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EXISTENCE, *A PRIORI* AND *A POSTERIORI* ERROR ESTIMATES FOR A NONLINEAR THREE-FIELD PROBLEM ARISING FROM OLDROYD-B VISCOELASTIC FLOWS *

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Abstract. In this paper, a nonlinear problem corresponding to a simplified Oldroyd-B model without convective terms is considered. Assuming the domain to be a convex polygon, existence of a solution is proved for small relaxation times. Continuous piecewise linear finite elements together with a Galerkin Least Square (GLS) method are studied for solving this problem. Existence and *a priori* error estimates are established using a Newton-chord fixed point theorem, *a posteriori* error estimates are also derived. An Elastic Viscous Split Stress (EVSS) scheme related to the GLS method is introduced. Numerical results confirm the theoretical predictions.

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1. Introduction

Numerical simulation of viscoelastic flows is of great importance for industrial processes involving plastics, paints or food. The modelling of viscoelastic flows generally consists in supplementing the mass and momentum equations with a rheological constitutive equation relating the velocity and the non Newtonian part of the stress.

When solving viscoelastic flows with finite element methods, the following points should be addressed, see for instance [1] for a review. Firstly, the finite element spaces used to approximate the velocity, pressure and extra-stress fields cannot be chosen arbitrarily, an inf-sup condition has to be satisfied [12, 13, 15, 24, 25]. Secondly, due to the presence of convective terms in both momentum and constitutive equations, adequate discretizations procedure must be used such as discontinuous finite elements, GLS stabilization procedures [5], or the characteristics method. Thirdly, the presence of nonlinear terms prevents numerical methods to converge at high Deborah numbers, this being consistent with theoretical [4, 19, 21, 23, 26] and experimental [22] studies.

In this paper, we focus on the first and last of these three points. In [6], a GLS method with continuous, piecewise linear finite elements was proposed for solving a three fields Stokes' problem. The method was stable even when the solvent viscosity was small compared to the polymer viscosity. The link with the EVSS method of [12] was proposed. The aim of this paper is to extend the work of [6] to a nonlinear model problem. Existence, *a priori* and *a posteriori* error estimates are derived. Numerical results confirm the theoretical predictions. Even though the model problem studied in this paper is simpler than those considered in [4, 19, 21, 23, 26], we believe that our results are interesting for the following reasons. Firstly, assuming the calculation domain to be a convex

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polygon, we propose an original, but natural, variational setting in order to prove existence of the continuous problem. More precisely, the velocity is in $W^{2,r}$, for some $r > 2$, whereas the pressure and the extra-stress are in $W^{1,r}$. Secondly, we consider an original stabilized finite element method and we prove existence, *a priori* and *a posteriori* error estimates. Finally, we present some numerical computations that i) confirm the optimality of our theoretical predictions ii) show that the method fails when the computational domain is not convex, thus suggesting that the problem is ill-posed.

The reader should note that the theoretical results presented in this paper can be seen as a first step towards the justification of some numerical methods used to perform mesoscopic calculations [7,8].

2. The model problem

Let Ω be a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. We consider the following problem. Given a force term $\mathbf{f}$, constant solvent and polymer viscosities $\eta_s \geqslant 0$ and $\eta_p > 0$, a constant relaxation time λ, find the velocity $\mathbf{u}$, pressure p and extra-stress $\boldsymbol{\sigma}$ such that

$$\begin{aligned} -2\eta_s \text{ div } \boldsymbol{\epsilon}(\mathbf{u}) + \nabla p - \text{div } \boldsymbol{\sigma} = \mathbf{f}&, \\ \text{div } \mathbf{u} = 0&, \\ \frac{1}{2\eta_p}\boldsymbol{\sigma} - \frac{\lambda}{2\eta_p}\Big((\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u}^T)\Big) - \boldsymbol{\epsilon}(\mathbf{u}) = 0&, \end{aligned} \tag{1}$$

in Ω, where the velocity $\mathbf{u}$ vanishes on $\partial\Omega$. Here $\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the rate of deformation tensor and $(\nabla\mathbf{u})\boldsymbol{\sigma}$ is the the matrix product between $\nabla\mathbf{u}$ and $\boldsymbol{\sigma}$.

This model problem is a simplification of viscoelastic models for polymeric liquids. The first equation corresponds to momentum conservation. The total stress is split into three contributions: the pressure $-p\mathbf{I}$, the stress due to the (Newtonian) solvent $2\eta_s\boldsymbol{\epsilon}(\mathbf{u})$, and the extra-stress $\boldsymbol{\sigma}$ due to the non Newtonian part of the fluid (for instance polymer chains). The third equation is a simplification of the Oldroyd-B constitutive relationship between the extra-stress and the velocity field. For the sake of simplicity, the convective terms in the first and last equations are removed. A future work should take these terms into account.

A theoretical result for Problem (1) with $\eta_s = 0$ and with convective terms in the first and last equation has been obtained in [23]. Using an iterative procedure, the solution $(\mathbf{u}, p, \boldsymbol{\sigma})$ was proved to be in $H^3 \times H^2 \times H^2$ provided $\mathbf{f} \in H^1$ was small enough and $\partial\Omega$ was sufficiently smooth. In this section, we shall prove that Problem (1) has a solution in $W^{2,r} \times W^{1,r} \times W^{1,r}$ for some $r > 2$, provided λ is small enough, when Ω is a convex polygonal domain.

Let $L^2_0(\Omega)$ be the space of $L^2(\Omega)$ functions having zero mean, let $L^2_s(\Omega)^4$ be the space of symmetric tensors having $L^2(\Omega)$ components. Given $\mathbf{f} \in H^{-1}(\Omega)^2$ and $\mathbf{g} \in L^2_s(\Omega)^4$, we first consider the following problem (we set $\lambda = 0$ and add a source term in the third equation of (1)): find $(\mathbf{u}, p, \boldsymbol{\sigma}) \in H^1_0(\Omega)^2 \times L^2_0(\Omega) \times L^2_s(\Omega)^4$ such that

$$\begin{aligned} -2\eta_s \text{ div } \boldsymbol{\epsilon}(\mathbf{u}) + \nabla p - \text{div } \boldsymbol{\sigma} = \mathbf{f}&, \\ \text{div } \mathbf{u} = 0&, \\ \frac{1}{2\eta_p}\boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u}) = \mathbf{g}&. \end{aligned} \tag{2}$$

Eliminating $\boldsymbol{\sigma}$ we obtain

$$\begin{aligned} -2(\eta_s + \eta_p)\text{div } \boldsymbol{\epsilon}(\mathbf{u}) + \nabla p = \mathbf{f} + 2\eta_p \text{div } \mathbf{g}&, \\ \text{div } \mathbf{u} = 0&. \end{aligned} \tag{3}$$

Since $\mathbf{g} \in L^2_s(\Omega)^4$, div $\mathbf{g} \in H^{-1}(\Omega)$, and Problem (3) is a classical Stokes problem with solution $(\mathbf{u}, p) \in H^1_0(\Omega)^2 \times L^2_0(\Omega)$. Setting $\boldsymbol{\sigma} = 2\eta_p(\boldsymbol{\epsilon}(\mathbf{u}) + \mathbf{g})$, then $(\mathbf{u}, p, \boldsymbol{\sigma})$ is solution of Problem (2). Thus we define the operator

$$\begin{aligned} T: H^{-1}(\Omega)^2 \times L^2_s(\Omega)^4 &\longrightarrow H^1_0(\Omega)^2 \times L^2_s(\Omega)^4 \\ (\mathbf{f}, \mathbf{g}) &\longrightarrow T(\mathbf{f}, \mathbf{g}) \underset{\text{def.}}{=} (\mathbf{u}, \boldsymbol{\sigma}), \end{aligned}$$

where $(\mathbf{u}, \boldsymbol{\sigma})$ is the solution of (2). Problem (1) can then be formally written as follows: find $(\mathbf{u}, \boldsymbol{\sigma})$ such that

$$(\mathbf{u}, \boldsymbol{\sigma}) = T\left(\mathbf{f}, \frac{\lambda}{2\eta_p}\Big((\nabla \mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{u})^T\Big)\right). \tag{4}$$

The question is now to find the correct spaces for the above problem. Indeed, since $(\mathbf{u}, \boldsymbol{\sigma})$ are *a priori* only in $H^1_0(\Omega)^2 \times L^2_s(\Omega)^4$, then $(\nabla \mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{u})^T$ is only in $L^1(\Omega)$, thus div $((\nabla \mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{u})^T) \notin H^{-1}(\Omega)$!

Let us recall some regularity properties of Stokes' problem. Let $\mu > 0$ and consider the solution $(\mathbf{w}, q) \in H^1_0(\Omega)^2 \times L^2_0(\Omega)$ of

$$\begin{aligned} -2\mu \text{div}\, \boldsymbol{\epsilon}(\mathbf{w}) + \nabla q &= \mathbf{f}, \\ \text{div}\, \mathbf{w} &= 0. \end{aligned}$$

According to Proposition 2.3 of [27], if the boundary $\partial\Omega$ is $\mathcal{C}^2$ and if $\mathbf{f} \in L^r(\Omega)^2$, then $\mathbf{w} \in W^{2,r}(\Omega)^2$, for all $1 < r < \infty$. According to Theorem 7.3.3.1 of [18] and Proposition 5.3 of [17], if Ω is a convex polygon, then there exists $r > 2$ (r depends on the largest angle of the polygon) such that, if $\mathbf{f} \in L^r(\Omega)^2$, then $\mathbf{w} \in W^{2,r}(\Omega)^2$ and $\|\mathbf{w}\|_{W^{2,r}} \leq C\|\mathbf{f}\|_{L^r}$. In the sequel, **this being crucial for our analysis, we will assume that Ω is a convex polygon** and we will consider r as above.

