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FINITE-ELEMENT DISCRETIZATIONS OF A TWO-DIMENSIONAL GRADE-TWO FLUID MODEL

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Abstract. We propose and analyze several finite-element schemes for solving a grade-two fluid model, with a tangential boundary condition, in a two-dimensional polygon. The exact problem is split into a generalized Stokes problem and a transport equation, in such a way that it always has a solution without restriction on the shape of the domain and on the size of the data. The first scheme uses divergence-free discrete velocities and a centered discretization of the transport term, whereas the other schemes use Hood-Taylor discretizations for the velocity and pressure, and either a centered or an upwind discretization of the transport term. One facet of our analysis is that, without restrictions on the data, each scheme has a discrete solution and all discrete solutions converge strongly to solutions of the exact problem. Furthermore, if the domain is convex and the data satisfy certain conditions, each scheme satisfies error inequalities that lead to error estimates.

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0. Introduction

This article is devoted to the numerical solution of the equations of a steady-state, two-dimensional grade-two fluid model:

$$-\nu\,\Delta\,\mathbf{u} + \mathbf{curl}(\mathbf{u} - \alpha\,\Delta\,\mathbf{u}) \times \mathbf{u} + \nabla\,p = \mathbf{f} \quad \text{in } \Omega\,, \tag{0.1}$$

with the incompressiblity condition:

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{0.2}$$

and the Dirichlet tangential boundary condition:

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \quad \text{with} \quad \mathbf{g}\cdot\mathbf{n} = 0\,, \tag{0.3}$$

where $\mathbf{n}$ denotes the exterior normal to the boundary $\partial\Omega$ of Ω, $\mathbf{u} = (u_1, u_2, 0)$,

$$\operatorname{div} \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\ ,\ \mathbf{curl}\,\mathbf{u} = (0, 0, \operatorname{curl} \mathbf{u})\ ,\ \operatorname{curl} \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\cdot$$

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A grade-two fluid is a non-Newtonian fluid and it is considered an appropriate model for the motion of a water solution of polymers, *cf.* Dunn and Rajagopal [17]. The parameter ν is the viscosity and the parameter α is a constant normal stress modulus, both divided by the density. When $\alpha = 0$, the constitutive equation reduces to that of the Navier-Stokes equation, owing to the identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{curl}\, \mathbf{u} \times \mathbf{u} + \frac{1}{2}\nabla(|\mathbf{u}|^2)\,,$$

where $|\cdot|$ denotes the vector Euclidean norm. Thus, p is not the pressure. In the case $\alpha = 0$, the pressure is given by $p - \frac{1}{2}|\mathbf{u}|^2$ and for $\alpha \neq 0$, the formula is more complex. However, for simplicity we refer to p as the "pressure" in the sequel.

Interestingly, the equations for the time-dependent version of this model, with $\nu = 0$, are recovered and studied by Holm *et al.* in [24], [25], under the name of averaged-Euler equations, where α is an averaged length scale.

According to the work of Dunn and Fosdick [16], to be consistent with thermodynamics, a grade-two fluid must satisfy

$$\nu \geq 0 \quad \text{and} \quad \alpha \geq 0\,.$$

The same property is derived independently in [24] and [25] for the averaged-Euler equations. The reader can refer to [17] for a thorough discussion on the sign of α. However, for the sake of generality, we shall not restrict its sign, because it has no influence on the mathematics of the steady-state problem.

This problem is difficult, even in two dimensions, because its nonlinear term involves a third-order derivative, whereas its elliptic term is only a Laplace operator, and for this reason, its dominating behavior is hyperbolic. It has been studied extensively by several authors, but the best proof of existence for both the time-dependent and steady-state grade-two fluid model in two and three dimensions, due to Cioranescu and Ouazar, dates back to 1981, *cf.* Ouazar [36] and Cioranescu and Ouazar [14, 15]. These authors proved existence of solutions, with H^3 regularity in space, by looking for a velocity $\mathbf{u}$ such that $\mathbf{z} = \mathbf{curl}(\mathbf{u} - \alpha\,\Delta\,\mathbf{u})$ has L^2 regularity in space, introducing $\mathbf{z}$ as an auxiliary variable and discretizing the equations of motion (in variational form) by Galerkin's method in the basis of the eigenfunctions of the operator $\mathbf{curl}\,\mathbf{curl}(\mathbf{u} - \alpha\,\Delta\,\mathbf{u})$. This choice of basis is optimal because it allows one to prove existence and regularity of solutions with minimal restrictions on the data and the domain.

The difficulties encountered in solving grade-two fluid models theoretically are amplified when solving them numerically, because of the high order of derivatives involved. In particular, the method of Cioranescu and Ouazar does not extend easily to discretizations, and we have chosen here the next best variant, that was already introduced in [36] for numerical purposes. The idea is to split (0.1–0.3) into a coupled generalized Stokes problem satisfied by $\mathbf{u}$:

$$-\nu\,\Delta\,\mathbf{u} + \mathbf{z} \times \mathbf{u} + \nabla\, p = \mathbf{f} \quad \text{in } \Omega\,, \tag{0.4}$$

and a transport equation satisfied by z, where $\mathbf{z} = (0, 0, z)$:

$$\nu\, z + \alpha\,\mathbf{u} \cdot \nabla\, z = \nu\,\mathrm{curl}\,\mathbf{u} + \alpha\,\mathrm{curl}\,\mathbf{f}\,. \tag{0.5}$$

With this approach, Girault and Scott in [20] prove that (0.1–0.3) always has a solution $\mathbf{u}$ in $H^1(\Omega)^2$ and p in $L^2(\Omega)$, on a Lipschitz-continuous domain, without restriction on the size of the data, provided $\mathrm{curl}\,\mathbf{f}$ belongs to $L^2(\Omega)$. (In fact, this result holds if $\mathrm{curl}\,\mathbf{f}$ belongs to $L^r(\Omega)$ for some $r > 1$, *cf.* Rem. 4.4). The formulation (0.4), (0.5) has a major advantage: by discretizing it with appropriate schemes, all the numerical analysis can be performed without having to derive a uniform $W^{1,\infty}$ estimate for the discrete velocity. Thus, our choices of finite-element schemes are dictated by three requisites, that mimick the situation of the exact problem:

- without restrictions on the data, the schemes must have a discrete solution in any Lipschitz polygon,

• again without restrictions, each discrete solution must converge strongly to some solution of the exact problem, as the mesh is refined,
• under suitable conditions on the data and the angles of the polygon, the discrete solutions must satisfy error inequalities leading to error estimates.

As expected, the difficulty lies in the derivation of an error inequality from the transport equation (0.5). To obtain this inequality, we find that we need either discrete velocities with exactly zero divergence, or we must compensate for a non-zero divergence by a suitable compatibility condition, or a suitable modification of the transport term. In the first case, following Scott and Vogelius [39] and Girault and Scott [21], it suffices to work with triangular finite elements of degree at least four in each triangle and we propose the following scheme, for a suitable approximation $\mathbf{g}_h$ of $\mathbf{g}$: *Find* $\mathbf{u}_h$ *in* $V_h + \mathbf{g}_h$ *and* $\mathbf{z}_h = (0,0,z_h)$ *with* z_h *in* Z_h, *such that*

$$\forall \mathbf{v}_h \in V_h\,,\ \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{z}_h \times \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)\,, \tag{0.6}$$

$$\forall \theta_h \in Z_h\,,\ \ \nu\,(z_h,\theta_h) + \alpha\,(\mathbf{u}_h\cdot\nabla\, z_h,\theta_h) =\ \nu\,(\operatorname{curl}\mathbf{u}_h,\theta_h) + \alpha\,(\operatorname{curl}\mathbf{f},\theta_h)\,, \tag{0.7}$$

where V_h is a finite-element space of continuous, vector-valued functions with zero divergence and zero trace on $\partial\Omega$, and Z_h is a finite-element space of continuous functions. The pressure is computed separately later.

In the second case, the pressure is retained in the formulation; here is the scheme, for another suitable approximation $\mathbf{g}_h$ of $\mathbf{g}$: *Find* $\mathbf{u}_h$ *in* $X_h + \mathbf{g}_h$, p_h *in* M_h *and* $\mathbf{z}_h = (0,0,z_h)$ *with* z_h *in* Z_h, *such that*

$$\forall \mathbf{v}_h \in X_h\,,\ \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{z}_h \times \mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div}\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)\,, \tag{0.8}$$

$$\forall q_h \in M_h\,,\ (q_h, \operatorname{div}\mathbf{u}_h) = 0\,, \tag{0.9}$$

$$\begin{aligned}\forall \theta_h \in Z_h\,,\ \ &\nu\,(z_h,\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h) + \alpha\,(\mathbf{u}_h\cdot\nabla\, z_h,\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h)\\ &=\ \nu\,(\operatorname{curl}\mathbf{u}_h,\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h) + \alpha\,(\operatorname{curl}\mathbf{f},\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h)\,,\end{aligned} \tag{0.10}$$

where δ is an arbitrary parameter such that the product $\alpha\,\delta$ is non-negative and is chosen to improve stability and accuracy, X_h, M_h and Z_h are finite-element spaces of continuous functions, and the functions of X_h vanish on $\partial\Omega$. The fact that $\mathbf{u}_h$ does not have exactly zero divergence is compensated by a compatibility condition between the spaces M_h and Z_h. It is deduced from Green's formula

$$\int_\Omega (\mathbf{u}_h\cdot\nabla\, z_h)\theta_h \,\mathrm{d}\mathbf{x} = -\int_\Omega (\mathbf{u}_h\cdot\nabla\,\theta_h) z_h \,\mathrm{d}\mathbf{x} - \int_\Omega (\operatorname{div}\mathbf{u}_h) z_h\theta_h \,\mathrm{d}\mathbf{x}\,.$$

In view of (0.9), we eliminate the last integral by asking that the product $z_h\theta_h$ belong to M_h. The streamline diffusion method (0.10) can be combined with the method (0.6) in order to enhance accuracy. But using the method (0.8) with (0.7) appears problematic.

In the third case, a compatibility condition between the spaces M_h and Z_h is not necessary, but the schemes are more complex. We obtain a centered scheme by complementing (0.8) and (0.9) with

$$\forall \theta_h \in Z_h\,,\ \nu\,(z_h,\theta_h) + \alpha\,(\mathbf{u}_h\cdot\nabla\, z_h,\theta_h) + \frac{\alpha}{2}((\operatorname{div}\mathbf{u}_h)z_h,\theta_h) = \nu\,(\operatorname{curl}\mathbf{u}_h,\theta_h) + \alpha\,(\operatorname{curl}\mathbf{f},\theta_h)\,. \tag{0.11}$$

And we obtain an upwind scheme by replacing (0.10) by

$$\begin{aligned}\forall \theta_h \in Z_h\,,\ \ &\nu\,(z_h,\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h) + \alpha\,(\mathbf{u}_h\cdot\nabla\, z_h,\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h) + \frac{\alpha}{2}((\operatorname{div}\mathbf{u}_h)z_h,\theta_h)\\ &=\ \nu\,(\operatorname{curl}\mathbf{u}_h,\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h) + \alpha\,(\operatorname{curl}\mathbf{f},\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h)\,.\end{aligned} \tag{0.12}$$

Note that (0.11) and (0.12) are generalizations of (0.7) and (0.10) respectively, since the extra term $((\operatorname{div} \mathbf{u}_h)z_h, \theta_h)$ vanishes in the functional setting of (0.7) and (0.10).

The reader can refer to Girault and Scott [22] for a different upwinding of the transport equation by a discontinuous Galerkin method.

Of course, there are other possibilities; in the first case, for instance, we might use discrete divergence-free velocities with less regularity than H^1; this will be the object of a forthcoming work of Amara, Bernardi and Girault [2]. There is another example in [5], where Baia and Sequeira use a formulation that is close to that of an Oldroyd B model, but in order to guarantee the convergence of their scheme, they must start with a first guess that cannot be obtained without knowing precisely the exact solution.

After this introduction, this article is organized as follows. In Section 1, we briefly discuss the equivalence between problem (0.1–0.3) and the mixed formulation (0.2–0.5). Sections 2, 3 and 4 are devoted to the centered scheme (0.6), (0.7). The upwind scheme (0.8–0.10) is analyzed in Section 5. Finally, Section 6 gives a brief analysis of the schemes using (0.11) and (0.12).

In the sequel, we shall use the following notation. Our problem will be set in a domain whose boundary is Lipschitz-continuous (*cf.* Grisvard [23]), referred to as a Lipschitz-continuous domain; but the discrete problem itself will be stated in a Lipschitz polygon, *i.e.* a poygonal domain with no slits. We denote by $\mathcal{D}(\Omega)$ the space of functions that have compact support in Ω and are indefinitely differentiable in Ω. Let (k_1, k_2) denote a pair of non-negative integers, set $|k| = k_1 + k_2$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2}} \cdot$$

Then, for any non-negative integer m and number $r \geq 1$, recall the classical Sobolev space (*cf.* Adams [1] or Nečas [34])

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega)\,;\, \partial^k v \in L^r(\Omega)\ \forall |k| \leq m\}\,,$$

equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v|^r \,\mathrm{d}\mathbf{x}\right]^{1/r},$$

and norm (for which it is a Banach space)

$$\|v\|_{W^{m,r}(\Omega)} = \left[\sum_{0 \leq k \leq m} |v|^r_{W^{k,r}(\Omega)}\right]^{1/r},$$

with the usual extension when $r = \infty$. The reader can refer to [23] for extensions of this definition to non-integral values of m. When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{u} = (u_1, u_2)$; then we set

$$\|\mathbf{u}\|_{L^r(\Omega)} = \left[\int_\Omega |\mathbf{u}(\mathbf{x})|^r \,\mathrm{d}\mathbf{x}\right]^{1/r}, \tag{0.13}$$

where $|\cdot|$ denotes the Euclidean vector norm.

For vanishing boundary values, we define

$$H_0^1(\Omega) = \{v \in H^1(\Omega)\,;\, v|_{\partial\Omega} = 0\} \cdot$$

We shall often use Sobolev's imbeddings: for any real number $p \geq 1$, there exists a constant S_p such that

$$\forall v \in H_0^1(\Omega)\ ,\ \|v\|_{L^p(\Omega)} \leq S_p |v|_{H^1(\Omega)}\,. \tag{0.14}$$

When $p = 2$, this reduces to Poincaré's inequality and S_2 is Poincaré's constant. For tangential boundary values, we define

$$H_T^1(\Omega) = \{\mathbf{v} \in H^1(\Omega)^2\,;\, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\ \cdot \tag{0.15}$$

A straightforward application of Peetre-Tartar's Theorem (*cf.* Peetre [37] and Tartar [42] or Girault and Raviart [19]) shows that the analogue of Sobolev's imbedding holds in $H_T^1(\Omega)$ for any real number $p \geq 1$:

$$\forall \mathbf{v} \in H_T^1(\Omega)\ ,\ \|\mathbf{v}\|_{L^p(\Omega)} \leq \tilde{S}_p |\mathbf{v}|_{H^1(\Omega)}\,. \tag{0.16}$$

In particular, for $p = 2$, the mapping $\mathbf{v} \mapsto |\mathbf{v}|_{H^1(\Omega)}$ is a norm on $H_T^1(\Omega)$, equivalent to the H^1 norm and $\tilde{S}_2$ is the analogue of Poincaré's constant. We shall also use the standard spaces for Navier-Stokes equations

$$V = \{\mathbf{v} \in H_0^1(\Omega)^2\,;\, \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}\,, \tag{0.17}$$

$$W = \{\mathbf{v} \in H_T^1(\Omega)\,;\, \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}\,, \tag{0.18}$$

$$L_0^2(\Omega) = \{q \in L^2(\Omega)\,;\, \int_\Omega q\,\mathrm{d}\mathbf{x} = 0\}\,,$$

and also the space

$$H(\operatorname{curl},\Omega) = \{\mathbf{v} \in L^2(\Omega)^2\,;\, \operatorname{curl}\mathbf{v} \in L^2(\Omega)\}\,\cdot$$

Finally, let us recall some properties of the stream-functions of vectors in W. For this, we must describe more precisely the geometry of the boundary $\partial\Omega$ of Ω. We denote by γ_i, $0 \leq i \leq R$, the connected components of $\partial\Omega$, with the convention that γ_0 is the exterior boundary of Ω, *i.e.* the boundary of the unbounded connected-component of $\mathbb{R}^2 \setminus \Omega$. With any $\mathbf{v} \in W$, we associate its unique stream-function $\varphi \in H^2(\Omega)$ that vanishes on γ_0, and is constant on each γ_i, for $1 \leq i \leq R$ (*cf.* [19]):

$$\mathbf{v} = \mathbf{curl}\,\varphi = (\frac{\partial\varphi}{\partial x_2}, -\frac{\partial\varphi}{\partial x_1})\ \cdot \tag{0.19}$$

1. A mixed formulation

The assumptions on the data are: Ω is a bounded domain in $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\partial\Omega$, $\mathbf{f}$ is a given function in $H(\operatorname{curl},\Omega)$, $\mathbf{g}$ is a given tangential vector field in $H^{1/2}(\partial\Omega)^2$, and $\nu > 0$ and α are two given real constants. The spaces for the unknowns $(\mathbf{u}, p)$ are $\mathbf{u} \in W^\alpha$ and $p \in L_0^2(\Omega)$, where

$$W^\alpha = \{\mathbf{v} \in W\,;\, \alpha\operatorname{curl}\Delta\,\mathbf{v} \in L^2(\Omega)\}\,, \tag{1.1}$$

and W^α reduces to W, the space of solutions of the Navier-Stokes equations, when $\alpha = 0$. This is consistent with the fact that the solutions of (0.1–0.3) converge to solutions of the Navier-Stokes equations when α tends to zero (*cf.* [20]). Our first lemma, established in [20] shows that, in the above spaces, problem (0.1–0.3) has the following equivalent mixed formulation, that for simplicity we denote as *Problem P*.

- *Problem P*: *Find* $(\mathbf{u}, p, z)$ *in* $H_T^1(\Omega) \times L_0^2(\Omega) \times L^2(\Omega)$ *solution of the generalized Stokes problem (0.4):*

$$-\nu\,\Delta\,\mathbf{u} + \mathbf{z}\times\mathbf{u} + \nabla\,p = \mathbf{f} \quad \text{in } \Omega\,,$$

where $\mathbf{z} = (0, 0, z)$ *and* $\mathbf{z} \times \mathbf{u} = (-zu_2, zu_1)$*, with the incompressibility condition (0.2):*

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega\,,$$

and the boundary condition (0.3):

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \quad \text{with} \quad \mathbf{g} \cdot \mathbf{n} = 0\,,$$

and the transport equation (0.5):

$$\nu\, z + \alpha\, \mathbf{u} \cdot \nabla\, z = \nu \operatorname{curl} \mathbf{u} + \alpha \operatorname{curl} \mathbf{f}\,.$$

Lemma 1.1. *Problem (0.1–0.3) with* $(\mathbf{u}, p)$ *in* $W^\alpha \times L^2_0(\Omega)$ *is equivalent to Problem P, i.e. (0.2–0.5).*

It is proven in [20] that *Problem P* has always at least one solution. Recall a standard lifting $\mathbf{w}_\mathbf{g}$ in W of $\mathbf{g}$: it is the solution of the non-homogeneous Stokes problem:

$$-\Delta\, \mathbf{w}_\mathbf{g} + \nabla\, p_\mathbf{g} = \mathbf{0} \;\text{ and }\; \operatorname{div} \mathbf{w}_\mathbf{g} = 0 \;\text{ in } \Omega\,,\; \mathbf{w}_\mathbf{g} = \mathbf{g} \text{ on } \partial\Omega\,. \tag{1.2}$$

It satisfies the bound (*cf.* for instance [19], Th. I.5.1):

$$|\mathbf{w}_\mathbf{g}|_{H^1(\Omega)} \le T \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\,. \tag{1.3}$$

Theorem 1.2. *Let* Ω *be Lipschitz-continuous. For all* $\nu > 0$*, all real numbers* α*, all* $\mathbf{f} \in H(\operatorname{curl}, \Omega)$ *and* $\mathbf{g} \in H^{1/2}(\partial\Omega)^2$ *satisfying* $\mathbf{g} \cdot \mathbf{n} = 0$*, Problem P has at least one solution* $(\mathbf{u}, p, z)$*. All solutions of Problem P satisfy the following estimates:*

$$|\mathbf{u}|_{H^1(\Omega)} \le \frac{S_2}{\nu} \|\mathbf{f}\|_{L^2(\Omega)} + T \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)} \big(1 + \frac{S_4 \tilde{S}_4}{\nu} \|z\|_{L^2(\Omega)}\big)\,, \tag{1.4}$$

$$\|p\|_{L^2(\Omega)} \le \frac{1}{\beta} (S_2 \|\mathbf{f}\|_{L^2(\Omega)} + \nu T \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)} + S_4 \tilde{S}_4 |\mathbf{u}|_{H^1(\Omega)} \|z\|_{L^2(\Omega)}\big)\,, \tag{1.5}$$

$$\|z\|_{L^2(\Omega)} \le \sqrt{2} |\mathbf{u}|_{H^1(\Omega)} + \frac{|\alpha|}{\nu} \|\operatorname{curl} \mathbf{f}\|_{L^2(\Omega)}\,, \tag{1.6}$$

$$\|\alpha\, \mathbf{u} \cdot \nabla\, z\|_{L^2(\Omega)} \le \nu \sqrt{2}\, |\mathbf{u}|_{H^1(\Omega)} + |\alpha|\, \|\operatorname{curl} \mathbf{f}\|_{L^2(\Omega)}\,, \tag{1.7}$$

where $\beta > 0$ *is the isomorphism constant of the divergence operator (cf. [19] or Brenner and Scott [8]),* S_p *and* $\tilde{S}_p$ *are defined in (0.14) and (0.16) respectively and* T *is defined in (1.3).*

If in addition, Ω *is a Lipschitz polygon, then for any* $t > \frac{1}{2}$*,*

$$\forall \varepsilon > 0\,,\; |\mathbf{u}|_{H^1(\Omega)} \le \frac{S_2}{\nu} \|\mathbf{f}\|_{L^2(\Omega)} + \frac{C}{\varepsilon^t} \|\mathbf{g}\|^{1+t}_{H^{1/2}(\partial\Omega)} + \frac{\varepsilon}{\nu} \|z\|_{L^2(\Omega)}\,, \tag{1.8}$$

$$\|z\|_{L^2(\Omega)} \leq 2\frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)} + 2\frac{\sqrt{2}}{\nu}S_2\|\mathbf{f}\|_{L^2(\Omega)} + \frac{C}{\nu^t}\|\mathbf{g}\|^{1+t}_{H^{1/2}(\partial\Omega)}\,, \tag{1.9}$$

where C depends only on t and Ω.

If Ω is Lipschitz-continuous and $\mathbf{g} \in W^{1-1/\lambda,\lambda}(\partial\Omega)^2$, for some $\lambda > 2$, then there exists a constant C that depends only on λ and Ω, such that

$$\forall\varepsilon > 0\,,\ |\mathbf{u}|_{H^1(\Omega)} \leq \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + \frac{C}{\varepsilon^{1/2}}\|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)} + \frac{\varepsilon}{\nu}\|z\|_{L^2(\Omega)}\,, \tag{1.10}$$

$$\|z\|_{L^2(\Omega)} \leq 2\frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)} + 2\frac{\sqrt{2}}{\nu}S_2\|\mathbf{f}\|_{L^2(\Omega)} + (2\sqrt{2})^{3/2}\frac{C}{\nu^{1/2}}\|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)}\,. \tag{1.11}$$

The first part of this theorem is established in [20], Theorem 2.5. The second part sharpens Theorem 2.5 of this reference by applying the construction of [21], Section 9.

2. A centered finite-element discretization

From now on, we assume that the domain Ω has a polygonal Lipschitz-continuous boundary, so it can be entirely triangulated. For an arbitrary triangle K, we denote by h_K the diameter of K and by ρ_K the radius of the ball inscribed in K. Let $h > 0$ be a discretization parameter and let $\mathcal{T}_h$ be a family of triangulations of $\overline{\Omega}$, consisting of triangles with maximum mesh size

$$h := \max_{K\in\mathcal{T}_h} h_K\,,$$

that is *non-degenerate* (also called *regular*):

$$\max_{K\in\mathcal{T}_h}\frac{h_K}{\rho_K} \leq \sigma_0\,, \tag{2.1}$$

with the constant σ_0 independent of h (*cf.* Ciarlet [12]). As usual, the triangulation is such that any two triangles are either disjoint or share a vertex or a complete side.

