon, we can assume that two fields of probability are given on Ω and Ω' respectively, therefore we can characterize then by the two entropies H(β) and H(b).

Definition 8. We shall say that the total amount of information which is contained in the NL : Ω → Ω' is H(b) + H(β). ∎

The reader may question why do we not rather consider the expression

$$H(\beta,b) = H(\beta) + H(b/\beta)$$
$$= H(b) + H(\beta/b) \ ?$$

This is merely due to the fact that, in a first stage, Ω and Ω' are defined independently by the observer who so describes them without taking account of any relation they may involve.

Definition 9. We shall say that the amount of uncertainty which is involved by the NL : $\Omega \to \Omega'$ is $H(b) - H(\beta)$. ∎

Indeed, despite a word may have several lexems, one has generally $H(b) \geq H(\beta)$ for natural languages. This being so, when $H(b) = H(\beta)$, we have a somewhat identification between syntax and semantics so that the message is completely determined by its syntax only. When $H(b) > H(\beta)$, we have not a one-to-one correspondence between Ω and Ω'; on the average a given word in Ω has $(H(b)-H(\beta))$ lexems, therefore an uncertainty which is specific to the structure of the NL.

9.6 Subjectivity, information, uncertainty

We are now in a position to introduce subjectivity in the communication process via natural languages.

In the absence of subjectivity, it seems right to assume that both $H(b) + H(\beta)$ and $H(b) - H(\beta)$ are constant. This assumption no longer holds in the presence of subjectivity. Indeed, in such a case, the structures of Ω and Ω' are not fixed as before, but they are rather changing with the subjectivity of the observer R. All takes place as if this latter were defining his own field of probability on Ω and Ω' respectively so that the various entropy functions above depends explicitely upon R via these probabilities, say $H(b/R)$ and $H(\beta/R)$. It is then difficult to assume again that $H(b/R) - H(\beta/R)$ are constant. But since it is a current matter that uncertainty decreases when information increases, and conversely; we shall assume that it is the product

$$(H(b/R) + H(\beta/R))(H(b/R) - H(\beta/R)) =$$

$$= H^2(b/R) - H^2(\beta/R)$$

which remains constant.

As a word of caution about the physical dimensions of H(b/R) and H(β/R), we shall introduce a constant c, depending upon the measurement unit, and such that the quantity

$$c^2H^2(b/R) - H^2(\beta/R)$$

is not affected by the subjectivity. Locally in syntax and semantics the differential geodesic

$$ds^2(L/R) = c^2dH^2(b/R) - dH^2(\beta/R)$$

is not affected by subjectivity.

So, according to the well known Lorentz equations, condition

$$c^2dH^2(b/R) - dH^2(B/R) = c^2dH^2(b) - dH^2(\beta)$$

holds when one has

$$H(\beta/R) = \rho[H(\beta) + uH(b)]$$

$$H(b/R) = \rho[H(b) + \frac{u}{c^2} H(\beta)]$$

with

$$\rho := (1 - \frac{u^2}{c^2})^{-1/2}$$

and this is exactly our relativistic fuzzy set model.

One of the most interesting suggestions of this derivation is the use of a relativistic framework to investigate societal systems. The central idea in this way would be to assume that they are subject to fields of communication, therefore the relativistic dynamics.

10. CONCLUSIONS

Fuziness is basically different from probability. Indeed, while probability defines the observed system itself which is stochastic in its nature in the sense that its state occurs randomly; by contrast, fuzziness refers to the observation process of a system by an observer, and therefore involves subjective factors which are essentially varying from an observer to another one.

A fuzziness theory should explicitely involve this relativistic feature, and it seems that fuzzy scientists fail to meet this requirement since fuzzy membership functions of fuzzy sets have been introduced also in the literature. Unfortunately, this has been done in a rather arbitrary way, and the problem of the dependence of this fuzzy membership function upon the observer remains unsolved.

In fact, the relativistic physics supply us with a mathematical model which explicitely involves observers, and applying it to "sets" and "variables" straightforwardly yields, in a natural way, the new concept of relativistic fuzziness function as basic concept.

The most striking characteristic of this relativistic approach is the composition law for fuzziness and observers which exhibits a saturating effect with respect to fuzziness, and it would be interesting to examine what it can yield in a quantitative approach to semantics and linguistics. For instance, consider the expression *very tall* and *likely tall* from where we derive *very likely tall* and *likely very tall*. Intuitively *very likely* is different from *likely very*, in other words, the product is not commutative, and it is exactly the case in our relativistic framework. Moreover, it seems that it is possible to consider *tall* as a set A, *very tall* as a relativistic fuzziness function $u_A(x/R')$ and similary *likely tall* as $u_A(x/R'')$ so that the composition law for observers would give the value of *very likely tall* and *likely very tall*.

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