We set

$$X = (W^{2,r}(\Omega) \cap H^1_0(\Omega))^2 \times W^{1,r}_s(\Omega)^4,$$

where $W^{1,r}_s(\Omega)^4$ is the space of symmetric tensors having components in $W^{1,r}(\Omega)$. We know that $W^{1,r}(\Omega)$, $r > 2$, is an algebra, namely if φ, $\psi \in W^{1,r}(\Omega)$ then the product $\varphi\psi \in W^{1,r}(\Omega)$ and there exists C such that

$$\|\varphi\psi\|_{W^{1,r}(\Omega)} \leq C\|\varphi\|_{W^{1,r}(\Omega)}\|\psi\|_{W^{1,r}(\Omega)} \qquad \forall \varphi, \psi \in W^{1,r}(\Omega).$$

Therefore, if $(\mathbf{u}, \boldsymbol{\sigma}) \in X$ then

$$\mathbf{g} = \frac{\lambda}{2\eta_p}\Big((\nabla \mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{u})^T\Big) \in W^{1,r}_s(\Omega)^4,$$

and div $\mathbf{g} \in L^r(\Omega)^2$. Thus, if $\mathbf{f} \in L^r(\Omega)^2$, then

$$T\left(\mathbf{f}, \frac{\lambda}{2\eta_p}\Big((\nabla \mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{u})^T\Big)\right) \in X.$$

From now on, we will restrict T to $L^r(\Omega)^2 \times W^{1,r}_s(\Omega)^4$ that is

$$\begin{aligned} T: L^r(\Omega)^2 \times W^{1,r}_s(\Omega)^4 &\longrightarrow X \\ (\mathbf{f}, \mathbf{g}) &\longrightarrow T(\mathbf{f}, \mathbf{g}) \underset{\text{def.}}{=} (\mathbf{u}, \boldsymbol{\sigma}), \end{aligned} \tag{5}$$

where $(\mathbf{u}, \boldsymbol{\sigma})$ is the solution of (2). The operator T is bounded since

$$\|T(\mathbf{f}, \mathbf{g})\|_X \leq C(\|\mathbf{f}\|_{L^r} + \|\mathbf{g}\|_{W^{1,r}}) \qquad \forall (\mathbf{f}, \mathbf{g}) \in L^r(\Omega)^2 \times W^{1,r}_s(\Omega)^4.$$

Finally, going back to (4), we are looking for $\mathbf{U} = (\mathbf{u}, \boldsymbol{\sigma}) \in X$ such that

$$F(\lambda, \mathbf{U}) = 0, \tag{6}$$

where $F : \mathbf{R} \times X \to X$ is defined by

$$F(\lambda, \mathbf{U}) = \mathbf{U} - T\Big(\mathbf{f}, \lambda S(\mathbf{U})\Big),$$

and $S : X \to W_s^{1,r}(\Omega)^4$ by

$$S(\mathbf{U}) = \frac{1}{2\eta_p}\Big((\nabla \mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{u})^T\Big). \tag{7}$$

We have the following result.

Lemma 2.1. *For all $\lambda \geq 0$, the operator $F(\lambda, \cdot) : X \to X$ is $\mathcal{C}^1$, with Frechet derivative given by*

$$D_{\mathbf{U}}F(\lambda, \mathbf{U})\mathbf{V} = (\mathbf{v}, \boldsymbol{\tau}) - T\left(\mathbf{0}, \frac{\lambda}{2\eta_p}\Big((\nabla \mathbf{u})\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla \mathbf{u})^T + (\nabla \mathbf{v})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{v})^T\Big)\right),$$

for all $\mathbf{U} = (\mathbf{u}, \boldsymbol{\sigma}) \in X$, $\mathbf{V} = (\mathbf{v}, \boldsymbol{\tau})$ in X.

Proof. It suffices to note that the operator $S : X \to W_s^{1,r}(\Omega)^4$ is $\mathcal{C}^1$ with Frechet derivative

$$DS(\mathbf{U})\mathbf{V} = (\nabla \mathbf{u})\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla \mathbf{u})^T + (\nabla \mathbf{v})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \mathbf{v})^T.$$

□

We now observe that, when $\lambda = 0$, (6) reduces to a classical three fields Stokes' problem. Thus, there is a unique $\mathbf{U}_0 = (\mathbf{u}_0, \boldsymbol{\sigma}_0) \in X$ such that $F(0, \mathbf{U}_0) = 0$. Using the implicit function theorem, we can then prove the following result.

Theorem 2.2. *There exists $\lambda_0 > 0$ and $\delta > 0$ such that, for all $\lambda \leq \lambda_0$, there exists a unique $\mathbf{U}(\lambda) = (\mathbf{u}(\lambda), \boldsymbol{\sigma}(\lambda)) \in X$ such that $F(\lambda, \mathbf{U}(\lambda)) = 0$ and $\|\mathbf{U}(\lambda) - \mathbf{U}_0\|_X \leq \delta$. Moreover, the mapping $\lambda \in [0, \lambda_0] \to \mathbf{U}(\lambda) \in X$ is continuous.*

Proof. When $\lambda = 0$ there is a unique $\mathbf{U}_0 = (\mathbf{u}_0, \boldsymbol{\sigma}_0) \in X$ such that $F(0, \mathbf{U}_0) = 0$. Moreover, using Lemma 2.1, we have $D_{\mathbf{U}}F(0, \mathbf{U}_0) = I$, which is an isomorphism onto X. The result is then an immediate consequence of the implicit function theorem. □

3. A Galerkin Least Square method

In [6], a GLS method with continuous, piecewise linear finite elements was studied for solving (1) in the linear case, *i.e.* when $\lambda = 0$. The method was proved to be stable and convergent even when the solvent viscosity η_s was small compared to the polymer viscosity η_p. *A priori* error estimates where derived in the space $W = H_0^1(\Omega)^2 \times L_0^2(\Omega) \times L_s^2(\Omega)^4$. Finally, this GLS method was shown to be equivalent to some EVSS method. The aim of the paper is to extend these results to the nonlinear case, that is when $\lambda \neq 0$.

For any $h > 0$, let $\mathcal{T}_h$ be a mesh of $\overline{\Omega}$ into triangles K with diameters h_K less than h. We assume that the mesh satisfies the regularity and inverse assumptions in the sense of [10]. We consider W_h the finite dimensional subspace of W consisting in continuous, piecewise linear velocities, pressures, and stresses on the mesh $\mathcal{T}_h$. More precisely $W_h \subset W$ is defined by $W_h = V_h \times Q_h \times M_h$ where

$$\begin{aligned} V_h &= \Big\{\mathbf{v} \in \mathcal{C}^0(\overline{\Omega})^2; \mathbf{v}_{|K} \in (\mathbb{P}_1)^2,\ \forall\, K \in \mathcal{T}_h\Big\} \cap H_0^1(\Omega)^2, \\ Q_h &= \Big\{q \in \mathcal{C}^0(\overline{\Omega}); q_{|K} \in \mathbb{P}_1,\ \forall\, K \in \mathcal{T}_h\Big\} \cap L_0^2(\Omega), \\ M_h &= \Big\{\boldsymbol{\tau} \in \mathcal{C}^0(\overline{\Omega})^4; \boldsymbol{\tau}_{|K} \in (\mathbb{P}_1)^4,\ \forall\, K \in \mathcal{T}_h\Big\} \cap L_s^2(\Omega)^4. \end{aligned} \tag{8}$$

In order to approach the solution of (1) we consider the following problem: find $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ such that

$$\begin{aligned} &2\eta_s(\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\epsilon}(\mathbf{v})) - (p_h, \text{div } \mathbf{v}) + (\boldsymbol{\sigma}_h, \boldsymbol{\epsilon}(\mathbf{v})) - (\mathbf{f}, \mathbf{v}) \\ &- (\text{div } \mathbf{u}_h, q) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(-2\eta_s \text{div } \boldsymbol{\epsilon}(\mathbf{u}_h) + \nabla p_h - \text{div } \boldsymbol{\sigma}_h - \mathbf{f}, \nabla q\Big)_K \\ &- \left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h - \frac{\lambda}{2\eta_p}(\nabla \mathbf{u}_h \boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h \nabla \mathbf{u}_h^T) - \boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\tau} + 2\eta_p \beta \boldsymbol{\epsilon}(\mathbf{v})\right) = 0, \end{aligned} \tag{9}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$. Here $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ scalar product for scalars, vectors and tensors, for instance

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \sum_{i,j=1}^{2} \int_\Omega \sigma_{ij}\ \tau_{ij}\ \mathrm{d}x \qquad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in L^2(\Omega)^4.$$

Also, $\alpha > 0$ and $0 < \beta < 2$ are dimensionless stabilization parameters. Note that, since the velocity is piecewise linear, then div $\boldsymbol{\epsilon}(\mathbf{u}_h)$ is zero in each of the mesh triangles.

In the next section, existence of a solution to (9) is proved, as well as *a priori* error estimates. In Section 5, *a posteriori* error estimates based on the equation residual are derived. In Section 6, an EVSS method related to the GLS method is presented. Numerical results on both GLS and EVSS schemes are reported in Section 7.

4. Existence and *a priori* error estimates

Let us consider the discrete counterpart of the operator T, namely the operator

$$\begin{aligned} T_h : L^2(\Omega)^2 \times L^2_s(\Omega)^4 &\longrightarrow H^1_0(\Omega)^2 \times L^2_s(\Omega)^4 \\ (\mathbf{f}, \mathbf{g}) \qquad &\longrightarrow T_h(\mathbf{f}, \mathbf{g}) \underset{\text{def.}}{=} (\mathbf{u}_h, \boldsymbol{\sigma}_h) \in V_h \times M_h, \end{aligned} \tag{10}$$

where $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ satisfies

$$\begin{aligned} &2\eta_s(\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\epsilon}(\mathbf{v})) - (p_h, \text{div } \mathbf{v}) + (\boldsymbol{\sigma}_h, \boldsymbol{\epsilon}(\mathbf{v})) - (\mathbf{f}, \mathbf{v}) \\ &- (\text{div } \mathbf{u}_h, q) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(-2\eta_s \text{div } \boldsymbol{\epsilon}(\mathbf{u}_h) + \nabla p_h - \text{div } \boldsymbol{\sigma}_h - \mathbf{f}, \nabla q\Big)_K \\ &- \left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h - \mathbf{g} - \boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\tau} + 2\eta_p \beta \boldsymbol{\epsilon}(\mathbf{v})\right) = 0, \end{aligned} \tag{11}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$. We then have the following result.

Lemma 4.1. *The operator T_h is well defined and is uniformly bounded with respect to h. Moreover, there exists $h_0 > 0$ and C such that, for all $(\mathbf{f}, \mathbf{g}) \in L^r(\Omega)^2 \times W^{1,r}_s(\Omega)^4$ we have*

$$\|T(\mathbf{f}, \mathbf{g}) - T_h(\mathbf{f}, \mathbf{g})\|_{H^1 \times L^2} \le Ch\Big(\|\mathbf{f}\|_{L^r} + \|\mathbf{g}\|_{W^{1,r}}\Big) \quad \forall h \le h_0.$$