Let us discretize *Problem P*: (0.2–0.5). To simplify the discussion, we shall first discretize it in a space V_h of divergence-free functions and leave the approximation of the pressure until the end of the section. We discretize z in the standard finite-element space $Z_h \subset H^1(\Omega)$:

$$Z_h = \{\theta \in \mathcal{C}^0(\overline{\Omega})\,;\, \forall K \in \mathcal{T}_h\,,\, \theta|_K \in \mathbb{P}_k\}\,, \tag{2.2}$$

for an integer $k \geq 1$, where $\mathbb{P}_k$ denotes the space of polynomials of degree less than or equal to k in two variables. Concerning the approximation properties of Z_h, there exists an approximation operator (*cf.* Clément [13], Scott and Zhang [40], Bernardi and Girault [6]) $R_h \in \mathcal{L}(W^{1,p}(\Omega); Z_h)$ for any number $p \geq 1$, such that, for $m = 0, 1$ and $0 \leq l \leq k$

$$\forall z \in W^{l+1,p}(\Omega)\ ,\ |R_h(z) - z|_{W^{m,p}(\Omega)} \leq C\,h^{l+1-m}|z|_{W^{l+1,p}(\Omega)}\,. \tag{2.3}$$

The space $V_h \subset V$ is constructed in Scott and Vogelius [39], but the approximation properties used here depend on [21]. Since we are in two dimensions, the zero divergence is achieved by discretizing the stream-function φ of $\mathbf{v}$ (*cf.* (0.19)) and as observed in Morgan and Scott [33], it is sufficient that the finite-element functions φ_h

be polynomials of degree at least five in each element. Therefore, it suffices that the finite-element functions of V_h have components of degree at least four in each element. Thus, for $r \geq 5$, we define

$$X_h = \{\mathbf{v} \in \mathcal{C}^0(\overline{\Omega})^2 \,;\, \forall K \in \mathcal{T}_h \,,\, \mathbf{v}|_K \in \mathbb{P}^2_{r-1}\}\,, \\ W_h = X_h \cap W \;,\; V_h = X_h \cap V\,. \tag{2.4}$$

Applying to φ the interpolation operator Π constructed in [21], we derive an approximation operator $P_h \in \mathcal{L}(W; W_h) \cap \mathcal{L}(V; V_h)$. In order to state its approximation properties, we shall need to distinguish between nonsingular and singular vertices of $\mathcal{T}_h$ (*cf.* [21, 33, 39]): a vertex of $\mathcal{T}_h$ is singular if all the edges of $\mathcal{T}_h$ meeting at this vertex fall on two straight lines. Otherwise, the vertex is nonsingular. In [21], the degrees of freedom at interior vertices are chosen so that if a nonsingular vertex becomes singular, as h tends to zero, the approximation properties of P_h are unaffected. In the case of a boundary vertex, this possible switching to nonsingularity is prevented by asking that, if three triangles meet at a nonconvex corner, then this vertex is always singular. With these assumptions, P_h satisfies the following approximation properties for any real number $p \geq 2$:

$$\forall \mathbf{v} \in W, \forall K \in \mathcal{T}_h \,,\; \|\mathbf{v} - P_h(\mathbf{v})\|_{L^p(K)} \leq C\, h_K^{2/p} |\mathbf{v}|_{H^1(S_K)}\,, \tag{2.5}$$

with a constant C independent of h_K, where S_K denotes a suitable macro-element surrounding K. When summed over all triangles $K \in \mathcal{T}_h$, this formula gives, with possibly different constants C, independent of h, for any real number $p \geq 2$:

$$\forall \mathbf{v} \in W \,,\; \|\mathbf{v} - P_h(\mathbf{v})\|_{L^p(\Omega)} \leq C\, h^{2/p} |\mathbf{v}|_{H^1(\Omega)}\,. \tag{2.6}$$

Similarly, when $\mathbf{v}$ belongs to $W \cap W^{s,p}(\Omega)^2$, for some real number $s \in [1, r]$ and number $p \geq 2$, we have, for $m = 0$ or 1,

$$|\mathbf{v} - P_h(\mathbf{v})|_{W^{m,p}(\Omega)} \leq C\, h^{s-m} |\mathbf{v}|_{W^{s,p}(\Omega)}\,. \tag{2.7}$$

Let $G_{h,T}$ denote the trace space of W_h and let $\mathbf{g}_h$ be the interpolation of $\mathbf{g}$ in $G_{h,T}$, constructed in [21]. It satisfies

$$\mathbf{g}_h = P_h(\mathbf{r})|_{\partial\Omega}\,, \tag{2.8}$$

for some lifting $\mathbf{r} \in W$ of $\mathbf{g}$. Note that, on one hand, $\mathbf{g}_h$ can be constructed from $\mathbf{g}$ intrinsically without knowing $\mathbf{r}$ and on the other hand, $\mathbf{g}_h$ does not depend on the choice of the particular lifting $\mathbf{r}$ because if $\tilde{\mathbf{r}}$ is another lifting of $\mathbf{g}$, then the fact that $\mathbf{r} - \tilde{\mathbf{r}}$ vanishes on the boundary implies that

$$P_h(\mathbf{r})|_{\partial\Omega} = P_h(\tilde{\mathbf{r}})|_{\partial\Omega}\,. \tag{2.9}$$

As written in the introduction, *Problem P* is discretized as follows: *Find* $\mathbf{u}_h$ *in* $V_h + \mathbf{g}_h$ *and* $\mathbf{z}_h = (0, 0, z_h)$ *with* z_h *in* Z_h, satisfying (0.6), (0.7):

$$\forall \mathbf{v}_h \in V_h \,, \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{z}_h \times \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \\ \forall \theta_h \in Z_h \,, \nu\,(z_h, \theta_h) + \alpha\, c(\mathbf{u}_h; z_h, \theta_h) = \nu\,(\operatorname{curl} \mathbf{u}_h, \theta_h) + \alpha\,(\operatorname{curl} \mathbf{f}, \theta_h)\,,$$

where by $V_h + \mathbf{g}_h$ we mean $V_h + \mathbf{w}_h$ for any extension $\mathbf{w}_h \in W_h$ of $\mathbf{g}_h$, and c denotes the trilinear form associated with a scalar advection term:

$$c(\mathbf{u}; z, \theta) = \sum_{i=1}^{2} \int_\Omega u_i \frac{\partial z}{\partial x_i} \theta \,\mathrm{d}\mathbf{x}\,. \tag{2.10}$$

It satisfies the important property, given by Green's formula

$$\forall \mathbf{u} \in W\,,\, \forall z \in H^1(\Omega)\ ,\ c(\mathbf{u}; z, z) = 0\,. \tag{2.11}$$

By means of the standard lifting $\mathbf{w_g} \in W$ defined by (1.2), it follows from (2.9) that by applying the trace theorem, (2.7) and (1.3), we obtain, with a constant C independent of h and $\mathbf{g}$,

$$\|\mathbf{g}_h\|_{H^{1/2}(\partial\Omega)} = \|P_h(\mathbf{w_g})\|_{H^{1/2}(\partial\Omega)} \le C\,|\mathbf{w_g}|_{H^1(\Omega)} \le C\,T\,\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\,. \tag{2.12}$$

2.1. **Existence of discrete solutions**

We shall use the discrete lifting $\mathbf{u}_{h,\mathbf{g}}$ constructed in [21] to prove existence of solutions when the quantities in (2.12) cannot be controlled by the first term of (0.6) because they are too large with respect to ν. It is a particular approximation of a variant of the classical Leray-Hopf lifting (*cf.* Leray [30], Hopf [26], Lions [31] or [19] for the proof). With respect to the Leray-Hopf lifting, it has the advantage that its gradient does not grow exponentially in a neighborhood of the boundary, an unrealistic behavior when it comes to numerical discretization.

For defining $\mathbf{u}_{h,\mathbf{g}}$, we need to distinguish the mesh size in a neighborhood of the boundary. We denote by Ω_ε the set

$$\Omega_\varepsilon = \{\mathbf{x} \in \Omega\,;\, d(\mathbf{x}) \le C_\Omega \varepsilon\}\,, \tag{2.13}$$

where d is the distance to the boundary, C_Ω is a suitable constant depending on Ω, and the parameter $\varepsilon > 0$ is small enough so that Ω_ε consists of mutually disjoint neighborhoods of the components γ_j. Then we denote by h_b the maximum diameter of the elements of $\mathcal{T}_h$ that intersect Ω_ε.

Theorem 2.1. *For any $\mathbf{g}_h \in G_{h,T}$, the trace space of W_h, and for any real number $\varepsilon > 0$, there exists a lifting of $\mathbf{g}_h$, $\mathbf{u}_{h,\mathbf{g}} \in V_h + \mathbf{g}_h$, such that, if $h_b < C_b\varepsilon$, for a constant $C_b > 0$ that depends only on Ω, then*

$$\|\mathbf{u}_{h,\mathbf{g}}\|_{L^s(\Omega)} \le C\varepsilon^{1/s-\delta}\|\mathbf{g}_h\|_{H^{1/2}(\partial\Omega)}\,,\ 1 \le s < \infty\,,\ 0 < \delta \le \frac{1}{s}\,, \tag{2.14}$$

$$|\mathbf{u}_{h,\mathbf{g}}|_{H^1(\Omega)} \le C\varepsilon^{-1/2-\delta}\|\mathbf{g}_h\|_{H^{1/2}(\partial\Omega)}\,,\ 0 < \delta \le \frac{1}{2}\,, \tag{2.15}$$

$$\forall \mathbf{v} \in H_0^1(\Omega)^2\,,\ 0 < \delta < 1\,,\ \big\||\mathbf{u}_{h,\mathbf{g}}|\,|\mathbf{v}|\big\|_{L^2(\Omega)} \le C\varepsilon^{1-\delta}\|\mathbf{g}_h\|_{H^{1/2}(\partial\Omega)}|\mathbf{v}|_{H^1(\Omega)}\,, \tag{2.16}$$

where the constants C depend on δ or on s and δ, but are independent of h, ε and $\mathbf{g}_h$. The norm expression for the vector functions in (2.16) is the Euclidean norm (cf. (0.13)).

As noted in [20], the form in (0.6) with fixed $\mathbf{z}$ in $L^2(\Omega)^3$ is both continuous and coercive as a bilinear form on $L^4(\Omega)^3$; in particular,

$$\forall \mathbf{v}_h \in X_h\,,\ (\mathbf{z}_h \times \mathbf{v}_h, \mathbf{v}_h) = 0\,. \tag{2.17}$$

Thus, since by construction $\mathbf{g}_h$ belongs to $G_{h,T}$, then for fixed $z_h \in Z_h$, problem (0.6) has a unique solution $\mathbf{u}_h = \mathbf{u}_h(z_h) \in V_h + \mathbf{g}_h$ and this solution satisfies the following *a priori* bounds.

Lemma 2.2. *For each $z_h \in Z_h$, (0.6) has a unique solution $\mathbf{u}_h \in V_h + \mathbf{g}_h$. This solution satisfies the estimate*

$$|\mathbf{u}_h|_{H^1(\Omega)} \le \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + \left(1 + \frac{S_4\tilde{S}_4}{\nu}\|z_h\|_{L^2(\Omega)}\right)C_1 T\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\,, \tag{2.18}$$

where C_1 is a constant independent of h, and T is the constant of (1.3). Moreover, there exists a constant $C_2 > 0$, independent of h, such that for all $\varepsilon > 0$, if for some $t > 1$,

$$h_b < C_2 \varepsilon^t \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}^{-t}, \tag{2.19}$$

then for any real number $s > \frac{t}{2}$,

$$|\mathbf{u}_h|_{H^1(\Omega)} \leq \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + \frac{C_3}{\varepsilon^s}\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}^{1+s} + \frac{\varepsilon}{\nu}\|z_h\|_{L^2(\Omega)}, \tag{2.20}$$

where the constant C_3 depends on s and t, but not on h, ν and ε.

Proof. The continuity and coercivity of the form in (0.6), implies that it has a unique solution $\mathbf{u}_h \in V_h + \mathbf{g}_h$. Similarly, let us define $\mathbf{w}_h \in V_h + \mathbf{g}_h$ by

$$\forall \mathbf{v}_h \in V_h, \ (\nabla \mathbf{w}_h, \nabla \mathbf{v}_h) = 0. \tag{2.21}$$

Clearly,

$$\forall \mathbf{v}_h \in V_h + \mathbf{g}_h, \ |\mathbf{w}_h|_{H^1(\Omega)} \leq |\mathbf{v}_h|_{H^1(\Omega)}.$$

Therefore, by choosing $\mathbf{v}_h = P_h(\mathbf{w}_\mathbf{g})$ defined by (1.2), we obtain

$$|\mathbf{w}_h|_{H^1(\Omega)} \leq |P_h(\mathbf{w}_\mathbf{g})|_{H^1(\Omega)} \leq C_1|\mathbf{w}_\mathbf{g}|_{H^1(\Omega)} \leq C_1 T\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}, \tag{2.22}$$

where C_1 is derived from (2.7). Then (2.18) follows easily by using $\mathbf{w}_h$ as lifting in (0.6).

To derive (2.20), we use the lifting $\mathbf{u}_{h,\mathbf{g}}$ of Theorem 2.1 with an arbitrary parameter $\mu > 0$. Assuming that $h_b < C_b\mu$ and applying (2.16) and (2.12), we obtain for any real number $t > 1$,

$$|\mathbf{u}_h|_{H^1(\Omega)} \leq \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + 2|\mathbf{u}_{h,\mathbf{g}}|_{H^1(\Omega)} + \frac{C'}{\nu}\mu^{1/t}\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\|z_h\|_{L^2(\Omega)}.$$

Then we recover (2.20) by setting

$$\varepsilon = \mu^{1/t} C' \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)},$$

and applying (2.15) and (2.12). □

The following theorem shows that this discretization of *Problem P* has at least one solution, with suitable restrictions on the size of the mesh.

Theorem 2.3. *The constant C_2 of (2.19) is such that for all $\nu > 0$ and $\alpha \in \mathbb{R}$, for all $\mathbf{f}$ in $H(curl, \Omega)$ and all $\mathbf{g}$ in $H^{1/2}(\partial\Omega)^2$ satisfying $\mathbf{g}\cdot\mathbf{n} = 0$, if h_b satisfies (2.19) with $\varepsilon = \frac{\nu}{2\sqrt{2}}$, i.e.*

$$h_b < C_2\Big(\frac{\nu}{2\sqrt{2}}\Big)^t \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}^{-t}, \ \textit{for some } t > 1, \tag{2.23}$$

then the discrete problem (0.6), (0.7) has at least one solution $\mathbf{u}_h \in V_h + \mathbf{g}_h$, $z_h \in Z_h$, and each solution satisfies the a priori estimate (2.18) and

$$\|z_h\|_{L^2(\Omega)} \leq \frac{2\sqrt{2}}{\nu}S_2\|\mathbf{f}\|_{L^2(\Omega)} + (2\sqrt{2})^{1+s}\frac{C_3}{\nu^s}\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}^{1+s} + 2\frac{|\alpha|}{\nu}\|\mathrm{curl}\,\mathbf{f}\|_{L^2(\Omega)}, \ \textit{for any } s > \frac{t}{2}, \tag{2.24}$$

where C_3 is the constant of (2.20).

Proof. It follows from Lemma 2.2 that problem (0.6), (0.7) is equivalent to: Find z_h in Z_h such that

$$\forall\theta_h \in Z_h\ ,\ \nu\,(z_h,\theta_h) + \alpha\, c(\mathbf{u}_h(z_h); z_h,\theta_h) = \nu\,(\operatorname{curl}\mathbf{u}_h(z_h),\theta_h) + \alpha\,(\operatorname{curl}\mathbf{f},\theta_h)\,, \tag{2.25}$$

where $\mathbf{u}_h(z_h) \in V_h + \mathbf{g}_h$ is the solution of (0.6). Let us solve (2.25) by Brouwer's Fixed Point Theorem. To this end, for fixed λ_h in Z_h, we define $H(\lambda_h)$ in Z_h by

$$\forall\mu_h \in Z_h\ ,\ (H(\lambda_h),\mu_h) = \nu\,(\lambda_h,\mu_h) + \alpha\, c(\mathbf{u}_h(\lambda_h);\lambda_h,\mu_h) - \nu\,(\operatorname{curl}\mathbf{u}_h(\lambda_h),\mu_h) - \alpha\,(\operatorname{curl}\mathbf{f},\mu_h)\,.$$

This finite-dimensional, square system of linear equations defines a continuous mapping $H : Z_h \to Z_h$. Moreover, the H^1 regularity of λ_h and the fact that $\mathbf{u}_h(\lambda_h)$ belongs to W, imply that, for all $\lambda_h \in Z_h$,

$$\begin{aligned}(H(\lambda_h),\lambda_h) &= \nu\,\|\lambda_h\|^2_{L^2(\Omega)} - \nu\,(\operatorname{curl}\mathbf{u}_h(\lambda_h),\lambda_h) - \alpha\,(\operatorname{curl}\mathbf{f},\lambda_h)\\ &\geq \nu\,\|\lambda_h\|^2_{L^2(\Omega)} - (\sqrt{2}\nu\,|\mathbf{u}_h(\lambda_h)|_{H^1(\Omega)} + |\alpha|\,\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)})\|\lambda_h\|_{L^2(\Omega)}\,.\end{aligned}$$

In view of Lemma 2.2, we apply (2.20) with $\varepsilon = \frac{\nu}{2\sqrt{2}}$: if h_b satisfies (2.23) then for all $\lambda_h \in Z_h$,

$$(H(\lambda_h),\lambda_h) \geq \frac{\nu}{2}\,\|\lambda_h\|^2_{L^2(\Omega)} - \big(\sqrt{2}S_2\|\mathbf{f}\|_{L^2(\Omega)} + |\alpha|\,\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)} + \sqrt{2}\nu\frac{C_3}{\varepsilon^s}\|\mathbf{g}\|^{1+s}_{H^{1/2}(\partial\Omega)}\big)\|\lambda_h\|_{L^2(\Omega)}\,.$$

Hence $(H(\lambda_h),\lambda_h) \geq 0$ for all λ_h in Z_h satisfying

$$\|\lambda_h\|_{L^2(\Omega)} = \frac{2\sqrt{2}}{\nu}S_2\|\mathbf{f}\|_{L^2(\Omega)} + 2\frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)} + 2\sqrt{2}\frac{C_3}{\varepsilon^s}\|\mathbf{g}\|^{1+s}_{H^{1/2}(\partial\Omega)}\,.$$

By Brouwer's Fixed Point Theorem this proves the existence of at least one solution z_h in Z_h of (2.25).

Finally, the imbedding of Z_h in $H^1(\Omega)$ implies that every solution of (0.6), (0.7) satisfies

$$\|z_h\|_{L^2(\Omega)} \leq \sqrt{2}|\mathbf{u}_h|_{H^1(\Omega)} + \frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)}\,. \tag{2.26}$$

Since $\mathbf{u}_h$ satisfies (2.20), the choice $\varepsilon = \frac{\nu}{2\sqrt{2}}$ yields (2.24). □

Remark 2.4. When $\mathbf{g}$ has a little more regularity (which will be the case for deriving error estimates), the statements of Lemma 2.2 and Theorem 2.3 simplify. Indeed, assume that there exists $\lambda > 2$ such that $\mathbf{g} \in W^{1-1/\lambda,\lambda}(\partial\Omega)^2$. Then we can prove that (2.19) and (2.20) are replaced by: there exists constants $C_2 > 0$ and C_3 such that for all $\varepsilon > 0$, if

$$h_b < C_2\varepsilon\|\mathbf{g}\|^{-1}_{W^{1-1/\lambda,\lambda}(\partial\Omega)}\,, \tag{2.27}$$

then

$$|\mathbf{u}_h|_{H^1(\Omega)} \leq \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + \frac{C_3}{\sqrt{\varepsilon}}\|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)} + \frac{\varepsilon}{\nu}\|z_h\|_{L^2(\Omega)}\,, \tag{2.28}$$

where C_2 and C_3 depend on λ, but not on h, ν and ε. Consequently, (2.23) and (2.24) are replaced by: if

$$h_b < C_2\frac{\nu}{2\sqrt{2}}\|\mathbf{g}\|^{-1}_{W^{1-1/\lambda,\lambda}(\partial\Omega)}\,, \tag{2.29}$$

then

$$\|z_h\|_{L^2(\Omega)} \leq \frac{2\sqrt{2}}{\nu} S_2 \|\mathbf{f}\|_{L^2(\Omega)} + 2\frac{|\alpha|}{\nu} \|\mathrm{curl}\, \mathbf{f}\|_{L^2(\Omega)} + \frac{C_3}{\sqrt{\nu}} (2\sqrt{2})^{3/2} \|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)}, \tag{2.30}$$

where C_2 and C_3 are the constants of (2.27) and (2.28). □

2.2. **Convergence**

Proposition 2.5. *Let $(\mathbf{u}_h, z_h) \in (V_h + \mathbf{g}_h) \times Z_h$ be any solution of the discrete problem (0.6), (0.7). We can extract a subsequence, still denoted by $(\mathbf{u}_h, z_h)$, such that*

$$\lim_{h\to 0} \mathbf{u}_h = \mathbf{u} \quad \textit{weakly in } W\,,$$
$$\lim_{h\to 0} z_h = z \quad \textit{weakly in } L^2(\Omega)\,,$$

where $(\mathbf{u}, z)$ is a solution of Problem P.

Proof. The uniform bounds (2.24) and (2.18) allow us to pass to the limit as (a subsequence of) h tends to zero and therefore there exist $\mathbf{u}$ in $H^1(\Omega)^2$ and z in $L^2(\Omega)$ such that $\mathbf{u}_h$ tends to $\mathbf{u}$ weakly in $H^1(\Omega)^2$ and z_h tends to z weakly in $L^2(\Omega)$. Clearly, $\mathrm{div}\, \mathbf{u} = 0$, and since, by a density argument and (2.7), $P_h(\mathbf{r})$ tends to $\mathbf{r}$ in $H^1(\Omega)^2$, we have $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$. In addition,

$$\lim_{h\to 0} \mathbf{u}_h = \mathbf{u} \;\text{ strongly in } L^4(\Omega)^2\,.$$

Let us prove that there exists p such that $(\mathbf{u}, p, z)$ is a solution of *Problem P*. To pass to the limit in (0.6), let $\mathbf{v}$ be any function in V and take $\mathbf{v}_h = P_h(\mathbf{v})$. Then $\mathbf{v}_h$ belongs to V_h and a density argument together with (2.7) implies that

$$\lim_{h\to 0} \mathbf{v}_h = \mathbf{v} \;\text{ strongly in } H^1(\Omega)^2 \;\text{ and in } L^4(\Omega)^2\,.$$

The above convergences allow us to pass to the limit in (0.6) and we obtain

$$\forall \mathbf{v} \in V\,,\; \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (\mathbf{z} \times \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})\,.$$

In turn, this implies that there exists a unique function p in $L^2_0(\Omega)$ such that

$$-\nu\Delta\, \mathbf{u} + \mathbf{z} \times \mathbf{u} + \nabla p = \mathbf{f} \;\text{ a.e. in } \Omega\,.$$

To pass to the limit in (0.7), let θ be any function in $W^{1,4}(\Omega)$ and take $\theta_h = R_h(\theta)$. Using again a density argument and (2.3), we find

$$\lim_{h\to 0} \theta_h = \theta \text{ strongly in } W^{1,4}(\Omega)\,.$$

As $\mathbf{u}_h$ belongs to W, and all functions here are sufficiently smooth, we can apply Green's formula:

$$c(\mathbf{u}_h; z_h, \theta_h) = -c(\mathbf{u}_h; \theta_h, z_h)\,,$$

and the strong convergence of $\mathbf{u}_h$ and $\nabla \theta_h$ in $L^4(\Omega)^2$ imply that

$$\lim_{h\to 0} c(\mathbf{u}_h; z_h, \theta_h) = -c(\mathbf{u}; \theta, z)\,.$$

Hence, for all θ in $W^{1,4}(\Omega)$, we obtain

$$\nu\,(z,\theta) - \alpha\, c(\mathbf{u};\theta,z) = \nu\,(\operatorname{curl}\mathbf{u},\theta) + \alpha\,(\operatorname{curl}\mathbf{f},\theta)\,,$$

and in the sense of distributions, this gives (0.5). □

In order to prove strong convergence, we need some sharp results on the transport equation (0.5), established in [20]. This equation is a particular case of: Find $z \in L^2(\Omega)$, such that

$$z + \mathcal{W}\,\mathbf{u}\cdot\nabla\, z = \ell \quad \text{ in } \Omega\,, \tag{2.31}$$

with ℓ given in $L^2(\Omega)$, $\mathcal{W}$ a given real parameter and $\mathbf{u}$ given in W. Strictly speaking, we should write $z = z(\mathbf{u})$, but for simplicity, we write only z. For fixed $\mathbf{u}$ in W, z belongs to the space

$$X_{\mathbf{u}} = \{z \in L^2(\Omega)\,;\, \mathbf{u}\cdot\nabla\, z \in L^2(\Omega)\}\,, \tag{2.32}$$

which is a Hilbert space equipped with the norm

$$\|z\|_{\mathbf{u}} = \Big(\|z\|^2_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla\, z\|^2_{L^2(\Omega)}\Big)^{1/2}\,. \tag{2.33}$$

While it is easy to construct a solution of (2.31), proving uniqueness of this solution is difficult, because of the low regularity of $\mathbf{u}$ and $\partial\Omega$. The following theorem and its corollaries, valid in any dimension $n \geq 2$, proven in [20], summarize the basic results we need on the transport equation.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ be Lipschitz-continuous. For all $\mathbf{u}$ in W, all ℓ in $L^2(\Omega)$ and all real numbers $\mathcal{W}$, the transport equation (2.31) has one and only one solution z in $X_{\mathbf{u}}$. It satisfies*

$$\|z\|_{L^2(\Omega)} \leq \|\ell\|_{L^2(\Omega)}\,. \tag{2.34}$$

Corollary 2.7. *Let $\Omega \subset \mathbb{R}^n$ be Lipschitz-continuous and let $\mathbf{u}$ be given in W. Then (2.11) extends to all z in $X_{\mathbf{u}}$:*

$$\forall z \in X_{\mathbf{u}}\ ,\ c(\mathbf{u};z,z) = 0\,. \tag{2.35}$$

As usual, (2.35) implies the anti-symmetry of c:

$$\forall \mathbf{u} \in W\,,\ \forall z,\theta \in X_{\mathbf{u}}\ ,\ c(\mathbf{u};z,\theta) = -c(\mathbf{u};\theta,z)\,. \tag{2.36}$$

Corollary 2.8. *Under the assumptions of Corollary 2.7, any ℓ in $L^2(\Omega)$ has the orthogonal decomposition:*

$$\ell = z + \mathbf{u}\cdot\nabla\, z \quad \text{in } \Omega\,,$$

where z belongs to $X_{\mathbf{u}}$, and

$$\|z\|^2_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla\, z\|^2_{L^2(\Omega)} = \|\ell\|^2_{L^2(\Omega)}\,. \tag{2.37}$$

Corollary 2.9. *Under the assumptions of Corollary 2.7, the space $\mathcal{D}(\Omega)$ is dense in $X_{\mathbf{u}}$.*

Now we can prove strong convergence of the discrete solution.