Proof. We introduce, as in [6], the bilinear form $B_h(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau})$ corresponding to the left hand side of the weak formulation (11) when $\mathbf{f}$ and $\mathbf{g}$ are zero. The operator T_h is then defined by $T_h(\mathbf{f}, \mathbf{g}) = (\mathbf{u}_h, \boldsymbol{\sigma}_h)$ where $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ satisfies

$$\begin{aligned} B_h(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) = (\mathbf{f}, \mathbf{v}) - \Big(\mathbf{g}, \boldsymbol{\tau} + 2\eta_p \beta \boldsymbol{\epsilon}(\mathbf{v})\Big) \\ - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}(\mathbf{f}, \nabla q)_K \end{aligned}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$. It is proved in [6] that $B_h : W_h \times W_h \to \mathbb{R}$ satisfies the uniform (with respect to h) inf-sup condition in the norm $\|\cdot\|_{H^1\times L^2\times L^2}$. Thus T_h is well defined and we have, for all $h > 0$

$$\|\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h\|_{H^1\times L^2\times L^2} \le C\|\mathbf{f}, \mathbf{g}\|_{L^2\times L^2},$$

and thus

$$\|T_h\|_{\mathcal{L}(L^2\times L^2, H^1\times L^2)} \le C,$$

where C does not depend on h. Moreover, if $(\mathbf{f}, \mathbf{g}) \in L^r(\Omega)^2 \times W_s^{1,r}(\Omega)^4$, then we can introduce $T(\mathbf{f}, \mathbf{g}) = (\mathbf{u}, \boldsymbol{\sigma}) \in X$, where $(\mathbf{u}, \boldsymbol{\sigma})$ is the solution of (2) and where T is defined in (5). We have

$$\begin{aligned}\|T(\mathbf{f}, \mathbf{g}) - T_h(\mathbf{f}, \mathbf{g})\|_{H^1\times L^2} &= \|\mathbf{u} - \mathbf{u}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H^1\times L^2} \\ &\le \|\mathbf{u} - r_h\mathbf{u}, \boldsymbol{\sigma} - r_h\boldsymbol{\sigma}\|_{H^1\times L^2} + \|r_h\mathbf{u} - \mathbf{u}_h, r_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H^1\times L^2},\end{aligned}$$

where r_h denotes Lagrange interpolant on scalars, vectors or tensors. Note that, considering $r_h\mathbf{u}$, r_hp and $r_h\boldsymbol{\sigma}$ has a meaning since $(\mathbf{u}, p, \boldsymbol{\sigma}) \in W^{2,r}(\Omega)^2 \times W^{1,r}(\Omega) \times W_s^{1,r}(\Omega)^4$, and thus $\mathbf{u}$, p and $\boldsymbol{\sigma}$ are continuous on $\overline{\Omega}$. Using classical interpolation results [10] together with the well posedness of the operator T, we have

$$\begin{aligned}\|\mathbf{u} - r_h\mathbf{u}, \boldsymbol{\sigma} - r_h\boldsymbol{\sigma}\|_{H^1\times L^2} &\le Ch\Big(|\mathbf{u}|_{H^2} + |\boldsymbol{\sigma}|_{H^1\cap\mathcal{C}^0}\Big) \\ &\le Ch\Big(\|\mathbf{u}\|_{W^{2,r}} + \|\boldsymbol{\sigma}\|_{W^{1,r}}\Big) \\ &\le Ch\Big(\|\mathbf{f}\|_{L^r} + \|\mathbf{g}\|_{W^{1,r}}\Big).\end{aligned} \tag{12}$$

On the other side, using the fact that B_h satisfies the uniform inf-sup condition on W_h, there is a constant C independent of h such that

$$\|r_h\mathbf{u} - \mathbf{u}_h, r_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H^1\times L^2} \le C \sup_{\mathbf{0}\ne(\mathbf{v},q,\boldsymbol{\tau})\in W_h} \frac{B_h(r_h\mathbf{u} - \mathbf{u}_h, r_hp - p_h, r_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau})}{\|\mathbf{v}, q, \boldsymbol{\tau}\|_{H^1\times L^2\times L^2}}\cdot \tag{13}$$

The scheme (11) is consistent in the sense of [14], that is

$$B_h(\mathbf{u} - \mathbf{u}_h, p - p_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) = 0 \qquad \forall(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h,$$

and thus

$$\|r_h\mathbf{u} - \mathbf{u}_h, r_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H^1\times L^2} \le C \sup_{\mathbf{0}\ne(\mathbf{v},q,\boldsymbol{\tau})\in W_h} \frac{B_h(\mathbf{u} - r_h\mathbf{u}, p - r_hp, \boldsymbol{\sigma} - r_h\boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau})}{\|\mathbf{v}, q, \boldsymbol{\tau}\|_{H^1\times L^2\times L^2}}\cdot$$

From the definition of B_h we have, for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$,

$$\begin{aligned}&B_h(\mathbf{u} - r_h\mathbf{u}, p - r_hp, \boldsymbol{\sigma} - r_h\boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau}) \\ &\quad = 2(\eta_s + \eta_p\beta)(\boldsymbol{\epsilon}(\mathbf{u} - r_h\mathbf{u}), \boldsymbol{\epsilon}(\mathbf{v})) \\ &\qquad - (p - r_hp, \operatorname{div}\mathbf{v}) + (1-\beta)(\boldsymbol{\sigma} - r_h\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\mathbf{v})) \\ &\qquad - (\operatorname{div}(\mathbf{u} - r_h\mathbf{u}), q) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}(\mathbf{f} - \nabla r_hp + \operatorname{div} r_h\boldsymbol{\sigma}, \nabla q)_K \\ &\qquad - \frac{1}{2\eta_p}(\boldsymbol{\sigma} - r_h\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\boldsymbol{\epsilon}(\mathbf{u} - r_h\mathbf{u}), \boldsymbol{\tau}).\end{aligned}$$

Using Cauchy-Schwarz and Young inequalities, there is a constant C independent of h such that

$$\begin{aligned}
&|B_h(\mathbf{u}-r_h\mathbf{u}, p-r_hp, \boldsymbol{\sigma}-r_h\boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau})| \\
&\quad \le C\Bigg(\|\mathbf{u}-r_h\mathbf{u}, p-r_hp, \boldsymbol{\sigma}-r_h\boldsymbol{\sigma}\|_{H^1\times L^2\times L^2}\ \|\mathbf{v}, q, \boldsymbol{\tau}\|_{H^1\times L^2\times L^2} \\
&\qquad + \sum_{K\in\mathcal{T}_h} h_K^2 \Big(\|\mathbf{f}\|_{L^2(K)} + \|r_hp\|_{H^1(K)} + \|r_h\boldsymbol{\sigma}\|_{H^1(K)} \Big) \|\nabla q\|_{L^2(K)} \Bigg).
\end{aligned}$$

We now use the inverse inequality $h_K\|\nabla q\|_{L^2(K)} \le C\|q\|_{L^2(K)}$, the fact that

$$\begin{aligned}
&\|r_hp\|_{H^1} \le C\|r_hp\|_{W^{1,r}} \le \tilde{C}\|p\|_{W^{1,r}} \\
&\|r_h\boldsymbol{\sigma}\|_{H^1} \le C\|r_h\boldsymbol{\sigma}\|_{W^{1,r}} \le \tilde{C}\|\boldsymbol{\sigma}\|_{W^{1,r}},
\end{aligned}$$

for h sufficiently small, and standard interpolation estimates [10] to obtain

$$\begin{aligned}
&|B_h(\mathbf{u}-r_h\mathbf{u}, p-r_hp, \boldsymbol{\sigma}-r_h\boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau})| \\
&\quad \le Ch\Bigg(\|\mathbf{u}, p, \boldsymbol{\sigma}\|_{W^{2,r}\times W^{1,r}\times W^{1,r}}\ \|\mathbf{v}, q, \boldsymbol{\tau}\|_{H^1\times L^2\times L^2} \\
&\qquad + \Big(\|\mathbf{f}\|_{L^r} + \|p\|_{W^{1,r}} + \|\boldsymbol{\sigma}\|_{W^{1,r}} \Big) \|q\|_{L^2} \Bigg).
\end{aligned}$$

Using the well posedness of the operator T we obtain

$$|B_h(\mathbf{u}-r_h\mathbf{u}, p-r_hp, \boldsymbol{\sigma}-r_h\boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau})| \le Ch\Big(\|\mathbf{f}\|_{L^r} + \|\mathbf{g}\|_{W^{1,r}} \Big) \|\mathbf{v}, q, \boldsymbol{\tau}\|_{H^1\times L^2\times L^2},$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$. The above inequality in (13), together with (12) finally yields

$$\|\mathbf{u}-\mathbf{u}_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H^1\times L^2} \le Ch\Big(\|\mathbf{f}\|_{L^r} + \|\mathbf{g}\|_{W^{1,r}} \Big),$$

which is the desired result. □

We now would like to rewrite the nonlinear GLS scheme (9) in an abstract framework, as we did in (6) for the continuous Problem (1). We now introduce $X_h = V_h \times M_h$ equipped with the $\|\cdot\|_{H^1\times L^2}$ norm and we want to rewrite the nonlinear GLS scheme (9) as

$$F_h(\lambda, \mathbf{U}_h) = 0 \qquad \text{with} \qquad \mathbf{U}_h = (\mathbf{u}_h, \boldsymbol{\sigma}_h). \tag{14}$$

Here $F_h : \mathbf{R} \times X_h \to X_h$ is defined by

$$F_h(\lambda, \mathbf{U}_h) = \mathbf{U}_h - T_h\Big(\mathbf{f}, \lambda S(\mathbf{U}_h)\Big),$$

where S is still formally defined by

$$S(\mathbf{U}_h) = S(\mathbf{u}_h, \boldsymbol{\sigma}_h) = \frac{1}{2\eta_p}\Big((\nabla\mathbf{u}_h)\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h(\nabla\mathbf{u}_h)^T \Big).$$

However, since $X_h \not\subset X$, we have to extend S on $W_0^{1,r}(\Omega)^2 \times W_s^{1,r}(\Omega)^4$, and since $W_s^{1,r}(\Omega)^4 \subset \mathcal{C}^0(\overline{\Omega})$, we can consider $S : W_0^{1,r}(\Omega)^2 \times W_s^{1,r}(\Omega)^4 \to L_s^r(\Omega)^4$. Clearly, this operator S is $\mathcal{C}^1$.

We have the following result.

Lemma 4.2. *Let* $\mathbf{U}(\lambda)$, $0 \le \lambda \le \lambda_0$, *be given by Theorem 2.2. There exists* C *such that, for all* $h > 0$, *for all* $\lambda \le \lambda_0$ *we have*

$$\|F_h(\lambda, r_h\mathbf{U}(\lambda))\|_{H^1\times L^2} \le Ch, \tag{15}$$

$$\|D_{\mathbf{U}}F_h(\lambda, r_h\mathbf{U}(\lambda)) - D_{\mathbf{U}}F_h(\lambda, \mathbf{V})\|_{\mathcal{L}(X_h)} \le C\frac{\lambda}{h}\|r_h\mathbf{U}(\lambda) - \mathbf{V}\|_{H^1\times L^2}, \tag{16}$$

for all $\mathbf{V} \in X_h$.