Theorem 2.10. *The subsequence of solutions* $(\mathbf{u}_h, z_h)$ *constructed in Proposition 2.5 converges strongly:*

$$\lim_{h\to 0} \mathbf{u}_h = \mathbf{u} \quad \textit{strongly in } W\,,$$
$$\lim_{h\to 0} z_h = z \quad \textit{strongly in } L^2(\Omega)\,.$$

Proof. Taking the difference between (0.6) and (0.4) and inserting $P_h(\mathbf{u})$, we obtain for all test functions $\mathbf{v}_h$ in V_h:

$$\nu(\nabla(\mathbf{u}_h - P_h(\mathbf{u})), \nabla\,\mathbf{v}_h) + (\mathbf{z}_h \times (\mathbf{u}_h - P_h(\mathbf{u})), \mathbf{v}_h) + ((\mathbf{z}_h - \mathbf{z}) \times P_h(\mathbf{u}), \mathbf{v}_h)$$
$$= \nu(\nabla(\mathbf{u} - P_h(\mathbf{u})), \nabla\,\mathbf{v}_h) + (\mathbf{z} \times (\mathbf{u} - P_h(\mathbf{u})), \mathbf{v}_h)\,.$$

By construction, $\mathbf{u}_h - P_h(\mathbf{u})$ vanishes on the boundary and therefore we can choose $\mathbf{v}_h = \mathbf{u}_h - P_h(\mathbf{u})$; this gives, using (2.17),

$$\nu\,|\mathbf{u}_h - P_h(\mathbf{u})|^2_{H^1(\Omega)} + ((\mathbf{z}_h - \mathbf{z}) \times P_h(\mathbf{u}), \mathbf{u}_h - P_h(\mathbf{u}))$$
$$= \nu(\nabla(\mathbf{u} - P_h(\mathbf{u})), \nabla(\mathbf{u}_h - P_h(\mathbf{u}))) + (\mathbf{z} \times (\mathbf{u} - P_h(\mathbf{u})), \mathbf{u}_h - P_h(\mathbf{u}))\,.$$

The convergences established by Proposition 2.5 and the properties (2.6) and (2.7) of P_h show that the last three terms in this equality tend to zero as h tends to zero. Therefore

$$\lim_{h\to 0} |\mathbf{u}_h - P_h(\mathbf{u})|^2_{H^1(\Omega)} = 0\,,$$

which implies the strong convergence of $\mathbf{u}_h$ to $\mathbf{u}$ in $H^1(\Omega)^2$.

To establish the strong convergence of z_h, we write

$$\|z_h - z\|^2_{L^2(\Omega)} = (z_h - z, z_h) - (z_h - z, z)\,,$$

and it suffices to study the first term. Taking the difference between (0.7) and (0.5), we obtain for all θ_h in Z_h:

$$\nu(z_h - z, \theta_h) + \alpha c(\mathbf{u}_h; z_h, \theta_h) - \alpha c(\mathbf{u}; z, \theta_h) = \nu(\mathrm{curl}(\mathbf{u}_h - \mathbf{u}), \theta_h)\,.$$

Applying (2.11), the choice $\theta_h = z_h$ gives:

$$(z_h - z, z_h) = \frac{\alpha}{\nu} c(\mathbf{u}; z, z_h) + (\mathrm{curl}(\mathbf{u}_h - \mathbf{u}), z_h)\,.$$

On one hand, the fact that z belongs to $X_{\mathbf{u}}$, the weak convergence of z_h, and Corollary 2.7 imply that

$$\lim_{h\to 0} c(\mathbf{u}; z, z_h) = c(\mathbf{u}; z, z) = 0\,.$$

On the other hand, the strong convergence of $\mathbf{u}_h$ in $H^1(\Omega)^2$ and the weak convergence of z_h imply that

$$\lim_{h\to 0} (\mathrm{curl}(\mathbf{u}_h - \mathbf{u}), z_h) = 0\,.$$

Hence

$$\lim_{h\to 0} (z_h - z, z_h) = 0\,,$$

thus proving the strong convergence of z_h. □

2.3. Approximation of the pressure

The discrete space for the pressure is determined by the range space of the divergence of the functions of $X_h \cap H_0^1(\Omega)^2$. This range space is thoroughly studied by Scott and Vogelius in [39]. Here is a short discussion of their results. For $r \geq 5$, let

$$Q_h = \{q \in L_0^2(\Omega)\,;\, \forall K \in \mathcal{T}_h,\ q|_K \in \mathbb{P}_{r-2}\}\,,$$

and let

$$M_h = \{q \in Q_h\,;\ \text{for any singular vertex } \mathbf{a},\ \sum_{i=1}^{k}(-1)^k q_i(\mathbf{a}) = 0\}\,, \tag{2.38}$$

where k is the number of triangles K_i of $\mathcal{T}_h$ meeting at $\mathbf{a}$, numbered from 1 to k, and q_i denotes $q|_{K_i}$. With this restriction, we have the following result.

Proposition 2.11.

$$M_h = \mathrm{div}(X_h \cap H_0^1(\Omega)^2)\,.$$

On one hand, the condition (2.38) on M_h is too restrictive for approximation, when either one or three triangles meet at a boundary singular vertex, because it does not allow one to approximate arbitrary continuous functions. Therefore, we must control the triangulation by asking that exactly two triangles meet at any boundary singular vertex. But since we have already imposed that a nonconvex boundary corner where three triangles meet is necessarily singular, then this new assumption forbids that three triangles meet at a nonconvex boundary corner. Then, as $r-2 \geq 3$, we can approximate the functions of $L_0^2(\Omega)$ by a Clément-type interpolator r_h similar to R_h (*cf.* [13], [6]), $r_h \in \mathcal{L}(L_0^2(\Omega); M_h \cap \mathcal{C}^0(\overline{\Omega}))$, such that for $0 \leq l \leq r-1$,

$$\forall q \in H^l(\Omega) \cap L_0^2(\Omega)\,,\ \|r_h(q) - q\|_{L^2(\Omega)} \leq C\, h^l |q|_{H^l(\Omega)}\,. \tag{2.39}$$

On the other hand, Proposition 2.11 is not sufficient to guarantee that M_h and $X_h \cap H_0^1(\Omega)^2$ satisfy a uniform discrete inf-sup condition. In particular, we must impose that the nonsingular vertices of $\mathcal{T}_h$ do not tend to singular vertices as h tends to zero. To this end, we introduce the quantities

$$D(\mathbf{a}) = \max\{|\theta_i + \theta_{i+1} - \pi|\,;\, 1 \leq i \leq k\}\,,$$

$$D(\mathcal{T}_h) = \min\{D(\mathbf{a})\,;\, \mathbf{a} \text{ is a nonsingular vertex of } \mathcal{T}_h\}\,, \tag{2.40}$$

where θ_i denotes the angle of the triangle K_i at the vertex $\mathbf{a}$, numbered modulo k. Note that $D(\mathbf{a})$ measures how close $\mathbf{a}$ is to being a singular vertex.

The uniform inf-sup condition below is established in [39] under the assumption that $\mathcal{T}_h$ is a uniformly regular family of triangulations: there exists a constant $\tau > 0$, independent of h, such that

$$\forall K \in \mathcal{T}_h\,,\ \tau\, h \leq h_K \leq \sigma_0 \rho_K\,. \tag{2.41}$$

Note that (2.1) is the second part of (2.41).

Theorem 2.12. *Let $\mathcal{T}_h$ satisfy (2.41) and assume that there exists a constant $\delta^* > 0$, independent of h, such that*

$$D(\mathcal{T}_h) \geq \delta^*\,. \tag{2.42}$$

Then the pair of spaces M_h and $X_h \cap H_0^1(\Omega)^2$, defined respectively by (2.38) and by (2.4), satisfies a uniform discrete inf-sup condition: there exists a constant $\beta^\star > 0$, independent of h, such that for all $q_h \in M_h$, there exists a unique $\mathbf{v}_h \in X_h \cap H_0^1(\Omega)^2$ satisfying

$$\begin{aligned} \forall \mathbf{w}_h \in V_h\,,\ (\nabla \mathbf{v}_h, \nabla \mathbf{w}_h) &= 0\,,\\ \operatorname{div} \mathbf{v}_h &= q_h\,,\\ |\mathbf{v}_h|_{H^1(\Omega)} &\leq \frac{1}{\beta^\star}\|q_h\|_{L^2(\Omega)}\,. \end{aligned} \tag{2.43}$$

This theorem has the following consequence. We skip the proof because it is straightforward.

Proposition 2.13. *Suppose that $\mathcal{T}_h$ satisfies (2.41) and (2.42). For each solution $(\mathbf{u}_h, z_h) \in (V_h + \mathbf{g}_h) \times Z_h$ of problem (0.6), (0.7), there exists a unique pressure $p_h \in M_h$ satisfying the equation*

$$\forall \mathbf{v}_h \in X_h \cap H_0^1(\Omega)^2\,,\ \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{z}_h \times \mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)\,. \tag{2.44}$$

Moreover, p_h has the following bound:

$$\|p_h\|_{L^2(\Omega)} \leq \frac{1}{\beta^*}(S_2\|\mathbf{f}\|_{L^2(\Omega)} + \nu C_1\, T\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)} + S_4\tilde{S}_4|\mathbf{u}_h|_{H^1(\Omega)}\|z_h\|_{L^2(\Omega)})\,, \tag{2.45}$$

where C_1 and T are the constants of (2.22).

Now, we can complete the statement of Theorem 2.10.

Theorem 2.14. *In addition to the assumptions of Proposition 2.13, suppose that exactly two triangles meet at any boundary singular vertex and three triangles are forbidden to meet at any nonconvex boundary corner of $\mathcal{T}_h$. Let $(\mathbf{u}_h, p_h, z_h)$ be a solution of (2.44), (0.7). Then we can extract a subsequence, denoted by $(\mathbf{u}_h, p_h, z_h)$, that converges strongly to a solution of Problem P.*

Proof. Subtracting (0.4) multiplied by a test function $\mathbf{v}_h \in X_h \cap H_0^1(\Omega)^2$ from (2.44) and inserting $r_h(p)$, we obtain

$$(p_h - r_h(p), \operatorname{div} \mathbf{v}_h) = (p - r_h(p), \operatorname{div} \mathbf{v}_h) + \nu(\nabla(\mathbf{u}_h - \mathbf{u}), \nabla \mathbf{v}_h) + ((\mathbf{z}_h - \mathbf{z}) \times \mathbf{u}_h, \mathbf{v}_h) + (\mathbf{z} \times (\mathbf{u}_h - \mathbf{u}), \mathbf{v}_h)\,.$$

Let us choose the function $\mathbf{v}_h$ associated by Theorem 2.12 with the function $q_h = p_h - r_h(p)$; this gives

$$\begin{aligned} \|p_h - r_h(p)\|_{L^2(\Omega)}^2 = (p - r_h(p), p_h - r_h(p)) + \nu(\nabla(P_h(\mathbf{u}) - \mathbf{u}), \nabla \mathbf{v}_h)\\ + ((\mathbf{z}_h - \mathbf{z}) \times \mathbf{u}_h, \mathbf{v}_h) + (\mathbf{z} \times (\mathbf{u}_h - \mathbf{u}), \mathbf{v}_h)\,. \end{aligned} \tag{2.46}$$

The desired convergence follows by taking the limit of the right-hand side of this equation and using the convergences of $\mathbf{u}_h$ and z_h, established by Theorem 2.10, (2.39), (2.43) and (2.45). □

3. Error estimates for the centered scheme

Throughout this section, we assume that Ω is a Lipschitz polygon, $\mathcal{T}_h$ satisfies (2.1), and $(\mathbf{u}_h, z_h) \in (V_h + \mathbf{g}_h) \times Z_h$ is a solution of (0.6), (0.7). To simplify the discussion, we forbid that three triangles meet at a nonconvex boundary corner. For deriving error estimates, it will be useful to have a uniform bound for $\mathbf{u}_h - \mathbf{u}$ in $L^\infty(\Omega)^2$ in terms of $z_h - z$ and $P_h(\mathbf{u}) - \mathbf{u}$. This is the object of the next subsection, where we shall derive estimates for $\mathbf{u}_h - \mathbf{u}$ in $W^{1,p}(\Omega)^2$.

3.1. Further estimates for the discrete velocity

Let us associate with z_h the solution $(\mathbf{w}(z_h), q(z_h))$ of the generalized Stokes problem:

$$-\nu\,\Delta\,\mathbf{w}(z_h) + \mathbf{z}_h \times \mathbf{w}(z_h) + \nabla\, q(z_h) = \mathbf{f} \quad \text{ in } \Omega\,, \tag{3.1}$$

$$\operatorname{div} \mathbf{w}(z_h) = 0 \quad \text{ in } \Omega\,, \tag{3.2}$$

$$\mathbf{w}(z_h) = \mathbf{g} \ \text{ on } \partial\Omega \quad \text{ with } \quad \mathbf{g}\cdot\mathbf{n} = 0\,. \tag{3.3}$$

If there is no ambiguity, we simply denote it by $(\mathbf{w}, q)$. Appropriate bounds for $\mathbf{u}_h - \mathbf{u}$ can be derived when $(\mathbf{w}, q)$ has sufficient regularity. Since $\mathbf{f} \in H(\text{curl}, \Omega)$ is already sufficiently smooth, all we need to do is impose higher regularity than $H^{1/2}$ to the boundary data $\mathbf{g}$. We introduce the following notation. For each connected component γ_j, $0 \le j \le R$, of $\partial\Omega$, we denote by Γ_i, for $1 \le i \le N$, the straight line segments of γ_j, with the convention that Γ_i is adjacent to Γ_{i+1} and Γ_{N+1} coincides with Γ_1. Also, we denote by $\mathbf{n}_i$ the unit normal to Γ_i pointing outside Ω, by $\mathbf{t}_i$ the unit tangent vector along Γ_i pointing in the clockwise direction, by $\mathbf{x}_i$ the common vertex of Γ_i and Γ_{i+1} and by ω_i the inner angle between them. Strictly speaking, we should use the notation Γ_i^j and N_j to specify the dependence on j, but we drop it to alleviate notation. The next two theorems are proven in [20].

Theorem 3.1. *Assume that all the inner angles of $\partial\Omega$ satisfy $0 < \omega_i < 2\pi$. If the boundary data $\mathbf{g} \in H^{1/2}(\partial\Omega)^2$ satisfies on each γ_j, $0 \le j \le R$,*

$$\mathbf{g} \in W^{5/4,4/3}(\Gamma_i)^2 \ \ \textit{for } 1 \le i \le N \quad , \quad \mathbf{g}\cdot\mathbf{n} = 0\,, \tag{3.4}$$

then the solution of problem (3.1–3.3) satisfies

$$\mathbf{w} \in W^{2,4/3}(\Omega)^2 \quad , \quad q \in W^{1,4/3}(\Omega)\,,$$

with continuous dependence on the data

$$\begin{aligned} \|\mathbf{w}\|_{W^{2,4/3}(\Omega)} + \|q\|_{W^{1,4/3}(\Omega)} \le B_1\big(&\|\mathbf{f}\|_{L^2(\Omega)} + [\mathbf{g}]_{W^{5/4,4/3}(\partial\Omega)} \\ &+ B_2(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)})\|z_h\|_{L^2(\Omega)} + B_3\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\|z_h\|^2_{L^2(\Omega)}\big)\,, \end{aligned} \tag{3.5}$$

where

$$[\mathbf{g}]_{W^{5/4,4/3}(\partial\Omega)} = \sum_{j=0}^{R} [\mathbf{g}]_{W^{5/4,4/3}(\gamma_j)}\ ,\ [\mathbf{g}]_{W^{5/4,4/3}(\gamma_j)} = \sum_{i=1}^{N} \|\mathbf{g}\|_{W^{5/4,4/3}(\Gamma_i)}\,.$$

Theorem 3.2. *We retain the hypotheses of Theorem 3.1 and in addition, we suppose Ω is a convex polygon and the boundary data $\mathbf{g} \in H^{1/2}(\partial\Omega)^2$ satisfies on each γ_j, $0 \le j \le R$,*

$$\mathbf{g} \in H^{3/2}(\Gamma_i)^2 \ \ \textit{for } 1 \le i \le N \quad , \quad \mathbf{g}\cdot\mathbf{n} = 0\,, \tag{3.6}$$

$$\int_0^{\varepsilon} \frac{1}{s} \left| \frac{\partial \mathbf{g}_{i+1}\cdot\mathbf{n}_i}{\partial \mathbf{t}_{i+1}}(\mathbf{x}_i + s\mathbf{t}_{i+1}) - \frac{\partial \mathbf{g}_i\cdot\mathbf{n}_{i+1}}{\partial \mathbf{t}_i}(\mathbf{x}_i - s\mathbf{t}_i) \right|^2 \mathrm{d}s < \infty\,, \tag{3.7}$$

where $\varepsilon = \min_{1\le i\le N}|\Gamma_i|$. Then the solution of problem (3.1–3.3) satisfies

$$\mathbf{w} \in H^2(\Omega)^2 \quad , \quad q \in H^1(\Omega)\,,$$

with continuous dependence on the data

$$\begin{aligned}\|\mathbf{w}\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)} \le B_1\big(\|\mathbf{f}\|_{L^2(\Omega)} + [\mathbf{g}]_{H^{3/2}(\partial\Omega)} + B_2(\|\mathbf{f}\|_{L^2(\Omega)} + [\mathbf{g}]_{W^{5/4,4/3}(\partial\Omega)})\|z_h\|_{L^2(\Omega)} \\ + B_3(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)})\|z_h\|^2_{L^2(\Omega)} + B_4\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\|z_h\|^3_{L^2(\Omega)}\big)\,,\end{aligned} \tag{3.8}$$

where

$$[\mathbf{g}]_{H^{3/2}(\partial\Omega)} = \sum_{j=0}^{R}[\mathbf{g}]_{H^{3/2}(\gamma_j)}\,,$$

$$[\mathbf{g}]^2_{H^{3/2}(\gamma_j)} = \sum_{i=1}^{N}\|\mathbf{g}\|^2_{H^{3/2}(\Gamma_i)} + \sum_{i=1}^{N}\int_0^{\varepsilon}\frac{1}{s}\left|\frac{\partial\mathbf{g}_{i+1}\cdot\mathbf{n}_i}{\partial\mathbf{t}_{i+1}}(\mathbf{x}_i + s\mathbf{t}_{i+1}) - \frac{\partial\mathbf{g}_i\cdot\mathbf{n}_{i+1}}{\partial\mathbf{t}_i}(\mathbf{x}_i - s\mathbf{t}_i)\right|^2 \mathrm{d}s\,.$$

Clearly, the system (0.6), (0.7) can only yield directly an upper bound for $\mathbf{u}_h - \mathbf{u}$ in $H^1(\Omega)^2$ in terms of $z_h - z$. But $\mathbf{u}_h$ is an approximation of $\mathbf{w}$, and considering that $\mathbf{w}$ satisfies the estimates (3.5) and (3.8) for correspondingly smooth $\mathbf{g}$, we may hope to obtain $W^{1,p}$ estimates for $\mathbf{u}_h - \mathbf{u}$ in terms of $z_h - z$ by exploiting more closely the relationship between $\mathbf{u}_h$ and $\mathbf{w}$. To simplify the formulas, we introduce the following notation:

$$K_1(h) = 1 + \frac{S_4\tilde{S}_4}{\nu}\|z_h\|_{L^2(\Omega)}\,, \tag{3.9}$$

and note that by virtue of (2.24), $K_1(h)$ is bounded independently of h.

Lemma 3.3. *Assume that* $\mathbf{g}$ *satisfies (3.4). There exists a constant* $C > 0$*, independent of* h*, such that*

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{H^1(\Omega)} \le C\,h^{1/2}K_1(h)|\mathbf{w}|_{H^{3/2}(\Omega)}\,. \tag{3.10}$$

If, in addition, Ω *is convex and* $\mathbf{g}$ *satisfies (3.6) and (3.7), then there exists another constant* $C > 0$ *independent of* h*, such that*

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{H^1(\Omega)} \le C\,hK_1(h)|\mathbf{w}|_{H^2(\Omega)}\,. \tag{3.11}$$

Proof. As in the proof of Theorem 2.10, we derive from (0.6) and (3.1), for all $\mathbf{v}_h \in V_h$,

$$\begin{aligned}&\nu(\nabla(\mathbf{u}_h - P_h(\mathbf{w})),\nabla\,\mathbf{v}_h) + (\mathbf{z}_h \times (\mathbf{u}_h - P_h(\mathbf{w})), \mathbf{v}_h) \\ &\qquad = \nu(\nabla(\mathbf{w} - P_h(\mathbf{w})), \nabla\,\mathbf{v}_h) + (\mathbf{z}_h \times (\mathbf{w} - P_h(\mathbf{w})), \mathbf{v}_h)\,.\end{aligned}$$

Then, choosing $\mathbf{v}_h = \mathbf{u}_h - P_h(\mathbf{w}) \in V_h$, we obtain

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{H^1(\Omega)} \le \left(1 + \frac{S_4\tilde{S}_4}{\nu}\|z_h\|_{L^2(\Omega)}\right)|\mathbf{w} - P_h(\mathbf{w})|_{H^1(\Omega)}\,, \tag{3.12}$$

and (3.10) follows from Theorem 3.1, the imbedding of $W^{2,4/3}(\Omega)$ into $H^{3/2}(\Omega)$, and (2.7) with $s = 3/2$ and $m = 1$.

Similarly, we derive (3.11) from (3.12) by applying Theorem 3.2 and (2.7) with $s = 2$ and $m = 1$. □

Remark 3.4. The statement of Lemma 3.3 does not involve q because $V_h \subset V$. In a more general context where V_h is not contained in V, but $\operatorname{div} \mathbf{v}_h$ satisfies a suitable orthogonality condition, such as in Section 5, (3.12) is replaced by

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{H^1(\Omega)} \le K_1(h)|\mathbf{w} - P_h(\mathbf{w})|_{H^1(\Omega)} + \frac{\sqrt{2}}{\nu}\|q - q_h\|_{L^2(\Omega)}\,, \tag{3.13}$$

for any q_h in a discrete pressure space. This does not change the order of the estimates corresponding to (3.10) and (3.11). □

Theorem 3.5. *Assume that $\mathcal{T}_h$ satisfies (2.41) and $\mathbf{g}$ satisfies (3.4). For any real number $p \in [2,4]$, there exists a constant C, depending on p but not on h, such that*

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{W^{1,p}(\Omega)} \le C\, h^{2/p-1/2} K_1(h)|\mathbf{w}|_{H^{3/2}(\Omega)}\,. \tag{3.14}$$

If in addition, Ω is a convex polygon and $\mathbf{g}$ satisfies (3.6) and (3.7), then for any number $2 \le p \le \infty$, there exists another constant C, depending on p but not on h, such that

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{W^{1,p}(\Omega)} \le C\, h^{2/p} K_1(h)|\mathbf{w}|_{H^2(\Omega)}\,. \tag{3.15}$$

Proof. The proof is a straightforward application of Lemma 3.3 and the following inverse inequality, valid for any number $2 \le p \le \infty$, in any finite-element space Θ_h associated with $\mathcal{T}_h$:

$$\forall v_h \in \Theta_h\ ,\ \|v_h\|_{L^p(\Omega)} \le C\, h^{\frac{2}{p}-1}\|v_h\|_{L^2(\Omega)}\,, \tag{3.16}$$

with a constant C depending on p but not on h. □

Remark 3.6. It may be possible to derive estimates analogous to (3.14) and (3.15) by adapting to problem (3.1–3.3) the arguments of Durán *et al.* [18]. But here such sharp arguments are not necessary because $\mathbf{w}$ is sufficiently smooth.