Proof. From the definition of F_h and since $F(\lambda, \mathbf{U}(\lambda)) = 0$, we have

$$F_h(\lambda, r_h\mathbf{U}(\lambda)) = r_h\mathbf{U}(\lambda) - \mathbf{U}(\lambda) - T_h\Big(\mathbf{f}, \lambda S(r_h\mathbf{U}(\lambda))\Big) + T\Big(\mathbf{f}, \lambda S(\mathbf{U}(\lambda))\Big),$$

so that

$$\begin{aligned}
&\|F_h(\lambda, r_h\mathbf{U}(\lambda))\|_{H^1\times L^2} \\
&\qquad \le \|\mathbf{U}(\lambda) - r_h\mathbf{U}(\lambda)\|_{H^1\times L^2} + \Big\|(T - T_h)\Big(\mathbf{f}, \lambda S(\mathbf{U}(\lambda))\Big)\Big\|_{H^1\times L^2} \\
&\qquad\quad + \Big\|T_h\Big(\mathbf{0}, \lambda[S(\mathbf{U}(\lambda)) - S(r_h\mathbf{U}(\lambda))]\Big)\Big\|_{H^1\times L^2}.
\end{aligned}$$

Using Lemma 4.1 for the second and third terms in the right hand side of the above inequality together with interpolation results, we obtain

$$\begin{aligned}
&\|F_h(\lambda, r_h\mathbf{U}(\lambda))\|_{H^1\times L^2} \\
&\qquad \le Ch\|\mathbf{U}(\lambda)\|_X + Ch\Big(\|\mathbf{f}\|_{L^r} + \lambda\|S(\mathbf{U}(\lambda))\|_{W^{1,r}}\Big) \\
&\qquad\quad + C\lambda\|S(\mathbf{U}(\lambda)) - S(r_h\mathbf{U}(\lambda))\|_{L^2},
\end{aligned} \tag{17}$$

C being independent of h and $\lambda \in [0, \lambda_0]$. A simple calculation shows that

$$\|S(\mathbf{U}(\lambda))\|_{W^{1,r}} \le C\|\mathbf{U}(\lambda)\|_X^2, \tag{18}$$

C being independent of h and $\lambda \in [0, \lambda_0]$. On the other hand, we also have

$$\begin{aligned}
&2\eta_p\Big(S(\mathbf{U}) - S(r_h\mathbf{U})\Big) \\
&\qquad = \nabla\mathbf{u}\boldsymbol{\sigma} + \boldsymbol{\sigma}\nabla\mathbf{u}^T - (\nabla r_h\mathbf{u})r_h\boldsymbol{\sigma} - r_h\boldsymbol{\sigma}(\nabla r_h\mathbf{u})^T \\
&\qquad = \nabla(\mathbf{u} - r_h\mathbf{u})\boldsymbol{\sigma} + (\nabla r_h\mathbf{u})(\boldsymbol{\sigma} - r_h\boldsymbol{\sigma}) + \boldsymbol{\sigma}\nabla(\mathbf{u} - r_h\mathbf{u})^T + (\boldsymbol{\sigma} - r_h\boldsymbol{\sigma})(\nabla r_h\mathbf{u})^T,
\end{aligned}$$

so that

$$\begin{aligned}
\|S(\mathbf{U}(\lambda)) - S(r_h\mathbf{U}(\lambda))\|_{L^2} &\le C\|\mathbf{U}(\lambda) - r_h\mathbf{U}(\lambda)\|_{H^1\times L^2}\|\mathbf{U}(\lambda)\|_X \\
&\le Ch\|\mathbf{U}(\lambda)\|_X^2,
\end{aligned} \tag{19}$$

C being independent of h and λ. Finally (19) and (18) in (17) yields (15) for $\lambda \in [0, \lambda_0]$.

Let us now prove (16). For all $\mathbf{V} = (\mathbf{v}, \boldsymbol{\tau}) \in X_h$, for all $\mathbf{W} = (\mathbf{w}, \boldsymbol{\gamma}) \in X_h$, we have

$$\Big(D_{\mathbf{U}}F_h(\lambda, r_h\mathbf{U}(\lambda)) - D_{\mathbf{U}}F_h(\lambda, \mathbf{V})\Big)\mathbf{W} = -T_h\Big(\mathbf{0}, \lambda\Big[DS(r_h\mathbf{U}(\lambda))\mathbf{W} - DS(\mathbf{V})\mathbf{W}\Big]\Big).$$

Using Lemma 4.1 we obtain

$$\left\|\Big(D_{\mathbf{U}}F_h(\lambda, r_h\mathbf{U}(\lambda)) - D_{\mathbf{U}}F_h(\lambda,\mathbf{V})\Big)\mathbf{W}\right\|_{H^1\times L^2} \leq C\lambda \left\|\Big(DS(r_h\mathbf{U}(\lambda)) - DS(\mathbf{V})\Big)\mathbf{W}\right\|_{L^2}.$$

We have

$$\begin{aligned}\left\|\Big(DS(r_h\mathbf{U}) - DS(\mathbf{V})\Big)\mathbf{W}\right\|_{L^2} &\\ &= \frac{1}{2\eta_p}\|\nabla(r_h\mathbf{u}-\mathbf{v})\boldsymbol{\gamma} + \boldsymbol{\gamma}\nabla(r_h\mathbf{u}-\mathbf{v})^T + \nabla\mathbf{w}(r_h\boldsymbol{\sigma}-\boldsymbol{\tau}) + (r_h\boldsymbol{\sigma}-\boldsymbol{\tau})\nabla\mathbf{w}^T\|_{L^2}\\ &\leq \frac{1}{\eta_p}\Big(\|\nabla(r_h\mathbf{u}-\mathbf{v})\|_{L^r}\|\boldsymbol{\gamma}\|_{L^q} + \|\nabla\mathbf{w}\|_{L^r}\|r_h\boldsymbol{\sigma}-\boldsymbol{\tau}\|_{L^q}\Big),\end{aligned}$$

with $q = \frac{2r}{r-2}$. For $p > 2$, the following inverse inequalities hold

$$\|v\|_{L^p} \leq \frac{C}{h^{\frac{p-2}{p}}}\|v\|_{L^2}, \qquad \|\nabla v\|_{L^p} \leq \frac{C}{h^{\frac{p-2}{p}}}\|\nabla v\|_{L^2},$$

for all continuous, piecewise linear function v, see [10]. This yields

$$\left\|\Big(DS(r_h\mathbf{U}) - DS(\mathbf{V})\Big)\mathbf{W}\right\|_{L^2} \leq C\frac{1}{h}\|r_h\mathbf{U}-\mathbf{V}\|_{H^1\times L^2}\|\mathbf{W}\|_{H^1\times L^2},$$

C being independent of λ and h. This last inequality yields (16). □

Before proving existence of a solution to (14) we still need to check that $D_{\mathbf{U}}F_h$ is invertible is the neighbourhood of $\mathbf{U}(\lambda)$.

Lemma 4.3. *Let* $\mathbf{U}(\lambda)$, $0 \leq \lambda \leq \lambda_0$, *be given by Theorem 2.2. There exists* $0 < \lambda_1 \leq \lambda_0$ *such that, for all* $\lambda \leq \lambda_1$ *and for all* h *we have*

$$\|D_{\mathbf{U}}F_h(\lambda, r_h\mathbf{U}(\lambda))^{-1}\|_{\mathcal{L}(X_h)} \leq 2.$$

Proof. By definition of F_h, we have

$$D_{\mathbf{U}}F_h(\lambda, r_h\mathbf{U}(\lambda)) = I - T_h\Big(\mathbf{0}, \lambda DS(r_h\mathbf{U}(\lambda))\Big),$$

where $DS(r_h\mathbf{U}(\lambda))$ is such that

$$DS(r_h\mathbf{U})\mathbf{V} = \frac{1}{2\eta_p}\Big(\nabla(r_h\mathbf{u})\boldsymbol{\tau} + \boldsymbol{\tau}\nabla(r_h\mathbf{u})^T + \nabla\mathbf{v}(r_h\boldsymbol{\sigma}) + (r_h\boldsymbol{\sigma})\nabla\mathbf{v}^T\Big),$$

for all $\mathbf{V} = (\mathbf{v}, \boldsymbol{\tau}) \in X_h$. Thus we can write

$$D_{\mathbf{U}}F_h(\lambda, r_h\mathbf{U}(\lambda)) = I - G_h \qquad \text{with} \qquad G_h = T_h(\mathbf{0}, \lambda DS(r_h\mathbf{U}(\lambda))).$$

If we prove that $\|G_h\|_{\mathcal{L}(X_h)} \leq 1/2$ for sufficiently small values of λ, then $D_{\mathbf{U}}F_h(\lambda, r_h\mathbf{U}(\lambda))$ is invertible and $\|D_{\mathbf{U}}F_h(\lambda, r_h\mathbf{U}(\lambda))^{-1}\|_{\mathcal{L}(X_h)} \leq 2$. Using Lemma 4.1, it suffices to prove that $DS(r_h\mathbf{U}(\lambda)) : X_h \to L^2(\Omega)^4$ is uniformly bounded with respect to h. Proceeding as in the proof of the previous Lemma we have

$$\begin{aligned}\|DS(r_h\mathbf{U})\mathbf{V}\|_{L^2} &= \frac{1}{2\eta_p}\left\|\nabla(r_h\mathbf{u})\boldsymbol{\tau} + \boldsymbol{\tau}\nabla(r_h\mathbf{u})^T + \nabla\mathbf{v}(r_h\boldsymbol{\sigma}) + (r_h\boldsymbol{\sigma})\nabla\mathbf{v}^T\right\|_{L^2}\\ &\leq C\Big(\|\nabla\mathbf{u}\|_{L^\infty}\|\boldsymbol{\tau}\|_{L^2} + \|\nabla\mathbf{v}\|_{L^2}\|\boldsymbol{\sigma}\|_{L^\infty}\Big)\\ &\leq \tilde{C}\Big(\|\mathbf{u}\|_{W^{2,r}}\|\boldsymbol{\tau}\|_{L^2} + \|\nabla\mathbf{v}\|_{L^2}\|\boldsymbol{\sigma}\|_{W^{1,r}}\Big),\end{aligned}$$

C, $\tilde{C}$ being independent of λ and h. In other words we have

$$\|DS(r_h\mathbf{U}(\lambda))\mathbf{V}\|_{L^2} \leq C\|\mathbf{U}(\lambda)\|_X\|\mathbf{V}\|_{H^1\times L^2},$$

and, going back to G_h we obtain

$$\|G_h(\mathbf{V})\|_{H^1\times L^2} \leq \tilde{C}\lambda\|\mathbf{U}(\lambda)\|_X\|\mathbf{V}\|_{H^1\times L^2}.$$

Finally, for λ sufficiently small we obtain $\|G_h\|_{\mathcal{L}(X_h)} \leq 1/2$ and the result follows. □

We are now in position to state the main result of the section

Theorem 4.4. *Let* $\mathbf{U}(\lambda)$, $0 \leq \lambda \leq \lambda_0$, *be given by Theorem 2.2. There exists* $0 < \overline{\lambda} \leq \lambda_0$, $\overline{h}$ *and* $\delta > 0$ *such that, for all* $0 \leq \lambda \leq \overline{\lambda}$, *for all* $0 < h \leq \overline{h}$, *there exists a unique* $\mathbf{U}_h(\lambda)$ *in the ball of* X_h *centered at* $r_h\mathbf{U}(\lambda)$ *with radius* δh *in the norm* $H^1 \times L^2$, *satisfying*

$$F_h(\lambda, \mathbf{U}_h(\lambda)) = 0.$$

Moreover, the mapping $\lambda \in [0, \overline{\lambda}] \to \mathbf{U}_h(\lambda) \in X_h$ *is continuous and there exists* $C > 0$ *such that the following a priori error estimate holds*

$$\|\mathbf{U}(\lambda) - \mathbf{U}_h(\lambda)\|_{H^1\times L^2} \leq Ch \qquad \forall\lambda \leq \overline{\lambda} \qquad \forall h \leq \overline{h}.$$

Remark 4.5. The statement of the above existence result is similar (although not the same) to those of [4,21], in which the convective term in the extra-stress constitutive equation are considered (see also [26] for analogous results on a second-grade fluid). However, a different technique is used in this paper, allowing *a priori* and *a posteriori* error estimates to be obtained more easily, with other assumptions.

In order to prove this theorem, we use the following abstract result.

Theorem 4.6 (Th. 2.1 of [9])**.** *Let* Y *and* Z *be two real Banach spaces with norms* $\|\cdot\|_Y$ *and* $\|\cdot\|_Z$ *respectively. Let* $G : Y \to Z$ *be a* $\mathcal{C}^1$ *mapping and* $v \in Y$ *be such that* $DG(v) \in \mathcal{L}(Y; Z)$ *is an isomorphism. We introduce the notations*

$$\begin{aligned}\epsilon &= \|G(v)\|_Z,\\ \gamma &= \|DG(v)^{-1}\|_{\mathcal{L}(Z;Y)},\\ L(\alpha) &= \sup_{x\in\overline{B}(v,\alpha)} \|DG(v) - DG(x)\|_{\mathcal{L}(Y;Z)},\\ &\quad \textit{with } \overline{B}(v,\alpha) = \{y \in Y; \|v - y\|_Y \leq \alpha\},\end{aligned}$$

and we are interested in finding $u \in Y$ *such that*

$$G(u) = 0. \tag{20}$$

We assume that $2\gamma L(2\gamma\epsilon) \leq 1$. *Then Problem (20) has a unique solution* u *in the ball* $\overline{B}(v, 2\gamma\epsilon)$ *and, for all* $x \in \overline{B}(v, 2\gamma\epsilon)$, *we have*

$$\|x - u\|_Y \leq 2\gamma\|G(x)\|_Z. \tag{21}$$

Proof of Theorem 4.4. We apply Theorem 4.6 with $Y = X_h$, $Z = X_h$, $G = F_h$, $v = r_h\mathbf{U}(\lambda)$ and the norm $\|\cdot\|_{H^1\times L^2}$ in X_h. The mapping $G : Y \to Z$ is $\mathcal{C}^1$ and, according to Lemma 4.2, for λ sufficiently small there is a constant C_1 independent of λ and h such that $\epsilon \leq C_1 h$. According to Lemma 4.3, for λ sufficiently small $\gamma \leq 2$.

According to Lemma 4.2, there is a constant C_2 independent of λ and h such that $L(\alpha) \leq C_2\alpha\lambda/h$. Thus, we have

$$2\gamma L(2\gamma\epsilon) \leq 2.2C_2(2.2.C_1h)\frac{\lambda}{h} = 16C_1C_2\lambda.$$

Thus, for λ sufficiently small $2\gamma L(2\gamma\epsilon) \leq 1$ and Theorem 4.6 applies. There exists a unique $\mathbf{U}_h(\lambda)$ in the ball $\overline{B}(v, 2\gamma\epsilon)$ such that $F_h(\lambda, \mathbf{U}_h(\lambda)) = 0$ and we have

$$\|r_h\mathbf{U}(\lambda) - \mathbf{U}_h(\lambda)\|_{H^1\times L^2} \leq 4C_1h.$$

It suffices to use the triangle inequality

$$\|\mathbf{U}(\lambda) - \mathbf{U}_h(\lambda)\|_{H^1\times L^2} \leq \|\mathbf{U}(\lambda) - r_h\mathbf{U}(\lambda)\|_{H^1\times L^2} + \|r_h\mathbf{U}(\lambda) - \mathbf{U}_h(\lambda)\|_{H^1\times L^2},$$

and standard interpolation [10] results to obtain the *a priori* estimates. The fact that the mapping $\lambda \to \mathbf{U}_h(\lambda)$ is continuous is a direct consequence of the implicit function theorem. □

5. *A POSTERIORI* ERROR ESTIMATES

Let us consider again the operator

$$\begin{aligned} T_h : L^2(\Omega)^2 \times L^2_s(\Omega)^4 &\longrightarrow H^1_0(\Omega)^2 \times L^2_s(\Omega)^4 \\ (\mathbf{f}, \mathbf{g}) \quad &\longrightarrow T_h(\mathbf{f}, \mathbf{g}) \underset{\text{def.}}{=} (\mathbf{u}_h, \boldsymbol{\sigma}_h) \in V_h \times M_h, \end{aligned}$$

where $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ satisfies (11). We now introduce a residual based error estimator for T_h. The notations are those of [2]. For any triangle K of the triangulation $\mathcal{T}_h$, let E_K be the set of its three edges. For each interior edge ℓ of $\mathcal{T}_h$, let us choose an arbitrary normal direction $\mathbf{n}$, let $[\cdot]_\ell$ denote the jump across edge ℓ. For each edge ℓ of $\mathcal{T}_h$ lying on the boundary $\partial\Omega$, we set $[\cdot]_\ell = 0$. The local error estimator corresponding to (11) is then defined by

$$\begin{aligned} \mu_K^2(\mathbf{f}, \mathbf{g}) = \frac{1}{\eta_s + \eta_p}\Bigg(& h_K^2\,\|-2\eta_s \text{ div } \boldsymbol{\epsilon}(\mathbf{u}_h) + \nabla p_h - \text{div } \boldsymbol{\sigma}_h - \mathbf{f}\|^2_{L^2(K)} \\ &+ \frac{1}{2}\sum_{\ell\in E_K} |\ell|\ \|[2\eta_s\boldsymbol{\epsilon}(\mathbf{u}_h)\mathbf{n}]_\ell\|^2_{L^2(\ell)}\Bigg) + (\eta_s + \eta_p)\|\text{div } \mathbf{u}_h\|^2_{L^2(K)} \\ &+ \eta_p\left\|\frac{1}{2\eta_p}\boldsymbol{\sigma}_h - \mathbf{g} - \boldsymbol{\epsilon}(\mathbf{u}_h)\right\|^2_{L^2(K)}. \end{aligned}$$

We have the following *a posteriori* error estimate for operator $T - T_h$.

Lemma 5.1. *There exists C and $h_0 > 0$ such that, for all $(\mathbf{f}, \mathbf{g}) \in L^r(\Omega)^2 \times W^{1,r}_s(\Omega)^4$ we have*

$$\|T(\mathbf{f}, \mathbf{g}) - T_h(\mathbf{f}, \mathbf{g})\|_{H^1\times L^2} \leq C\left(\sum_{K\in\mathcal{T}_h} \mu_K^2(\mathbf{f}, \mathbf{g})\right)^{1/2} \qquad \forall h \leq h_0.$$

Proof. We set $T(\mathbf{f}, \mathbf{g}) = (\mathbf{u}, \boldsymbol{\sigma})$ where $(\mathbf{u}, p, \boldsymbol{\sigma})$ is the solution of (2), we also set $T_h(\mathbf{f}, \mathbf{g}) = (\mathbf{u}_h, \boldsymbol{\sigma}_h)$ where $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h)$ is the solution of (11). We recall that $W = H^1_0(\Omega)^2 \times L^2_0(\Omega) \times L^2_s(\Omega)^4$. Let $B : W \times W \to \mathbb{R}$ be the bilinear form corresponding to the weak formulation of (2), namely

$$\begin{aligned} B(\mathbf{u}, p, \boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau}) = 2\eta_s(\boldsymbol{\epsilon}(\mathbf{u}), \boldsymbol{\epsilon}(\mathbf{v})) - (p, \text{div } \mathbf{v}) + (\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\mathbf{v})) \\ - (\text{div } \mathbf{u}, q) - \frac{1}{2\eta_p}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\boldsymbol{\epsilon}(\mathbf{u}), \boldsymbol{\tau}), \end{aligned}$$

for all $(\mathbf{u}, p, \boldsymbol{\sigma})$ and $(\mathbf{v}, q, \boldsymbol{\tau})$ in W. In Lemma 2 of [6], it is proved that B satisfies an inf-sup condition, therefore we have

$$\|\mathbf{u}-\mathbf{u}_h, p-p_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H^1\times L^2\times L^2} \leq C \sup_{\mathbf{0}\neq(\mathbf{v},q,\boldsymbol{\tau})\in W} \frac{B(\mathbf{u}-\mathbf{u}_h, p-p_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau})}{\|\mathbf{v}, q, \boldsymbol{\tau}\|_{H^1\times L^2\times L^2}},$$

C being independent of h. For all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W$ we have

$$B(\mathbf{u}-\mathbf{u}_h, p-p_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) = (\mathbf{f}, \mathbf{v}) - (\mathbf{g}, \boldsymbol{\tau}) - B(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}).$$

Introducing $B_h : W_h \times W_h \to \mathbb{R}$ as in the proof of Lemma 4.1, we have

$$\begin{aligned} B(\mathbf{u}-\mathbf{u}_h, p-p_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) &= (\mathbf{f}, \mathbf{v}-\mathbf{v}_h) - (\mathbf{g}, \boldsymbol{\tau}-\boldsymbol{\tau}_h) - B(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}-\mathbf{v}_h, q-q_h, \boldsymbol{\tau}-\boldsymbol{\tau}_h) \\ &\quad + (B_h - B)(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}_h, q_h, \boldsymbol{\tau}_h), \end{aligned}$$

for all $(\mathbf{v}_h, q_h, \boldsymbol{\tau}_h) \in W_h$, that is:

$$\begin{aligned} &B(\mathbf{u}-\mathbf{u}_h, p-p_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) \\ &\quad = -2\eta_s(\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\epsilon}(\mathbf{v}-\mathbf{v}_h)) + (p_h, \text{div } (\mathbf{v}-\mathbf{v}_h)) - (\boldsymbol{\sigma}_h, \boldsymbol{\epsilon}(\mathbf{v}-\mathbf{v}_h)) + (\mathbf{f}, \mathbf{v}-\mathbf{v}_h) \\ &\quad\quad + (\text{div } \mathbf{u}_h, q-q_h) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(-2\eta_s \text{div } \boldsymbol{\epsilon}(\mathbf{u}_h) + \nabla p_h - \text{div } \boldsymbol{\sigma}_h - \mathbf{f}, \nabla q_h\Big)_K \\ &\quad\quad + \left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h - \mathbf{g} - \boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\tau}-\boldsymbol{\tau}_h - 2\eta_p\beta\boldsymbol{\epsilon}(\mathbf{v}_h)\right). \end{aligned}$$