Nevertheless, if we replace $P_h(\mathbf{w})$ by the Stokes projection of $\mathbf{w}$, $S_h(\mathbf{w}) \in V_h + \mathbf{g}_h$, defined by:

$$\forall \mathbf{v}_h \in V_h\,,\ (\nabla(S_h(\mathbf{w}) - \mathbf{w}), \nabla\, \mathbf{v}_h) = 0\,,$$

we can improve (3.10) and (3.11). Indeed, instead of (3.12), we obtain for any real number $r > 2$:

$$\begin{aligned} |\mathbf{u}_h - S_h(\mathbf{w})|_{H^1(\Omega)} &\le \frac{S_q}{\nu}\|z_h\|_{L^2(\Omega)}\|\mathbf{w} - S_h(\mathbf{w})\|_{L^r(\Omega)} \\ &\le \frac{S_q}{\nu}\|z_h\|_{L^2(\Omega)}\big(\|\mathbf{w} - P_h(\mathbf{w})\|_{L^r(\Omega)} + \|P_h(\mathbf{w}) - S_h(\mathbf{w})\|_{L^r(\Omega)}\big)\,, \end{aligned} \tag{3.17}$$

where $q = \frac{2r}{r-2}$. The second term is bounded first by the inverse inequality (3.16), provided $\mathcal{T}_h$ satisfies (2.41), and next by duality:

$$\|P_h(\mathbf{w}) - S_h(\mathbf{w})\|_{L^r(\Omega)} \le Ch^{2/r-1}\|P_h(\mathbf{w}) - S_h(\mathbf{w})\|_{L^2(\Omega)}\,, \tag{3.18}$$

$$\|P_h(\mathbf{w}) - S_h(\mathbf{w})\|_{L^2(\Omega)} \le \|P_h(\mathbf{w}) - \mathbf{w}\|_{L^2(\Omega)} + Ch^{1/2}|S_h(\mathbf{w}) - \mathbf{w}|_{H^1(\Omega)} + C\|\mathbf{g}_h - \mathbf{g}\|_{L^2(\partial\Omega)}\,.$$

Therefore if $\mathbf{g}$ satisfies (3.4), we obtain for any real number $r > 2$:

$$|\mathbf{u}_h - S_h(\mathbf{w})|_{H^1(\Omega)} \le \frac{C}{\nu} h^{2/r}\|z_h\|_{L^2(\Omega)}|\mathbf{w}|_{H^{3/2}(\Omega)}\,. \tag{3.19}$$

When Ω is convex and $\mathbf{g}$ satisfies (3.6) and (3.7), we derive from (3.18) and a duality argument:

$$|\mathbf{u}_h - S_h(\mathbf{w})|_{H^1(\Omega)} \leq \frac{C}{\nu} h^{2/r+1} \|z_h\|_{L^2(\Omega)} |\mathbf{w}|_{H^2(\Omega)} \,. \tag{3.20}$$

Similarly, we can improve (3.14) and (3.15). For any real number $p \in [2,4]$ and for any $\varepsilon > 0$, under the assumptions of the first part of Theorem 3.5, we have

$$|\mathbf{u}_h - S_h(\mathbf{w})|_{W^{1,p}(\Omega)} \leq \frac{C}{\nu} h^{2/p-\varepsilon} \|z_h\|_{L^2(\Omega)} |\mathbf{w}|_{H^{3/2}(\Omega)} \,. \tag{3.21}$$

For any number $2 \leq p \leq \infty$ and for any $\varepsilon > 0$, under the assumptions of the second part of Theorem 3.5, we have

$$|\mathbf{u}_h - S_h(\mathbf{w})|_{W^{1,p}(\Omega)} \leq \frac{C}{\nu} h^{1+2/p-\varepsilon} \|z_h\|_{L^2(\Omega)} |\mathbf{w}|_{H^2(\Omega)} \,. \tag{3.22}$$

Here C denote constants that depend on p and r or ε, but are independent of h. □

In view of Remark 2.4, Lemma 2.2 and the approximation properties of P_h, we have the following corollary.

Corollary 3.7. *Let h satisfy (2.29). Under the assumptions of the first part of Theorem 3.5, there exists a constant C, that depends on the data $\mathbf{f}$ and $\mathbf{g}$, but is independent of h, such that any solution $\mathbf{u}_h$ of (0.6), (0.7) satisfies the uniform bound:*

$$|\mathbf{u}_h|_{W^{1,4}(\Omega)} \leq C \,. \tag{3.23}$$

Moreover under the assumptions of the second part of Theorem 3.5, for each real number $p \geq 2$, there exists another constant C, depending on p but not on h, such that

$$|\mathbf{u}_h|_{W^{1,p}(\Omega)} \leq C \,. \tag{3.24}$$

We assume that h satisfies (2.29) and not just h_b, because if (2.41) and (2.29) hold simultaneously, then h must also satisfy (2.29).

Remark 3.8. Note that, so far, we cannot extend (3.24) to $p = \infty$ because we have no bound for $\mathbf{w}$ in $W^{1,\infty}(\Omega)^2$. Such a bound would require an estimate for z_h in $L^r(\Omega)$ for some $r > 2$ and so far this appears to be an open problem. □

Remark 3.9. As a consequence, $\mathbf{u}_h$ is uniformly bounded in $L^\infty(\Omega)^2$, but for this, it is not necessary that $\mathbf{g}$ satisfy (3.4). Indeed, suppose that $\mathbf{g} \in H^{1/2+s}(\partial\Omega)^2$ for some $s \in (0, 1/2)$; then $\mathbf{w} \in H^{1+s}(\Omega)^2$ and, for $\lambda = \frac{2}{1-s}$, we have $\mathbf{g} \in W^{1-1/\lambda,\lambda}(\partial\Omega)^2$ and $\mathbf{w} \in W^{1,\lambda}(\Omega)^2$. Therefore (3.16) yields:

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{W^{1,\lambda}(\Omega)} \leq C_\lambda h^{2/\lambda-1} |\mathbf{u}_h - P_h(\mathbf{w})|_{H^1(\Omega)} \,.$$

Then (3.12), (2.7) and the above regularity of $\mathbf{w}$ imply

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{W^{1,\lambda}(\Omega)} \leq C K_1(h) |\mathbf{w}|_{H^{1+s}(\Omega)} \,,$$

whence the existence of another constant C, independent of h, such that

$$\|\mathbf{u}_h\|_{L^\infty(\Omega)} \leq C \,.$$

The following lemma compares $\mathbf{u}$ and $\mathbf{w}$. Note that its statement is independent of the particular functions z and z_h. It is valid for any pair of solutions of the generalized Stokes problem (0.4), (0.2), (0.3) associated with any pair of functions z and z_h in $L^2(\Omega)$.

Lemma 3.10. *There exists a constant C_1, independent of h, such that*

$$\|\mathbf{u}-\mathbf{w}\|_{W^{2,4/3}(\Omega)} \le \frac{C_1}{\nu}\|z-z_h\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^4(\Omega)}\big(1+\frac{S_4^2}{\nu}\|z\|_{L^2(\Omega)}\big)\,. \tag{3.25}$$

Proof. Subtracting (3.1) from (0.4), we find

$$-\nu\,\Delta(\mathbf{u}-\mathbf{w})+\mathbf{z}\times(\mathbf{u}-\mathbf{w})+\nabla(p-q)=-(\mathbf{z}-\mathbf{z}_h)\times\mathbf{w}\,,$$

$$\operatorname{div}(\mathbf{u}-\mathbf{w})=0 \;\text{ in } \Omega\,,\;\; \mathbf{u}-\mathbf{w}=\mathbf{0} \;\text{ on } \partial\Omega\,.$$

Therefore

$$|\mathbf{u}-\mathbf{w}|_{H^1(\Omega)} \le \frac{S_4}{\nu}\|z-z_h\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^4(\Omega)}\,, \tag{3.26}$$

which implies that

$$\|\mathbf{z}\times(\mathbf{u}-\mathbf{w})\|_{L^{4/3}(\Omega)} \le \|z\|_{L^2(\Omega)}\|\mathbf{u}-\mathbf{w}\|_{L^4(\Omega)}+\frac{S_4^2}{\nu}\|z\|_{L^2(\Omega)}\|z-z_h\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^4(\Omega)}\,.$$

This inequality and the triangle inequality give

$$\|\mathbf{z}\times(\mathbf{u}-\mathbf{w})+(\mathbf{z}-\mathbf{z}_h)\times\mathbf{w}\|_{L^{4/3}(\Omega)} \le (1+\frac{S_4^2}{\nu}\|z\|_{L^2(\Omega)})\|z-z_h\|_{L^2(\Omega)}\|\mathbf{w}\|_{L^4(\Omega)}\,,$$

and (3.25) then follows from the regularity of the homogeneous Stokes problem. ☐

Let us define

$$K_2(h)=\frac{1}{\nu}\|\mathbf{w}\|_{L^4(\Omega)}\big(1+\frac{S_4^2}{\nu}\|z\|_{L^2(\Omega)}\big)\,, \tag{3.27}$$

and note that as z_h is bounded in $L^2(\Omega)$, $K_2(h)$ is also bounded with respect to h. Then Lemma 3.10 implies in particular that there exists a constant C_∞, independent of h, such that

$$\|\mathbf{u}-\mathbf{w}\|_{L^\infty(\Omega)} \le C_\infty K_2(h)\|z-z_h\|_{L^2(\Omega)}\,. \tag{3.28}$$

Note also that when Ω is convex and $\mathbf{w}\in L^\infty(\Omega)^2$, which is the case if $\mathbf{g}\in H^{1+s}(\partial\Omega)^2$ for some $s\in(0,1/2)$, then we can replace (3.25) by

$$\|\mathbf{u}-\mathbf{w}\|_{H^2(\Omega)} \le C_1' K_3(h)\|z-z_h\|_{L^2(\Omega)}\,, \tag{3.29}$$

where

$$K_3(h)=\frac{1}{\nu}\left(\|\mathbf{w}\|_{L^\infty(\Omega)}+C_\infty K_2(h)\|z\|_{L^2(\Omega)}\right)\,. \tag{3.30}$$

Now we turn to $\mathbf{u}_h-\mathbf{u}$.

Theorem 3.11. *Under the assumptions of the first part of Theorem 3.5, we have for any real number $p\in[2,4]$ and with constants C that depend on p, but not on h:*

$$\begin{aligned}|\mathbf{u}_h-\mathbf{u}|_{W^{1,p}(\Omega)} \le{}& |P_h(\mathbf{u})-\mathbf{u}|_{W^{1,p}(\Omega)}+Ch^{2/p-1}K_1(h)|P_h(\mathbf{u})-\mathbf{u}|_{H^1(\Omega)}\\ &+CK_2(h)\big(1+Ch^{2/p-1/2}K_1(h)\big)\|z-z_h\|_{L^2(\Omega)}\,.\end{aligned} \tag{3.31}$$

Proof. Let us write

$$|\mathbf{u}_h - \mathbf{u}|_{W^{1,p}(\Omega)} \leq |\mathbf{u}_h - P_h(\mathbf{w})|_{W^{1,p}(\Omega)} + |P_h(\mathbf{w}-\mathbf{u})|_{W^{1,p}(\Omega)} + |P_h(\mathbf{u}) - \mathbf{u}|_{W^{1,p}(\Omega)}\,.$$

We apply (3.16) and (3.12) to the first term and (2.7) to the second term:

$$|\mathbf{u}_h - \mathbf{u}|_{W^{1,p}(\Omega)} \leq C\, h^{2/p-1} K_1(h) |\mathbf{w} - P_h(\mathbf{w})|_{H^1(\Omega)} + C|\mathbf{w}-\mathbf{u}|_{W^{1,p}(\Omega)} + |P_h(\mathbf{u}) - \mathbf{u}|_{W^{1,p}(\Omega)}\,. \tag{3.32}$$

Next we write:

$$|\mathbf{w} - P_h(\mathbf{w})|_{H^1(\Omega)} \leq |\mathbf{w} - \mathbf{u} - P_h(\mathbf{w}-\mathbf{u})|_{H^1(\Omega)} + |P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)}\,,$$

and we apply (2.7) to the first term, considering that $\mathbf{w}-\mathbf{u} \in W^{2,4/3}(\Omega)^2 \subset H^{3/2}(\Omega)^2$:

$$|\mathbf{w} - P_h(\mathbf{w})|_{H^1(\Omega)} \leq C h^{1/2} \|\mathbf{w}-\mathbf{u}\|_{H^{3/2}(\Omega)} + |P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)}\,.$$

Then (3.25) yields

$$|\mathbf{w} - P_h(\mathbf{w})|_{H^1(\Omega)} \leq \frac{C}{\nu} h^{1/2} \|z - z_h\|_{L^2(\Omega)} \|\mathbf{w}\|_{L^4(\Omega)} (1 + \frac{S_4^2}{\nu}\|z\|_{L^2(\Omega)}) + |P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)}\,,$$

and we derive (3.31) by substituting this inequality and applying again (3.25) into (3.32). □

Remark 3.12. The statement of the previous theorem can be viewed as a decoupling *a priori* error estimate for the velocity. Under the assumptions of the second part of Theorem 3.5, it also holds for any real number $p \geq 2$; (3.31) becomes

$$|\mathbf{u}_h - \mathbf{u}|_{W^{1,p}(\Omega)} \leq |P_h(\mathbf{u}) - \mathbf{u}|_{W^{1,p}(\Omega)} + C' h^{2/p-1} K_1(h) |P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} + C' K_3(h)\big(1 + K_1(h) h^{2/p}\big) \|z - z_h\|_{L^2(\Omega)}\,,$$

where the constants C' depend on p, but not on h. Similarly, under the same assumptions, we obtain

$$\begin{aligned}\big(\sum_{K\in\mathcal{T}_h} |\mathbf{u}_h - \mathbf{u}|^2_{H^2(K)}\big)^{1/2} \leq & \big(\sum_{K\in\mathcal{T}_h} |P_h(\mathbf{u}) - \mathbf{u}|^2_{H^2(K)}\big)^{1/2} \\ & + \frac{C'}{h} K_1(h) |P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} + C' K_3(h)\big(1 + K_1(h)\big) \|z - z_h\|_{L^2(\Omega)}\,,\end{aligned}$$

with other constants C' independent of h. □

Remark 3.13. The bound (3.31) with some $p > 2$ will be applied to estimate $\|\mathbf{u}_h - \mathbf{u}\|_{L^\infty(\Omega)}$ in (3.56), under the assumption that the domain is convex. But for this, it is not necessary that $\mathcal{T}_h$ be quasi-uniform and (2.41) can be substantially relaxed. Indeed, we can sharpen (3.16) and write:

$$\forall v_h \in \Theta_h\,,\ \|v_h\|_{L^p(\Omega)} \leq \frac{C}{\rho_{\min}^{1-2/p}} \|v_h\|_{L^2(\Omega)}\,,$$

where $\rho_{\min}$ denotes the minimum of ρ_K. With this, (3.25), (3.29) and (3.30), taking into account the convexity of Ω, we can replace (3.31) by:

$$\begin{aligned}\|\mathbf{u}_h - \mathbf{u}\|_{L^\infty(\Omega)} \leq & \|P_h(\mathbf{u}) - \mathbf{u}\|_{L^\infty(\Omega)} + \frac{C K_1(h)}{\rho_{\min}^{1-2/p}} |P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} \\ & + \big(C K_2(h) + K_1(h) K_3(h) \frac{C h}{\rho_{\min}^{1-2/p}}\big) \|z - z_h\|_{L^2(\Omega)}\,,\end{aligned} \tag{3.33}$$

where C denote various constants independent of h. Thus, by taking, say, $p = 2 + 1/4$, if we want the factor

$$\frac{h}{\rho_{\min}^{1-2/p}}$$

to be of the order of $h^{1/4}$ (for instance), it suffices that $\mathcal{T}_h$ satisfy: there exists a constant $\gamma > 0$ independent of h, such that

$$\forall K \in \mathcal{T}_h \, , \; \rho_K \geq \gamma \, h^6 \, , \tag{3.34}$$

a condition that allows refinements appropriate for singularities introduced by domain geometry. In this case, we have:

$$\begin{aligned} \|\mathbf{u}_h - \mathbf{u}\|_{L^\infty(\Omega)} \leq \|P_h(\mathbf{u}) - \mathbf{u}\|_{L^\infty(\Omega)} + \frac{C_1 K_1(h)}{\rho_{\min}^{1/9}} |P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} \\ + \big(C_2 K_2(h) + C_3 h^{1/4} K_1(h) K_3(h)\big) \|z - z_h\|_{L^2(\Omega)} \, , \end{aligned} \tag{3.35}$$

with constants C_i independent of h. Note that, since the degree of the elements approximating $\mathbf{u}$ is at least two, then for $\mathbf{u} \in H^3(\Omega)^2$, the term involving $\rho_{\min}$ is of the order of $h^{2-1/9}$. □

3.2. Additional regularity and uniqueness of the exact solution

So far, we have examined the regularity of $\mathbf{u}$ for z in $L^2(\Omega)$. But in view of the transport equation (0.5) and its discretization (0.7), we shall be led to investigate the case where z belongs to $H^1(\Omega)$, and this holds in particular if $\mathbf{u}$ belongs to $W^{1,\infty}(\Omega)^2 \cap H^2(\Omega)^2$ and $\operatorname{curl} \mathbf{f}$ belongs to $H^1(\Omega)$. This point is easily explained by reverting to the basic transport equation (2.31). Formally, ∇z satisfies

$$\nabla z + \mathcal{W} \nabla (\mathbf{u} \cdot \nabla z) = \nabla \ell \, ,$$

i.e.

$$\nabla z + \mathcal{W} \nabla \mathbf{u} \cdot \nabla z + \mathcal{W} \mathbf{u} \cdot \nabla (\nabla z) = \nabla \ell \, .$$

Whence

$$\|\nabla z\|_{L^2(\Omega)} \big(1 - |\mathcal{W}| \|\nabla \mathbf{u}\|_{L^\infty(\Omega)}\big) \leq \|\nabla \ell\|_{L^2(\Omega)} \, . \tag{3.36}$$

When the domain Ω is convex, this inequality is derived rigorously by discretizing (2.31) in the basis of eigenfunctions of the Laplace operator, with a Neumann boundary condition:

$$-\Delta v_k = \lambda_k v_k \;\text{ in } \Omega \, , \; \frac{\partial v_k}{\partial \mathbf{n}} = 0 \;\text{ on } \partial\Omega \, , \; \int_\Omega v_k \, d\mathbf{x} = 0 \, .$$

The convexity of Ω guarantees that v_k belongs to $H^2(\Omega)$.

In the present situation, the following proposition, proven in [20], gives a bound for $\mathbf{u}$ in $W^{1,\infty}(\Omega)^2$.

Proposition 3.14. *Let Ω be a convex polygon. There exists a real number $r_0 > 2$, depending on the inner angles of $\partial\Omega$, such that: if for some real number r with $2 < r < r_0$, and on each γ_j, $0 \leq j \leq R$,*

$$\mathbf{g} \in W^{2-1/r,r}(\Gamma_i)^2 \;\; \textit{for } 1 \leq i \leq N \; , \; \mathbf{g} \cdot \mathbf{n} = 0 \, , \tag{3.37}$$

$$\left(\frac{\partial \mathbf{g}_{i+1} \cdot \mathbf{n}_i}{\partial \mathbf{t}_{i+1}} - \frac{\partial \mathbf{g}_i \cdot \mathbf{n}_{i+1}}{\partial \mathbf{t}_i} \right) (\mathbf{x}_i) = 0 \;\; \textit{for } 1 \leq i \leq N \, , \tag{3.38}$$

then any solution $\mathbf{u} \in W^\alpha$ *of (0.1–0.3) belongs to* $W^{2,r}(\Omega)^2$ *and*

$$|\alpha|\|\mathbf{u}\|_{W^{2,r}(\Omega)} \leq C_r\big(\|\operatorname{curl}\mathbf{u}\|_{L^2(\Omega)} + \frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)} + |\alpha|[\mathbf{g}]_{W^{2-1/r,r}(\partial\Omega)}\big)\,, \tag{3.39}$$

where C_r *is a constant independent of* α *and* ν *and*

$$[\mathbf{g}]_{W^{2-1/r,r}(\partial\Omega)} = \sum_{j=0}^{R}[\mathbf{g}]_{W^{2-1/r,r}(\gamma_j)}\ ,\ [\mathbf{g}]_{W^{2-1/r,r}(\gamma_j)} = \sum_{i=1}^{N}\|\mathbf{g}\|_{W^{2-1/r,r}(\Gamma_i)}\,.$$

Proposition 3.15. *Under the assumptions and notation of Proposition 3.14, we have*

$$|\alpha|\|\nabla\,\mathbf{u}\|_{L^\infty(\Omega)} \leq C_r(C_1(\mathbf{f}) + C_2(\mathbf{g}))\,, \tag{3.40}$$

where C_r *is a constant independent of* α *and* ν*,*

$$C_1(\mathbf{f}) = \frac{1}{\nu}(1+2\sqrt{2})(\sqrt{2}S_2\|\mathbf{f}\|_{L^2(\Omega)} + |\alpha|\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)})\,, \tag{3.41}$$

$$C_2(\mathbf{g}) = \sqrt{2}(1+(2\sqrt{2})^{3/2})\frac{C}{\sqrt{\nu}}\|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)} + |\alpha|[\mathbf{g}]_{W^{2-1/r,r}(\partial\Omega)}\,, \tag{3.42}$$

C *and* λ *are the constants of (1.11).*

Proof. The assumptions on $\mathbf{g}$ guarantee that $\mathbf{g} \in W^{1-1/\lambda,\lambda}(\partial\Omega)^2$ for some $\lambda > 2$. We apply (1.10) with $\varepsilon = \nu$ and we substitute the estimate (1.11) into it:

$$|\mathbf{u}|_{H^1(\Omega)} \leq \frac{S_2}{\nu}(1+2\sqrt{2})\|\mathbf{f}\|_{L^2(\Omega)} + 2\frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)} + (1+(2\sqrt{2})^{3/2})\frac{C}{\sqrt{\nu}}\|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)}\,, \tag{3.43}$$

where C is the constant of (1.11). But (3.39) implies

$$|\alpha|\|\nabla\,\mathbf{u}\|_{L^\infty(\Omega)} \leq C_r\big(\sqrt{2}|\mathbf{u}|_{H^1(\Omega)} + \frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)} + |\alpha|[\mathbf{g}]_{W^{2-1/r,r}(\partial\Omega)}\big)\,,$$

with another constant C_r, and (3.40) follows by substituting (3.43) into this inequality. □

Applying (3.36) to the present situation, we obtain:

Proposition 3.16. *We retain the assumptions and notation of Propositions 3.14 and 3.15, and we suppose that the data* $\mathbf{f}$ *and* $\mathbf{g}$ *are small: there exists a constant* η_1 *with* $0 < \eta_1 < \nu$ *such that*

$$C_r(C_1(\mathbf{f}) + C_2(\mathbf{g})) = \nu - \eta_1\,. \tag{3.44}$$

Then, if $\operatorname{curl}\mathbf{f}$ *belongs to* $H^1(\Omega)$*, any solution* $(\mathbf{u}, z)$ *of Problem P satisfies* $z \in H^1(\Omega)$ *and*

$$|z|_{H^1(\Omega)} \leq \frac{1}{\eta_1}(\nu|\operatorname{curl}\mathbf{u}|_{H^1(\Omega)} + |\alpha||\operatorname{curl}\mathbf{f}|_{H^1(\Omega)})\,. \tag{3.45}$$

Remark 3.17. The fact that z belongs to $H^1(\Omega)$ gives some information on the continuity of the solution z of (0.5) with respect to $\mathbf{u}$. Indeed, for $\mathbf{u}$ and $\mathbf{v}$ given in W, let z_1 and z_2 in $L^2(\Omega)$ be defined by:

$$\nu\,z_1 + \alpha\,\mathbf{u}\cdot\nabla\,z_1 = \nu\operatorname{curl}\mathbf{u} + \alpha\operatorname{curl}\mathbf{f}\,,$$

$$\nu\, z_2 + \alpha(\mathbf{u}+\mathbf{v})\cdot\nabla\, z_2 = \nu\,\mathrm{curl}(\mathbf{u}+\mathbf{v}) + \alpha\,\mathrm{curl}\,\mathbf{f}\,.$$

Then $z_2 - z_1$ satisfies

$$\nu(z_2 - z_1) + \alpha(\mathbf{u}+\mathbf{v})\cdot\nabla(z_2 - z_1) = \nu\,\mathrm{curl}\,\mathbf{v} - \alpha\,\mathbf{v}\,\cdot\nabla\, z_1\,.$$

Hence, assuming that Ω is convex, $\mathbf{u}$ belongs to $(W^{1,\infty}(\Omega)\cap H^2(\Omega))^2$, $\mathrm{curl}\,\mathbf{f}$ belongs to $H^1(\Omega)$ and

$$|\alpha|\|\nabla\,\mathbf{u}\|_{L^\infty(\Omega)} \le \nu - \eta \quad \text{for some } \eta > 0\,,$$

then we have $z_1 \in H^1(\Omega)$ and

$$\|z_2 - z_1\|_{L^2(\Omega)} \le \|\mathrm{curl}\,\mathbf{v}\|_{L^2(\Omega)} + \frac{|\alpha|}{\nu}\|\mathbf{v}\|_{L^\infty(\Omega)}|z_1|_{H^1(\Omega)}\,.$$

Therefore if $\mathbf{v}$ tends to zero in $(L^\infty(\Omega)\cap H^1(\Omega))^2$, then $z_2 - z_1$ tends to zero in $L^2(\Omega)$. □

Now, we investigate uniqueness of the exact solution. A sufficient condition for uniqueness is given in [20], but it is based on the fact that $\alpha\mathrm{curl}\Delta\,\mathbf{u} \in L^2(\Omega)$, a property that is not available in the discrete case. So let us derive here another sufficient condition, possibly less sharp, but better adapted to the formulation of *Problem P*. Thus, let $(\mathbf{u}_1, z_1)$ and $(\mathbf{u}_2, z_2)$ be any two solutions of *Problem P* (we eliminate the pressure, since it is determined by the other variables). Then arguing as in Lemma 3.10, we easily derive

$$|\mathbf{u}_1 - \mathbf{u}_2|_{H^1(\Omega)} \le \frac{S_4}{\nu}\|\mathbf{u}_2\|_{L^4(\Omega)}\|z_1 - z_2\|_{L^2(\Omega)}\,, \tag{3.46}$$

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{W^{2,4/3}(\Omega)} \le C_1 K_2\|z_1 - z_2\|_{L^2(\Omega)}\,, \tag{3.47}$$

where C_1 is the constant of (3.25) and

$$K_2 = \frac{1}{\nu}\|\mathbf{u}_2\|_{L^4(\Omega)}\big(1 + \frac{S_4^2}{\nu}\|z_1\|_{L^2(\Omega)}\big)\,. \tag{3.48}$$

As a consequence, $\mathbf{u}_1 - \mathbf{u}_2 \in L^\infty(\Omega)^2$ and

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(\Omega)} \le C_\infty K_2\|z_1 - z_2\|_{L^2(\Omega)}\,, \tag{3.49}$$

where C_∞ is the constant of (3.28).