We then proceed as in [3,28], integrate by parts on each triangle $K \in \mathcal{T}_h$ the first three terms in the right hand side of the above equation, and choose $\mathbf{v}_h = \mathbf{R}_h\mathbf{v}$ (where R_h is Clément's interpolant [11]), $q_h = 0$, $\boldsymbol{\tau}_h = 0$ to conclude. □

We are now in position to state *a posteriori* error estimates for the solution to (9). Let us briefly recall the notations. Problem (1) is written as $F(\lambda, \mathbf{U}) = 0$ with $\mathbf{U} = (\mathbf{u}, \boldsymbol{\sigma}) \in X$ and $F(\lambda, \mathbf{U}) = \mathbf{U} - T(\mathbf{f}, \lambda S(\mathbf{U}))$, the operators T and S being defined in (5) and (7). Problem (9) is written as $F_h(\lambda, \mathbf{U}_h) = 0$ with $\mathbf{U}_h = (\mathbf{u}_h, \boldsymbol{\sigma}_h) \in X_h \not\subset X$ and $F_h(\lambda, \mathbf{U}_h) = \mathbf{U}_h - T_h(\mathbf{f}, \lambda S(\mathbf{U}_h))$, the operator T_h being defined in (10). According to Theorem 2.2, for λ sufficiently small, there is a unique $\mathbf{U}(\lambda)$ such that $F(\lambda, \mathbf{U}(\lambda)) = 0$. According to Theorem 4.4, for h and λ sufficiently small, there exists a unique $\mathbf{U}_h(\lambda)$ in a neighbourhood of $r_h\mathbf{U}(\lambda)$ depending on h (in the norm $\|\cdot\|_{H^1\times L^2}$) such that $F_h(\lambda, \mathbf{U}_h(\lambda)) = 0$. Moreover, when h goes to zero, $\mathbf{U}_h(\lambda)$ converges to $\mathbf{U}(\lambda)$ in the norm $\|\cdot\|_{H^1\times L^2}$.

Theorem 5.2. *There exists λ_0, h_0 and $C > 0$ such that, for all $\lambda \leq \lambda_0$, for all $h \leq h_0$,*

$$\|\mathbf{U}(\lambda) - \mathbf{U}_h(\lambda)\|_{H^1\times L^2} \leq C \left(\sum_{K\in\mathcal{T}_h} \mu_K^2\Big(\mathbf{f}, \lambda S(\mathbf{U}_h(\lambda))\Big)\right)^{1/2}. \tag{22}$$

Proof. Using the definition of F and F_h we have

$$\begin{aligned} \mathbf{U} - \mathbf{U}_h &= T\Big(\mathbf{f}, \lambda S(\mathbf{U})\Big) - T_h\Big(\mathbf{f}, \lambda S(\mathbf{U}_h)\Big) \\ &= T\Big(\mathbf{0}, \lambda\Big(S(\mathbf{U}) - S(\mathbf{U}_h)\Big)\Big) + (T - T_h)\Big(\mathbf{f}, \lambda S(\mathbf{U}_h)\Big). \end{aligned} \tag{23}$$

We now bound the first term in the right hand side of (23). If $\mathbf{V} = (\mathbf{v}, \boldsymbol{\tau})$ is defined by

$$\mathbf{V} = T\Big(\mathbf{0}, \lambda\Big(S(\mathbf{U}) - S(\mathbf{U}_h)\Big)\Big),$$

then there exists $q \in L_0^2(\Omega)$ such that $(\mathbf{v}, q) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ satisfies

$$\begin{aligned} -2(\eta_s + \eta_p)\text{div } \boldsymbol{\epsilon}(\mathbf{v}) + \nabla q &= \lambda \text{div } \Big(S(\mathbf{U}) - S(\mathbf{U}_h)\Big), \\ \text{div } \mathbf{v} &= 0. \end{aligned}$$

We then have the following estimates

$$\begin{aligned} \|\mathbf{V}\|_{H^1\times L^2} = \|\mathbf{v}, q\|_{H^1\times L^2} &\leq C\lambda \Big\|\text{div } \Big(S(\mathbf{U}) - S(\mathbf{U}_h)\Big)\Big\|_{H^{-1}} \\ &\leq \tilde{C}\lambda \|S(\mathbf{U}) - S(\mathbf{U}_h)\|_{L^2}. \end{aligned}$$

Now we have

$$\begin{aligned} &2\eta_p\Big(S(\mathbf{U}) - S(\mathbf{U}_h)\Big) \\ &\quad = \nabla\mathbf{u}\boldsymbol{\sigma} + \boldsymbol{\sigma}\nabla\mathbf{u}^T - (\nabla\mathbf{u}_h)\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h(\nabla\mathbf{u}_h)^T \\ &\quad = \nabla(\mathbf{u} - \mathbf{u}_h)\boldsymbol{\sigma} + (\nabla\mathbf{u}_h)(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + \boldsymbol{\sigma}\nabla(\mathbf{u} - \mathbf{u}_h)^T + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(\nabla\mathbf{u}_h)^T, \end{aligned}$$

so that

$$\|S(\mathbf{U}) - S(\mathbf{U}_h)\|_{L^2} \leq \frac{1}{\eta_p}\Big(\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2}\|\boldsymbol{\sigma}\|_{L^\infty} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}\|\nabla\mathbf{u}_h\|_{L^\infty}\Big).$$

We now bound the last term of the above inequality. We have

$$\begin{aligned} \|\nabla\mathbf{u}_h\|_{L^\infty} &\leq \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^\infty} + \|\nabla\mathbf{u}\|_{L^\infty} \\ &\leq \|\nabla(\mathbf{u} - r_h\mathbf{u})\|_{L^\infty} + \|\nabla(r_h\mathbf{u} - \mathbf{u}_h)\|_{L^\infty} + \|\nabla\mathbf{u}\|_{L^\infty}. \end{aligned}$$

We then use standard interpolation estimates [10], an inverse estimate, and a Sobolev imbedding theorem to obtain

$$\|\nabla\mathbf{u}_h\|_{L^\infty} \leq C\Big(\|\mathbf{u}\|_{W^{2,r}} + \frac{1}{h}\|\nabla(r_h\mathbf{u} - \mathbf{u}_h)\|_{L^2} + \|\mathbf{u}\|_{W^{2,r}}\Big),$$

with C independent of λ and h. Finally, applying Theorem 4.4 to the above estimate we have, for $h \leq \overline{h}$ and $\lambda \leq \overline{\lambda}$, $\|\nabla\mathbf{u}_h\|_{L^\infty} \leq C$ so that

$$\|S(\mathbf{U}) - S(\mathbf{U}_h)\|_{L^2} \leq C\|\mathbf{U} - \mathbf{U}_h\|_{H^1\times L^2},$$

C being independent of h and $\lambda \in [0, \overline{\lambda}]$. Thus, we have shown that

$$\Big\|T\Big(\mathbf{0}, \lambda\Big(S(\mathbf{U}) - S(\mathbf{U}_h)\Big)\Big)\Big\|_{H^1\times L^2} \leq C\lambda\|\mathbf{U} - \mathbf{U}_h\|_{H^1\times L^2}, \tag{24}$$

where C does not depend on h and $\lambda \in [0, \overline{\lambda}]$.

In order to bound the second term in the right hand side of (23), we use Lemma 5.1. There is a constant C independent of h and λ such that we have, for h sufficiently small

$$\left\|(T-T_h)\Big(\mathbf{f},\lambda S(\mathbf{U}_h)\Big)\right\|_{H^1\times L^2} \le C\left(\sum_{K\in\mathcal{T}_h}\mu_K^2\Big(\mathbf{f},\lambda S(\mathbf{U}_h)\Big)\right)^{1/2}. \tag{25}$$

Finally, estimates (24) and (25) in (23) yield the result provided λ is small enough. □

Remark 5.3. Proceeding as in [2,28], we can prove a lower bound similar to the upper bound of Lemma 5.1. Then, we can also prove a lower bound similar to the upper bound of Theorem 5.2. Thus, the error estimator is equivalent to the true error. We did not include such a result in this paper in order to shorten the presentation.

Remark 5.4. We can show a sharper estimate than (22). Indeed, let us introduce as in [6] the norm $\|\cdot\|_W$ defined, for all $(\mathbf{v},q,\boldsymbol{\tau})\in W$, by

$$\|\mathbf{v},q,\boldsymbol{\tau}\|_W^2 \;=\; 2(\eta_s+\eta_p)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2+\frac{1}{2(\eta_s+\eta_p)}\|q\|^2+\frac{1}{2\eta_p}\|\boldsymbol{\tau}\|^2. \tag{26}$$

Then, the following estimate also holds

$$\|\mathbf{u}-\mathbf{u}_h,p-p_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_W\le C\left(\sum_{K\in\mathcal{T}_h}\mu_K^2\Big(\mathbf{f},\lambda S(\mathbf{u}_h,\boldsymbol{\sigma}_h)\Big)\right)^{1/2},$$

where C is independent of λ, h and η_s, η_p. Here $(\mathbf{u},p,\boldsymbol{\sigma})$ is the solution of (1), $(\mathbf{u}_h,p_h,\boldsymbol{\sigma}_h)$ is the solution of (9). Thus, our *a posteriori* error estimates also hold when the solvent viscosity is small.

6. Link with an EVSS formulation

The Elastic Viscous Split Stress (EVSS) formulation corresponding to (9) is obtained from the following differential problem: find the velocity $\mathbf{u}$, pressure p, extra-stresses $\boldsymbol{\sigma}$ and $\mathbf{D}$ such that

$$\begin{aligned}-2(\eta_s+\eta_p)\mathrm{div}\;\boldsymbol{\epsilon}(\mathbf{u})+\nabla p-\mathrm{div}\;(\boldsymbol{\sigma}-2\eta_p\mathbf{D})=\mathbf{f},\\ \mathrm{div}\;\mathbf{u}=0,\\ \frac{1}{2\eta_p}\boldsymbol{\sigma}-\frac{\lambda}{2\eta_p}\Big((\nabla\mathbf{u})\boldsymbol{\sigma}+\boldsymbol{\sigma}(\nabla\mathbf{u}^T)\Big)-\boldsymbol{\epsilon}(\mathbf{u})=0,\\ \mathbf{D}-\boldsymbol{\epsilon}(\mathbf{u})=0,\end{aligned} \tag{27}$$

in Ω, where the velocity $\mathbf{u}$ vanishes on $\partial\Omega$. Obviously, at the continuous level (27) is equivalent to (1).