On the other hand, arguing as in Remark 3.17 and assuming that $z_2 \in H^1(\Omega)$, we obtain

$$\|z_1 - z_2\|_{L^2(\Omega)} \le \|\mathrm{curl}(\mathbf{u}_1 - \mathbf{u}_2)\|_{L^2(\Omega)} + \frac{|\alpha|}{\nu}\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(\Omega)}|z_2|_{H^1(\Omega)}\,.$$

Therefore, substituting (3.46) and (3.49) into this inequality, we derive

$$\|z_1 - z_2\|_{L^2(\Omega)} \le \frac{1}{\nu}\big(\sqrt{2}S_4\|\mathbf{u}_2\|_{L^4(\Omega)} + |\alpha|C_\infty K_2|z_2|_{H^1(\Omega)}\big)\|z_1 - z_2\|_{L^2(\Omega)}\,.$$

Hence we have proven the following proposition.

Proposition 3.18. *In addition to the assumptions of Proposition 3.16, we suppose that the data* $\mathbf{f}$ *and* $\mathbf{g}$ *are small enough so that there exists a constant* η_2 *with* $0 < \eta_2 < \nu$ *such that any solution* $(\mathbf{u}, z)$ *of Problem P satisfies*

$$\sqrt{2}S_4\|\mathbf{u}\|_{L^4(\Omega)} + |\alpha|C_\infty K_2|z|_{H^1(\Omega)} \le \nu - \eta_2\,. \tag{3.50}$$

Then Problem P has a unique solution.

3.3. Error bounds

From the exact *Problem P* and the discrete problem (0.6), (0.7), we readily obtain, for all $\mathbf{v}_h$ in V_h and all θ_h in Z_h:

$$\nu(\nabla(\mathbf{u}_h - \mathbf{u}), \nabla \mathbf{v}_h) + ((\mathbf{z}_h - \mathbf{z}) \times \mathbf{u}_h, \mathbf{v}_h) + (\mathbf{z} \times (\mathbf{u}_h - \mathbf{u}), \mathbf{v}_h) = 0\,, \tag{3.51}$$

$$\nu(z_h - z, \theta_h) + \alpha\, c(\mathbf{u}_h - \mathbf{u}; z_h, \theta_h) + \alpha\, c(\mathbf{u}; z_h - z, \theta_h) = \nu(\mathrm{curl}(\mathbf{u}_h - \mathbf{u}), \theta_h)\,. \tag{3.52}$$

From (3.51) and (2.46), we easily derive the following lemma.

Lemma 3.19. *Let* $(\mathbf{u}_h, z_h)$ *be a solution of (0.6), (0.7) and let* $(\mathbf{u}, z)$ *be a solution of Problem P. We have:*

$$|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} \le 2|\mathbf{u} - P_h(\mathbf{u})|_{H^1(\Omega)} + \frac{S_4}{\nu}\|P_h(\mathbf{u})\|_{L^4(\Omega)}\|z - z_h\|_{L^2(\Omega)} + \frac{S_4}{\nu}\|z\|_{L^2(\Omega)}\|\mathbf{u} - P_h(\mathbf{u})\|_{L^4(\Omega)}\,. \tag{3.53}$$

If $\mathcal{T}_h$ *satisfies (2.41) and (2.42) and exactly two triangles meet at any boundary singular vertex, then the pressure* p_h *defined by Proposition 2.13 satisfies:*

$$\begin{aligned}\|p - p_h\|_{L^2(\Omega)} &\le 2\,\|p - r_h(p)\|_{L^2(\Omega)} + \frac{\nu}{\beta^*}|\mathbf{u} - P_h(\mathbf{u})|_{H^1(\Omega)} \\ &\quad + \frac{S_4\tilde{S}_4}{\beta^*}\left(\|z\|_{L^2(\Omega)}|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} + |\mathbf{u}_h|_{H^1(\Omega)}\|z - z_h\|_{L^2(\Omega)}\right)\,.\end{aligned} \tag{3.54}$$

Now, let us examine (3.52).

Lemma 3.20. *Let* $(\mathbf{u}_h, z_h)$ *be a solution of (0.6), (0.7) and let* $(\mathbf{u}, z)$ *be a solution of Problem P. For any* λ_h *in* Z_h*, we have*

$$\|z - z_h\|_{L^2(\Omega)} \le 2\,\|z - \lambda_h\|_{L^2(\Omega)} + \|\mathrm{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)} + \frac{|\alpha|}{\nu}\left(\|(\mathbf{u} - \mathbf{u}_h)\cdot\nabla\lambda_h\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla(z - \lambda_h)\|_{L^2(\Omega)}\right)\,. \tag{3.55}$$

Proof. Inserting any $\lambda_h \in Z_h$ into (3.52), we derive for all $\theta_h \in Z_h$

$$\begin{aligned}\nu(z_h - \lambda_h, \theta_h) + \alpha c(\mathbf{u}_h; z_h - \lambda_h, \theta_h) &= \nu(\mathrm{curl}(\mathbf{u}_h - \mathbf{u}), \theta_h) \\ &\quad + \nu(z - \lambda_h, \theta_h) + \alpha c(\mathbf{u}; z - \lambda_h, \theta_h) + \alpha c(\mathbf{u} - \mathbf{u}_h; \lambda_h, \theta_h)\,.\end{aligned}$$

Then (3.55) follows by choosing $\theta_h = z_h - \lambda_h$ and applying (2.11). □

Note that the statement of this lemma requires no particular regularity assumption on the data and the domain. However, If we want to deduce from it a useful error inequality, we must assume that z belongs to $H^1(\Omega)$.

Corollary 3.21. *Let* $(\mathbf{u}_h, z_h)$ *be a solution of (0.6), (0.7) and* $(\mathbf{u}, z)$ *a solution of Problem P. Under the assumptions of Proposition 3.16 (so that* $z \in H^1(\Omega)$*), we have*

$$\begin{aligned}\|z - z_h\|_{L^2(\Omega)} &\le 2\,\|z - R_h(z)\|_{L^2(\Omega)} + \sqrt{2}|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} \\ &\quad + \frac{|\alpha|}{\nu}\left(\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(\Omega)}|R_h(z)|_{H^1(\Omega)} + \|\mathbf{u}\|_{L^\infty(\Omega)}|z - R_h(z)|_{H^1(\Omega)}\right)\,.\end{aligned} \tag{3.56}$$

By substituting (3.35) and (3.53) into (3.56), we immediately derive the main result of this section.

Theorem 3.22. *Let Ω be a convex polygon and assume that $\mathcal{T}_h$ satisfies (3.34) and h_b satisfies (2.29). Suppose that for some r with $2 < r < r_0$, the boundary data $\mathbf{g}$ satisfies (3.37) and (3.38),* $\operatorname{curl}\mathbf{f}$ *belongs to $H^1(\Omega)$, and suppose that there exist constants $\eta_1 > 0$ and $\eta_2 > 0$ such that (3.44) and (3.50) both hold. Then, if the data are small enough so that*

$$\sqrt{2}E_1 S_4 |\mathbf{u}|_{H^1(\Omega)} + |\alpha| E_2 |z|_{H^1(\Omega)} (C_2 K_2(h) + C_3 h^{1/4} K_1(h) K_3(h)) \le \frac{\nu}{2}\,, \tag{3.57}$$

where $K_1(h)$, $K_2(h)$, $K_3(h)$, C_2 and C_3 are the constants of (3.9), (3.27), (3.30) and (3.35), and E_1 and E_2 are the constants of the inequalities

$$\|P_h(\mathbf{u})\|_{L^4(\Omega)} \le E_1 |\mathbf{u}|_{H^1(\Omega)}\ ,\ |R_h(z)|_{H^1(\Omega)} \le E_2 |z|_{H^1(\Omega)}\,,$$

we have the following error estimate:

$$\begin{aligned}\|z - z_h\|_{L^2(\Omega)} \le{}& 4\|z - R_h(z)\|_{L^2(\Omega)} + 4\sqrt{2}|\mathbf{u} - P_h(\mathbf{u})|_{H^1(\Omega)} \\ &+ \frac{2}{\nu}\sqrt{2}S_4 \|z\|_{L^2(\Omega)} \|\mathbf{u} - P_h(\mathbf{u})\|_{L^4(\Omega)} + 2\frac{|\alpha|}{\nu}(\|\mathbf{u}\|_{L^\infty(\Omega)} |z - R_h(z)|_{H^1(\Omega)} \\ &+ E_2 |z|_{H^1(\Omega)} (\|\mathbf{u} - P_h(\mathbf{u})\|_{L^\infty(\Omega)} + \frac{C_1 K_1(h)}{\rho_{\min}^{1/9}} |\mathbf{u} - P_h(\mathbf{u})|_{H^1(\Omega)}))\,,\end{aligned} \tag{3.58}$$

where C_1 is also the constant of (3.35).

Note that the left-hand sides of (3.57) and (3.50) have related structures. Note also that the statement of this theorem remains valid when α tends to zero.

The conclusion that we can draw from (3.53), (3.54), (3.58), and Remarks 3.12 and 3.13 is: if the domain is convex and $\mathcal{T}_h$ satisfies (3.34), if z belongs to $H^2(\Omega)$ and $\mathbf{u}$ to $H^3(\Omega)^2$, then $\|z - z_h\|_{L^2(\Omega)}$ and $|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)}$ are of the order of h. If (3.34) is replaced by:

$$\forall K \in \mathcal{T}_h\,,\ \rho_K \ge \gamma\, h^2\,, \tag{3.59}$$

which still allows for a wide range of refinements, then the same is true for $|\mathbf{u} - \mathbf{u}_h|_{W^{1,p}(\Omega)}$ for all $p \in [2, 4]$. The condition on the triangulation becomes more and more restrictive until we need the quasi-uniformity of $\mathcal{T}_h$ in order to prove that

$$\Big(\sum_{K\in\mathcal{T}_h} |\mathbf{u} - \mathbf{u}_h|^2_{H^2(K)}\Big)^{1/2} \quad \text{and } \|p - p_h\|_{L^2(\Omega)}$$

are also of the order of h, for p in $H^1(\Omega)$. Of course, when the solution is very smooth, any order of accuracy can be attained by using polynomials of high enough degree. The first result is disappointing considering that the error for z is measured only in the L^2 norm. This loss of accuracy is due mainly to the hyperbolic character of the problem, but partly also to the fact that we are using a centered scheme. The upwind schemes studied in the last two sections will allow us to improve a little this result.

We end this section with a remark on uniqueness of the discrete solution. The proof of uniqueness of the discrete solution is still an open problem, if we want to keep the regularity of the exact solution compatible with a polygonal domain. Indeed, any pair of solutions $(\mathbf{u}_h, z_h)$, $(\mathbf{u}'_h, z'_h)$ of (0.6), (0.7) in $(V_h + \mathbf{g}_h) \times Z_h$ satisfies: $\mathbf{u}_h - \mathbf{u}'_h \in V_h$, $z_h - z'_h \in Z_h$,

$$\forall \mathbf{v}_h \in V_h\ ,\ \nu(\nabla(\mathbf{u}_h - \mathbf{u}'_h), \nabla\,\mathbf{v}_h) + (\mathbf{z}'_h \times (\mathbf{u}_h - \mathbf{u}'_h), \mathbf{v}_h) = -((\mathbf{z}_h - \mathbf{z}'_h) \times \mathbf{u}_h, \mathbf{v}_h)\,, \tag{3.60}$$

$$\forall \theta_h \in Z_h\ ,\ \nu(z_h - z'_h, \theta_h) + \alpha\, c(\mathbf{u}_h; z_h - z'_h, \theta_h) + \alpha\, c(\mathbf{u}_h - \mathbf{u}'_h; z'_h, \theta_h) = \nu(\operatorname{curl}(\mathbf{u}_h - \mathbf{u}'_h), \theta_h)\,. \tag{3.61}$$

Therefore

$$\|z_h - z_h'\|_{L^2(\Omega)} \le \frac{|\alpha|}{\nu}\|(\mathbf{u}_h - \mathbf{u}_h')\cdot\nabla z_h'\|_{L^2(\Omega)} + \sqrt{2}|\mathbf{u}_h - \mathbf{u}_h'|_{H^1(\Omega)}\,.$$

The difficulty comes from the first term in the right-hand side of this inequality: we can easily derive from (3.60) a bound for $\|\mathbf{u}_h - \mathbf{u}_h'\|_{L^\infty(\Omega)}$, but we have no bound for $|z_h'|_{H^1(\Omega)}$. So far, the only way in which we can estimate this term is by writing that

$$|z_h'|_{H^1(\Omega)} \le |z_h' - R_h(z)|_{H^1(\Omega)} + |R_h(z)|_{H^1(\Omega)} \le \frac{C}{h}\|z_h' - R_h(z)\|_{L^2(\Omega)} + |R_h(z)|_{H^1(\Omega)}\,.$$

In view of (3.58), this gives a bound for $|z_h'|_{H^1(\Omega)}$, if we assume that $z \in H^2(\Omega)$, but this assumption is very restrictive on the angles of $\partial\Omega$.

4. Successive approximations

The mixed *Problem P* is easily linearized by successive approximations. Starting from an arbitrary z^0 in $L^2(\Omega)$, we define the sequence $(\mathbf{u}^n, p^n, z^n) \in W \times L_0^2(\Omega) \times L^2(\Omega)$ for $n \ge 1$, by:

$$-\nu\Delta\,\mathbf{u}^n + \mathbf{z}^{n-1}\times\mathbf{u}^n + \nabla\,p^n = \mathbf{f}\ ,\ \operatorname{div}\mathbf{u}^n = 0\ \text{ in }\Omega\,, \tag{4.1}$$

$$\mathbf{u}^n = \mathbf{g}\ \text{ on }\partial\Omega\,, \tag{4.2}$$

$$\nu\,z^n + \alpha\,\mathbf{u}^n\cdot\nabla\,z^n = \nu\operatorname{curl}\mathbf{u}^n + \alpha\operatorname{curl}\mathbf{f}\ \text{ in }\Omega\,. \tag{4.3}$$

Theorem 2.6 shows that, in a Lipschitz-continuous domain Ω, z^0 defines uniquely this sequence for all $\nu > 0$, all real numbers α, all $\mathbf{f} \in H(\operatorname{curl},\Omega)$ and $\mathbf{g} \in H^{1/2}(\partial\Omega)^2$ satisfying $\mathbf{g}\cdot\mathbf{n} = 0$. The next lemma shows that this sequence satisfies the same bounds as each solution of *Problem P*. To simplify, we assume that $\mathbf{g}$ has a little more regularity than $H^{1/2}$.

Lemma 4.1. *If Ω is Lipschitz-continuous and $\mathbf{g} \in W^{1-1/\lambda,\lambda}(\partial\Omega)^2$, for some $\lambda > 2$, then for all $\nu > 0$, all real numbers α, all $\mathbf{f} \in H(\operatorname{curl},\Omega)$ and all starting functions $z^0 \in L^2(\Omega)$, the solution $(\mathbf{u}^n, p^n, z^n)$ of (4.1–4.3) is bounded as follows:*

$$\|z^n\|_{L^2(\Omega)} \le \mathcal{K}_1 := \operatorname{Max}(\|z^0\|_{L^2(\Omega)}, \mathcal{K}_0)\,, \tag{4.4}$$

where

$$\mathcal{K}_0 = 2\frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(\Omega)} + 2\frac{\sqrt{2}}{\nu}S_2\|\mathbf{f}\|_{L^2(\Omega)} + (2\sqrt{2})^{3/2}\frac{C}{\nu^{1/2}}\|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)}\,, \tag{4.5}$$

and C is the constant of (1.11),

$$|\mathbf{u}^n|_{H^1(\Omega)} \le \mathcal{K}_2 := \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + T\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\big(1 + \frac{S_4\tilde{S}_4}{\nu}\mathcal{K}_1\big)\,, \tag{4.6}$$

$$\|p^n\|_{L^2(\Omega)} \le \frac{1}{\beta}(S_2\|\mathbf{f}\|_{L^2(\Omega)} + \nu T\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)} + S_4\tilde{S}_4\mathcal{K}_2\mathcal{K}_1)\,. \tag{4.7}$$

Proof. First observe that $\mathbf{u}^n$ is related to z^{n-1} as $\mathbf{u}$ is related to z, and therefore (4.6) and (4.7) follow from Theorem 1.2. Moreover $\mathbf{u}^n$ satisfies (1.10):

$$\forall\varepsilon > 0\,,\ |\mathbf{u}^n|_{H^1(\Omega)} \le \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + \frac{C}{\varepsilon^{1/2}}\|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)} + \frac{\varepsilon}{\nu}\|z^{n-1}\|_{L^2(\Omega)}\,.$$

Now the proof proceeds by induction on n. Clearly z^0 satisfies (4.4); therefore assume that z^{n-1} satisfies (4.4). Considering that z^n satisfies

$$\|z^n\|_{L^2(\Omega)} \leq \sqrt{2}|\mathbf{u}^n|_{H^1(\Omega)} + \frac{|\alpha|}{\nu}\|\mathrm{curl}\,\mathbf{f}\|_{L^2(\Omega)}\,,$$

and $\mathbf{u}^n$ satisfies (1.10), we obtain

$$\|z^n\|_{L^2(\Omega)} \leq \sqrt{2}\frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + \sqrt{2}\frac{C}{\varepsilon^{1/2}}\|\mathbf{g}\|^{3/2}_{W^{1-1/\lambda,\lambda}(\partial\Omega)} + \sqrt{2}\frac{\varepsilon}{\nu}\|z^{n-1}\|_{L^2(\Omega)} + \frac{|\alpha|}{\nu}\|\mathrm{curl}\,\mathbf{f}\|_{L^2(\Omega)}\,.$$

But either $\|z^n\|_{L^2(\Omega)} < \|z^{n-1}\|_{L^2(\Omega)}$ or $\|z^{n-1}\|_{L^2(\Omega)} \leq \|z^n\|_{L^2(\Omega)}$. Then the result follows by choosing $\varepsilon = \frac{\nu}{2\sqrt{2}}$. □

By imposing convexity on the domain and smallness assumptions on the data, we can prove that this algorithm is contracting. This is the object of the next theorem; we skip the proof because it is an easy adaptation of the arguments of Section 3.

Theorem 4.2. *We retain the assumptions and notation of Proposition 3.16, and we suppose in addition that the data are sufficently small so that*

$$\frac{1}{\nu}\|\mathbf{u}\|_{L^4(\Omega)}\big(\sqrt{2}S_4 + C\frac{|\alpha|}{\nu}|z|_{H^1(\Omega)}(1+\mathcal{K}_0\frac{S_4^2}{\nu})\big) = \theta < 1\,,$$

where the constant C depends only on Ω, and $\mathcal{K}_0$ is the constant of (4.5). Then, for any $n \geq 1$,

$$\|z^n - z\|_{L^2(\Omega)} \leq \theta\|z^{n-1} - z\|_{L^2(\Omega)}\,.$$

When discretized, (4.1–4.3) gives the following algorithm: Starting from an arbitrary $z_h^0 \in Z_h$, find $(\mathbf{u}_h^n, p_h^n, z_h^n)$ in $(V_h + \mathbf{g}_h) \times M_h \times Z_h$ such that,

$$\begin{aligned}\forall \mathbf{v}_h \in & X_h \cap H_0^1(\Omega)^2\,,\\ &\nu(\nabla\,\mathbf{u}_h^n, \nabla\,\mathbf{v}_h) + (\mathbf{z}_h^{n-1} \times \mathbf{u}_h^n, \mathbf{v}_h) - (p_h^n, \mathrm{div}\,\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)\,,\end{aligned} \tag{4.8}$$

$$\forall \theta_h \in Z_h\ ,\ \nu\,(z_h^n, \theta_h) + \alpha\,c(\mathbf{u}_h^n; z_h^n, \theta_h) = \nu\,(\mathrm{curl}\,\mathbf{u}_h^n, \theta_h) + \alpha\,(\mathrm{curl}\,\mathbf{f}, \theta_h)\,. \tag{4.9}$$

A straightforward variant of Lemma 4.1 shows that if h_b satisfies (2.29), then for all $n \geq 0$,

$$\|z_h^n\|_{L^2(\Omega)} \leq \mathrm{Max}(\|z_h^0\|_{L^2(\Omega)}, \tilde{\mathcal{K}})\,, \tag{4.10}$$

where $\tilde{\mathcal{K}}$ is the expression in the left-hand side of (2.30). Of course, $\mathbf{u}_h^n$ and p_h^n satisfy the estimates (2.18) and (2.45) with respect to z_h^{n-1}. If z^n belongs to $H^1(\Omega)$, we can establish an error estimate of the form (3.58) for $z^n - z_h^n$, under assumptions similar to those of Theorem 3.22. The proof follows closely the steps developed in Section 3, with the exception of Proposition 3.14, because the relationship between $\mathbf{u}^n$ and z^n is no longer given by $z^n = \mathrm{curl}(\mathbf{u}^n - \alpha\,\Delta\,\mathbf{u}^n)$ and we can only use (4.1). However, this difficulty can be by-passed by solving a transport equation in $L^r(\Omega)$ for some $r > 2$. As in Theorem 2.6, this is done in arbitrary dimension k.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^k$ be bounded and Lipschitz-continuous and let $r \geq 2$ be a real number. For all $\mathbf{u}$ in W, all ℓ in $L^r(\Omega)$, and all real numbers $\mathcal{W}$, the transport equation (2.31) has one and only one solution z in $L^r(\Omega)$, with $\mathbf{u}\cdot\nabla\,z$ in $L^r(\Omega)$. It satisfies*

$$\|z\|_{L^r(\Omega)} \leq \|\ell\|_{L^r(\Omega)}\,. \tag{4.11}$$

Proof. Take $2 < r < \infty$. Let $B \supset \overline{\Omega}$ be a smooth ball and let

$$V(B) = \{\mathbf{v} \in H_0^1(B)^k\,;\, \operatorname{div} \mathbf{v} = 0\}\ ,\ \mathcal{V}(B) = \{\mathbf{v} \in \mathcal{D}(B)^k\,;\, \operatorname{div} \mathbf{v} = 0\}\cdot$$

As $\mathbf{u} \in W$, it has an extension, say $\tilde{\mathbf{u}}$ in $V(B)$ (*cf.* [19]); let $\tilde{\ell}$ be the extension of ℓ by zero outside Ω. Since $\mathcal{V}(B)$ is dense in $V(B)$ (*cf.* [19]), there exists a sequence $\mathbf{u}_m \in \mathcal{V}(B)$ such that $\mathbf{u}_m$ converges to $\tilde{\mathbf{u}}$ strongly in $H^1(B)^k$. Then we consider the transport equation in B: Find $z_m \in L^r(B)$ solution of

$$z_m + \mathcal{W}\mathbf{u}_m \cdot \nabla z_m = \tilde{\ell} \ \text{ in } B\,.$$

Ortega establishes in [35] that this equation has a unique solution $z_m \in L^r(B)$ and

$$\|z_m\|_{L^r(B)} \leq \|\tilde{\ell}\|_{L^r(B)} = \|\ell\|_{L^r(\Omega)}\,.$$

Therefore, a subsequence still denoted by z_m converges to a function z, weakly in $L^r(B)$ and

$$\|z\|_{L^r(B)} \leq \|\ell\|_{L^r(\Omega)}\,.$$

An easy argument shows that z is a solution of

$$z + \mathcal{W}\tilde{\mathbf{u}} \cdot \nabla z = \tilde{\ell} \ \text{ in } B\,,$$

and hence $z|_\Omega$ is a solution of (2.31) in Ω and it satisfies (4.11). Its uniqueness follows trivially from the uniqueness of this solution in $L^2(\Omega)$. □

As a consequence, if both $\operatorname{curl} \mathbf{f}$ and $\operatorname{curl} \mathbf{u}$ belong to $L^r(\Omega)$, then the solution z of (0.5) belongs to $L^r(\Omega)$ and

$$\|z\|_{L^r(\Omega)} \leq \nu \|\operatorname{curl} \mathbf{u}\|_{L^r(\Omega)} + |\alpha| \|\operatorname{curl} \mathbf{f}\|_{L^r(\Omega)}\,. \tag{4.12}$$

To simplify, we start the algorithm (4.1–4.3) with $z^0 = 0$. Then (4.4) implies that

$$\|z^n\|_{L^2(\Omega)} \leq \mathcal{K}_1\,.$$

Hence, assuming the hypotheses of Theorem 3.1, we have $\mathbf{u}^n \in W^{2,4/3}(\Omega)^2$ and for $s \leq 4$

$$\|\mathbf{u}^n\|_{W^{1,s}(\Omega)} \leq \mathcal{K}_2\,,$$

where $\mathcal{K}_2$ and all constants below are independent of n. Therefore, if for some r with $2 < r \leq 4$, $\operatorname{curl} \mathbf{f} \in L^r(\Omega)$, (4.12) implies,

$$\|z^n\|_{L^r(\Omega)} \leq \mathcal{K}_3\,.$$

Finally, assume we are in the situation of Proposition 3.14 and suppose that $\mathbf{f} \in L^r(\Omega)^2$ and $\operatorname{curl} \mathbf{f} \in L^r(\Omega)$, where r is given by this proposition. Then applying the regularity argument that is used in proving Proposition 3.14, we have $\mathbf{u}^n \in W^{2,r}(\Omega)^2$ and

$$\|\mathbf{u}^n\|_{W^{2,r}(\Omega)} \leq \mathcal{K}_4\,.$$

Since all the constants above are independent of n, and are small if the data $\mathbf{f}$ and $\mathbf{g}$ are small, this proves Proposition 3.15 for $\mathbf{u}^n$ and Proposition 3.16 for z^n, with an appropriate change in the coefficients $C_1(\mathbf{f})$ and $C_2(\mathbf{g})$. Hence the statement of Theorem 3.22 holds for $z^n - z_h^n$, with other constants, independent of h and n.

Remark 4.4. The fact that Theorem 4.3 holds true for $r > 2$ implies by transposition and duality (*cf.* Lions and Magenes [32]) that if ℓ belongs to $L^r(\Omega)$ with $r > 1$ when $k = 2$ or $r \geq 2k/(k+2)$ when $k \geq 3$, then (2.31) has one and only one solution z in $L^r(\Omega)$ satisfying (4.11). Then, in two dimensions, a fixed-point argument shows that problem (0.1–0.3) has at least one solution for $\mathbf{f} \in L^2(\Omega)^2$ with $\operatorname{curl} \mathbf{f} \in L^r(\Omega)$ for some $r > 1$. □

5. An upwind, streamline diffusion, scheme

The upwinding in the finite-element scheme (0.8–0.10) is obtained by streamline diffusion in the transport equation. This technique was first introduced by Hugues in [27] and studied by Johnson *et al.* in [29], (*cf.* also Johnson [28] and Pironneau [38]). We shall see below that the use of streamline diffusion allows one to derive an estimate for $\mathbf{u}_h \cdot \nabla z_h$, that could not be obtained with a centered scheme. The analysis of this upwind scheme uses several results established in the preceding sections, and therefore we shall only sketch most of the proofs.

To begin with, let $X_{h,T}$, M_h and Z_h be finite-element spaces such that $X_{h,T} \subset H^1_T(\Omega)$, $M_h \subset L^2_0(\Omega)$ and $Z_h \subset H^1(\Omega)$. Let $\mathbf{u}_h$ have approximately zero divergence, *i.e.*

$$\forall q_h \in M_h\,,\ \int_\Omega q_h \operatorname{div} \mathbf{u}_h \,\mathrm{d}\mathbf{x} = 0\,; \tag{5.1}$$

note that the boundary value implies necessarily that

$$\int_\Omega \operatorname{div} \mathbf{u}_h \,\mathrm{d}\mathbf{x} = 0\,.$$

Then Green's formula gives

$$c(\mathbf{u}_h; z_h, \theta_h) = -c(\mathbf{u}_h; \theta_h, z_h) - \int_\Omega (\operatorname{div} \mathbf{u}_h) z_h \theta_h \,\mathrm{d}\mathbf{x}\,.$$

But we cannot bound this last integral because we have no *a priori* estimate for $\|\operatorname{div} \mathbf{u}_h\|_{L^\infty(\Omega)}$; it appears to stem from a $W^{1,\infty}$ *a priori* estimate for $\mathbf{u}_h$, which at the present stage is still an open problem (*cf.* Rem. 3.8). Thus, in view of (5.1), one way to eliminate it, is to ask that the product $z_h\theta_h$ belong to $M_h + \mathbb{R}$ (without enforcing the zero mean-value). Considering that we choose the functions of Z_h to be globally continuous, they must be polynomials of degree at least one in each triangle; this means that the functions of M_h must be polynomials of degree at least two in each triangle. This suggests to use a Hood-Taylor scheme with velocities that are polynomials of degree at least three (*cf.* Brezzi and Falk [10] and Brezzi and Fortin [11]).

Accordingly, we assume that $\mathcal{T}_h$ satisfies (2.1). Since we shall be dealing with Lagrange interpolants, we shall no longer distinguish between singular and nonsingular vertices, with one exception: for the inf-sup condition (5.7) below, we shall forbid triangles with two sides on the boundary. Now, we define the finite-element spaces of lowest degree:

$$X_{h,T} = \{\mathbf{v} \in C^0(\overline{\Omega})^2\,;\, \forall K \in \mathcal{T}_h\,,\ \mathbf{v}|_K \in \mathbb{P}_3^2, \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}\,,\ X_h = X_{h,T} \cap H^1_0(\Omega)^2\,, \tag{5.2}$$

$$M_h = \{q \in C^0(\overline{\Omega})\,;\, \forall K \in \mathcal{T}_h\,,\ q|_K \in \mathbb{P}_2, \int_\Omega q \,\mathrm{d}\mathbf{x} = 0\}\,, \tag{5.3}$$

$$W_h = \{\mathbf{v} \in X_{h,T}\,;\, \forall q \in M_h\,,\ \int_\Omega q \operatorname{div} \mathbf{v} \,\mathrm{d}\mathbf{x} = 0\}\ ,\ V_h = W_h \cap H^1_0(\Omega)^2\,, \tag{5.4}$$

$$Z_h = \{\theta \in C^0(\overline{\Omega})\,;\, \forall K \in \mathcal{T}_h\,,\ \theta|_K \in \mathbb{P}_1\} \cdot \tag{5.5}$$

By construction, Green's formula implies

$$\forall \mathbf{v}_h \in W_h\,,\ \forall z_h, \theta_h \in Z_h\ ,\ c(\mathbf{v}_h; z_h, \theta_h) = -c(\mathbf{v}_h; \theta_h, z_h)\,. \tag{5.6}$$

5.1. A local approximation operator preserving the discrete divergence

We know from [10] that, on a non-degenerate triangulation such that no triangle has two sides on the boundary, the pair of spaces (X_h, M_h) satisfies a uniform discrete inf-sup condition: there exists a constant $\beta^* > 0$, independent of h, such that for all $q_h \in M_h$,

$$\sup_{\mathbf{v}_h \in X_h} \frac{\int_\Omega q_h \operatorname{div} \mathbf{v}_h \, \mathrm{d}\mathbf{x}}{|\mathbf{v}_h|_{H^1(\Omega)}} \geq \beta^* \|q_h\|_{L^2(\Omega)} . \tag{5.7}$$

This yields automatically the existence of an approximation operator preserving the discrete divergence and stable in the H^1 norm. However, such operators are defined globally because they stem from the solution of a discrete Stokes problem in Ω. Now, for the subsequent analysis, we shall require an operator satisfying sharp estimates in L^p and $W^{1,p}$ and this is better established via a local construction, thereby avoiding a duality argument. The proof of (5.7) in [10] uses global arguments, but part of it can be retained and combined with the approach of Boland and Nicolaides [7] and Stenberg [41] (*cf.* [19]) in order to yield local estimates.

So, we propose to construct $P_h \in \mathcal{L}(H_0^1(\Omega)^2; X_h) \cap \mathcal{L}(H_T^1(\Omega); X_{h,T})$, satisfying

$$\forall \mathbf{v} \in H_T^1(\Omega) \, , \; \|\mathbf{v} - P_h(\mathbf{v})\|_{L^p(\Omega)} \leq C\, h^{2/p} |\mathbf{v}|_{H^1(\Omega)} \, , \tag{5.8}$$

$$\forall \mathbf{v} \in H_T^1(\Omega) \cap W^{s,p}(\Omega)^2 \, , \; |\mathbf{v} - P_h(\mathbf{v})|_{W^{m,p}(\Omega)} \leq C\, h^{s-m} |\mathbf{v}|_{W^{s,p}(\Omega)} \, , \tag{5.9}$$

for all numbers p such that $2 \leq p \leq \infty$, all real numbers s with $1 \leq s \leq 4$, $m = 0, 1$, and

$$\forall \mathbf{w} \in H_T^1(\Omega) \, , \; \forall q_h \in M_h \, , \; \int_\Omega q_h \operatorname{div}(P_h(\mathbf{w}) - \mathbf{w}) \, \mathrm{d}\mathbf{x} = 0 \, . \tag{5.10}$$

The construction of P_h proceeds in two steps: first, we construct a suitable approximation operator Π_h satisfying

$$\forall K \in \mathcal{T}_h \, , \; \int_K \operatorname{div}(\Pi_h(\mathbf{w}) - \mathbf{w}) \, \mathrm{d}\mathbf{x} = 0 \, , \tag{5.11}$$

and set

$$P_h(\mathbf{w}) = \Pi_h(\mathbf{w}) + \mathbf{c}_h \, , \tag{5.12}$$

where $\mathbf{c}_h \in X_h$ is an appropriate correction satisfying $\mathbf{c}_h \cdot \mathbf{n} = 0$ on ∂K for all K, and for all $q_h \in M_h$:

$$\int_\Omega q_h \operatorname{div} \mathbf{c}_h \, \mathrm{d}\mathbf{x} = \int_\Omega q_h \operatorname{div}(\mathbf{w} - \Pi_h(\mathbf{w})) \, \mathrm{d}\mathbf{x} \, . \tag{5.13}$$

Note that by virtue of (5.11) and the boundary condition on $\mathbf{c}_h$, this equality remains true if we add any constant to q_h on any K.

For the first step, let Π_h be a regularization operator similar to the one defined by Scott and Zhang in [40] for the cubic element with the following degrees of freedom in each triangle K: values at the vertices of K, integral moments of order 0 and 1 on each edge, integral moment of order 0 on K. In other words, denoting the vertices of K by $\mathbf{a}_i$ and its edges by K_i', $1 \leq i \leq 3$, the degrees of freedom of a polynomial p of $\mathbb{P}_3$ are:

$$p(\mathbf{a}_i) \, , \; 1 \leq i \leq 3 \, ,$$

$$\int_{K_i'} p(s) \, \mathrm{d}s \, , \; \int_{K_i'} p(s) s \, \mathrm{d}s \, , \; 1 \leq i \leq 3 \, ,$$

$$\int_K p(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

In particular, only the point values must be regularized, and we proceed as follows. For each vertex $\mathbf{a}$, we choose a side $\kappa'_{\mathbf{a}}$ with $\mathbf{a}$ as one end-point and such that $\kappa'_{\mathbf{a}}$ lies on the boundary $\partial\Omega$ if $\mathbf{a}$ belongs to $\partial\Omega$. On this side $\kappa'_{\mathbf{a}}$, we define $\psi_{\mathbf{a}}$, the dual basis function of the Lagrange basis functions on $\kappa'_{\mathbf{a}}$, associated with these degrees of freedom (there are four of them in this case), and we set:

$$\Pi_h v(\mathbf{a}) = \int_{\kappa'_{\mathbf{a}}} v(s)\psi_{\mathbf{a}}(s)\,\mathrm{d}s\,.$$

The other degrees of freedom are:

$$\int_K \Pi_h v(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_K v(\mathbf{x})\,\mathrm{d}\mathbf{x}\,,$$

$$\int_{K'} \Pi_h v(s)q(s)\,\mathrm{d}s = \int_{K'} v(s)q(s)\,\mathrm{d}s\,,\ \text{ for all sides } K' \text{ of } K\,,\ \text{ for all } q \in \mathbb{P}_1\,.$$

Then necessarily, for any f in $W^{1,1}(\Omega)$ and any edge K' of K, we have

$$\int_{K'} (\Pi_h(f) - f)\mathrm{d}s = 0\,,$$

so that (5.11) is satisfied. Furthermore, by construction, Π_h preserves two different kinds of boundary conditions: the zero normal component (because the normal vector $\mathbf{n}$ is constant on each segment Γ_i of $\partial\Omega$), *i.e.* $\Pi_h \in \mathcal{L}(H^1_T(\Omega); X_{h,T})$ and the zero trace, *i.e.* $\Pi_h \in \mathcal{L}(H^1_0(\Omega)^2; X_h)$. Finally, Π_h satisfies locally (5.8) and (5.9).

For the second step, to construct $\mathbf{c}_h$, we shall prove an inf-sup condition in each K. First, we associate with each interior edge of $\mathcal{T}_h$, a unit tangent vector whose direction is fixed once and for all. Let $\mathbf{a}_i$, $1 \le i \le 3$, be the vertices of K, e_i the edge opposite $\mathbf{a}_i$ and $\mathbf{t}_i$ the unit tangent vector we have chosen along e_i. Following [10], we define on each edge of K, $e_k = [\mathbf{a}_i, \mathbf{a}_j]$, the two nodes

$$\mathbf{a}_{iij} = (\frac{1}{2} + \theta)\mathbf{a}_i + (\frac{1}{2} - \theta)\mathbf{a}_j\ ,\ \theta = \frac{1}{\sqrt{12}}\,,$$

and we denote by $\mathbf{a}_{123}$ the centroid of K. This choice of nodes is motivated by the quadrature formula established in [10]:

$$\forall p \in \mathbb{P}_4\,,\ \int_K p\,\mathrm{d}\mathbf{x} = |K| \left(\frac{9}{20} p(\mathbf{a}_{123}) - \frac{1}{60} \sum_{i=1}^{3} p(\mathbf{a}_i) + \frac{1}{10} \sum_{i\neq j} p(\mathbf{a}_{iij}) \right)\,. \tag{5.14}$$

Now, let q_h be any polynomial in $\mathbb{P}_2 \cap L^2_0(K)$, and let us associate with q_h a vector-valued polynomial, $\mathbf{v}_h$ of $\mathbb{P}^2_3$, such that

$$\mathbf{v}_h(\mathbf{a}_i) = \mathbf{0}\ ,\ 1 \le i \le 3\,,$$

$$\mathbf{v}_h(\mathbf{a}_{123}) = -|K|\nabla\, q_h(\mathbf{a}_{123})\,, \tag{5.15}$$

and on any side e_k of K that is not on $\partial\Omega$:

$$\mathbf{v}_h \cdot \mathbf{t}_k(\mathbf{a}_{iij}) = -|e_k|^2 (\nabla\, q_h \cdot \mathbf{t}_k)(\mathbf{a}_{iij})\ ,\ \mathbf{v}_h \cdot \mathbf{n}_k(\mathbf{a}_{iij}) = 0\,, \tag{5.16}$$

where $\mathbf{n}_k$ is the normal to e_k, and if e_k lies on $\partial\Omega$, we set $\mathbf{v}_h(\mathbf{a}_{iij}) = \mathbf{0}$.

Lemma 5.1. *Assume that $\mathcal{T}_h$ is non-degenerate and each triangle K has at most one edge on $\partial\Omega$. Then for any triangle K and for all $q_h \in \mathbb{P}_2 \cap L^2_0(K)$, the function $\mathbf{v}_h$ defined above satisfies:*

$$\forall p \geq 2\,,\ \int_K q_h \mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} \geq \hat{c}\|q_h\|_{L^p(K)}\|q_h\|_{L^{p'}(K)}\ ,\ \frac{1}{p}+\frac{1}{p'}=1\,, \tag{5.17}$$

$$\forall p \geq 2\,,\ \|\mathbf{v}_h\|_{L^p(K)} \leq \hat{c} h_K^{2/p}\|q_h\|_{L^2(K)}\,, \tag{5.18}$$

$$\forall p \geq 2\,,\ |\mathbf{v}_h|_{W^{1,p}(K)} \leq \hat{c}\|q_h\|_{L^p(K)}\,, \tag{5.19}$$

where $\hat{c}$ denote several constants, depending possibly on p, but independent of h, K, q_h and $\mathbf{v}_h$. Note that in these three inequalities, the exponents p are independent of each other and can be infinite.

Proof. Let us consider the case where K has one side, say e_3, on the boundary, the general case of an interior triangle being similar. Since by definition, $\mathbf{v}_h \cdot \mathbf{n} = 0$ on ∂K, we have

$$\int_K q_h \mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} = -\int_K \mathbf{v}_h \cdot \nabla\, q_h \,\mathrm{d}\mathbf{x}\,,$$

and since the integrand belongs to $\mathbb{P}_4$, the quadrature formula (5.14) gives

$$\int_K q_h \mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} = |K|\left(\frac{9}{20}|K|\|\nabla\, q_h(\mathbf{a}_{123})\|^2 + \frac{1}{10}|e_1|^2\sum_{i\neq j\neq 1}|\nabla\, q_h\cdot \mathbf{t}_1(\mathbf{a}_{iij})|^2 + \frac{1}{10}|e_2|^2\sum_{i\neq j\neq 2}|\nabla\, q_h\cdot \mathbf{t}_2(\mathbf{a}_{iij})|^2\right).$$

Let us pass to the reference triangle $\hat{K}$ in such a way that e_1 and e_2 are mapped respectively on the $\hat{x}_1$ and $\hat{x}_2$ axes. To simplify, set $\mathbf{w}_h = \nabla\, q_h$. Considering that the tangent is preserved by affine transformations, the above formula becomes

$$\int_K q_h \mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} = |K|\left(\frac{9}{20}|K|\|\hat{\mathbf{w}}(\hat{\mathbf{a}}_{123})\|^2 + \frac{1}{10}|e_1|^2\sum_{i\neq j\neq 1}|\hat{w}_1(\hat{\mathbf{a}}_{iij})|^2 + \frac{1}{10}|e_2|^2\sum_{i\neq j\neq 2}|\hat{w}_2(\hat{\mathbf{a}}_{iij})|^2\right). \tag{5.20}$$

But $\hat{\mathbf{w}} \in \mathbb{P}_1^2$; therefore the above right-hand side vanishes if and only if $\hat{\mathbf{w}} = \mathbf{0}$ in $\hat{K}$. Note that here we use the fact that K has at most one side on $\partial\Omega$; otherwise, $\hat{\mathbf{w}}$ does not necessarily vanish on $\hat{K}$. Thus the above right-hand side defines a norm on $\mathbb{P}_1^2$, and since on this space all norms are equivalent, this implies

$$\int_K q_h \mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} \geq \hat{c}\rho_K^2|K|\|\widehat{\nabla\, q_h}\|^2_{L^2(\hat{K})} \geq \hat{c}\rho_K^2|K|((B^tB)^{-1}\hat{\nabla}\,\hat{q}, \hat{\nabla}\,\hat{q})_{\hat{K}} \geq \hat{c}|K|\|\hat{\nabla}\,\hat{q}\|^2_{L^2(\hat{K})}\,,$$

where B denotes the matrix of the transformation from $\hat{K}$ onto K. But $\hat{q}$ has zero mean-value in $\hat{K}$ and hence

$$\int_K q_h \mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} \geq \hat{c}|K|\|\hat{q}\|_{L^p(\hat{K})}\|\hat{q}\|_{L^{p'}(\hat{K})} \geq \hat{c}\|q_h\|_{L^p(K)}\|q_h\|_{L^{p'}(K)}\ ,\ \frac{1}{p}+\frac{1}{p'}=1\,.$$

Finally (5.18) and (5.19) are easily derived from (5.15) and (5.16). □

Remark 5.2. Note that every function $\mathbf{v}_h \in \mathbb{P}_k^2$ such that

$$\mathbf{v}_h \cdot \mathbf{n} = 0 \text{ on } \partial K\,,$$

satisfies

$$\|\mathbf{v}_h\|_{L^p(K)} \leq \hat{c} h_K^{2/p} |\mathbf{v}_h|_{H^1(K)} \,. \tag{5.21}$$

Indeed, since B is an invertible mapping, the fact that $\mathbf{v}_h \cdot \mathbf{n} = 0$ on ∂K implies that if $\hat{\mathbf{v}} = \mathbf{c}$ on $\hat{K}$ then $\mathbf{c} = \mathbf{0}$. Therefore, as $\hat{\mathbf{v}}$ belongs to a finite-dimensional space,

$$\|\mathbf{v}_h\|_{L^p(K)} \leq \hat{c} |K|^{1/p} \|\hat{\mathbf{v}}\|_{L^p(\hat{K})} \leq \hat{c} |K|^{1/p} |\hat{\mathbf{v}}|_{H^1(\hat{K})} \leq \hat{c} h_K^{2/p} |\mathbf{v}_h|_{H^1(K)} \,.$$

□

For any $q_h \in M_h$, let $\tilde{q}_h$ be defined in each K by

$$\tilde{q}_h = q_h - \frac{1}{|K|} \int_K q_h \, \mathrm{d}\mathbf{x} \,,$$

and let

$$\begin{aligned} \mathcal{M}_h &= \{\tilde{q}_h \,;\, q_h \in M_h\} \,, \\ \mathcal{X}_h &= \{\mathbf{v}_h \in X_h \,;\, \forall K \in \mathcal{T}_h \,,\, \mathbf{v}_h \cdot \mathbf{n} = 0 \text{ on } \partial K\} \,, \\ \mathcal{V}_h &= \{\mathbf{v}_h \in \mathcal{X}_h \,;\, \forall q_h \in \mathcal{M}_h \,,\, \int_\Omega q_h \mathrm{div}\, \mathbf{v}_h \, \mathrm{d}\mathbf{x} = 0\} \cdot \end{aligned}$$

Note that $\mathcal{M}_h$ is a vector space and

$$\mathcal{V}_h = \{\mathbf{v}_h \in \mathcal{X}_h \,;\, \forall q_h \in M_h \,,\, \int_\Omega q_h \mathrm{div}\, \mathbf{v}_h \, \mathrm{d}\mathbf{x} = 0\} \,,$$

owing to the trace condition on the functions of $\mathcal{X}_h$. Clearly, if $\tilde{q}_h \in \mathcal{M}_h$, the function $\mathbf{v}_h$ constructed for Lemma 5.1 in each K belongs globally to $\mathcal{X}_h$ because $\nabla \tilde{q}_h \cdot \mathbf{t} = \nabla q_h \cdot \mathbf{t}$ is continuous across the edges K'.

Let $\mathbf{a}$ be an interior vertex of $\mathcal{T}_h$ and let $\Omega_\mathbf{a}$ be the union of all triangles of $\mathcal{T}_h$ sharing this vertex. Let $\mathcal{N}_h$ denote a set of interior vertices of $\mathcal{T}_h$, chosen so that

$$\overline{\Omega} = \cup_{\mathbf{a} \in \mathcal{N}_h} \overline{\Omega}_\mathbf{a} \,.$$

Now, assume that each triangle K has at most one edge on $\partial\Omega$; then the triangles of $\Omega_\mathbf{a}$ also have at most one edge on $\partial\Omega_\mathbf{a}$. We can define the analogues, $\mathcal{X}_h(\Omega_\mathbf{a})$, $\mathcal{M}_h(\Omega_\mathbf{a})$ and $\mathcal{V}_h(\Omega_\mathbf{a})$, of $\mathcal{X}_h$, $\mathcal{M}_h$ and $\mathcal{V}_h$, on $\Omega_\mathbf{a}$ instead of Ω, namely

$$\begin{aligned} \mathcal{X}_h(\Omega_\mathbf{a}) &= \{\mathbf{v}_h \in \mathcal{X}_h \,;\, \mathbf{v}_h = 0 \text{ on } \partial\Omega_\mathbf{a}, \text{ extended by zero outside}\} \,, \\ \mathcal{M}_h(\Omega_\mathbf{a}) &= \{\tilde{q}_h|\Omega_\mathbf{a} \,;\, \tilde{q}_h \in \mathcal{M}_h\} \,, \\ \mathcal{V}_h(\Omega_\mathbf{a}) &= \{\mathbf{v}_h \in \mathcal{X}_h(\Omega_\mathbf{a}) \,;\, \forall q_h \in \mathcal{M}_h(\Omega_\mathbf{a}) \,,\, \int_\Omega q_h \mathrm{div}\, \mathbf{v}_h \, \mathrm{d}\mathbf{x} = 0\} \cdot \end{aligned} \tag{5.22}$$

Note that

$$\mathcal{V}_h(\Omega_\mathbf{a}) = \{\mathbf{v}_h \in \mathcal{X}_h(\Omega_\mathbf{a}) \,;\, \forall q_h \in M_h \,,\, \int_\Omega q_h \mathrm{div}\, \mathbf{v}_h \, \mathrm{d}\mathbf{x} = 0\} \,, \tag{5.23}$$

because the functions in $\mathcal{X}_h(\Omega_\mathbf{a})$ satisfy $\mathbf{v}_h \cdot \mathbf{n} = 0$ on ∂K for all triangles K contained in $\Omega_\mathbf{a}$. We have the following extension of Lemma 5.1 on $\Omega_\mathbf{a}$.

Lemma 5.3. *We retain the assumptions of Lemma 5.1. Let* $\mathbf{a} \in \mathcal{N}_h$*, let* L *be the number of triangles in* $\Omega_{\mathbf{a}}$*, and let* p *be any number with* $2 \le p \le \infty$*; for each* $q_h \in \mathcal{M}_h(\Omega_{\mathbf{a}})$*, there exists* $\mathbf{v}_h \in \mathcal{X}_h(\Omega_{\mathbf{a}})$ *such that*

$$\int_{\Omega_{\mathbf{a}}} q_h \mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} \ge \hat{c}(p,L)\|q_h\|_{L^p(\Omega_{\mathbf{a}})}\|q_h\|_{L^{p'}(\Omega_{\mathbf{a}})}\ ,\ \frac{1}{p}+\frac{1}{p'}=1\,, \tag{5.24}$$

$$|\mathbf{v}_h|_{W^{1,p}(\Omega_{\mathbf{a}})} \le \|q_h\|_{L^p(\Omega_{\mathbf{a}})}\,, \tag{5.25}$$

where the constant $\hat{c}(p,L)$ *depends on* p *and* L*, but is independent of* h*,* $\mathbf{a}$*,* $\Omega_{\mathbf{a}}$*,* q_h *and* $\mathbf{v}_h$*.*

Proof. We briefly sketch the proof, which is an easy variant of that of Lemma 5.1. Formula (5.20) is applied in each K and summed over all triangles of $\Omega_{\mathbf{a}}$. But $\Omega_{\mathbf{a}}$ is the image by a piecewise affine mapping of a reference macro-element $\hat{\Omega}_{\mathbf{a}}$ that is a regular L-gon (*cf.* for instance [6]). Hence, the equivalences of norms used in proving Lemma 5.1 hold on $\hat{\Omega}_{\mathbf{a}}$, with constants that depend on L and p, but not on h and $\mathbf{a}$. This allows us to apply the final arguments of Lemma 5.1 first on $\hat{\Omega}_{\mathbf{a}}$ and next on $\Omega_{\mathbf{a}}$ The constant one in (5.25) is obtained by scaling $\mathbf{v}_h$. □

Theorem 5.4. *If* $\mathcal{T}_h$ *is non-degenerate and each triangle* K *has at most one edge on* $\partial\Omega$*, then there exists an operator* $P_h \in \mathcal{L}(H^1_0(\Omega)^2; X_h) \cap \mathcal{L}(H^1_T(\Omega); X_{h,T})$ *satisfying (5.8), (5.9) and (5.10).*

Proof. Given a triangle K of $\mathcal{T}_h$, let $\mathbf{a} = \mathbf{a}(K)$ be a vertex of $\mathcal{N}_h$ chosen such that K belongs to $\Omega_{\mathbf{a}}$. Then for $\mathbf{a} \in \mathcal{N}_h$, let $\Delta_{\mathbf{a}}$ denote the union of the triangles associated by this mapping with $\mathbf{a}$. On one hand,

$$\Delta_{\mathbf{a}} \subset \Omega_{\mathbf{a}}\,,$$

and on the other hand, the set $\{\Delta_{\mathbf{a}}\,;\, \mathbf{a} \in \mathcal{N}_h\}$ is a partition of Ω, since by this mapping, each triangle is counted exactly once.