In order to solve this problem, we consider the following EVSS scheme: find $(\mathbf{u}_h,p_h,\boldsymbol{\sigma}_h,\mathbf{D}_h)\in V_h\times Q_h\times M_h\times M_h$ such that

$$\begin{aligned}&2(\eta_s+\eta_p)(\boldsymbol{\epsilon}(\mathbf{u}_h),\boldsymbol{\epsilon}(\mathbf{v}))-(p_h,\mathrm{div}\;\mathbf{v})+(\boldsymbol{\sigma}_h-2\eta_p\mathbf{D}_h,\boldsymbol{\epsilon}(\mathbf{v}))-(\mathbf{f},\mathbf{v})\\ &-(\mathrm{div}\;\mathbf{u}_h,q)-\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(-2\eta_s\mathrm{div}\;\boldsymbol{\epsilon}(\mathbf{u}_h)+\nabla p_h-\mathrm{div}\;\boldsymbol{\sigma}_h-\mathbf{f},\nabla q\Big)_K\\ &-\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h-\frac{\lambda}{2\eta_p}(\nabla\mathbf{u}_h\boldsymbol{\sigma}_h+\boldsymbol{\sigma}_h\nabla\mathbf{u}_h^T)-\boldsymbol{\epsilon}(\mathbf{u}_h),\boldsymbol{\tau}\right),\\ &+(\mathbf{D}_h-\boldsymbol{\epsilon}(\mathbf{u}_h),\mathbf{E})=0,\end{aligned} \tag{28}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}, \mathbf{E}) \in V_h \times Q_h \times M_h \times M_h$. In the above scheme, only the stabilization terms corresponding to the first equation of (27) have been added in order to avoid spurious oscillations for the pressure. However, no special attention is paid to the extra-stress $\boldsymbol{\sigma}_h$ and the stabilization term

$$\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h - \frac{\lambda}{2\eta_p}(\nabla\mathbf{u}_h\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h\nabla\mathbf{u}_h^T) - \boldsymbol{\epsilon}(\mathbf{u}_h), 2\eta_p\beta\boldsymbol{\epsilon}(\mathbf{v})\right)$$

present in (9) is missing in (28). The extra-stress $\mathbf{D}_h$ is added only for stability purposes and thus (28) can be interpreted as an EVSS scheme for solving (1).

In the linear case (*i.e.* when $\lambda = 0$), the GLS scheme (9) with $\beta = 1$ is equivalent to the EVSS formulation (28), as explained in [6]. Indeed, $\mathbf{D}_h \in M_h$ is such that

$$(\mathbf{D}_h - \boldsymbol{\epsilon}(u_h), \mathbf{E}) = 0 \qquad \forall \mathbf{E} \in M_h,$$

and, when $\lambda = 0$, $\boldsymbol{\sigma}_h \in M_h$ is such that

$$\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h - \boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\tau}\right) = 0 \qquad \forall \boldsymbol{\tau} \in M_h.$$

Thus, $\mathbf{D}_h = \frac{1}{2\eta_p}\boldsymbol{\sigma}_h$ and (28) reduces to (9) with $\beta = 1$.

In the nonlinear case, the two GLS and EVSS schemes are not equivalent anymore. Indeed, when $\lambda \neq 0$, we have

$$\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h + \frac{\lambda}{2\eta_p}(\nabla\mathbf{u}_h\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h\nabla\mathbf{u}_h^T) - \mathbf{D}_h, \boldsymbol{\tau}\right) = 0 \qquad \forall \boldsymbol{\tau} \in M_h,$$

and since $(\nabla\mathbf{u}_h\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h\nabla\mathbf{u}_h^T) \notin M_h$ we cannot conclude.

Proceeding as we did for the GLS scheme (9), and under the same assumptions, we can prove that (28) has a solution converging to the solution of (27).

7. Numerical results

We now discuss iterative decoupling schemes for solving (9) with $\beta = 1$ and (28).

Our iterative procedures are an extension of those presented in [6] and [12] and allow velocity and pressure computations to be decoupled from extra-stresses computations.

Let us start with the GLS scheme (9) and $\beta = 1$. Let $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)$ be the known approximation of $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h)$ after n steps. Step $(n+1)$ of the algorithm consists in first computing $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ by using the mass and momentum equations, then under-relaxing with parameter $0 \leq \omega \leq 1$, and finally computing $\boldsymbol{\sigma}_h^{n+1}$ with the constitutive relationship. Thus iteration $(n+1)$ consists in finding $(\tilde{\mathbf{u}}_h^{n+1}, \tilde{p}_h^{n+1}) \in V_h \times Q_h$ such that

$$\begin{aligned}
&2\eta_s(\boldsymbol{\epsilon}(\tilde{\mathbf{u}}_h^{n+1}), \boldsymbol{\epsilon}(\mathbf{v})) - (\tilde{p}_h^{n+1}, \text{div } \mathbf{v}) + (\boldsymbol{\sigma}_h^n, \boldsymbol{\epsilon}(\mathbf{v}))\\
&- \left(\boldsymbol{\sigma}_h^n - \lambda(\nabla\mathbf{u}_h^n\boldsymbol{\sigma}_h^n + \boldsymbol{\sigma}_h^n(\nabla\mathbf{u}_h^n)^T) - 2\eta_p\boldsymbol{\epsilon}(\tilde{\mathbf{u}}_h^{n+1}), \boldsymbol{\epsilon}(\mathbf{v})\right) - (\mathbf{f}, \mathbf{v})\\
&- (\text{div } \tilde{\mathbf{u}}_h^{n+1}, q)\\
&- \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\left(-2\eta_s \text{div } \boldsymbol{\epsilon}(\tilde{\mathbf{u}}_h^{n+1}) + \nabla\tilde{p}_h^{n+1} - \text{div } \boldsymbol{\sigma}_h^n - \mathbf{f}, \nabla q\right)_K = 0,
\end{aligned} \tag{29}$$

for all $(\mathbf{v}, q) \in V_h \times Q_h$, then updating $\mathbf{u}_h^{n+1}$ and p_h^{n+1} as following

$$\begin{aligned}
\mathbf{u}_h^{n+1} &= \omega\tilde{\mathbf{u}}_h^{n+1} + (1-\omega)\mathbf{u}_h^n,\\
p_h^{n+1} &= \omega\tilde{p}_h^{n+1} + (1-\omega)p_h^n,
\end{aligned}$$

and finally finding $\boldsymbol{\sigma}_h^{n+1} \in M_h$ such that

$$\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h^{n+1} - \frac{\lambda}{2\eta_p}(\nabla \mathbf{u}_h^n \boldsymbol{\sigma}_h^n + \boldsymbol{\sigma}_h^n (\nabla \mathbf{u}_h^n)^T) - \boldsymbol{\epsilon}(\mathbf{u}_h^{n+1}), \boldsymbol{\tau}\right) = 0 \qquad \forall \boldsymbol{\tau} \in M_h. \tag{30}$$

Let us now turn to the EVSS scheme (28). Let $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n, \mathbf{D}_h^n)$ be the known approximation of $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \mathbf{D}_h)$ after n steps. Iteration $(n+1)$ consists in finding $(\tilde{\mathbf{u}}_h^{n+1}, \tilde{p}_h^{n+1}) \in V_h \times Q_h$ such that

$$\begin{aligned}
&2(\eta_s + \eta_p)(\boldsymbol{\epsilon}(\tilde{\mathbf{u}}_h^{n+1}), \boldsymbol{\epsilon}(\mathbf{v})) - (\tilde{p}_h^{n+1}, \operatorname{div} \mathbf{v}) + (\boldsymbol{\sigma}_h^n - 2\eta_p \mathbf{D}_h^n, \boldsymbol{\epsilon}(\mathbf{v})) - (\mathbf{f}, \mathbf{v}) \\
&- (\operatorname{div} \tilde{\mathbf{u}}_h^{n+1}, q) \\
&- \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \left(-2\eta_s \operatorname{div} \boldsymbol{\epsilon}(\tilde{\mathbf{u}}_h^{n+1}) + \nabla \tilde{p}_h^{n+1} - \operatorname{div} \boldsymbol{\sigma}_h^n - \mathbf{f}, \nabla q\right)_K = 0,
\end{aligned} \tag{31}$$

for all $(\mathbf{v}, q) \in V_h \times Q_h$, then updating $\mathbf{u}_h^{n+1}$ and p_h^{n+1} as following

$$\begin{aligned}
\mathbf{u}_h^{n+1} &= \omega \tilde{\mathbf{u}}_h^{n+1} + (1-\omega)\mathbf{u}_h^n, \\
p_h^{n+1} &= \omega \tilde{p}_h^{n+1} + (1-\omega) p_h^n,
\end{aligned}$$

and finally finding $\boldsymbol{\sigma}_h^{n+1} \in M_h$ such that

$$\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h^{n+1} - \frac{\lambda}{2\eta_p}(\nabla \mathbf{u}_h^n \boldsymbol{\sigma}_h^n + \boldsymbol{\sigma}_h^n (\nabla \mathbf{u}_h^n)^T) - \boldsymbol{\epsilon}(\mathbf{u}_h^{n+1}), \boldsymbol{\tau}\right) = 0 \qquad \forall \boldsymbol{\tau} \in M_h, \tag{32}$$

and $\mathbf{D}_h^{n+1} \in M_h$ such that

$$(\mathbf{D}_h^{n+1} - \boldsymbol{\epsilon}(\mathbf{u}_h^{n+1}), \mathbf{E}) = 0, \qquad \forall \mathbf{E} \in M_h. \tag{33}$$

The computational effort required to compute the velocity and pressure thus corresponds to solving a Stokes' problem with a conventional GLS method, whereas the computations of the extra-stresses are explicit provided the mass matrices are lumped. The decoupled EVSS procedure is more interesting than the GLS one from the implementation point of view. Indeed, if the constitutive relationship between the extra-stress and the velocity field is replaced by a more realistic equation (for instance the Phan-Thien-Tanner or the FENE-P models), then only the portion of the code corresponding to (32) should be updated. This is not the case of the GLS scheme since some of the terms present in (29) are due to the constitutive equation.

The EVSS scheme is particularly interesting for molecular models, the constitutive relationship being replaced by a stochastic differential equation for the dumbbells elongations, see for instance [8,16,20].

7.1. **A simple test case**

In order to investigate numerically the rate of convergence of both GLS and EVSS schemes we have set

$$\mathbf{u}(x_1, x_2) = \begin{pmatrix} u_1(x_2) \\ u_2(x_1) \end{pmatrix} = \begin{pmatrix} \sin(\pi x_2) e^{x_2} \\ \sin(\pi x_1) e^{x_1} \end{pmatrix}, \qquad p(x_1, x_2) = 0.$$

The constitutive relationship (the last equation of (1)) then leads to

$$\sigma_{11} = 2\eta_p \lambda u_1' \gamma, \qquad \sigma_{12} = \eta_p \gamma, \qquad \sigma_{22} = 2\eta_p \lambda u_2' \gamma,$$

where γ is defined by

$$\gamma(x_1, x_2) = \frac{u_1'(x_2) + u_2'(x_1)}{1 - 4\lambda^2 u_1'(x_2) u_2'(x_1)},$$

whenever $\lambda < 1/\sqrt{4 \max |u_1' u_2'|} = \frac{1}{2\pi e} \simeq 0.058$. Then, the source term in the momentum equation is given by

$$\mathbf{f} = -\eta_s \begin{pmatrix} u_1'' \\ u_2'' \end{pmatrix} - \eta_p \begin{pmatrix} 2\lambda u_1' \partial\gamma/\partial x_1 + \partial\gamma/\partial x_2 \\ \partial\gamma/\partial x_1 + 2\lambda u_2' \partial\gamma/\partial x_2 \end{pmatrix}.$$

The calculation domain was the unit square cut into several squares, each square being cut into two triangles along one of its diagonal. The viscosities were $\eta_s = 0.01$, $\eta_p = 1$, the GLS stabilization parameter was $\alpha = 0.01$, the relaxation parameter was $\omega = 0.5$, the elastic time scale was $\lambda = 0.02$. In Table 1 we have reported the L^2 error of the scalar unknowns (velocity components u_1, u_2, pressure p, extra-stress components σ_{11},...) with several meshes. Clearly, the order of convergence with respect to h is close to two for the velocity and one for the pressure and extra-stress.

TABLE 1. L^2 error and number of iterations to achieve convergence with several meshes, top: GLS scheme, bottom EVSS scheme.

Mesh	u_1	u_2	p	σ_{11}	σ_{12}	σ_{22}	iterations
10×10	0.00041	0.00041	0.36	0.19	0.41	0.19	22
20×20	0.00012	0.00012	0.17	0.066	0.14	0.066	22
40×40	0.000037	0.000037	0.082	0.023	0.048	0.023	23
80×80	0.000010	0.000010	0.040	0.0079	0.017	0.0079	23

Mesh	u_1	u_2	p	σ_{11}	σ_{12}	σ_{22}	iterations
10×10	0.00049	0.00049	0.38	0.19	0.41	0.19	22
20×20	0.00016	0.00016	0.18	0.066	0.14	0.066	22
40×40	0.000052	0.000052	0.091	0.022	0.047	0.022	23
80×80	0.000015	0.000015	0.045	0.0078	0.016	0.0078	23

The stopping criterion for the iterative procedures (29–33) to reach convergence with respect to n was 10^{-6} on the relative discrepancy. From Table 2, we can see that, for a fixed λ, the number of iterations does not depend on the mesh size nor on the solvent viscosity. However, when λ increases, the number of iterations increases.

TABLE 2. Number of iterations to achieve convergence with respect to n on a 20×20 mesh. Left table: with several solvent viscosities ($\lambda = 0.02$). Right table: with several relaxation times ($\eta_s = 0.01$). The xx symbol means that the scheme was divergent.

η_s	GLS	EVSS
1	22	22
0.01	22	22
0	22	22

λ	GLS	EVSS
0.02	22	22
0.03	22	22
0.04	22	22
0.05	36	35
0.055	88	62
0.06	xx	225
0.065	xx	xx

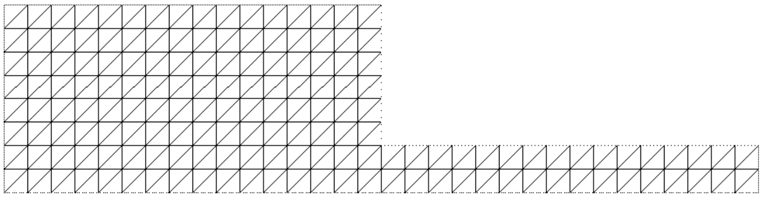

FIGURE 1. The coarsest mesh (mesh 1) used for the computations (201 vertices). Mesh 2 is obtained by cutting the mesh size by two (721 vertices), mesh 3 by four (2721 vertices) and mesh 4 by eight (10561 vertices).

7.2. **The 4:1 planar contraction**

The 4:1 planar contraction is a classical test case in the frame of non Newtonian flows [22]. This test case is not covered by our theory since the calculation domain is not convex, and the velocity gradient is not bounded at the reentrant corner. From the physical point of view, instabilities are observed at high Deborah numbers. From the numerical point of view, most viscoelastic codes fail to converge at high Deborah numbers. Moreover, as reported in several papers (see for instance Sect. 7.2 of [1] for details), the maximum attainable Deborah number seems to decrease with mesh size. The well posedness of the problem is still an open question.

We have performed computations on half of the contraction, with four different meshes, the coarsest mesh being the one of Figure 1. The inlet and outlet velocities were imposed to be parabolic, with maximum velocity one at the inlet. The viscosities were $\eta_s = 0.01$, $\eta_p = 1$, the GLS stabilization parameter was $\alpha = 0.01$, the elastic time scale λ was ranging from 0.01 to 0.04. In Table 3 we have reported the number of iterations to reach convergence for several mesh sizes and several values of λ when using the EVSS scheme (similar results were obtained with the GLS scheme). It seems that, the more the mesh is refined, the smaller the maximum attainable Deborah number. Moreover, increasing the solvent viscosity does not help convergence.

There are (at least) two possible reasons to explain failure of the numerical procedure (31–33).

- The decoupled procedure (31–33) used to obtain the solution of (28) is not appropriate. Newton's method should be implemented to check if convergence could be obtained with higher Deborah numbers, and several meshes.
- Problem (1) has many solutions or no solution at all for the contraction flow.

TABLE 3. The 4:1 planar contraction flow. Number of iterations to achieve convergence with four meshes (see caption of Fig. 1) and several values of λ. Left: $\lambda = 0.04$, middle: $\lambda = 0.02$, right: $\lambda = 0.01$. The xx symbol means that the scheme was divergent.

Mesh	iterations
1	17
2	xx
3	xx
4	xx

Mesh	iterations
1	16
2	16
3	17
4	xx

Mesh	iterations
1	15
2	15
3	16
4	18

8. CONCLUSION AND PERSPECTIVES

In this paper, existence, *a priori* and *a posteriori* error estimates have been obtained for the GLS approximation of an Oldroyd-B problem without convection. An EVSS method is also introduced and the link with the GLS method is shown. Numerical results are presented.

An important point is now to extend these theoretical predictions to the case when the convective terms are added to the momentum equation and to the constitutive equation. Also, we are looking forward to extending this work to the case of mesoscopic models in order to justify the computations performed in [7, 8].

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References

[1] F.P.T. Baaijens, Mixed finite element methods for viscoelastic flow analysis: a review. *J. Non-Newtonian Fluid Mech.* **79** (1998) 361–385.

[2] I. Babuska, R. Duran, and R. Rodriguez, Analysis of the efficiency of an *a posteriori* error estimator for linear triangular finite elements. *SIAM J. Numer. Anal.* **29** (1992) 947–964.

[3] J. Baranger and H. El-Amri, Estimateurs *a posteriori* d'erreur pour le calcul adaptatif d'écoulements quasi-newtoniens. *RAIRO Modél. Math. Anal. Numér.* **25** (1991) 31–48.

[4] J. Baranger and D. Sandri, Finite element approximation of viscoelastic fluid flow. *Numer. Math.* **63** (1992) 13–27.

[5] M. Behr, L. Franca, and T. Tezduyar, Stabilized finite element methods for the velocity-pressure-stress formulation of incompressible flows. *Comput. Methods Appl. Mech. Engrg.* **104** (1993) 31–48.

[6] J. Bonvin, M. Picasso and R. Stenberg, GLS and EVSS methods for a three-field Stokes problem arising from viscoelastic flows. *Comput. Methods Appl. Mech. Engrg.* **190** (2001) 3893–3914.

[7] J.C. Bonvin, *Numerical simulation of viscoelastic fluids with mesoscopic models.* Ph.D. thesis, Département de Mathématiques, École Polytechnique Fédérale de Lausanne (2000).

[8] J.C. Bonvin and M. Picasso, Variance reduction methods for CONNFFESSIT-like simulations. *J. Non-Newtonian Fluid Mech.* **84** (1999) 191–215.

[9] G. Caloz and J. Rappaz, Numerical analysis for nonlinear and bifurcation problems, in *Handbook of Numerical Analysis. Vol. V: Techniques of Scientific Computing (Part 2)*, P.G. Ciarlet and J.L. Lions, Eds., Elsevier, Amsterdam (1997) 487–637.

[10] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*. North-Holland Publishing Company, Amsterdam, New York, Oxford (1978).

[11] P. Clément, Approximation by finite elements using local regularization. *RAIRO Anal. Numér.* **8** (1975) 77–84.

[12] M. Fortin, R. Guénette, and R. Pierre, Numerical analysis of the modified EVSS method. *Comput. Methods Appl. Mech. Engrg.* **143** (1997) 79–95.

[13] M. Fortin and R. Pierre, On the convergence of the mixed method of Crochet and Marchal for viscoelastic flows. *Comput. Methods Appl. Mech. Engrg.* **73** (1989) 341–350.

[14] L. Franca, S. Frey, and T.J.R. Hughes, Stabilized finite element methods: Application to the advective-diffusive model. *Comput. Methods Appl. Mech. Engrg.* **95** (1992) 253–276.

[15] L. Franca and R. Stenberg, Error analysis of some GLS methods for elasticity equations. *SIAM J. Numer. Anal.* **28** (1991) 1680–1697.

[16] X. Gallez, P. Halin, G. Lielens, R. Keunings, and V. Legat, The adaptative Lagrangian particle method for macroscopic and micro-macro computations of time-dependent viscoelastic flows. *Comput. Methods Appl. Mech. Engrg.* **180** (199) 345–364.

[17] V. Girault and L.R. Scott, Analysis of a 2nd grade-two fluid model with a tangential boundary condition. *J. Math. Pures Appl.* **78** (1999) 981–1011.

[18] P. Grisvard, *Elliptic Problems in Non Smooth Domains*. Pitman, Boston (1985).

[19] C. Guillopé and J.-C. Saut, Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.* **15** (1990) 849–869.

[20] M.A. Hulsen, A.P.G. van Heel, and B.H.A.A. van den Brule, Simulation of viscoelastic clows using Brownian configuration Fields. *J. Non-Newtonian Fluid Mech.* **70** (1997) 79–101.

[21] K. Najib and D. Sandri, On a decoupled algorithm for solving a finite element problem for the approximation of viscoelastic fluid flow. *Numer. Math.* **72** (1995) 223–238.

[22] L.M. Quinzani, R.C. Armstrong, and R.A. Brown, Birefringence and Laser-Doppler velocimetry studies of viscoelastic flow through a planar contraction. *J. Non-Newtonian Fluid Mech.* **52** (1994) 1–36.

[23] M. Renardy, Existence of slow steady flows of viscoelastic fluids with differential constitutive equations. *Z. Angew. Math. Mech.* **65** (1985) 449–451.

[24] V. Ruas, Finite element methods for the three-field stokes system. *RAIRO Modél. Math. Anal. Numér.* **30** (1996) 489–525.

[25] D. Sandri, Analysis of a three-fields approximation of the stokes problem. *RAIRO Modél. Math. Anal. Numér.* **27** (1993) 817–841.

[26] A. Sequeira and M. Baia, A finite element approximation for the steady solution of a second-grade fluid model. *J. Comput. Appl. Math.* **111** (1999) 281–295.

[27] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*. North-Holland Publishing Company, Amsterdam, New York, Oxford (1984).

[28] R. Verfürth, *A posteriori* error estimators for the Stokes equations. *Numer. Math.* **55** (1989) 309–325.