As $\mathcal{T}_h$ is non-degenerate, it is locally quasi-uniform and therefore $\Omega_{\mathbf{a}}$ has at most L triangles, where L is independent of h and $\mathbf{a}$. It follows from Lemma 5.3 that the pair of spaces $(\mathcal{X}_h(\Omega_{\mathbf{a}}), \mathcal{M}_h(\Omega_{\mathbf{a}}))$ satisfies a family of inf-sup conditions, with different norms, but uniform with respect to h and $\mathbf{a}$. Therefore, for any $f \in L^2(\Omega)$ such that $f \in L^2_0(K)$ on all K, and for each $\mathbf{a} \in \mathcal{N}_h$, there exists a unique element $\mathbf{c}_{h,\mathbf{a}} \in \mathcal{V}_h(\Omega_{\mathbf{a}})^\perp$, the orthogonal complement of $\mathcal{V}_h(\Omega_{\mathbf{a}})$ in $\mathcal{X}_h(\Omega_{\mathbf{a}})$ for the scalar product $(\nabla\,\mathbf{u}, \nabla\,\mathbf{v})$, solution of

$$\forall \tilde{q}_h \in \mathcal{M}_h(\Omega_{\mathbf{a}})\,,\ \int_{\Omega_{\mathbf{a}}} \tilde{q}_h \mathrm{div}\, \mathbf{c}_{h,\mathbf{a}} \,\mathrm{d}\mathbf{x} = \int_{\Delta_{\mathbf{a}}} f\, \tilde{q}_h \,\mathrm{d}\mathbf{x}\,. \tag{5.26}$$

Furthermore, the Babuška-Brezzi Theorem (*cf.* Babuška [4], Brezzi [9] or [19]) and Lemma 5.3 with $p = 2$ yield

$$|\mathbf{c}_{h,\mathbf{a}}|_{H^1(\Omega_{\mathbf{a}})} \le \frac{1}{\hat{c}(2,L)}\|f\|_{L^2(\Delta_{\mathbf{a}})}\,, \tag{5.27}$$

where $\hat{c}(2,L)$ is the constant of (5.24) with $p = 2$. In turn (5.21) and (5.27) give, for any number p such that $2 \le p \le \infty$:

$$\|\mathbf{c}_{h,\mathbf{a}}\|_{L^p(\Omega_{\mathbf{a}})} \le \hat{c} h^{2/p}\|f\|_{L^2(\Delta_{\mathbf{a}})}\,. \tag{5.28}$$

In addition, we easily derive that, for any number p with $2 \le p \le \infty$,

$$|\mathbf{c}_{h,\mathbf{a}}|_{W^{1,p}(\Omega_{\mathbf{a}})} \le \hat{c}\frac{{h_{\mathbf{a}}}^{2/p}}{\rho_{\mathbf{a}}}|\mathbf{c}_{h,\mathbf{a}}|_{H^1(\Omega_{\mathbf{a}})}\,, \tag{5.29}$$

where

$$h_{\mathbf{a}} = \sup_{K \in \Omega_{\mathbf{a}}} h_K \ , \ \rho_{\mathbf{a}} = \inf_{K \in \Omega_{\mathbf{a}}} \rho_K \, ,$$

and the constant $\hat{c}$ depends on p but is independent of $\mathbf{a}$ and h. The non-degeneracy of $\mathcal{T}_h$ implies in particular that $h_{\mathbf{a}}/\rho_{\mathbf{a}} \le \sigma_0$, *i.e.* it satisfies the analogue of (2.1).

Now, let us choose $f = \operatorname{div}(\mathbf{w} - \Pi_h(\mathbf{w}))$ (that belongs indeed to $L^2_0(K)$ in each K) and let p be any number such that $2 \le p \le \infty$. On one hand, (5.28) and the local approximation properties of Π_h imply directly:

$$\|\mathbf{c}_{h,\mathbf{a}}\|_{L^p(\Omega_{\mathbf{a}})} \le \hat{c}\, h^{2/p} |\mathbf{w}|_{H^1(\tilde{\Delta}_{\mathbf{a}})} \, , \tag{5.30}$$

and on the other hand (5.29), (5.27), (2.1), the fact that $\Omega_{\mathbf{a}}$ has at most L triangles with L independent of h, and the local approximation properties of Π_h imply:

$$|\mathbf{c}_{h,\mathbf{a}}|_{W^{1,p}(\Omega_{\mathbf{a}})} \le \hat{c} \frac{{h_{\mathbf{a}}}^{2/p}}{\rho_{\mathbf{a}}} |\mathbf{w} - \Pi_h(\mathbf{w})|_{H^1(\Delta_{\mathbf{a}})} \le \hat{c} |\mathbf{w}|_{W^{1,p}(\tilde{\Delta}_{\mathbf{a}})} \, , \tag{5.31}$$

where $\tilde{\Delta}_{\mathbf{a}}$ denotes the macro-element required for defining $\Pi_h(\mathbf{w})$ in $\Delta_{\mathbf{a}}$, and again $\tilde{\Delta}_{\mathbf{a}}$ is the union of at most $\tilde{L}$ triangles, with $\tilde{L}$ independent of $\mathbf{a}$ and h. More generally, we can easily prove in the same way that

$$|\mathbf{c}_{h,\mathbf{a}}|_{W^{m,p}(\Omega_{\mathbf{a}})} \le \hat{c}\, h^{s-m} |\mathbf{w}|_{W^{s,p}(\tilde{\Delta}_{\mathbf{a}})} \, , \tag{5.32}$$

for $s \in [1, 4]$ and $m = 0, 1$.

Finally, let us extend $\mathbf{c}_{h,\mathbf{a}}$ by zero outside $\Omega_{\mathbf{a}}$ and set

$$\mathbf{c}_h = \sum_{\mathbf{a} \in \mathcal{N}_h} \mathbf{c}_{h,\mathbf{a}} \, .$$

Then $\mathbf{c}_h \in \mathcal{X}_h$ and summing (5.26) over all $\mathbf{a} \in \mathcal{N}_h$, we obtain for all $\tilde{q}_h \in \mathcal{M}_h$:

$$\int_\Omega \tilde{q}_h \operatorname{div} \mathbf{c}_h \, \mathrm{d}\mathbf{x} = \sum_{\mathbf{a} \in \mathcal{N}_h} \int_{\Omega_{\mathbf{a}}} \tilde{q}_h \operatorname{div} \mathbf{c}_{h,\mathbf{a}} \, \mathrm{d}\mathbf{x} = \sum_{\mathbf{a} \in \mathcal{N}_h} \int_{\Delta_{\mathbf{a}}} f\, \tilde{q}_h \, \mathrm{d}\mathbf{x} = \int_\Omega f\, \tilde{q}_h \, \mathrm{d}\mathbf{x} \, ,$$

i.e.

$$\forall q_h \in M_h \, , \int_\Omega q_h \operatorname{div} \mathbf{c}_h \, \mathrm{d}\mathbf{x} = \int_\Omega f\, q_h \, \mathrm{d}\mathbf{x} \, .$$

Then setting $P_h(\mathbf{w}) = \Pi_h(\mathbf{w}) + \mathbf{c}_h$, (5.8) and (5.9) follow by summing (5.30) and (5.32) over all $\mathbf{a} \in \mathcal{N}_h$ and using the fact that the maximum number of occurrences of a given triangle K in the set of all $\Omega_{\mathbf{a}}$ and all $\tilde{\Delta}_{\mathbf{a}}$ is bounded by a fixed constant independent of h. □

Remark 5.5. Since the construction of P_h is local, the distance between support($P_h(\mathbf{v})$) and support($\mathbf{v}$) is of the order of h. □

5.2. Existence of discrete solutions

Problem P is discretized as follows: *Find* $(\mathbf{u}_h, p_h, z_h)$ *in* $(X_h + \mathbf{g}_h) \times M_h \times Z_h$ *satisfying (0.8–0.10):*

$$\forall \mathbf{v}_h \in X_h \, , \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{z}_h \times \mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \, ,$$

$$\forall q_h \in M_h \, , (q_h, \operatorname{div} \mathbf{u}_h) = 0 \, ,$$

$$\forall \theta_h \in Z_h\,,\ \nu\,(z_h,\theta_h + \delta \mathbf{u}_h \cdot \nabla\,\theta_h) + \alpha\,(\mathbf{u}_h \cdot \nabla\, z_h, \theta_h + \delta \mathbf{u}_h \cdot \nabla\,\theta_h)$$
$$= \nu\,(\mathrm{curl}\,\mathbf{u}_h, \theta_h + \delta \mathbf{u}_h \cdot \nabla\,\theta_h) + \alpha\,(\mathrm{curl}\,\mathbf{f}, \theta_h + \delta \mathbf{u}_h \cdot \nabla\,\theta_h)\,,$$

where $\mathbf{z}_h = (0,0,z_h)$, δ is an arbitrary parameter (to be chosen later) such that the product $\alpha\,\delta$ is non-negative and, as in Section 3, $\mathbf{g}_h = P_h(\mathbf{r})$ where $\mathbf{r}$ is any lifting of $\mathbf{g}$ in W. Note that here also, on one hand, $\mathbf{g}_h$ can be constructed directly by interpolating $\mathbf{g}$ on $\partial\Omega$ with the analogue of Π_h, and on the other hand, $\mathbf{g}_h$ does not depend on the particular lifting chosen; in addition, $\mathbf{g}_h$ satisfies (2.12).

Unfortunately, as the divergence of the discrete functions is not exactly zero, we cannot apply Theorem 2.1, whose proof is derived by expressing the discrete lifting as a **curl**. We shall revert instead to the following result established in [21], for the lifting of the exact solution, and take advantage of the sharp estimates for P_h.

Theorem 5.6. *Let Ω be a Lipschitz polygon. For any $\mathbf{g} \in H^{1/2}(\partial\Omega)^2$ satisfying $\mathbf{g}\cdot\mathbf{n} = 0$ and for any real number $\varepsilon > 0$, there exists a function $\mathbf{u}_\mathbf{g} \in W$, whose support is contained in Ω_ε (cf. (2.13)), depending continuously on $\mathbf{g}$, such that $\mathbf{u}_\mathbf{g} = \mathbf{g}$ on $\partial\Omega$,*

$$\|\mathbf{u}_\mathbf{g}\|_{L^s(\Omega)} \le C\varepsilon^{1/s-\delta}\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\,,\ 1 \le s < \infty\,,\ 0 < \delta \le \frac{1}{s}\,, \tag{5.33}$$

$$|\mathbf{u}_\mathbf{g}|_{H^1(\Omega)} \le C\varepsilon^{-1/2-\delta}\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\,,\ 0 < \delta \le \frac{1}{2}\,, \tag{5.34}$$

$$\forall \mathbf{v} \in H_0^1(\Omega)^2\,,\ \big\||\mathbf{u}_\mathbf{g}|\,|\mathbf{v}|\big\|_{L^2(\Omega)} \le C\varepsilon^{1-\delta}\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}|\mathbf{v}|_{H^1(\Omega)}\,,\ 0 < \delta < 1\,, \tag{5.35}$$

where C denotes various constants that depend on δ or on s and δ, but are independent of ε and $\mathbf{g}$.

Then Theorem 2.1 is replaced by:

Lemma 5.7. *Under the assumptions of Theorem 5.4, for any $\mathbf{g} \in H^{1/2}(\partial\Omega)^2$ such that $\mathbf{g}\cdot\mathbf{n} = 0$ and for any real number $\varepsilon > 0$, the lifting $\mathbf{u}_{h,\mathbf{g}} = P_h(\mathbf{u}_\mathbf{g})$, with the function $\mathbf{u}_\mathbf{g}$ of Theorem 5.6, satisfies*

$$|\mathbf{u}_{h,\mathbf{g}}|_{H^1(\Omega)} \le C\varepsilon^{-1/2-\delta}\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\,,\ 0 < \delta \le \frac{1}{2}\,, \tag{5.36}$$

and if $h_b < \varepsilon$, where h_b is the maximum diameter of triangles intersecting Ω_ε, then for all $\mathbf{v} \in H_0^1(\Omega)^2$ and $0 < \delta < \frac{1}{2}$,

$$\big\||\mathbf{u}_{h,\mathbf{g}}|\,|\mathbf{v}|\big\|_{L^2(\Omega)} \le C\varepsilon^{1/2-\delta}\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}|\mathbf{v}|_{H^1(\Omega)}\,, \tag{5.37}$$

where the constants C depend on δ, but are independent of h, ε and $\mathbf{g}$.

Proof. The first inequality follows immediately from (5.9) with $s = m = 1$, and (5.34). The second result is obtained by applying first Hölder's inequality and next (5.8) with $p = 2 + 2/t$ for any real number $0 < t < \infty$,

$$\begin{aligned}\big\||\mathbf{u}_{h,\mathbf{g}}|\,|\mathbf{v}|\big\|_{L^2(\Omega)} &\le \|\mathbf{u}_{h,\mathbf{g}}\|_{L^{2+2/t}(\Omega)}\|\mathbf{v}\|_{L^{2+2t}(\Omega)}\\ &\le \|\mathbf{v}\|_{L^{2+2t}(\Omega)}\big(\|\mathbf{u}_{h,\mathbf{g}} - \mathbf{u}_\mathbf{g}\|_{L^{2+2/t}(\Omega)} + \|\mathbf{u}_\mathbf{g}\|_{L^{2+2/t}(\Omega)}\big)\\ &\le \|\mathbf{v}\|_{L^{2+2t}(\Omega)}\big(Ch_b^{\frac{t}{1+t}}|\mathbf{u}_\mathbf{g}|_{H^1(\Omega)} + \|\mathbf{u}_\mathbf{g}\|_{L^{2+2/t}(\Omega)}\big)\,,\end{aligned}$$

because the support of $P_h(\mathbf{u}_\mathbf{g})$ is contained in $\Omega_{\varepsilon+h_b}$. Then (5.37) follows by substituting (5.33) and (5.34) into this inequality. □

Now, we have the analogue of Lemma 2.2, with a similar proof.

Lemma 5.8. *Under the assumptions of Theorem 5.4, for each $z_h \in Z_h$, (0.8), (0.9) has a unique solution $\mathbf{u}_h \in X_h + \mathbf{g}_h$. This solution satisfies the estimate (2.18):*

$$|\mathbf{u}_h|_{H^1(\Omega)} \leq \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + K_1(h)C_1T\|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}\,,$$

where C_1 is a constant independent of h, and T is the constant of (1.3). Moreover, there exists a constant $C_2 > 0$, independent of h, such that for all $\varepsilon > 0$, if for some $t > 0$,

$$h_b < C_2\varepsilon^{2+t}\|\mathbf{g}\|^{-2-t}_{H^{1/2}(\partial\Omega)}\,, \tag{5.38}$$

then for any real number $s > \frac{t}{2}$,

$$|\mathbf{u}_h|_{H^1(\Omega)} \leq \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(\Omega)} + \frac{C_3}{\varepsilon^{1+s}}\|\mathbf{g}\|^{2+s}_{H^{1/2}(\partial\Omega)} + \frac{\varepsilon}{\nu}\|z_h\|_{L^2(\Omega)}\,, \tag{5.39}$$

where the constant C_3 depends on s and t, but not on h, ν and ε.

This result gives us the following existence theorem.

Theorem 5.9. *Let the triangulation be as in Theorem 5.4. The constant C_2 of (5.38) is such that for all $\nu > 0$ and $\alpha \in \mathbb{R}$, for all $\mathbf{f}$ in $H(curl,\Omega)$ and all $\mathbf{g}$ in $H^{1/2}(\partial\Omega)^2$ satisfying $\mathbf{g}\cdot\mathbf{n} = 0$, if*

$$h_b < C_2\nu^{2+t}\|\mathbf{g}\|^{-2-t}_{H^{1/2}(\partial\Omega)}\,,\ \text{for some } t > 0\,, \tag{5.40}$$

then the discrete problem (0.8–0.10) with any $\delta \in \mathbb{R}$ such that $\alpha\,\delta > 0$, has at least one solution $(\mathbf{u}_h, p_h, z_h)$ in $(X_h + \mathbf{g}_h) \times M_h \times Z_h$ and every solution satisfies the a priori estimates (2.18), (5.39) with the same constant C_3, and (2.45). Moreover, for any $s > \frac{t}{2}$,

$$\frac{\nu}{4}\|z_h\|^2_{L^2(\Omega)} + \frac{\alpha\,\delta}{2}\|\mathbf{u}_h\cdot\nabla z_h\|^2_{L^2(\Omega)} \leq C_4\big(\frac{S_\delta}{\alpha}\big)^{2+s}\nu^{-1-2s}\|\mathbf{g}\|^{4+2s}_{H^{1/2}(\partial\Omega)} + S_\delta\big(\frac{\alpha}{\nu}\|\mathrm{curl}\,\mathbf{f}\|^2_{L^2(\Omega)} + \frac{6}{\alpha}\frac{S_2^2}{\nu}\|\mathbf{f}\|^2_{L^2(\Omega)}\big)\,, \tag{5.41}$$

where $S_\delta = \alpha + \delta\nu$ has the same sign as α.

Proof. The only part of the proof that differs from that of Theorem 2.3 is the estimate (5.41). By choosing $\theta_h = z_h$ in (0.10) and applying (5.6), we readily obtain

$$\frac{\nu}{2}\|z_h\|^2_{L^2(\Omega)} + \frac{\alpha\,\delta}{2}\|\mathbf{u}_h\cdot\nabla z_h\|^2_{L^2(\Omega)} \leq S_\delta(2\frac{\nu}{\alpha}|\mathbf{u}_h|^2_{H^1(\Omega)} + \frac{\alpha}{\nu}\|\mathrm{curl}\,\mathbf{f}\|^2_{L^2(\Omega)})\,. \tag{5.42}$$

Then (5.41) is derived by substituting (5.39) into (5.42) with the choice

$$\varepsilon = \frac{\nu}{2\sqrt{6}}\sqrt{\frac{\alpha}{S_\delta}}\ ,\ i.e.\ \varepsilon = O(\nu)\,.$$

Note that the proof of this theorem requires (5.6). This is one of the reasons why we choose compatible spaces M_h and Z_h. □

Remark 5.10. Note that, in contrast to (2.24), (5.41) does not allow α to tend to zero. □

5.3. Convergence

As far as convergence is concerned, the discussion splits according to the choice of δ. Assume first that δ is independent of h. The above uniform bounds allow us to prove that (a subsequence of) the sequences $\mathbf{u}_h$, p_h, z_h and $\mathbf{u}_h \cdot \nabla z_h$ converge weakly to $\mathbf{u}$ in W, to p in $L^2_0(\Omega)$, to z in $L^2(\Omega)$ and to w in $L^2(\Omega)$ respectively as h tends to zero. For proving that $w = \mathbf{u} \cdot \nabla z$, we consider $c(\mathbf{u}_h; z_h, R_h(\varphi))$ for $\varphi \in \mathcal{D}(\Omega)$. On one hand,

$$\lim_{h \to 0} c(\mathbf{u}_h; z_h, R_h(\varphi)) = (w, \varphi)\,,$$

and on the other hand, (5.6), the strong convergence of $\mathbf{u}_h$ in $L^4(\Omega)^2$ and the fact that z belongs to $X_{\mathbf{u}}$ imply that

$$\lim_{h \to 0} c(\mathbf{u}_h; z_h, R_h(\varphi)) = -\lim_{h \to 0} c(\mathbf{u}_h; R_h(\varphi), z_h) = -c(\mathbf{u}; \varphi, z) = (\mathbf{u} \cdot \nabla z, \varphi)\,.$$

For proving that $(\mathbf{u}, p, z)$ is a solution of *Problem P*, we proceed as in Section 2. Choosing again $\theta_h = R_h(\varphi)$ for $\varphi \in \mathcal{D}(\Omega)$ and passing to the limit in (0.10), we find

$$\forall \varphi \in \mathcal{D}(\Omega)\,,\ \nu\,(z, \varphi + \delta \mathbf{u} \cdot \nabla \varphi) + \alpha\,(\mathbf{u} \cdot \nabla z, \varphi + \delta \mathbf{u} \cdot \nabla \varphi) = \nu\,(\operatorname{curl} \mathbf{u}, \varphi + \delta \mathbf{u} \cdot \nabla \varphi) + \alpha\,(\operatorname{curl} \mathbf{f}, \varphi + \delta \mathbf{u} \cdot \nabla \varphi)\,. \tag{5.43}$$

Now, let θ be any function in $X_{\mathbf{u}}$. Corollary 2.9 states that there exists a sequence of functions φ_m contained in $\mathcal{D}(\Omega)$, that converges strongly to θ in $X_{\mathbf{u}}$. Thus taking $\varphi = \varphi_m$ in (5.43), and taking the limit with respect to m, this convergence and the fact that z belongs to $X_{\mathbf{u}}$ yield (5.43) for any function $\theta \in X_{\mathbf{u}}$. Then in view of Corollary 2.8, we recover (0.5).

The strong convergence of $\mathbf{u}_h$ is proven as in Section 2 and it suffices to examine z_h. Again choosing z_h for test function in (0.10), and using the strong convergence of $\mathbf{u}_h$, we obtain:

$$\lim_{h \to 0} (\nu \|z_h\|^2_{L^2(\Omega)} + \alpha\,\delta \|\mathbf{u}_h \cdot \nabla z_h\|^2_{L^2(\Omega)}) = (\nu \operatorname{curl} \mathbf{u} + \alpha \operatorname{curl} \mathbf{f}, z + \delta \mathbf{u} \cdot \nabla z)\,.$$

Next, substituting (0.5) in the right-hand side, this becomes

$$\lim_{h \to 0} (\nu \|z_h\|^2_{L^2(\Omega)} + \alpha\,\delta \|\mathbf{u}_h \cdot \nabla z_h\|^2_{L^2(\Omega)}) = \nu \|z\|^2_{L^2(\Omega)} + \alpha\,\delta \|\mathbf{u} \cdot \nabla z\|^2_{L^2(\Omega)}\,. \tag{5.44}$$

Therefore

$$\lim_{h \to 0} \big(\nu \|z_h - z\|^2_{L^2(\Omega)}\big) = \alpha\,\delta \big(\|\mathbf{u} \cdot \nabla z\|^2_{L^2(\Omega)} - \lim_{h \to 0} \|\mathbf{u}_h \cdot \nabla z_h\|^2_{L^2(\Omega)}\big) \le 0\,.$$

Hence

$$\lim_{h \to 0} \big(\nu \|z_h - z\|^2_{L^2(\Omega)}\big) = 0\,.$$

Together with (5.44), this also implies the strong convergence of $\mathbf{u}_h \cdot \nabla z_h$.

Now, let us adapt the convergence analysis to the case where $\delta = \operatorname{sign}(\alpha) h$, which will be our future choice. Then (5.42) implies that $\sqrt{\alpha\delta}(\mathbf{u}_h \cdot \nabla z_h)$ converges weakly to some function λ in $L^2(\Omega)$, the other convergences being unchanged. Let us pass to the limit in (0.10) with $\theta_h = R_h(\varphi)$ for $\varphi \in \mathcal{D}(\Omega)$. Then since $\sqrt{\alpha\delta}(\mathbf{u}_h \cdot \nabla R_h(\varphi))$ tends to zero strongly in $L^2(\Omega)$, we have

$$\lim_{h \to 0} \nu(z_h, \delta \mathbf{u}_h \cdot \nabla R_h(\varphi)) = 0\ ,\ \lim_{h \to 0} \alpha\delta(\mathbf{u}_h \cdot \nabla z_h, \mathbf{u}_h \cdot \nabla R_h(\varphi)) = 0\,.$$

Therefore we obtain in the limit

$$\forall \varphi \in \mathcal{D}(\Omega)\,,\ \nu(z, \varphi) + \alpha(\mathbf{u} \cdot \nabla z, \varphi) = \nu(\operatorname{curl} \mathbf{u}, \varphi) + \alpha(\operatorname{curl} \mathbf{f}, \varphi)\,.$$

Finally, the strong convergence of $\mathbf{u}_h$ is unchanged and to establish the strong convergence of z_h and $\sqrt{\alpha\delta}(\mathbf{u}_h \cdot \nabla z_h)$, we take the difference between (0.10) with test function z_h and (0.5) multiplied by $z_h + \delta\mathbf{u}_h \cdot \nabla z_h$:

$$\nu\|z_h\|^2_{L^2(\Omega)} + \alpha\delta\|\mathbf{u}_h \cdot \nabla z_h\|^2_{L^2(\Omega)} = \nu(z, z_h + \delta\mathbf{u}_h \cdot \nabla z_h) + \alpha(\mathbf{u} \cdot \nabla z, z_h + \delta\mathbf{u}_h \cdot \nabla z_h) + \nu(\operatorname{curl}(\mathbf{u}_h - \mathbf{u}), z_h + \delta\mathbf{u}_h \cdot \nabla z_h)\,.$$

By passing to the limit, this gives

$$\lim_{h\to 0}(\nu\|z_h\|^2_{L^2(\Omega)} + \alpha\delta\|\mathbf{u}_h \cdot \nabla z_h\|^2_{L^2(\Omega)}) = \nu\|z\|^2_{L^2(\Omega)}\,.$$

Hence $\lim_{h\to 0} \nu\|z_h\|^2_{L^2(\Omega)} \le \nu\|z\|^2_{L^2(\Omega)}$, thus implying first that

$$\lim_{h\to 0} \nu\|z_h\|^2_{L^2(\Omega)} = \nu\|z\|^2_{L^2(\Omega)}\,,$$

and next

$$\lim_{h\to 0} \sqrt{\alpha\delta}\|\mathbf{u}_h \cdot \nabla z_h\|_{L^2(\Omega)} = 0\,.$$

The next theorem collects these convergence results.

Theorem 5.11. *For any $\delta \in \mathbb{R}$ such that $\alpha\,\delta > 0$, we can extract a subsequence of solutions $(\mathbf{u}_h, p_h, z_h)$ of the upwind scheme (0.8–0.10) that converges strongly to a solution of Problem P:*

$$\lim_{h\to 0} \mathbf{u}_h = \mathbf{u} \quad \textit{strongly in } W\,,$$
$$\lim_{h\to 0} p_h = p \quad \textit{strongly in } L^2(\Omega)\,,$$
$$\lim_{h\to 0} z_h = z \quad \textit{strongly in } L^2(\Omega)\,.$$

If δ is independent of h, we have also

$$\lim_{h\to 0} \mathbf{u}_h \cdot \nabla z_h = \mathbf{u} \cdot \nabla z \quad \textit{strongly in } L^2(\Omega)\,,$$

and if $\delta = \operatorname{sign}(\alpha)h$, then

$$\lim_{h\to 0} \sqrt{\alpha\delta}(\mathbf{u}_h \cdot \nabla z_h) = 0 \quad \textit{strongly in } L^2(\Omega)\,.$$

5.4. **Error estimates**

Now, we turn to error estimates; we retain the same generalized Stokes problem (3.1–3.3) and we retain the assumptions of Theorem 5.4. First, the estimates of Section 3.1 readily extend here. For instance, (3.12) is replaced by (3.13):

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{H^1(\Omega)} \le K_1(h)|\mathbf{w} - P_h(\mathbf{w})|_{H^1(\Omega)} + \frac{\sqrt{2}}{\nu}\|q - r_h(q)\|_{L^2(\Omega)}\,.$$

Note that the occurrence of the last term involving the approximation error of q spoils the good estimates that were obtained in Remark 3.6, so that there is no point here in replacing $P_h(\mathbf{w})$ by the Stokes projection $S_h(\mathbf{w})$. When $\mathbf{g}$ satisfies (3.4), we obtain

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{H^1(\Omega)} \le C\,h^{1/2}\big(K_1(h)|\mathbf{w}|_{H^{3/2}(\Omega)} + \frac{1}{\nu}|q|_{H^{1/2}(\Omega)}\big)\,. \tag{5.45}$$

If in addition, $\mathcal{T}_h$ satisfies (2.41), then for any real number $p \in [2, 4]$, there exists a constant C_p, independent of h, such that

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{W^{1,p}(\Omega)} \le C_p h^{2/p-1/2}\big(K_1(h)|\mathbf{w}|_{H^{3/2}(\Omega)} + \frac{1}{\nu}|q|_{H^{1/2}(\Omega)}\big)\,. \tag{5.46}$$

If Ω is convex and $\mathbf{g}$ satisfies (3.6) and (3.7), the bound (5.45) becomes:

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{H^1(\Omega)} \le C\,h\big(K_1(h)|\mathbf{w}|_{H^2(\Omega)} + \frac{1}{\nu}|q|_{H^1(\Omega)}\big)\,. \tag{5.47}$$

Thus, if in this case $\mathcal{T}_h$ satisfies (2.41), then for any real number $p \ge 2$, there exists another constant C_p, independent of h, such that

$$|\mathbf{u}_h - P_h(\mathbf{w})|_{W^{1,p}(\Omega)} \le C_p h^{2/p}\big(K_1(h)|\mathbf{w}|_{H^2(\Omega)} + \frac{1}{\nu}|q|_{H^1(\Omega)}\big)\,. \tag{5.48}$$

On the other hand, we always have

$$\|p - q\|_{L^2(\Omega)} \le \frac{\nu}{\beta} S_4 K_2(h)\|z - z_h\|_{L^2(\Omega)}\,,$$

and if $\mathbf{g}$ satisfies (3.4), then

$$\|p - q\|_{W^{1,4/3}(\Omega)} \le \nu C K_2(h)\|z - z_h\|_{L^2(\Omega)}\,.$$

In this case,

$$\|q - r_h(q)\|_{L^2(\Omega)} \le \|p - r_h(p)\|_{L^2(\Omega)} + \nu C h^{1/2} K_2(h)\|z - z_h\|_{L^2(\Omega)}\,. \tag{5.49}$$

From (5.46), (5.49), and (5.9), we easily derive the following extension of Theorem 3.11:

Theorem 5.12. *Under the assumptions of Theorem 5.4 and the first part of Theorem 3.5, we have for any real number* $r \in [2, 4]$*:*

$$\begin{aligned} |\mathbf{u}_h - \mathbf{u}|_{W^{1,r}(\Omega)} \le\ & |P_h(\mathbf{u}) - \mathbf{u}|_{W^{1,r}(\Omega)} + C h^{2/r-1}(K_1(h)|P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} \\ & + \|p - r_h(p)\|_{L^2(\Omega)}) + C K_2(h)\big(1 + h^{2/r-1/2}(1 + K_1(h))\big)\|z - z_h\|_{L^2(\Omega)}\,, \end{aligned} \tag{5.50}$$

where C *denote constants that depend on* r*, but not on* h*.*

Remark 5.13. As in Remark 3.13, if the domain is convex and $\mathcal{T}_h$ satisfies (3.34), then we have the analogue of (3.35):

$$\begin{aligned} \|\mathbf{u}_h - \mathbf{u}\|_{L^\infty(\Omega)} \le\ & \|P_h(\mathbf{u}) - \mathbf{u}\|_{L^\infty(\Omega)} + \frac{C_1}{\rho_{\min}^{1/9}}(K_1(h)|P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} + \frac{\sqrt{2}}{\nu}\|p - r_h(p)\|_{L^2(\Omega)}) \\ & + \|z - z_h\|_{L^2(\Omega)}\big(C_2 K_2(h) + C_3 h^{1/4} K_3(h)(1 + K_1(h))\big)\,, \end{aligned} \tag{5.51}$$

with constants C_i independent of h. If $\mathcal{T}_h$ satisfies (3.59), then

$$\begin{aligned} |\mathbf{u}_h - \mathbf{u}|_{W^{1,4}(\Omega)} \le\ & \|P_h(\mathbf{u}) - \mathbf{u}\|_{W^{1,4}(\Omega)} + \frac{C_1}{\rho_{\min}^{1/2}}(K_1(h)|P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} + \frac{\sqrt{2}}{\nu}\|p - r_h(p)\|_{L^2(\Omega)}) \\ & + \|z - z_h\|_{L^2(\Omega)}\big(C_2 K_2(h) + C_3 K_3(h)(1 + K_1(h))\big)\,, \end{aligned} \tag{5.52}$$

with other constants C_i independent of h. □

Now, an error inequality is more easily derived from the upwinded transport equation (0.10) than from (0.7), because its structure yields directly an upper bound for $\sqrt{\alpha\delta}\mathbf{u}_h\cdot\nabla(z_h-\lambda_h)$, with any choice of λ_h. Furthermore, the parameter δ can be chosen so as to enhance the convergence. As will be explained in the next proposition, the choice is: $\delta = \text{sign}(\alpha)h$.

Proposition 5.14. *Let $(\mathbf{u}_h, p_h, z_h)$ be any solution of (0.8–0.10) with the choice*

$$\delta = \text{sign}(\alpha)h\,, \tag{5.53}$$

and let $(\mathbf{u}, p, z)$ be a solution of Problem P such that $z \in H^1(\Omega)$. We have the following estimate for $z_h - \lambda_h$, for any λ_h in Z_h:

$$\begin{aligned}\frac{\nu}{2}\|z_h-\lambda_h\|^2_{L^2(\Omega)} + \frac{|\alpha|h}{2}\|\mathbf{u}_h\cdot\nabla(z_h-\lambda_h)\|^2_{L^2(\Omega)} &\le \frac{5}{2}|\alpha|hC_\infty^2|z-\lambda_h|^2_{H^1(\Omega)}\\ &+ (2\nu+\frac{5}{2}(\frac{|\alpha|}{h}+\nu^2\frac{h}{|\alpha|}))\|z-\lambda_h\|^2_{L^2(\Omega)} + 4\frac{\alpha^2}{\nu}\|z-\lambda_h\|^2_{L^4(\Omega)}|\mathbf{u}-\mathbf{u}_h|^2_{W^{1,4}(\Omega)}\\ &+ \nu(4+5\nu\frac{h}{|\alpha|})|\mathbf{u}-\mathbf{u}_h|^2_{H^1(\Omega)} + |\alpha|(\frac{5}{2}h+2\frac{|\alpha|}{\nu})\|\mathbf{u}-\mathbf{u}_h\|^2_{L^\infty(\Omega)}|z|^2_{H^1(\Omega)}\,,\end{aligned} \tag{5.54}$$

where C_∞ is the constant of:

$$\|\mathbf{u}_h\|_{L^\infty(\Omega)} \le C_\infty\,.$$

Proof. By taking the difference between (0.10) and (0.5) multiplied by the test function $\theta_h+\delta\mathbf{u}_h\cdot\nabla\,\theta_h$, inserting λ_h and choosing $\theta_h = z_h-\lambda_h$, we obtain

$$\begin{aligned}&\nu\|z_h-\lambda_h\|^2_{L^2(\Omega)} + \alpha\delta\|\mathbf{u}_h\cdot\nabla(z_h-\lambda_h)\|^2_{L^2(\Omega)} = -\nu(\lambda_h-z, z_h-\lambda_h+\delta\mathbf{u}_h\cdot\nabla(z_h-\lambda_h))\\ &\quad-\alpha(\mathbf{u}_h\cdot\nabla(\lambda_h-z), z_h-\lambda_h+\delta\mathbf{u}_h\cdot\nabla(z_h-\lambda_h)) - \alpha((\mathbf{u}_h-\mathbf{u})\cdot\nabla\,z, z_h-\lambda_h+\delta\mathbf{u}_h\cdot\nabla(z_h-\lambda_h))\\ &\quad+\nu(\text{curl}(\mathbf{u}_h-\mathbf{u}), z_h-\lambda_h+\delta\mathbf{u}_h\cdot\nabla(z_h-\lambda_h))\,.\end{aligned} \tag{5.55}$$

The estimates for all the terms in the right-hand side of (5.55) are standard except for the second term because it involves the gradient of $\lambda_h - z$, and the upper bound for this term is only of the order of h. Applying Green's formula, we have

$$-\alpha(\mathbf{u}_h\cdot\nabla(\lambda_h-z), z_h-\lambda_h) = \alpha(\mathbf{u}_h\cdot\nabla(z_h-\lambda_h), \lambda_h-z) + \alpha(\text{div}(\mathbf{u}_h-\mathbf{u})(z_h-\lambda_h), \lambda_h-z)\,.$$

Thus, for any $\gamma > 0$ and $\varepsilon > 0$,

$$\begin{aligned}|\alpha(\mathbf{u}_h\cdot\nabla(\lambda_h-z), z_h-\lambda_h)| &\le \frac{\alpha}{2}[\gamma\delta\|\mathbf{u}_h\cdot\nabla(z_h-\lambda_h)\|^2_{L^2(\Omega)} + \frac{1}{\gamma\delta}\|\lambda_h-z\|^2_{L^2(\Omega)}]\\ &+ \frac{1}{2}[\nu\varepsilon\|z_h-\lambda_h\|^2_{L^2(\Omega)} + \frac{\alpha^2}{\nu\varepsilon}\|\text{div}(\mathbf{u}_h-\mathbf{u})\|^2_{L^4(\Omega)}\|\lambda_h-z\|^2_{L^4(\Omega)}]\,.\end{aligned}$$

Therefore, for any $\zeta > 0$, $\gamma > 0$ and $\varepsilon > 0$,

$$\begin{aligned}|\alpha(\mathbf{u}_h\cdot\nabla(\lambda_h-z), z_h-\lambda_h+\delta\mathbf{u}_h\cdot\nabla(z_h-\lambda_h))| &\le \frac{\alpha}{2}[\gamma\delta\|\mathbf{u}_h\cdot\nabla(z_h-\lambda_h)\|^2_{L^2(\Omega)} + \frac{1}{\gamma\delta}\|\lambda_h-z\|^2_{L^2(\Omega)}]\\ &+ \frac{1}{2}[\nu\varepsilon\|z_h-\lambda_h\|^2_{L^2(\Omega)} + \frac{\alpha^2}{\nu\varepsilon}\|\text{div}(\mathbf{u}_h-\mathbf{u})\|^2_{L^4(\Omega)}\|\lambda_h-z\|^2_{L^4(\Omega)}]\\ &+ \frac{\alpha\delta}{2}[\zeta\|\mathbf{u}_h\cdot\nabla(z_h-\lambda_h)\|^2_{L^2(\Omega)} + \frac{C_\infty^2}{\zeta}|\lambda_h-z|^2_{H^1(\Omega)}]\,.\end{aligned} \tag{5.56}$$

By choosing $\lambda_h = R_h(z)$, the leading term in (5.56) is $|\lambda_h - z|^2_{H^1(\Omega)}$, because it is of the order of h^2. Since it has the factor δ, by choosing $\delta = \mathrm{sign}(\alpha)h$, both products $\delta|\lambda_h - z|^2_{H^1(\Omega)}$ and $\frac{1}{\delta}\|\lambda_h - z\|^2_{L^2(\Omega)}$ have the order of h^3. This accounts for the choice of δ. □

By substituting (5.50) and the following inequality, for any $q_h \in M_h$,

$$|\mathbf{u}_h - \mathbf{u}|_{H^1(\Omega)} \le 2|P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} + \frac{\sqrt{2}}{\nu}\|q_h - p\|_{L^2(\Omega)} + \frac{S_4}{\nu}(\|z\|_{L^2(\Omega)}\|P_h(\mathbf{u}) - \mathbf{u}\|_{L^4(\Omega)} + \|z_h - z\|_{L^2(\Omega)}\|P_h(\mathbf{u})\|_{L^4(\Omega)}), \quad (5.57)$$

into (5.54), we derive for small enough data and smooth enough solutions, if the domain is convex and if $\mathcal{T}_h$ satisfies (3.59) (for simplicity we do not detail the constants):

$$\nu\|z_h - z\|^2_{L^2(\Omega)} + |\alpha|h\|\mathbf{u}_h \cdot \nabla(z_h - z)\|^2_{L^2(\Omega)} \le C(\frac{1}{h}\|z - R_h(z)\|^2_{L^2(\Omega)} + h|z - R_h(z)|^2_{H^1(\Omega)} + |\mathbf{u} - P_h(\mathbf{u})|^2_{L^\infty(\Omega)} + \frac{1}{\rho_{\min}^{2/9}}(|\mathbf{u} - P_h(\mathbf{u})|^2_{H^1(\Omega)} + \|p - r_h(p)\|^2_{L^2(\Omega)})). \quad (5.58)$$

The conclusion that we can draw from this error inequality is that, under the above assumptions, this Taylor-Hood method is of order $O(h^{3/2})$.

Of course, the same upwinding scheme can be used with the divergence-zero discretization studied in the preceding sections, and it also permits to gain a factor of $\sqrt{h}$ in the error estimates.

6. Schemes with an antisymmetric transport term

In this section, we study very briefly a centered scheme and an upwind scheme where the transport term is antisymmetrized. The techniques of proof are combinations of the ones introduced in the preceding sections.

Here, we choose for both schemes the standard Hood-Taylor spaces for the velocity and pressure:

$$X_{h,T} = \{\mathbf{v} \in C^0(\overline{\Omega})^2\,;\, \forall K \in \mathcal{T}_h\,,\, \mathbf{v}|_K \in \mathbb{P}_2^2, \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}\,,\; X_h = X_{h,T} \cap H^1_0(\Omega)^2\,, \quad (6.1)$$

$$M_h = \{q \in C^0(\overline{\Omega})\,;\, \forall K \in \mathcal{T}_h\,,\, q|_K \in \mathbb{P}_1, \int_\Omega q\,\mathrm{d}\mathbf{x} = 0\}\,, \quad (6.2)$$

and the space (5.5) for z_h:

$$Z_h = \{\theta \in C^0(\overline{\Omega})\,;\, \forall K \in \mathcal{T}_h\,,\, \theta|_K \in \mathbb{P}_1\}.$$

The argument of the preceding section allows us to construct a local approximation operator $P_h \in \mathcal{L}(H^1_0(\Omega)^2; X_h) \cap \mathcal{L}(H^1_T(\Omega); X_{h,T})$ satisfying (5.8), (5.9) and (5.10).

Let us consider first a centered scheme: *Find $\mathbf{u}_h$ in $X_h + \mathbf{g}_h$, p_h in M_h and $\mathbf{z}_h = (0,0,z_h)$ with z_h in Z_h, satisfying (0.8), (0.9) and (0.11)*

$$\forall \mathbf{v}_h \in X_h\,,\, \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{z}_h \times \mathbf{u}_h, \mathbf{v}_h) - (p_h, \mathrm{div}\,\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)\,,$$

$$\forall q_h \in M_h\,,\, (q_h, \mathrm{div}\,\mathbf{u}_h) = 0\,,$$

$$\forall \theta_h \in Z_h\,,\, \nu\,(z_h, \theta_h) + \alpha\,(\mathbf{u}_h \cdot \nabla z_h, \theta_h) + \frac{\alpha}{2}((\mathrm{div}\,\mathbf{u}_h)z_h, \theta_h) = \nu\,(\mathrm{curl}\,\mathbf{u}_h, \theta_h) + \alpha\,(\mathrm{curl}\,\mathbf{f}, \theta_h)\,.$$

From the above properties of P_h, we deduce that $\mathbf{u}_h$ satisfies the statement of Lemma 5.8, with a different constant C_3. In view of Green's formula, we have the identity

$$\int_\Omega (\mathbf{u}_h \cdot \nabla z_h) z_h \,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega (\operatorname{div} \mathbf{u}_h) z_h^2 \,\mathrm{d}\mathbf{x} = 0\,. \tag{6.3}$$

Therefore, if

$$h_b < C\nu^{2+t} \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}^{-2-t}\,, \text{ for some } t > 0\,, \tag{6.4}$$

where C is a suitable constant, then (0.8), (0.9) and (0.11) has at least one solution and each solution satisfies the *a priori* estimate for any $s > t/2$:

$$\|z_h\|_{L^2(\Omega)} \leq \frac{2\sqrt{2}}{\nu} S_2 \|\mathbf{f}\|_{L^2(\Omega)} + (2\sqrt{2})^{2+s} \frac{C}{\nu^{1+s}} \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)}^{2+s} + 2\frac{|\alpha|}{\nu} \|\operatorname{curl} \mathbf{f}\|_{L^2(\Omega)}\,, \text{ for any } s > \frac{t}{2}\,. \tag{6.5}$$

This shows first the weak convergence of z_h and $\mathbf{u}_h$ and next the strong convergence of $\mathbf{u}_h$. From these two convergences, we deduce that the limit functions satisfy the exact equation and finally, we prove the strong convergence of z_h.

The error estimate is a little more complex because of the extra nonlinear term $\frac{\alpha}{2}((\operatorname{div} \mathbf{u}_h) z_h, \theta_h)$. Its contribution to the error bound appears as:

$$A = \frac{\alpha}{2}(\operatorname{div}(\mathbf{u}_h - \mathbf{u}) R_h(z), z_h - R_h(z))\,;$$

and we bound A as follows:

$$|A| \leq C\frac{|\alpha|}{2} \|\operatorname{div}(\mathbf{u}_h - \mathbf{u})\|_{L^{2+1/4}(\Omega)} |R_h(z)|_{H^1(\Omega)} \|z_h - R_h(z)\|_{L^2(\Omega)}\,,$$

where C is a Sobolev imbedding constant. Therefore, (3.56) is replaced by

$$\begin{aligned}\|z - z_h\|_{L^2(\Omega)} \leq\ & 2\,\|z - R_h(z)\|_{L^2(\Omega)} + \sqrt{2}|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} \\ & + \frac{|\alpha|}{\nu}(|R_h(z)|_{H^1(\Omega)}(\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(\Omega)} + C\frac{\sqrt{2}}{2}|\mathbf{u} - \mathbf{u}_h|_{W^{1,2+1/4}(\Omega)}) \\ & + \|\mathbf{u}\|_{L^\infty(\Omega)} |z - R_h(z)|_{H^1(\Omega)})\,.\end{aligned}$$

Since the bounds for $|\mathbf{u} - \mathbf{u}_h|_{W^{1,2+1/4}(\Omega)}$ and $\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(\Omega)}$ differ only by the term $|\mathbf{u} - P_h(\mathbf{u})|_{W^{1,2+1/4}(\Omega)}$, we conclude that if Ω is a convex polygon, $\mathcal{T}_h$ satisfies (3.34) and h_b satisfies (6.4), if z belongs to $H^2(\Omega)$ and $\mathbf{u}$ to $H^3(\Omega)^2$, then under a smallness condition on the data that is very similar to (3.57), $\|z - z_h\|_{L^2(\Omega)}$ and $|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)}$ are of the order of h.

Finally, we consider an upwind scheme: *Find* $\mathbf{u}_h$ *in* $X_h + \mathbf{g}_h$, p_h *in* M_h *and* $\mathbf{z}_h = (0, 0, z_h)$ *with* z_h *in* Z_h, *satisfying (0.8), (0.9) and (0.12)*:

$$\begin{aligned}\forall \theta_h \in Z_h\,,\ & \nu\,(z_h, \theta_h + \delta \mathbf{u}_h \cdot \nabla \theta_h) + \alpha\,(\mathbf{u}_h \cdot \nabla z_h, \theta_h + \delta \mathbf{u}_h \cdot \nabla \theta_h) + \frac{\alpha}{2}((\operatorname{div} \mathbf{u}_h) z_h, \theta_h) \\ &= \nu\,(\operatorname{curl} \mathbf{u}_h, \theta_h + \delta \mathbf{u}_h \cdot \nabla \theta_h) + \alpha\,(\operatorname{curl} \mathbf{f}, \theta_h + \delta \mathbf{u}_h \cdot \nabla \theta_h)\,.\end{aligned}$$

In contrast to the preceding schemes, owing to the term $\nu\delta(z_h, \mathbf{u}_h \cdot \nabla z_h)$, we need a smallness condition on δ in order to establish existence of a discrete solution. More precisely, we choose $\delta = \operatorname{sign}(\alpha) h$, in order to improve accuracy, and we prove existence of solutions provided h_b satisfies (6.4) with a suitable constant C, and h satisfies:

$$h \leq \frac{|\alpha|}{9\nu} \cdot \tag{6.6}$$

In this case, we have the analogue of (5.41) with $S_h = |\alpha| + h\,\nu$, a suitable constant C and any $s > \frac{t}{2}$:

$$\frac{\nu}{2}\|z_h\|^2_{L^2(\Omega)} + |\alpha|\, h\|\mathbf{u}_h \cdot \nabla\, z_h\|^2_{L^2(\Omega)} \leq 3\frac{S_h}{\nu}(|\alpha|\|\mathrm{curl}\,\mathbf{f}\|^2_{L^2(\Omega)} + \frac{6}{|\alpha|}S_2^2\|\mathbf{f}\|^2_{L^2(\Omega)}) + C(\frac{S_h}{\alpha})^{2+s}\nu^{-1-2s}\|\mathbf{g}\|^{4+2s}_{H^{1/2}(\partial\Omega)} \cdot$$

Under the additional restriction (6.6), we derive from this estimate that the scheme is strongly convergent. From the above argument about the extra nonlinear term $\frac{\alpha}{2}((\mathrm{div}\,\mathbf{u}_h)z_h, \theta_h)$, we deduce that for small enough data and smooth enough solutions, if the domain is convex and if $\mathcal{T}_h$ satisfies (3.59), then the error of this scheme is of order $O(h^{3/2})$.

Remark 6.1. By comparing the four schemes studied here, we see on one hand that the divergence-zero scheme (0.6), (0.7) imposes substantially less restriction on the meshsize near the boundary than the other schemes (compare (2.23) with (5.40)). On the other hand, the bounds and error estimates proved for both centered schemes (0.6), (0.7) and (0.8), (0.9) and (0.11) remain valid as α tends to zero. This is not the case for the two upwind schemes (0.8)–(0.10) and (0.8), (0.9), (0.12). In view of (6.6), this last scheme seems to be the least adapted to small values of α. □

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